ON THE COEFFICIENTS AND THE GROWTH OF GAP POWER SERIES*

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1. Introduction and outline of general method.

1.1. Introduction. Assume that we are given an entire function $f$ with a gap power series expansion, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{with} \quad a_n = 0 \quad \text{for} \quad n \neq \lambda_k, \quad k = 1, 2, \ldots,$$

where $\{\lambda_k\}$ is a certain sequence of natural numbers, $0 < \lambda_1 < \lambda_2 < \cdots$. It is well-known that under suitable conditions on $\{\lambda_k\}$ the function $f$ has, roughly speaking, about the same rate of growth as $z^{\alpha}$ in different directions. For example [11, p. 622], if $\{\lambda_k\}$ has density $D \rightarrow 0$, then in every angle $\alpha \leq \arg z \leq \beta$ with $\beta - \alpha > 2\pi D$, $f$ will be of the same order and type as in the full plane. In particular, if the power series has Fabry gaps:

$$D = 0, \quad \text{or equivalently} \quad \frac{\lambda_k}{k} \rightarrow \infty, \quad k \rightarrow \infty,$$

we can conclude from order and type in every angle $\alpha \leq \arg z \leq \beta$, $\beta > \alpha$, to the order and type of $f$ in the full plane.

In this paper we are interested in the limiting case, in which not the behavior of $f$ in an angle, but only on a radius, for example for $z = x > 0$, is known. Fabry gaps no longer suffice to get information about the growth of $m(r) = \max_{|z| \leq r} |f(z)|$, since already Pólya pointed out [11, p. 636] that there exist entire functions with Fabry gaps (even $\lambda_k/k \geq \log \log k$), which are bounded for $x > 0$.

Instead, it will be seen that the slightly stronger gap condition

$$a_n = 0 \quad \text{for} \quad n \neq \lambda_k, \quad k = 1, 2, \ldots, \quad \text{with} \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty,$$

will be the proper one to conclude from the growth of $f$ on $z = x > 0$ to a similar growth of $m(r)$. One particular case is known:

**Theorem** [9, p. 286]. *If the entire function (1.1) satisfies the gap condition (1.2), and $f(z)$ is bounded for $z = x > 0$, then $f$ is a constant; in fact, $f = 0$ since $a_0 = 0$.*

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One of our results will be:

**Theorem 6.** If the entire function (1.1) satisfies the gap condition (1.2) and if \( f(x) = O(e^{\alpha x}) \), \( x \to +\infty \), \( \alpha > 0 \), then \( f \) is at most of order \( \alpha \) and type 1.

A theorem of this type was recently proved by the author [5]. There the growth of \( f \) on an arbitrary Jordan arc from 0 to \( \infty \) (instead of \( z = x > 0 \)) was prescribed, but the gap condition (1.2) had to be strengthened to

\[
\frac{\lambda_k}{k (\log k)^{1+\epsilon}} \to \infty \quad k \to \infty, \text{ some } \epsilon > 0,
\]

in order to apply the Wiman-Valiron theory. The case \( \alpha = 1 \) plays a decisive role in the proof of the unrestricted high indices theorem for Borel summability and motivated our investigations.

A main step in our proof will be the derivation of a representation formula for the coefficients \( a_n \) of \( f \) (see (1.11)), and the radial growth of \( f \) will be reflected in an estimate of \( |a_n| \), which in turn can be used to estimate \( m(r) \). Such estimates of \( |a_n| \) are typical for high indices theorems for power series, but our complex variable method does not give such fine estimates as \( a_n = O(1) \) for a Hadamard gap power series in \( |z| < 1 \) which is bounded on \((0, 1)\). We quote one of our results in this direction.

**Theorem 11.** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is regular in \( |z| < 1 \) and has Hadamard gaps, i.e.,

\[
a_n = 0 \text{ for } n \neq \lambda_k, \quad \text{where } \frac{\lambda_{k+1}}{\lambda_k} \geq \theta > 1,
\]

then each of the conditions

\[
f(x) = s + O((1 - x)^\alpha), x \to 1 - 0, \alpha > 0, \text{ or } f' \in L_p(0, 1), p > 1,
\]

implies \( \sum_{n=0}^{\infty} |a_n|^\epsilon < \infty \) for every \( \epsilon > 0 \).

### 1.2. Lemma on functions of exponential type.

The following lemma of Phragmén-Lindelöf type will be used.

**Lemma 1** [12, p. 36], [2, p. 82]. Let \( f \) be regular and of exponential type in \( \Re z \geq 0, |f(z)| \leq M \) for \( z = iy \), and

\[
h_f(0) = \limsup_{x \to +\infty} \frac{\log |f(x)|}{x} \leq c.
\]

Then

\[
|f(z)| \leq Me^{cx}, \quad z = x + iy, \quad x \geq 0.
\]

### 1.3. Outline of general method.

Assume that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) converges for \( |z| < T_0, 0 < T_0 \leq \infty \), and that \( a_0 = 0 \). For any fixed \( T \) with \( 0 < T < T_0 \) we shall study the auxiliary function

\[
H(z; T) = \int_0^T f(t)t^{z-1} \, dt, \quad z = x + iy.
\]
This transformation (with $T = \infty$) has been employed by Edrei [4, p. 121] in the case that $f(t)$ was bounded on $t > 0$, but the corresponding transformation for Dirichlet series (see (4.3)) goes back to V. Bernstein [1, p. 111] who used it for different purposes. Since $f(t)/t$ is regular on $\langle 0, T \rangle$, $H(z; T)$ will be defined for Re $z < 1$, and will represent there a regular function.

On the imaginary axis we have

\[(1.5) \quad |H(z; T)| \leq \int_0^\infty \frac{|f(t)|}{t} \, dt = M(T), \quad z = iy.\]

In order to obtain the analytic continuation of $H(z; T)$ beyond Re $z = 1$, we write for $z$ in Re $z < 0$ (so that $|e^{-z}|$ is bounded on $\langle 0, T \rangle$)

\[
H(z; T) = \int_0^T \sum_{n=1}^\infty a_n t^{n-z-1} \, dt = \sum_{n=1}^\infty a_n \int_0^T t^{n-z-1} \, dt = \sum_{n=1}^\infty \frac{a_n T^{n-z}}{n-z},
\]

so that $H(z; T)$ has the alternate representation in Re $z < 0$:

\[(1.6) \quad H(z; T) = -T^{-z} \cdot \sum_{n=1}^\infty \frac{a_n T^n}{z-n}.\]

However, since $D(T) = \sum_{n=1}^\infty |a_n| T^n < \infty$, the series in (1.6) converges uniformly for all $z$ with $|z - n| \geq \eta > 0$, i.e., $H(z; T)$ is a meromorphic function with possible simple poles at $z = n$, at which $H(z; T)$ has residues $-a_n$, $n = 1, 2, \ldots$.

As for the growth of $H$, we immediately obtain from (1.6)

\[(1.7) \quad |H(z; T)| \leq T^{-x} \cdot \frac{D(T)}{\eta} \quad \text{if} \quad |z - n| \geq \eta > 0 \quad \text{for} \quad n = 1, 2, \ldots .\]

From now on we shall assume that

\[a_n = 0 \quad \text{for} \quad n \neq \lambda_k, \quad \text{with} \quad \sum_{k=1}^\infty \frac{1}{\lambda_k} < \infty.\]

We form the Blaschke product

\[(1.8) \quad B(z) = \prod_{k=1}^\infty \frac{\lambda_k - z}{\lambda_k + z} = \prod_{k=1}^\infty \left(1 - \frac{2z}{\lambda_k + z}\right),\]

which converges for every $z \neq -\lambda_k$, $k = 1, 2, \ldots$. $B(z)$ has simple zeros at $z = \lambda_k$, simple poles at $z = -\lambda_k$, and

\[(1.9) \quad |B(z)| \leq 1 \quad \text{for Re} \ z \geq 0 \quad \text{and} \quad |B(z)| = 1 \quad \text{for Re} \ z = 0.\]

Therefore

\[\Phi(z; T) = B(z) \cdot H(z; T)\]
is regular in \( \text{Re } z \geq 0 \), and \( |\Phi(z; T)| \leq M(T) \) for \( z = iy \). With regards to the growth of \( \Phi \) in \( \text{Re } z \geq 0 \), we get from (1.7)

\[
|\Phi(z; T)| \leq 2D(T) \cdot T^{-\varepsilon} \text{ if } |z - n| \geq \frac{1}{2}, \quad n = 1, 2, \ldots ;
\]

thus on the circle \( |z - n| = 1/2 \) we have

\[
|\Phi(z; T)| \leq 2D(T)T^{-n} \cdot \begin{cases} T^{1/2} & \text{if } T \geq 1, \\ T^{-1/2} & \text{if } T < 1, \end{cases}
\]

which by the maximum principle holds also inside that circle. This implies (1.10)

\[
|\Phi(z; T)| \leq D'(T) \cdot T^{-\varepsilon}, \quad \text{Re } z \geq 0,
\]

with \( D'(T) = 2D(T) \cdot T^{\varepsilon+1} \), depending on whether \( T \geq 1 \) or \( T < 1 \).

This estimate of \( |\Phi| \), which contains the "bad" constant \( D(T) = \sum |a_n| T^n \), can be improved by an application of Lemma 1. First, (1.10) shows that \( \Phi \) is of exponential type in \( \text{Re } z \geq 0 \), and furthermore

\[
h_\Phi(0) = \limsup_{x \to +\infty} \frac{\log |\Phi(x; T)|}{x} \leq -\log T.
\]

Lemma 1 yields therefore \( |\Phi(z; T)| \leq M(T) e^{(-\log T)z} \), \( \text{Re } z = x \geq 0 \), and for \( z = n = \lambda_m \) we obtain, in particular,

\[
|\Phi(n; T)| = |a_n| \cdot |B'(\lambda_m)| \leq M(T) \cdot T^{-n},
\]

(1.11)

\[
n = \lambda_m, \quad m = 1, 2, \ldots,
\]

valid for every \( T \) in \( 0 < T < T_0 \).

This formula is the basis of our results: The growth of \( f(x), x > 0 \), reflects in \( M(T) \), assumptions on the gap exponents \( \lambda_k \) enter into \( |B'(\lambda_m)| \), and combining both we obtain information about \( |a_n| \).

By way of an example, if \( T_0 = \infty \) and \( f(x) \) is bounded for \( x > 0 \), we have \( M(T) = O(\log T), T \to +\infty \), and we see that the right-hand side of (1.11) tends to zero for \( T \to +\infty \) and every fixed \( n > 0 \). Since \( B'(\lambda_m) \neq 0 \) we get \( a_n = 0, n > 0 \); hence \( f \) is constant; this is Macintyre's result mentioned above.

2. On the derivative of Blaschke products. Let \( \lambda = \{\lambda_k\} \) be a sequence of positive numbers, \( 0 < \lambda_1 < \lambda_2 < \cdots \), with

\[
\begin{align*}
(a) \quad & \lambda_{k+1} - \lambda_k \geq \delta > 0, \\
(b) \quad & \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty.
\end{align*}
\]

In order to estimate \( |a_n| \) by (1.11), it is necessary to obtain information about
\[(2.2) \quad \left| B'(\lambda_m) \right|^{-1} = 2\lambda_m p_m \quad \text{with} \quad p_m = \prod_{k \leq m} \left| \frac{\lambda_k + \lambda_m}{\lambda_k - \lambda_m} \right|.\]

We see that \(p_m > 1\) for all \(m\), and in §2 we shall discuss estimates of \(p = \{p_m\}\) from above for various choices of \(\lambda\).

### 2.1. The general case.

**Theorem 1.** If \(\lambda\) satisfies (2.1), we have

\[(2.3) \quad 0 < \log p_m = o(\lambda_m), \quad m \to \infty.\]

**Proof.** We write \(p_m = \Pi_1\Pi_2\Pi_3\), where \(\Pi_1\) contains the factors with \(k < m\), \(\Pi_2\) those with \(\lambda_m < \lambda_k < 2\lambda_m\), and \(\Pi_3\) those with \(\lambda_k \geq 2\lambda_m\). For \(\Pi_1\) we have \(\lambda_m - \lambda_k \geq (m - k)\delta\), \(k = 1, 2, \ldots, m - 1\), and therefore

\[\Pi_1 = \prod_{k < m} \frac{\lambda_m + \lambda_k}{\lambda_m - \lambda_k} \leq \left(\frac{2\lambda_m}{\delta}\right)^{m-1} \cdot \frac{1}{(m - 1)!} \leq \left(\frac{2\lambda_m e}{\delta(m - 1)}\right)^{m-1},\]

since \(n^n/n! \leq e^n\). This implies

\[\log \Pi_1 \leq \lambda_m \cdot \frac{m - 1}{\lambda_m} \left[ \log \frac{\lambda_m}{m - 1} + C \right] = o(\lambda_m), \quad m \to \infty,
\]

since \(\lambda_m/m \to \infty\), which is a consequence of (2.1b).

Assume \(\Pi_2\) contains \(N\) factors (if \(N = 0\), put \(\Pi_2 = 1\)). Then as above

\[\Pi_2 \leq \left(\frac{3\lambda_m}{\delta}\right)^N \cdot \frac{1}{N!} \leq \left(\frac{3\lambda_m e}{\delta N}\right)^N\]

and hence

\[\log \Pi_2 \leq \lambda_m \cdot \frac{N}{\lambda_m} \left[ \log \frac{\lambda_m}{N} + C \right] = o(\lambda_m), \quad m \to \infty,
\]

since \(m + N = o(\lambda_m + N) = o(\lambda_m)\), hence \(N = o(\lambda_m)\) or \(\lambda_m/N \to \infty\).

In \(\Pi_3\) we finally have \(\lambda_k \geq 2\lambda_m\) or \((\lambda_k - \lambda_m)^{-1} \leq 2/\lambda_k\), so that

\[\frac{\lambda_k + \lambda_m}{\lambda_k - \lambda_m} = 1 + \frac{2\lambda_m}{\lambda_k - \lambda_m} \leq 1 + 4 \frac{\lambda_m}{\lambda_k}\]

and therefore

\[\log \Pi_3 \leq \sum \log \left(1 + 4 \frac{\lambda_m}{\lambda_k}\right) < 4\lambda_m \cdot \sum_{\lambda_k \geq 2\lambda_m} \frac{1}{\lambda_k} = o(\lambda_m), \quad m \to \infty.
\]

Combining our results, we arrive at (2.3).

### 2.2. Hadamard sequences.

The sequence \(\lambda = \{\lambda_k\}\) is called a Hadamard sequence if

\[\lambda_{k+1}/\lambda_k \geq \theta, \quad k = 1, 2, \ldots, \quad \text{for some} \ \theta > 1.\]
Theorem 2. The sequence \( p = \{p_m\} \) is bounded if and only if \( \lambda \) is a Hadamard sequence. If \( \lambda_{k+1}/\lambda_k \to \infty, k \to \infty \), then \( p_m \to 1, m \to \infty \).

Proof. First let \( \lambda \) be a Hadamard sequence. We write \( p_m = \Pi_1 \Pi_2 \), where

\[
\Pi_1 = \prod_{k<m} \frac{\lambda_m + \lambda_k}{\lambda_m - \lambda_k} = \prod_{j=1}^{m-1} \frac{1 + \lambda_{m-j}/\lambda_m}{1 - \lambda_{m-j}/\lambda_m} \leq \prod_{j=1}^{m-1} \frac{1 + \theta^{-j}}{1 - \theta^{-j}},
\]

since \( \lambda_{m-j}/\lambda_m \leq \theta^{-j}, j = 1, 2, \ldots, m - 1 \). Similarly,

\[
\Pi_2 = \prod_{k>m} \frac{\lambda_k + \lambda_m}{\lambda_k - \lambda_m} = \prod_{j=1}^{\infty} \frac{1 + \lambda_m/\lambda_{m+j}}{1 - \lambda_m/\lambda_{m+j}} \leq \prod_{j=1}^{\infty} \frac{1 + \theta^{-j}}{1 - \theta^{-j}},
\]

and therefore

\[
(2.4) \quad 1 < p_m \leq C(\theta), \quad m = 1, 2, \ldots.
\]

If \( \lambda \) is not a Hadamard sequence, there exists a sequence of indices \( k \) for which \( \lambda_{k+1}/\lambda_k \to 1 \). Since every factor in \( p_m \) is greater than 1, we have for these indices

\[
p_k \geq \frac{\lambda_{k+1} + \lambda_k}{\lambda_{k+1} - \lambda_k} \geq \frac{\lambda_{k+1}/\lambda_k + 1}{\lambda_{k+1}/\lambda_k - 1} \to \infty.
\]

Now let \( \lambda_{k+1}/\lambda_k \to \infty, k \to \infty \), so that for given \( \epsilon, 0 < \epsilon \leq 1/2 \), there exists \( N = N(\epsilon) \) such that \( \lambda_{k-1}/\lambda_k \leq \epsilon, k > N \). Observing

\[
\log \frac{1 + x}{1 - x} \leq 3x \quad \text{in} \quad 0 \leq x \leq \frac{1}{2},
\]

we get for \( m > N \),

\[
\log \prod_{k>m} \frac{\lambda_k + \lambda_m}{\lambda_k - \lambda_m} = \sum_{k>m} \log \frac{1 + \lambda_m/\lambda_k}{1 - \lambda_m/\lambda_k} \leq 3 \sum_{j=1}^{\infty} \frac{\lambda_m}{\lambda_m+j} \leq 3 \frac{\epsilon}{1 - \epsilon},
\]

since \( \lambda_m/\lambda_{m+j} \leq \epsilon^j, j = 1, 2, \ldots \). On the other hand,

\[
\log \prod_{k<m} \frac{\lambda_m + \lambda_k}{\lambda_m - \lambda_k} = \sum_{k<m} \log \frac{1 + \lambda_k/\lambda_m}{1 - \lambda_k/\lambda_m} \leq 3 \sum_{k<N} \frac{\lambda_k}{\lambda_m} + 3 \sum_{N \leq k < m} \sum_{j=1}^{\infty} \frac{\lambda_k}{\lambda_{m+j}}
\]

notice that \( \lambda_k/\lambda_m \leq \lambda_{m-1}/\lambda_m \leq \epsilon \leq 1/2 \). For \( N \leq k < m \) we have

\[
\lambda_k \leq \epsilon^j \lambda_{k+j}, \quad \text{i.e.,} \quad \lambda_k \leq \epsilon^{m-k} \lambda_m,
\]

so that the last sum is equal to or less than

\[
\sum_{N \leq k < m} \epsilon^{m-k} < \frac{\epsilon}{1 - \epsilon},
\]

whereas \( \sum_{k<N} \epsilon < \epsilon \) for \( m \) large enough. This proves \( p_m \to 0, m \to \infty \).

2.3. The case \( \lambda_k = k^\alpha, \alpha > 1 \). We shall need the following result.
Lemma 2. For every \( \alpha > 1 \),

\[
J(\alpha) = \int_{0}^{\infty} \log \left| \frac{x^\alpha + 1}{x^\alpha - 1} \right| \, dx = \pi \tan \frac{\pi}{2\alpha}.
\]

**Proof.** Let \( C \) be the path consisting of

\[
0 \leq x \leq R; \quad |z| = R, \quad 0 \leq \arg z \leq \frac{\pi}{2\alpha}; \quad z = re^{i\pi/(2\alpha)}, \quad 0 \leq r \leq R,
\]

indented by circular arcs of radii \( \rho \) around \( z_0 \) and \( z_1 \). Then

\[
\int_{C} \log \frac{z^\alpha + 1}{z^\alpha - 1} \, dz = 0.
\]

Letting \( \rho \to 0, R \to \infty \), and separating real and imaginary parts, we arrive at (2.5).

Now we shall study the sequence \( p = \{p_m\} \) in the case \( \lambda_k = k^\alpha, \alpha > 1 \).

**Theorem 3.** If \( k = \lambda^\alpha, k = 1, 2, \ldots, \alpha > 1 \), then

\[
0 < \log p_m < J(\alpha)m = J(\alpha)\lambda_m^{1/\alpha}, \quad m = 1, 2, \ldots,
\]

where \( J(\alpha) \) is defined in (2.5). The constant \( J(\alpha) \) is best possible.

**Proof.** We have for all \( m = 1, 2, \ldots, \)

\[
\frac{1}{m} \log p_m = \frac{1}{m} \sum_{k \neq m} \log \left| \frac{\lambda_k/\lambda_m + 1}{\lambda_k/\lambda_m - 1} \right| = \frac{1}{m} \sum_{k \neq m} \log \left| \frac{(k/m)^\alpha + 1}{(k/m)^\alpha - 1} \right|,
\]

which can be interpreted as the lower Riemann sum for the function

\[
h(x) = \log \left| \frac{x^\alpha + 1}{x^\alpha - 1} \right|, \quad 0 \leq x < \infty,
\]

and \( \Delta x = 1/m \). This implies (2.6).

That actually \( m^{-1} \log p_m \to J(\alpha), m \to \infty \), follows from the fact that the Riemann sums are integrals over step functions \( h_m(x) \) for which

\[
h_m(x) \to h(x), m \to \infty, 0 \leq x < \infty, \text{ and } h_m(x) \leq h(x).
\]

The Lebesgue convergence theorem asserts that

\[
\frac{1}{m} \log p_m = \int_{0}^{\infty} h_m(x) \, dx \to \int_{0}^{\infty} h(x) \, dx = J(\alpha), \quad m \to \infty.
\]

**Remark.** Since \( h(x) \) is monotonic and convex in each of the intervals \((0, 1)\) and \((1, \infty)\), it is easy to see that even

\[
m^{-1} \log p_m \to J(\alpha), \quad m \to \infty.
\]

Now we ask ourselves whether (2.6) still holds if \( \lambda_k = k^\alpha \) is replaced by
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The sequence \( m^{-1} \log p_m \) may be unbounded if only \( \lambda_k \equiv k^\alpha, k \to \infty \), is assumed. We show this for \( \alpha = 2 \), defining \( \{\lambda_k\} \) from a certain index on as blocks of consecutive integers. In the \( q \)th block

\[
\lambda_{k+q} = k_q^2 \quad \text{and} \quad \lambda_{k+q+j} = k_q^2 + j, \quad 0 < j \leq j(q) = \left\lfloor \frac{k_q}{\log \log k_q} \right\rfloor.
\]

If \( k_{q+1} = k_q + j(q) + 1 \), we have \( k_{q+1}^2 > k_q^2 + j(q) + 1 \), so that the \( q \)-block and the \((q + 1)\)-block do not overlap. We now have

\[
\frac{\lambda_k}{k^2} = \frac{\lambda_{k+q+j}}{(k_q + j)^2} = \frac{k_q^2 + j}{(k_q + j)^2} = \frac{1 + j/k_q^2}{(1 + j/k_q)^2} \to 1, \quad k \to \infty,
\]

and on the other hand, for \( m = k_q \),

\[
p_m = \prod_{k \neq m} \left| \frac{\lambda_k + \lambda_m}{\lambda_k - \lambda_m} \right| \geq \prod_{j=1}^{j(q)} \frac{\lambda_{m+j} + \lambda_m}{\lambda_{m+j} - \lambda_m} \geq \frac{\lambda_k^{j(q)}}{j(q)!} = k_q^{j(q)} \cdot \frac{k_q^{j(q)}}{j(q)!} \geq k_q^{j(q)},
\]

since \( k_q \geq j(q) \). Therefore

\[
\frac{1}{m} \log p_m \geq \frac{1}{k_q} j(q) \log k_q \approx \frac{\log k_q}{\log \log k_q} \to \infty, \quad q \to \infty.
\]

2.4. The case \( \lambda_{k+1} - \lambda_k \geq \theta \lambda_k^\sigma, \quad 0 < \sigma < 1, \quad \theta > 0 \). We first remark that this condition implies

\[
\log \lambda_{k+1} - \lambda_k \geq A = A(\theta, \sigma) > 0 \quad \text{with} \quad \tau = 1 - \sigma, \quad \text{for} \quad k = 1, 2, \ldots.
\]

Because we have

\[
\lambda_{k+1} \geq \lambda_k^{\tau} \left( 1 + \frac{\theta}{\lambda_k^{1-\sigma}} \right) \geq \lambda_k^{\tau} \left( 1 + \frac{\tau \theta}{2} \lambda_k^{\sigma-1} \right), \quad k > k_0(\sigma, \theta),
\]

so that \( \lambda_{k+1}^\tau - \lambda_k^\tau \geq \tau \theta/2, \quad k > k_0(\sigma, \theta) \), which implies (2.7).

**Theorem 4.** If \( \lambda_{k+1} - \lambda_k \geq \theta \lambda_k^\sigma \) for \( 0 < \sigma < 1 \) and some \( \theta > 0 \), then

\[
0 < \log p_m < \frac{J(\tau^{-1})}{A} \cdot \lambda_m^{\tau}, \quad m = 1, 2, \ldots,
\]

where \( \tau = 1 - \sigma, A \) is the constant in (2.7), and \( J \) is the integral defined in (2.5).

**Remark.** In the special case \( \lambda_{k+1} - \lambda_k \geq \theta \sqrt{\lambda_k} \) we therefore obtain

\[
\log p_m = O(\sqrt{\lambda_m}), \quad m \to \infty.
\]

If \( \lambda_k = k^\alpha, \quad \alpha > 1 \), we have \( \lambda_{k+1} - \lambda_k \sim k^{\alpha-1} = k^{1-1/\alpha} \), and (2.8) gives

\[
\log p_m = O(\lambda_m^{1/\alpha}), \quad m \to \infty,
\]

as already seen in (2.6).

**Proof.** Notice that \( \lambda_k \geq (\text{const.}) \cdot k^{1+\sigma} \), so that \( \sum \lambda_k^{-1} < \infty \). Now
2.5. Generalization in the case of Hadamard sequences. Let $\lambda = \{\lambda_k\}$ be a Hadamard sequence, $\lambda_{k+1}/\lambda_k \geq \theta > 1$. In a generalization of our method outlined in 1.3 we shall need information about the sequence $q = \{q_m\}$ defined by

$$q_m = \prod_{k \neq m} \left| \frac{\lambda_k + \lambda_m + 2\gamma_m}{\lambda_k - \lambda_m} \right|, \quad \text{where} \quad \gamma_m = \frac{\lambda_m}{m}.$$  

**Theorem 5.** If $\lambda$ is a Hadamard sequence, $q = \{q_m\}$ is bounded.

**Proof.** Since $\lambda_m \leq \theta^{-j} \lambda_{m+j}$, $j = 1, 2, \cdots$, we have

$$\prod_{k \neq m} \leq \prod_{k > m} \frac{1 + 3\lambda_m/\lambda_k}{1 - \lambda_m/\lambda_k} \leq \prod_{j=1}^{\infty} \frac{1 + 3\theta^{-j}}{1 - \theta^{-j}}.$$  

Furthermore

$$\prod_{k < m} (1 - \lambda_k/\lambda_m) \geq \prod_{j=1}^{\infty} (1 - \theta^{-j}) \neq 0,$$

and finally

$$\prod_{k < m} \left(1 + \frac{\lambda_k + 2\gamma_m}{\lambda_m} \right) < \exp \left\{ \sum_{k < m} \frac{\lambda_k}{\lambda_m} + \frac{2\gamma_m \cdot m}{\lambda_m} \right\} < \exp \left\{ \sum_{j=1}^{\infty} \theta^{-j} + 2 \right\};$$

note $1 + x < e^x$. Therefore $\{q_m\}$ is bounded.
3. Application of general method to entire functions.

3.1. The general gap condition. First we assume that an entire function
\( f(z) = \sum a_n z^n \) is given, the coefficients of which satisfy the general gap
condition

\[
(3.1) \quad a_n = 0 \quad \text{for} \quad n \neq \lambda_k, \quad k = 1, 2, \ldots, \quad \text{where} \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty;
\]

here the \( \lambda_k \) are integers with \( 0 < \lambda_1 < \lambda_2 < \cdots \), so that in particular \( a_0 = 0 \).

Using (1.11) and (2.2) we obtain the coefficient estimate

\[
(3.2) \quad |a_n| \leq 2np_m \cdot M(T) \cdot T^{-n}, \quad n = \lambda_m, m = 1, 2, \ldots,
\]

where \( \{p_m\} \) is the sequence studied in §2, and where

\[
M(T) = \int_{0}^{T} \frac{|f(t)|}{t} \, dt, \quad 0 < T < \infty.
\]

Note that (3.2) is valid for every \( T \) in \( 0 < T < \infty \).

If \( f \) is of polynomial growth on the positive axis, \( f(x) = O(x^\alpha), x \to +\infty, \alpha > 0 \), we get \( M(T) = O(T^\alpha), T \to \infty \), so that the right-hand side of
(3.2) tends to zero for \( T \to \infty \) and every fixed \( n > \alpha \). Hence \( a_n = 0 \),
\( n > \alpha \), i.e., \( f \) is a polynomial of degree \( \leq \alpha \).

In the more interesting case of exponential growth

\[
f(x) = O(e^{x^\alpha}), \quad x \to +\infty, \quad \alpha > 0,
\]

we have

\[
\int_{1}^{T} e^{ta} \, dt = \frac{1}{\alpha} \int_{1}^{T^\alpha} u e^{u} \, du < \frac{2}{\alpha} \int_{1}^{T^\alpha/2} u e^{u} \, du < \frac{4}{\alpha T^\alpha} e^{T^\alpha}, \quad T^\alpha > 2,
\]

and therefore

\[
M(T') = O\left(\frac{e^{T^\alpha}}{T^\alpha}\right), \quad T' \to \infty.
\]

If we choose \( T > 0 \) so that \( T^\alpha = 1 + n/\alpha \), we obtain

\[
M(T)T^{-n} = O(1)e^{1+n/\alpha} \left(1 + \frac{n}{\alpha}\right)^{-(1+n/\alpha)}
\]

\[
= O(1) \cdot \left[\sqrt{n} \Gamma \left(1 + \frac{n}{\alpha}\right)\right]^{-1}
\]

by Stirling's formula, and (3.2) yields
Combining this with Theorem 1, we obtain:  

**Theorem 6.** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is an entire function satisfying the gap condition (3.1) and if \( f(x) = O(e^{x^\alpha}), x \to +\infty, \alpha > 0 \), then

\[
a_n = O(1) \frac{e^{\epsilon n}}{\Gamma\left(1 + \frac{n}{\alpha}\right)}, \quad n \to \infty,
\]

for every \( \epsilon > 0 \). In particular, \( f \) is at most of type 1 of order \( \alpha \).

The last statement is proved by writing

\[
|f(z)| \leq \sum_{n=0}^{\infty} |a_n| \cdot |z|^n = O(1) \sum_{n=0}^{\infty} \frac{y^n}{\Gamma\left(1 + \frac{n}{\alpha}\right)} \text{ with } y = e^\epsilon |z|,
\]

and observing that Mittag-Leffler's function

\[
E_\gamma(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1 + \gamma n)} = O(\exp y^{1/\gamma}), \quad y \to +\infty, \gamma > 0,
\]

(see, for instance, [6, p. 198]). Thus

\[
|f(z)| = O[\exp (e^{\epsilon |z|})], \quad |z| \to \infty,
\]

for every \( \epsilon > 0 \), and the result follows.

The case \( \alpha = 1 \) is of particular importance in the theory of Borel summability. If \( B = \sum_{n=0}^{\infty} c_n = s \), one easily finds that

\[
\sum_{n=0}^{\infty} \frac{c_n x^n}{n!} = o(e^x), \quad x \to +\infty,
\]

so that by Theorem 6, with \( \alpha = 1 \),

\[
c_n = O(e^{\epsilon n}), \quad n \to \infty,
\]

for every \( \epsilon > 0 \), provided the \( c_n \) satisfy the gap hypothesis (3.1).

---

1 Professor Korevaar pointed out to me that Theorem 6 is closely related to work on the Müntz-Szasz approximation theorem done by Clarkson-Erdős [3] and Korevaar [7]. The first authors proved ([3, pp. 6-7], see also [7, p. 756]):

\[
(*) \quad \inf \| x^\lambda - P(x) \| \geq (1 + \epsilon)^{-\lambda_n}, \quad \epsilon > 0, \quad n > n_0(\epsilon),
\]

where \( \| \| \) is the \( L_2 \) norm in \((0, 1)\), and where the infimum ranges over all \( P(x) = \sum_{k \neq n} a_k x^k \). It is easy to see that (*) furnishes another proof of Theorem 6.
Theorem 7. If \( \sum_{n=0}^{\infty} c_n \) is Borel summable and satisfies the gap condition (3.1), the power series \( \sum_{n=0}^{\infty} c_n z^n \) will converge in \( |z| < 1 \).

If the gap condition (3.1) is strengthened to

\[
(3.4) \quad c_n = 0 \text{ for } n \neq \lambda_k, \quad k = 1, 2, \ldots, \text{ where } \lambda_{k+1} - \lambda_k \geq \theta \sqrt{\lambda_k}, \quad \theta > 0,
\]

we may apply a result of Meyer-König and Zeller \([10, \text{p. 205}]\) to obtain the convergence of \( \sum_{n=0}^{\infty} c_n \) from its Borel summability. This is the **unrestricted high indices theorem for Borel summability** proved earlier by the author \([5]\).

### 3.2. Precision of the gap condition in Theorem 6

We shall now see that the gap condition (3.1) is best possible in Theorem 6.

Theorem 8. For every sequence \( \{\lambda_k\} \) of integers with

\[
(3.5) \quad 0 < \lambda_1 < \lambda_2 < \cdots \text{ and } \sum_{k=1}^{\infty} \lambda_k^{-1} = \infty
\]

there exists an entire function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with \( a_n = 0 \) for \( n \neq \lambda_k \), which is \( O(1) \) for \( z = x \to +\infty \), but of infinite order.

This result is essentially due to Macintyre. We need:

**Lemma 3.** If \( \{\lambda_k\} \) is a sequence of integers for which (3.5) holds, there exists a subsequence \( \{\lambda_{k_m}\} \) such that

\[
(3.6) \quad (a) \sum_{m=1}^{\infty} \lambda_{k_m}^{-1} = \infty, \quad (b) \lambda_{k_m}/m \to \infty, \quad m \to \infty.
\]

**Proof.** Let \( \{\epsilon_k\} \) be a monotonic null sequence for which \( \sum_{k=1}^{\infty} \epsilon_k \lambda_k^{-1} = \infty \); we may assume \( \epsilon_1 \leq 1 \). Put \( j_m = [\epsilon^{-1}_m], \) \( m = 1, 2, \ldots \), so that \( 1 \leq j_m \to \infty, \) and put

\[
k_1 = 1, \quad k_{m+1} = k_m + j_m, \quad m \geq 1, \text{ so that } k_m = 1 + \sum_{r=1}^{m-1} j_r, \quad m \geq 1.
\]

Thereby the subsequence is determined, and we claim (3.6) to be fulfilled.

First, we have for monotony reasons

\[
\epsilon_i \lambda_i^{-1} \leq \epsilon_k \lambda_k^{-1} \quad \text{in } i_m : k_m \leq k < k_{m+1},
\]

therefore

\[
j_m \epsilon_k \lambda_k^{-1} \geq \sum_{i_m}^{i_{m+1}} \epsilon_k \lambda_k^{-1}, \quad m = 1, 2, \ldots,
\]

and if we take the sum over \( m \) and observe \( j_m \leq j_{k_m} \leq \epsilon_m^{-1}, \) we get

\[
\sum_{m} \lambda_{k_m}^{-1} \geq \sum_{k} \epsilon_k \lambda_k^{-1} = \infty.
\]
Property (3.6b) follows from
\[ \frac{\lambda_{km}}{m} = \frac{\lambda_{km}}{k_m} \cdot \frac{k_m}{m} \geq 1 \cdot \frac{k_m}{m} > \frac{1}{m} \sum_{j=1}^{m-1} j, \quad m \to \infty. \]

Proof of Theorem 8. Extract from \( \{\lambda_k\} \) a subsequence \( \{\lambda_{km}\} \) satisfying (3.6). According to Macintyre [9, pp. 287–290] there exists an entire transcendental function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with \( a_n = 0 \) for \( n \neq \lambda_{km} \), \( m = 1, 2, \cdots \), hence certainly \( a_n = 0 \) for \( n \neq \lambda_k \), \( k = 1, 2, \cdots \), which is bounded for \( z = x > 0 \). Property (3.6b) shows that \( f \) has Fabry gaps, and by a result of Pólya [11, p. 631] \( f \) cannot be of finite order.

3.3. Entire functions with Hadamard gaps. We now consider the case that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with
\[ (3.7) \quad a_n = 0 \text{ for } n \neq \lambda_k, \quad k = 1, 2, \cdots, \text{ where } \lambda_{k+1}/\lambda_k \geq \theta > 1. \]

Lemma 4. If \( \{n_k\} \) is a sequence of natural numbers with \( n_{k+1}/n_k \geq \theta > 1 \), then
\[ (3.8) \quad g(x) = \sum_{k=1}^{\infty} \frac{\sqrt[n_k]{n_k}}{n_k!} x^{n_k} = O(e^x), \quad x \to +\infty. \]

Proof. Let \( \gamma > 0 \) be so small that \((1 + \gamma)/(1 - \gamma) < \theta \). Given \( x > 0 \), there is at most one of the numbers \( n_k \) in the interval \((1 - \gamma)x, (1 + \gamma)x\); otherwise \( n_{k+1}/n_k \leq (1 + \gamma)/(1 - \gamma) < \theta \) for some \( k \). Assuming that \( n_k \) is the integer in that interval, we have
\[ e^{-x} g(x) = \sum e^{-x} \frac{\sqrt[n_k]{n_k}}{n_k!} x^{n_k} + e^{-x} \frac{\sqrt[n_{k'}]{n_{k'}}}{n_{k'}!} x^{n_{k'}}, \]
where the sum ranges over \( n_k \) with \( |n_k - x| > \gamma x \). This sum is therefore less than
\[ \sum_{|n-x| > \gamma x} e^{-x} \frac{\sqrt[n]{n}}{n!} x^n = o(1), \quad x \to +\infty, \]
(see [6, pp. 200–201]). Moreover, for all \( n = 0, 1, 2, \cdots \) and \( x \geq 0 \),
\[ e^{-x} \frac{\sqrt[n]{n}}{n!} x^n \leq e^{-x} \frac{\sqrt[n]{n}}{n!} n^n \leq C, \]
by Stirling’s formula. This proves (3.8).

Theorem 9. If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is an entire function satisfying the gap condition (3.7), and \( f(x) = O(e^x), \quad x \to +\infty \), then
\[ a_n = O(1) \frac{\sqrt[n]{n}}{n!}, \quad n \to \infty, \]
and \( O(1) \) cannot be replaced by \( o(1) \). Furthermore \( f(z) = O(e^{\lambda |z|}), \quad |z| \to \infty \).
Proof. The estimate of $a_n$ follows from (3.3) and Theorem 2. Lemma 4 shows that $O(1)$ cannot be replaced by $o(1)$, and that

$$|f(z)| \leq \sum_{n=0}^{\infty} |a_n| \cdot |z|^n = O(1) \cdot \sum_{k=1}^{\infty} \frac{\sqrt{\lambda_k}}{\lambda_k^s} |z|^{\lambda_k} = O(e^{\epsilon |z|}), \quad |z| \to \infty.$$  

4. On the coefficients of Dirichlet series. We shall now derive results similar to those of §3, but for the coefficients of Dirichlet series the growth of which is known as $z$ approaches the boundary of convergence. We assume that

$$(1.1) \quad f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges for $\text{Re} \ z = x > 0$, and that

$$(1.2) \quad 0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_{n+1} - \lambda_n \geq \delta > 0, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty.$$  

The second of these conditions implies that (1.1) converges absolutely in $x > 0$ (see, for example, [1, p. 4]).

4.1. Modification of general method. Instead of (1.4) we start from the transformation

$$(4.1) \quad H(z; T) = \int_{T}^{\infty} f(t) e^{t} dt, \quad z = x + iy,$$  

for fixed $T > 0$. Since $f(t) = O(e^{-\lambda_1 t})$, $t \to +\infty$, the integral converges for $x < \lambda_1$ representing an analytic function in that halfplane. On $x = 0$ we have

$$(4.2) \quad |H(z; T)| \leq \int_{T}^{\infty} |f(t)| dt, \quad z = iy.$$  

The analytic continuation of $H(z; T)$ beyond $\text{Re} \ z = \lambda_1$ can be obtained by inserting (4.1) into (4.3) and reversing the order of integration and summation for $\text{Re} \ z < 0$. We obtain

$$(4.3) \quad H(z; T) = -e^{-z \cdot} \sum_{n=1}^{\infty} \frac{a_n}{z - \lambda_n} e^{-\lambda_n z}, \quad \text{Re} \ z < 0.$$  

However, since $\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n z} < \infty$, the series in (4.3) converges uniformly for all $z$ with $|z - \lambda_n| \geq \eta > 0$, so that $H(z; T)$ is meromorphic with simple poles at $z = \lambda_n$, at which $H(z; T)$ has residues $-a_n$, $n = 1, 2, \cdots$. The poles are removed by considering

$$\Phi(z; T) = B(z) \cdot H(z; T)$$

with the Blaschke product $B(z)$ of (1.8), and it is found in the same way.
as in §1.3 that
\[ |\Phi(z; T)| \leq e^{Tz} \int_{T}^{\infty} |f(t)| \, dt, \quad \text{Re} \, z = x \geq 0. \]

Inserting \( z = \lambda_n \) we obtain
\[ |a_n| \cdot |B'(\lambda_n)| \leq e^{\lambda_n^\alpha} \int_{T}^{\infty} |f(t)| \, dt, \quad n = 1, 2, \ldots, \]
and therefore
\[ |a_n| \leq 2\lambda_n p_n e^{\lambda_n^\alpha} \int_{T}^{\infty} |f(t)| \, dt, \quad n = 1, 2, \ldots, \tag{4.6} \]
for every \( T > 0 \), where \( \{p_n\} \) is the sequence defined in (2.2) and studied in §2.

### 4.2. Applications of (4.6) in special cases.

(a) If \( f \in L(0, 1) \), let \( T \to 0 \) in (4.6).

**Theorem 10.** Let \( \{\lambda_n\} \) satisfy (4.2), and let \( f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \) converge for \( \text{Re} \, z = x > 0 \). If \( f \in L(0, 1) \), we have
\[ |a_n| \leq 2\lambda_n p_n \int_{0}^{\infty} |f(t)| \, dt, \quad n = 1, 2, \ldots, \]
where \( \{p_n\} \) is the sequence studied in §2.

If we specialize \( \lambda_n = n^\alpha \), \( \alpha > 1 \), and use our results in §2.3, we obtain the estimate
\[ |a_n| \leq 2n^\alpha e^{J(\alpha)n} \int_{0}^{\infty} |f(t)| \, dt, \quad n = 1, 2, \ldots, \]
with \( J(\alpha) = \pi \tan (\pi/2\alpha) \). This includes a result of Kuttner [8, p. 124], according to which \( \lim_{x \to 0^+} f(x) = s \) implies \( a_n = O(e^{\rho n}) \) for every \( \rho > J(\alpha) \). His proof depends on other work of Miss Cartwright. Kuttner also notes that the estimate of \( a_n \) is not true for \( \rho < J(\alpha) \).

(b) If \( f(t) = O(1/t) \), \( t \to 0^+ \), we put \( T = \lambda_n^{-1} \) in (4.6) to obtain
\[ a_n = O(\lambda_n p_n \log \lambda_n), \quad n \to \infty. \]

(c) If \( f(t) = O(1/t^\beta) \), \( t \to 0^+, \beta > 1 \), we again put \( T = \lambda_n^{-1} \) in (4.6) to obtain
\[ a_n = O(\lambda_n^\beta p_n), \quad n \to \infty. \]

(d) If \( f \) is of bounded variation in \( (0, 1) \), i.e., \( f' \in L(0, 1) \), we get
\[ |a_n| \leq 2p_n \int_{0}^{\infty} |f'(t)| \, dt, \quad n = 1, 2, \ldots. \]

To see this, apply Theorem 10 to \( f'(z) = -\sum_{n=1}^{\infty} a_n \lambda_n e^{-\lambda_n z} \).
4.3. Refinement of method in the case of Hadamard gaps. Our results of §4.2 can be improved somewhat if \( \lambda_n \) is a Hadamard sequence:

\[
\lambda_{n+1}/\lambda_n \geq \theta, \quad n = 1, 2, \ldots, \quad \text{for some } \theta > 1.
\]

First we improve (4.6). To this end consider the function \( H(z; T) \) of (4.3) not in \( \text{Re} z \geq 0 \) but in \( \text{Re} z \geq -\gamma \), where \( \gamma \geq 0 \) is fixed for the moment. On \( \text{Re} z = -\gamma \) we have

\[
|H(z; T)| \leq \int_{T}^{\infty} |f(t)| e^{-\gamma t} \, dt, \quad z = -\gamma + iy.
\]

Instead of \( B(z) \) we now use

\[
B(z; \gamma) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{\lambda_k + z + 2\gamma};
\]

note that \( |B(z; \gamma)| = 1 \) on \( \text{Re} z = -\gamma \) and \( |B(z; \gamma)| \leq 1 \) in \( \text{Re} z \geq -\gamma \). The function

\[
\Phi(z; T; \gamma) = B(z; \gamma) \cdot H(z; T), \quad \text{Re} z \geq -\gamma,
\]

is of exponential type in \( \text{Re} z \geq -\gamma \) with \( h(0) \leq T \), so that by Lemma 1,

\[
|\Phi(z; T; \gamma)| \leq e^{T(\gamma + \epsilon)} \int_{T}^{\infty} |f(t)| e^{-\gamma t} \, dt, \quad \text{Re} z = x \geq -\gamma.
\]

Putting \( z = \lambda_n \) we obtain

\[
|a_n| \cdot |B'(\lambda_n; \gamma)| \leq e^{T(\lambda_n + \gamma)} \int_{T}^{\infty} |f(t)| e^{-\gamma t} \, dt, \quad n = 1, 2, \ldots,
\]

valid for all \( \gamma \geq 0, \, T > 0 \). In view of §2.5 we choose \( \gamma = \gamma_n = \lambda_n/n \), and we obtain

\[
(4.7) \quad |a_n| \leq 2(\lambda_n + \gamma_n) q_n \cdot e^{T(\lambda_n + \gamma_n)} \int_{T}^{\infty} |f(t)| e^{-\gamma_n t} \, dt, \quad n = 1, 2, \ldots,
\]

where \( q_n \) are the numbers defined by (2.11) which are bounded if \( \{\lambda_n\} \) is a Hadamard sequence (Theorem 5).

We now restrict the behavior of \( f(t) \) as \( t \to 0+ \). Assume first that \( f \in L(0, 1) \); this always implies \( f \in L(0, \infty) \) since \( f(t) = O(e^{-\lambda t}), \, t \to \infty \). Letting \( T \to 0 \), (4.7) gives

\[
(4.8) \quad a_n = O(\lambda_n) \cdot \int_{0}^{\infty} |f(t)| e^{-\gamma_n t} \, dt, \quad n \to \infty,
\]

where, as always, \( \gamma_n = \lambda_n/n \). If only \( f \in L(0, 1) \) is known we just get \( a_n = o(\lambda_n) \). If, however, \( f \in L_p(0, 1), \, p > 1 \), or if \( f(t) = O(1/t^\beta), \, t \to 0+, \, 0 < \beta < 1 \), we obtain by simple calculation
we have put $p = \beta^{-1}$ and observed that $n = O(\log \lambda_n)$.

In the case that $f(t)$ remains bounded as $t \to 0+$, our method gives $a_n = O(n)$ compared to the main step $a_n = O(1)$ in Ingham’s proof of the high indices theorem (see [6, p. 173]). If, however, we assume slightly more, our method yields $\sum_{n=1}^{\infty} |a_n| < \infty$.

**Theorem 11.** Let $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ with $\lambda_{n+1}/\lambda_n \geq \theta > 1$ converge for $\text{Re} \ z = x > 0$, and assume that

$$f(x) = s + O(x^\alpha), \quad x \to 0+, \text{ some } \alpha > 0,$$

or

$$f' \in L_p(0, 1), \text{ some } p > 1.$$

Then $a_n = O(q^n)$ for some $q < 1$; in particular, $\sum_{n=1}^{\infty} |a_n|^\epsilon < \infty$ for every $\epsilon > 0$.

Again, our method does not give Zygmund’s result: $\sum_{n=1}^{\infty} |a_n| < \infty$ if $f' \in L(0, 1)$ (see [13, p. 197]).

**Proof.** We may take $\alpha$ in $0 < \alpha \leq 1$, and then may assume $s = 0$ in (4.10); otherwise consider $f^*(z) = f(z) - s e^{\lambda_1 z}$. Apply (4.8):

$$a_n = O(n^{1-\alpha} \lambda_n^{-\alpha}) = O(n^{1+\alpha \log n}), \quad n \to \infty,$$

since $\lambda_n \geq \theta^{n-1} \lambda_1$. From this our conclusion follows.

If (4.11) is assumed, an application of (4.9), with $\beta = \alpha^{-1}$, to $f'(z) = -\sum_{n=1}^{\infty} a_n \lambda_n e^{-\lambda_n z}$ yields

$$a_n = O(n^{1-\beta} \lambda_n^{-\beta}) = O(n^{1-\beta \log n}),$$

and Theorem 11 is proved.

**Corollary.** If $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ with $\lambda_{n+1}/\lambda_n \geq \theta > 1$ converges for $\text{Re} \ z = x > 0$ and is strongly continuous as $z = x \to 0+$ in the sense of (4.10), then $f$ is of bounded variation in $(0, 1)$.

This follows from

$$\int_0^1 |f'(x)| \, dx = \int_0^1 \left| \sum_{n=1}^{\infty} a_n \lambda_n e^{-\lambda_n x} \right| \, dx$$

$$\leq \int_0^1 \sum_{n=1}^{\infty} |a_n| \lambda_n e^{-\lambda_n x} \, dx = \sum_{n=1}^{\infty} |a_n| \int_0^1 e^{-\lambda_n x} \, dx \leq \sum_{n=1}^{\infty} |a_n|$$

and Theorem 11.

Finally we remark that $f(t) = O(1/t), \ t \to 0+$, implies

$$a_n = O(\lambda_n \log n), \quad n \to \infty,$$

which is slightly better than the result in §4.2(b); this is obtained by
putting $T = \lambda_n^{-1}$ in (4.7). If $f$ grows faster than $t^{-1}$, $t \to 0+$, our refined method does not improve the results obtained in §4.2.

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