Tunneling density of states, pair correlation, and Josephson current in spin-incoherent Luttinger-liquid/superconductor hybrid systems

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(Received 24 March 2008; published 18 April 2008)

We study a hybrid system consisting of a spin-incoherent Luttinger liquid adjoined at one or both ends to a superconductor. We find that the tunneling density of states diverges at low energies and exhibits a universal frequency dependence independent of the strength of the interactions in the system. We show that in spite of exponentially decaying pair correlations with distance into the spin-incoherent Luttinger liquid, the Josephson current remains robust. Compared to the zero temperature Luttinger-liquid case, there is a factor of 2 reduction in the critical current and a halving of the period in the phase difference between the superconductors. We hope these results motivate a class of experiments in the spin-incoherent regime of one-dimensional systems.

DOI: 10.1103/PhysRevB.77.140505 PACS number(s): 74.50.+r, 71.10.Pm, 73.21.--b, 73.23.--b

The low energy behavior of gapless one-dimensional systems, which may be realized by electrons in quantum wires or nanotubes, is described by the Luttinger-liquid (LL) theory.\textsuperscript{1} The elementary excitations of the interacting theory are decoupled bosonic charge and spin modes that propagate with different velocities, a phenomenon known as spin-charge separation that has already been experimentally observed.\textsuperscript{2} Strong repulsive interactions tend to suppress the spin velocity while enhancing the charge one, thereby accentuating the spin-charge separation. For strong enough interactions, a window of energy opens at finite temperature where the spin sector consists of thermally excited (randomized) states while the charge sector is essentially at zero temperature. A one-dimensional (1D) system in this regime is known as a spin-incoherent Luttinger liquid (SILL).\textsuperscript{3}

While the theory of the SILL has rapidly progressed,\textsuperscript{4} the challenge of reaching the desired window of energy has slowed experiment. To date, the best experimental evidence has appeared in momentum resolved tunneling on gated quantum wires.\textsuperscript{5} Unfortunately, the analysis of the experiments is somewhat involved\textsuperscript{3} and it has become highly desirable to propose (and carry out) experiments to probe the SILL. Recent experimental progress has made it possible to fabricate devices consisting of nanotubes\textsuperscript{6} or quantum wires\textsuperscript{7} between two superconductors (SCs). Through gating to modulate the electron density and interaction strength, such devices open the possibility of studying the SILL in SILL-SC hybrid structures.

In this work, we address the theory of SILL-SC hybrid structures. Various aspects of LL-SC structures have been discussed in the literature, including the tunneling density of states,\textsuperscript{8} pair correlations,\textsuperscript{9} and Josephson current.\textsuperscript{9,10} Compared to the LL case, the SILL-SC structures exhibit a number of remarkable features. In particular, we find that the tunneling density of states of a SILL contacted to a SC diverges at low frequencies with a universal form independent of the strength of the interactions in the system. By contrast, for the isolated LL, SILL, and the LL-SC system, the energy dependence of the tunneling density of states depends on the strength of the interactions. We also compute the decay of the pair correlations into the SILL from the SC and, as might be expected from the highly excited spin states, they decay exponentially fast with a length scale set by the interparticle spacing. However, the Josephson current that results from the coherent propagation of the Cooper pairs through a finite length SILL remains robust. Moreover, the critical current shows the same scaling with length as in the case of a LL but its value is reduced by a factor of 2. As a function of phase between the two superconductors, the period of the Josephson current is halved. Both the tunneling density of states and the Josephson current should be experimentally accessible. Observation of the results described here would be a smoking gun signature of the SILL in these hybrid structures.

A schematic of our model is shown in Fig. 1. We assume that the SILL is adiabatically connected to the SC so that the scattering at the interface is in the Andreev limit. In all our calculations, we assume that $k_B T \ll \Delta$, where $k_B$ is Boltzmann’s constant, $T$ is the temperature, and $\Delta$ is the magnitude of the superconducting gap. Bosonization procedures have been developed for LL-SC hybrid systems\textsuperscript{9,11} valid for energies $\omega \ll \Delta \ll E_F$ that are also small compared to the characteristic spin and charge energies. At the lowest energy scales, the Hamiltonian is given by $H=H_p+H_g$, where

\begin{align*}
H_p &= \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \frac{1}{2} \sum_{\mathbf{k}_{1}\mathbf{k}_{2}} \Delta \langle \phi_{1} + \phi_{2} \rangle_{\mathbf{k}_{1}\mathbf{k}_{2}} c_{\mathbf{k}_{1}\sigma}^\dagger c_{\mathbf{k}_{2}\sigma}^\dagger c_{\mathbf{k}_{2}(-\mathbf{k}_{1})\sigma} c_{\mathbf{k}_{1}(-\mathbf{k}_{2})\sigma}, \\
H_g &= \sum_{\mathbf{k}\sigma} \sum_{\sigma'} \xi_{\mathbf{k}} \sum_{\mathbf{q}} \langle \phi_{1} + \phi_{2} \rangle_{\mathbf{k}\mathbf{q}} \langle \phi_{1} + \phi_{2} \rangle_{\mathbf{k}+\mathbf{q}\sigma'}^\dagger c_{\mathbf{k}+\mathbf{q}\sigma}^\dagger c_{\mathbf{k}\sigma} c_{\mathbf{k}+\mathbf{q}\sigma'}^\dagger c_{\mathbf{k}\sigma'}.
\end{align*}

FIG. 1. (Color online) Schematic of the model we study. A spin-incoherent Luttinger liquid (SILL) of length $L$ is adiabatically connected to two superconductors (SCs) with phase $\chi_1$ and $\chi_2$ and the same superconducting gap $\Delta$. The tunneling density of states with a distance $x$ from the end of the SILL can be probed with electron tunneling from a metallic lead such as scanning tunneling microscope (STM) tip. A Josephson current flows through the SILL when $\chi = \chi_1 - \chi_2 \neq 0$. 

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\[ H_\rho = v_\rho \int \frac{dx}{2\pi} \left[ \frac{1}{g_\rho} \left[ \frac{\partial}{\partial x} \theta_\rho(x) \right]^2 + g_{\rho} \left[ \frac{\partial}{\partial x} \phi_\rho(x) \right]^2 \right], \]  
and \( H_\rho \) has the same form only with \( \rho \) replaced by \( \sigma \). Here, \( \theta_\rho(x) \) and \( \phi_\rho(x) \) are bosonic fields representing charge and current density fluctuations, \( g_\rho \) measures the strength of the interactions \( [g_\rho = 1 \text{ for noninteracting systems, } g_\rho < 1 \text{ for repulsive interactions, and } g_\sigma = 1 \text{ for SU(2) invariant spin interactions}] \), and \( v_\rho \) is the velocity of the charge modes. The mode expansions of the bosonic fields for a LL between two superconductors separated by a distance \( L \) are\(^9\)

\[
\theta_\rho(x) = \phi_\rho(x) = \phi_\rho^{(0)} + \sum_{q > 0} q_\rho \cos(q_\rho x)(b_{q\rho}^\dagger + b_{q\rho}^\dagger),
\]

\[
\phi_\rho(x) = \frac{\pi}{\sqrt{2}} M^\frac{x}{L} + \sum_{q > 0} q_\rho \sin(q_\rho x)(b_{q\rho}^\dagger - b_{q\rho}),
\]

where \( q_\rho = \sqrt{\frac{2\pi}{L}} e^{-\alpha q} \) with \( \alpha \) as a small distance cutoff on the scale of the interparticle spacing, \([b_{q\rho}, b_{q'\rho}^\dagger] = \delta_{q, q'} \delta_{q, q'}^\dagger\], for \( \beta = \rho, \sigma \), and \( \chi \) is the phase difference of the order parameter of the two superconductors (assumed to have the same \( \Delta \)). The integers \( J' \) and \( M \) are the topological (zero mode) numbers\(^12\) that are related to excess charge and spin densities. They obey the constraint \( J' + M = \text{even} \). Substituting the expansion Eq. (2) into Eq. (1), leads to

\[
H = \frac{\pi}{4L} \left[ v_\rho \rho \left( J' + \frac{\chi}{\pi} \right)^2 + \frac{\alpha M^2}{g_\rho} \right] + \sum_{q > 0} q_\rho (v_\rho \rho \rho \rho + v_\rho \rho \rho \rho) - \sum_{q > 0} q_\rho (v_\rho \rho \rho \rho \rho + v_\rho \rho \rho \rho \rho),
\]

While the form of the energy contribution from the nonzero modes \( (q > 0) \) is valid only for low energies relative to the spin and charge energy, the zero mode contribution is valid at all energies, in particular, in the spin-incoherent regime defined by the condition \( E_{\text{spin}} \ll k_B T \ll E_{\text{charge}} \), where \( E_{\text{spin/charge}} = \frac{\hbar^2}{2L} \) for zero mode properties and \( E_{\text{spin/charge}} = \frac{\hbar^2}{c_0} \) for fluctuating quantities such as the Green’s function and pair correlations. In terms of fields (2), the bosonized electron annihilation operator is \( \psi_\rho(x, t) = -\frac{1}{\sqrt{2}} e^{i \theta_\rho(x, t)} \phi_\rho(x, t) \) and \( \phi_\rho \) is \( \phi_\rho(x, t) = 0 \). The charge fields \( \theta_\rho = (\theta_\rho + \theta_\rho^\dagger)/\sqrt{2} \) and the spin fields \( \sigma_\rho = (\sigma_\rho + \sigma_\rho^\dagger)/\sqrt{2} \) with identical definitions for \( \phi_\rho \) and \( \phi_\rho \).

**Single-particle Green’s function.** We compute the single-particle Green’s function

\[
G_\rho(x, x'; \tau, \tau') = \langle T_\rho \theta_\rho(x, \tau) \phi_\rho^\dagger(x', \tau') \rangle
\]

for a SILL connected to a single SC. Fiete and Balents\(^13\) have developed a simple but powerful method to evaluate such expectation values and we employ it here. Let us first consider the trace over the spin sector, which we assume consists of highly thermalized random spins. The dominant contribution comes from the terms where there are effectively no exchanges of particles for all the particles between \((\chi', \tau')\) and \((x, \tau)\). For the single-particle Green’s function, this implies that all spins have the same orientation between \((\chi', \tau')\) and \((x, \tau)\), and this occurs with a probability of \( 2^{-|N(x, \tau; x', \tau')|} \), where \( N(x, \tau; x', \tau') \) is the number of electrons between the two points. A factor \( -|N(x, \tau; x', \tau')| \) arises from the permutation of the propagating electron with the other electrons in the SILL. The general result is\(^13\)

\[
G_\rho(x, x'; \tau, \tau') = \sum_{m = -\infty}^{\infty} \langle \delta m - N(x, \tau; x', \tau') \rangle (\tau - \tau')^{2|m|} - \ln(2)
\]

\[
\times \exp\left( -\pi \frac{|\langle m \rangle - \langle m \rangle^2|}{\langle \Theta_\rho^2 \rangle} \right) \exp\left( -\pi \frac{|\langle m \rangle - \langle m \rangle^2|}{\langle \Theta_\rho^2 \rangle} \right)
\]

where \( \langle \Theta_\rho \rangle = \frac{1}{\alpha} \left[ 2 \ln(\frac{\alpha}{\tau}) + \ln\left( \frac{v_\rho \tau^2}{\alpha} + 2\pi \right) \right] \) and \( \langle \Theta_\rho^2 \rangle = \frac{\alpha}{\tau} \left[ 4 \ln(\frac{\alpha}{\tau}) - 2 \ln\left( \frac{\alpha}{\tau} \right) - \ln\left( \frac{v_\rho \tau^2}{\alpha} + 2\pi \right) \right] \). The resulting Green’s function (valid for \( x, v_\rho \tau \gg \alpha \)) is

\[
G_\rho(x, x; \tau, \tau) = \frac{1}{\sqrt{g_\rho}} \left[ \ln\left( \frac{(v_\rho \tau)^2}{\alpha^2} \right) - \frac{1}{2} \ln\left( \frac{(v_\rho \tau)^2}{\alpha^2} + \frac{v_\rho \tau^2}{2\pi} \right) \right]
\]

\[
\times \left( \frac{\alpha^2}{(2\pi)^2} \right)^{1/8} \exp\left( -\frac{(v_\rho \tau^2)^2}{\alpha^2} \right) \cdot \left( \frac{v_\rho \tau^2}{\alpha^2} \right)^{-1/8} \exp\left( -\frac{(v_\rho \tau^2)^2}{\alpha^2} \right),
\]

where we have kept only the dominant \( m = 0 \) term.\(^13\)

**Tunneling density of states.** The local tunneling density of states, \( A(x, \omega) \), can be computed by Fourier transforming the Green’s function. For \( v_\rho \tau \gg x \), the Green’s function \( G_\rho(x, x; \tau, 0) \) is

\[
\sim \left( \frac{\alpha^2}{(2\pi)^2} \right)^{1/8} \exp\left( -\frac{(v_\rho \tau^2)^2}{\alpha^2} \right) \cdot \left( \frac{v_\rho \tau^2}{\alpha^2} \right)^{-1/8} \exp\left( -\frac{(v_\rho \tau^2)^2}{\alpha^2} \right),
\]

which implies
TUNNELING DENSITY OF STATES, PAIR...
$E_c/E_p$. In the limit $E_c/E_p \ll 1$, this zero moves to $\pm \pi/2$. As Fig. 2 shows, already for $E_c/E_p=0.1$, the limit $E_c/E_p \ll 1$ is approached. The condition $E_c/E_p \ll 1$ is a prerequisite for SILL physics, so $J(\chi)$ can establish whether the 1D system has large enough spin-charge separation for SILL physics. In the extreme limit $E_p \gg k_B T \gg E_c \to 0$, the exact form of the spin-incoherent Josephson current can be analytically obtained,

$$
\frac{J(\chi)}{(ev_g r_p L)} = \begin{cases} 
\frac{\chi}{\pi} + 1, & -\pi < \chi < -\frac{\pi}{2} \\
\frac{\chi}{\pi}, & -\frac{\pi}{2} < \chi < \frac{\pi}{2} \\
\frac{\chi}{\pi} - 1, & \frac{\pi}{2} < \chi < \pi.
\end{cases}
$$ (11)

Compared to the LL case at $T=0$, both the period and the critical current are halved in the SILL regime. The latter is reminiscent of Matveev’s result for the conductance of a quantum wire in the SILL regime adiabatically connected to the Fermi liquid leads, where the conductance reduces to $e^2/h$ per mode rather than the $T=0$ universal value of $2e^2/h$. Thus, the observation of a Josephson current that follows Eq. (11) is a clear indication of SILL behavior. It is remarkable that in spite of the exponentially decaying pair correlations [Eq. (9)], the Josephson current remains robust, only with a factor of 2 reduction in the critical current compared to the $T=0$ LL result. Physically, this is because the SC phase difference $\chi$ couples only to the charge degrees of freedom which remain coherent in the SILL. Note also that the critical current in the SILL regime scales as $\sim 1/L$ as general arguments require.9

In summary, we have determined the properties expected for a SILL adiabatically connected to one or two superconductors. The tunneling density of states exhibits a universal frequency dependence independent of interactions in the system and the Josephson current has a sawtooth form with a factor of 2 reduction in the critical current and a halving of the period. If the contacts of the SILL are nonideal (nonadiabatic), the adiabatic regime may still be obtained at low energies as impurities are irrelevant in the SILL for $g_p > 1/2$.16

In this sense, the adiabatic model is even more relevant for the SILL than for the LL.

We are grateful to Oleg Starykh for helpful discussions and for financial support from the NSF under Grants No. PHY05-51164, and No. DMR-0606489, the Lee A. DuBridge Foundation, and the Welch Foundation.

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