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Approach of a vortex pair to a rigid free surface in viscous fluid

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The motion in a viscous incompressible fluid of a vortex pair toward a rigid plane wall on which slip is allowed is considered. It is shown that the centroids of vorticity do not approach the wall monotonically, and there is some rebound at a rate depending upon the viscosity and initial separation of the vortices.

Saffman[1] considered the perpendicular approach of a symmetrical vortex pair of equal and opposite two-dimensional vortices toward a plane wall in inviscid incompressible fluid. It was shown that the vortices must approach the wall monotonically in the absence of viscous effects. Suppose the wall is $y = 0$ and the vortices are in the first and second quadrants. The vorticity $\omega(x,y,t)$ is antisymmetrical about the line of symmetry $x = 0$, i.e.,

$$\omega(x,y,t) = -\omega(-x,y,t)\,. \tag{1}$$

Let Γ and $\bar{y}$, both positive, be the total strength and height of the centroid above the wall of the vorticity in the first quadrant as follows:

$$\Gamma = \int_0^\infty \int_0^\infty \omega\, dx\, dy\,, \tag{2}$$

$$\bar{y} = \frac{1}{\Gamma}\int_0^\infty \int_0^\infty \omega y\, dx\, dy\,. \tag{3}$$

In the absence of viscosity,

$$\frac{d\Gamma}{dt} = 0\,, \tag{4}$$

and it was shown that

$$\Gamma\frac{d\bar{y}}{dt} = -\frac{1}{2}\int_0^\infty v(0,y)^2\, dy\,, \tag{5}$$

where $v(x,y)$ is the y component of velocity. Thus $d\bar{y}/dt < 0$ for all time and the vortex pair approaches the wall monotonically.

It is known[2] that this result is false in a viscous fluid with no slip at the wall. A boundary layer is induced, which separates, injecting vorticity into the fluid that causes the approaching vortex pair to bounce back from the wall in a manner that depends only weakly on the viscosity. There is a question of whether the vortex pair rebounds in a viscous fluid if the surface is free, i.e., the fluid can slip but the tangential stress is zero. This describes a vortex pair incident vertically upward on an air–water interface. In the limit of zero Froude number, the surface can be modeled by a rigid plane. The purpose of this Brief Communication is to show theoretically that in this limit there is some rebound and that the vorticity centroid does not approach the wall monotonically, even though no boundary layers are created, and confirm the numerical evidence for rebound, e.g., Peace and Riley.[3] (The effect of surface waves at finite Froude number is beyond the scope of this work.)

When the wall is rigid but stress free, the boundary conditions on $y = 0$ are

$$v = 0, \quad \frac{\partial u}{\partial y} = 0\,, \tag{6}$$

where u is the x component of velocity. Slip at the wall, i.e., $u \neq 0$, is allowed. From symmetry, we have on $x = 0$,

$$u = 0, \quad \frac{\partial v}{\partial x} = 0\,. \tag{7}$$

It follows from (6) and (7) that[3]

$$\omega = 0, \quad \text{on } x = 0 \text{ and } y = 0\,. \tag{8}$$

Consider I_x, the horizontal component of the hydrodynamic impulse of the vorticity in the first quadrant,

$$I_x = \Gamma\bar{y} = \int_0^\infty \int_0^\infty \omega y\, dx\, dy\,. \tag{9}$$

The vorticity evolves according to the equation

$$\frac{\partial\omega}{\partial t} = -u\frac{\partial\omega}{\partial x} - v\frac{\partial\omega}{\partial x} + \nu\nabla^2\omega\,. \tag{10}$$

The fluid is supposed incompressible so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{11}$$

On differentiating (9), we have, using (10) and (11) and the boundary conditions on u, v, and ω,

$$\begin{aligned}\Gamma\frac{d\bar{y}}{dt} + \bar{y}\frac{d\Gamma}{dt} &= \iint\left[-y\left(u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y}\right) + \nu y\nabla^2\omega\right]dx\,dy \\ &= \iint\left(-\frac{\partial}{\partial x}(yu\omega) - \frac{\partial}{\partial y}(yv\omega) + v\omega + \nu y\nabla^2\omega\right)dx\,dy \\ &= \iint\left(v\omega + \nu y\frac{\partial^2\omega}{\partial x^2} + \nu y\frac{\partial^2\omega}{\partial y^2}\right)dx\,dy\,, \end{aligned} \tag{12}$$

where the double integrals are over the first quadrant. It is assumed that ω vanishes exponentially at infinity, so that there are no contributions from boundary integrals at infinity.

Consider now, separately, the three terms in the last line of (12). First,

$$\iint v\omega \, dx\, dy$$

$$= \iint \left(\frac{\partial}{\partial x} \frac{1}{2} v^2 - \frac{\partial}{\partial y} (uv) - \frac{\partial}{\partial x} \frac{1}{2} u^2 \right) dx\, dy$$

$$= -\frac{1}{2} \int_0^\infty v_0^2 \, dy \,, \tag{13}$$

where $v_0 = v(0, y)$.

Second,

$$\iint \nu y \frac{\partial^2 \omega}{\partial x^2} dx\, dy = -\nu \int_0^\infty y \frac{\partial \omega}{\partial x}\bigg|_{x=0} dy < 0 \,, \tag{14}$$

since vorticity must diffuse out of the first quadrant and there cannot be a local minimum so that $\partial\omega/\partial x > 0$ on $x = 0$, $y > 0$.

Third

$$\iint \nu y \frac{\partial^2 \omega}{\partial y^2} dx\, dy = \nu \iint \left[\frac{\partial}{\partial y}\left(y \frac{\partial \omega}{\partial y} \right) - \frac{\partial \omega}{\partial y} \right] dx\, dy = 0 \tag{15}$$

on integrating with respect to y. Thus

$$\frac{d}{dt}(\Gamma \bar{y}) = -\frac{1}{2} \int_0^\infty \left(v^2 + 2\nu y \frac{\partial \omega}{\partial x} \right)_{x=0} dy < 0 \,, \tag{16}$$

and the vertical component of the first moment of the vorticity distribution decreases monotonically.

However, it does not follow from (16) that $d\bar{y}/dt < 0$, because

$$\frac{d\Gamma}{dt} = -\nu \int_0^\infty \frac{\partial \omega}{\partial x}\bigg|_{x=0} dy - \nu \int_0^\infty \frac{\partial \omega}{\partial y}\bigg|_{y=0} dx < 0 \,, \tag{17}$$

since the integrands are positive. Hence $d\bar{y}/dt$ is the sum of terms of opposite sign, and there is no reason why the height of the centroid above the wall should decrease monotonically.

Indeed, we expect that $\bar{y}$ will asymptotically increase. To see this, note that in a way similar to that by which (16) was derived, we can show that

$$\frac{d}{dt}(\Gamma \bar{x}) = \frac{1}{2} \int_0^\infty \left(u^2 + 2\nu x \frac{\partial \omega}{\partial y} \right)_{y=0} dx > 0 \,. \tag{18}$$

Hence the vorticity moves asymptotically away from the centerline $x = 0$, and the right-hand side of (16) will tend to zero. It follows that as $t \to \infty$,

$$\Gamma \bar{y} \sim \text{const} \tag{19}$$

and the continuing decrease in Γ due to vorticity diffusion through the wall will lead to an eventual increase in $\bar{y}$ and an apparent rebound from the wall in a way which will depend upon the value of the viscosity and the initial distribution of vorticity. Note that the inviscid flow field satisfies the boundary conditions (6) and (7), so that no boundary layers appear.

An estimate of the trajectory after rebound is obtained by making the rough approximations

$$\Gamma \bar{y} = K, \quad \frac{d\Gamma}{dt} = -\frac{\nu \Gamma}{\bar{y}^2}, \quad \frac{d\bar{x}}{dt} = \frac{\Gamma}{4\pi \bar{y}} \,, \tag{20}$$

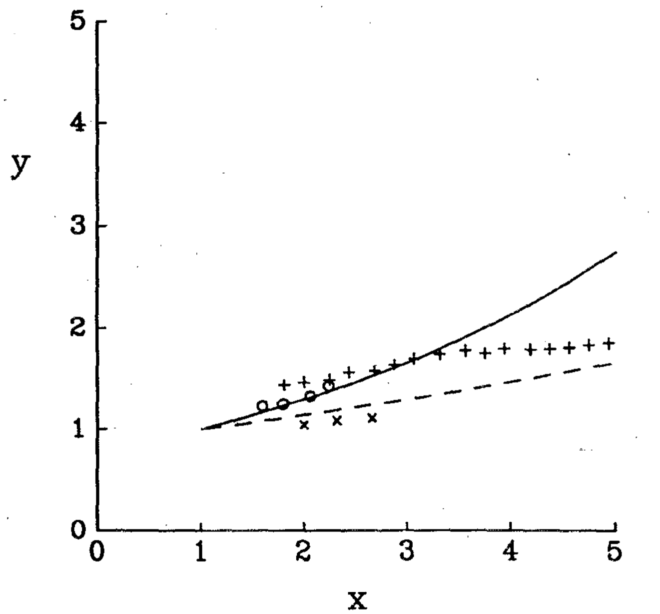

FIG. 1. Rebounding trajectories given by asymptotic formulas of Eq. (21); —, Re = 50, - - -, Re = 100. Peace and Riley: ○, Re = 50, ×, Re = 100. Ohring and Lugt: +, Re = 50, Fr = 0.356.

where $K = \Gamma_0 \bar{x}_0$, and Γ_0, $\bar{x}_0$ are the initial values of Γ and $\bar{x}$. These give

$$\bar{y} \sim \bar{x}_0 \left(1 + \frac{2\nu t}{\bar{x}_0^2} \right)^{1/2}, \quad \bar{x} \sim \bar{x}_0 + \frac{\Gamma_0 \bar{x}_0}{8\pi\nu} \log\left(1 + \frac{2\nu t}{\bar{x}_0^2} \right). \tag{21}$$

The curves given by Eq. (21) are plotted in Fig. 1 for $t > 0$, corresponding to rebound with the minimum distance from the wall assumed to occur at $\bar{y} = \bar{x} = \bar{x}_0$, for two Reynolds numbers $\text{Re} = \Gamma_0/\nu = 50$ and 100. Distances have been scaled so that $\bar{x}_0 = 1$. Also shown on the curve are numerical data given by Peace and Riley[3] for the same Reynolds numbers taken from Fig. 4 of their paper. The agreement is remarkably good. Results from a numerical study[4] allowing for surface deformation at a Reynolds number of 50 and a Froude number of 0.356 are also plotted. Note that the numerical data plot the point of vorticity maximum, which does *not* necessarily coincide with the vorticity centroid, and is plotted for the vortices moving away from the wall.

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