General charge conjugation operators in simple Lie groups

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A description of particular elements ("charge conjugation operators") found in any compact simple Lie group \( K \) is presented. Such elements \( R_i \) transform a physical state (weight vector of a basis of a representation space) into others with opposite "charge" (ith component of the weight), sometimes changing also the sign of the state. It is demonstrated that exploitation of these elements and the finite subgroup \( N \) of \( K \) generated by them offer new powerful methods for computing with representations of the Lie group. Their application to construction of bases in representation spaces is considered in detail. It represents a completely new direction to the problem.

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I. INTRODUCTION

In this article we study certain elements \( R \) of order four, i.e., \( R^4 = 1 \), in connected compact simple Lie groups in order to demonstrate that they provide a new and powerful tool for applications. Although their importance has long been understood in the theory of Lie groups,\(^4\) these elements have so far not been used in physics literature except for Refs. 2 and 3 (which are based entirely on this work) and they appear here for the first time in what might be called the theory of computation with Lie groups.

Intuitively these elements can be viewed in the following way: Given a simple Lie group \( K \) of rank \( l \), then in a description of relevant physical states \( \{ \lambda_1, \lambda_2, \ldots, \lambda_l \} \), which are weight vectors in a representation space \( V \) of \( K \), an important role is played by "quantum numbers" or "charges" \( \lambda_i, i = 1, 2, \ldots, l \), which are defined as eigenvalues of \( I \) suitably chosen linearly independent "diagonal" elements of the Lie algebra of \( K \). The subject of our article is the elements \( R_i, i = 1, 2, \ldots, l \), of \( K \) which permute the weight vectors of the same \( K \)-multiplet in such a way that \( R_i \) \( \rightarrow \) \( - \lambda_i \) if \( \lambda_i \neq 0 \). For lack of any better name we call \( R_i \) the charge conjugation operators (CCO) although it is only in special situations that one of them may coincide with the usual operator reversing electric charge.\(^4\) It turns out that the action of \( R_i \) on \( \{ \lambda_1, \lambda_2, \ldots, \lambda_l \} \) is quite nontrivial. Besides reversing the charges (components of weights) they sometimes reverse the sign of the state or permute several states with the same "quantum numbers" (weights) when \( \lambda_i = 0 \). Let us underline the fact that there are no charge conjugating elements in \( K \) which would be of order 2 in all finite-dimensional representations of \( K \).

The role which \( R \) may play in applications far exceeds the charge conjugation. In that respect Refs. 2 and 3, where they provide the main tool of the approach, are only modest illustrations of the possibilities. There all nonzero Clebsch–Gordon coefficients arising in a tensor product of two irreducible representation spaces of \( K \) are given by a small representative subset of them and any other coefficient is identity with one of the subset using CCO. Fortunately, the economy made this way rapidly increases with the rank \( l \) of \( K \) roughly being proportional to the order \( |W| \) of the Weyl group \( W \) of \( K \). Thus for instance, in the case of rank one group \( SU(2) \), the saving made by using CCO is the smallest because \( |W| = 2 \). It is equivalent to the well-known fact that from each pair of \( SU(2) \) Clebsch–Gordon coefficients \( C([j_1, j_2; m_1, m_2]) \) and \( C([j_1, j_2; -m_1, -m_2]) \) it suffices to calculate only one of them. However, for \( SU(n) \) the economy provided by CCO increases as \( n! \).

In particle physics it is conceivable that the usual requirements of invariance of a (grand unified) model under the action of a reductive Lie subgroup \( K' \) of a semisimple group \( K \) is too strong and that all the conclusions drawn from the model would follow requiring only the invariance under the action of a finite subgroup \( F \) of \( K \) generated by \( R \), and possibly some other elements of finite order in \( K \). In general, the finite subgroup \( N \) of \( K \) generated by \( R \), is of importance whenever \( K \) appears, even if its role so far has not been fully appreciated.

So far the possibilities of building \( N \)-invariant models which are not \( K' \)-invariant remain completely unexplored. They would closely resemble the \( K' \)-invariant ones in that the \( K' \)-multiplets would be formed as direct sums of \( N \)-multiplets, but they would be simpler because \( N \) as a finite group has only finitely many irreducible representations.

Questions of this type motivate our undertaking although we do not address them directly in the article.

The first objective of the paper is to bring together what is known about CCO in a coherent way.

The second objective is related to the problem of construction of bases in representation spaces. Until now, in spite of the obvious importance of the problem, there is no satisfactory general method of construction. [Note added in proof: Daya-Nand Verma has produced an as yet unproved algorithm for constructing bases by the first of the methods below. It appears very promising.] Indeed, there are three well-known ways how to construct a basis. The first is a multiple application of generators to one basis vector. Although this is a general method in principle, practically it is so unruly that it is of use in spaces of low dimension only. Even sophisticated versions\(^3,6\) developed for purely theoreti-

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cal reasons did not make it any more useful in applications. The second way is to use chains of reductive subgroups. In special cases this produces perhaps the most explicit and desirable form of representation theory. Unfortunately, in most cases it offers at best only a simplification of the construction. The third method based on subalgebras and their centralizers is practically restricted to low rank groups or to very particular classes of representations. Systematic exploitation of CCO and exploitation of the finite group \( N \subset K \) generated by them leads to a new approach to the problem, where the group \( N \) plays the role of the Weyl group of \( K \) "lifted" into the representation space.

Let us emphasize that we are concerned here with methods which apply to any simple (and by an obvious extension to any reductive) Lie group \( K \) and therefore we ignore existing vast literature applicable only to groups of particular type(s).

The most obvious obstacle to construction of basis for representation spaces for anything beyond those of very low dimension is the sheer enormity of the number of vectors to be written down. The natural way out of this is to compute only the dominant weight spaces (which in general make only a tiny fraction of the entire space) and to use the group \( N \) to move outside them when necessary for some problem at hand. This approach leads to two fundamental problems

(i) Build a "good" basis for each of the dominant weight spaces.

(ii) Describe how to move about in the rest of the representation space.

Consequently, the second objective of our article is to describe an approach to basis construction, at least as far as it is possible at this time, and to illustrate various aspects of it by numerous examples, because in many cases of practical interest it offers a considerable help already in its present form.

The problem (i) of building orthonormal bases in dominant weight spaces is the truly difficult part of the construction. Our examples in Sec. VI illustrate two approaches to solving it. The first uses tensor products of simple spaces (practical in many situations), the second one involves the representation theory of subgroups of \( N \) (eigenspace decomposition of stabilizers of dominant weight vectors). A third approach would be exploitation of various subgroups of \( K \).

Whenever a particular (reductive) subgroup \( K' \subset K \) is of importance in an application, it should be reflected in the basis construction. That is, the bases in dominant weight subspaces [problem (i)] above have to be built using \( N' \) of \( K' \) rather than \( N' \) of \( K \). Naturally, in the simple situation when the corresponding branching rule for \( K' \subset K \) contains each \( K' \)-irreducible component at most once, there are the usual shortcuts so that one faces the same problem but for smaller representations of the smaller group \( K' \).

Moving between weight spaces of the representation [problem (ii)] is accomplished by two processes: (a) moving along \( N \)-orbits of the space; (b) crossing orbits. The first process is carried out by the group \( N \) whose action is completely described in Sec. III. The second process is carried out by transforming dominant weight vectors of one space to another dominant weight subspace. This involves computing the action of a few generators between a few subspaces once for all. The Figs. 3 and 4 illustrate the succinct way in which this information can be presented.

In Sec. II we work out in detail and with only simple means the CCO in the SU(2) case. This serves as an introduction to the general situation described in Secs. III and IV, followed by some examples (Secs. V and VI).

II. THE CHARGE CONJUGATION OPERATOR OF SU(2)

In order to specify an irreducible representation of SU(2) of dimension \( L+1 \) we use \( (L) \), where \( L \) is related to the highest weight \( \Lambda = j \) of the representation by

\[
L = 2(\Lambda, \beta) + j = 2j.
\]  

(2.1)

Hence \( j \) is the familiar "angular momentum." A convenient orthonormal basis consists of the vectors (angular momentum states) denoted by \( |j, k \rangle \), such that

\[
M = \frac{1}{2}(\mu, \beta) + j |j, \mu \rangle,
\]

where \( \mu = M^{2} / 2 \) is a weight of the weight system \( O_{L} \) of \( (L) \). Specifically,

\[
M = 2(A - j_{\mu} \beta, \beta) + j \epsilon_{k} \in \{0, 1, \ldots, L\}.
\]

(2.2)

Thus

\[
M \in \{L, L - 2, \ldots, -L\}.
\]

(2.3)

Assuming \( (\beta, \beta) = 2 \), the action of the Lie algebra spanned by generators \( e_{\pm} \) and \( h \) satisfying the commutation relations

\[
[e_{+}, e_{-}] = h, \quad [h, e_{\pm}] = \pm 2e_{\pm}
\]

(2.4)
on the basis vectors \( |j, k \rangle \) of the space \( V^{L} \) is given by

\[
h |j, k \rangle = (\mu, \beta) |j, k \rangle = M |j, k \rangle,
\]

(2.5)

\[
e_{\pm} |j, k \rangle = \frac{1}{2} \sqrt{(L^{2} - M^{2})} |L \mp M \mp j\pm 2\rangle |j, k \pm 2\rangle.
\]

There is up to an inversion only one CCO in the rank one group SU(2). It is defined by

\[
R = \exp e_{-} \exp(-e_{+}) \exp e_{-}
\]

(2.6)

from which it follows that

\[
R |j, k \rangle = (-1)^{L-M/2} |j, k \rangle.
\]

(2.7)

Let us illustrate (2.7) by an example:

\[
R |j, k \rangle = \exp_{-} \exp(-e_{+}) \left( 1 + e_{+} + \frac{1}{2} e_{+}^{2} + \cdots \right) |j, k \rangle
\]

\[
= \exp_{-} \left( 1 - e_{+} + \frac{1}{2} e_{+}^{2} - \frac{1}{2} e_{+}^{3} - \cdots \right)
\]

\[
\times \left( |j, k \rangle + 2 |j, k \rangle + \left( \frac{3}{2} |j, k \rangle \right) + \left( \frac{5}{2} |j, k \rangle \right) + \cdots \right)
\]

\[
= \left( 1 + e_{-} + \frac{1}{2} e_{-}^{2} + \frac{1}{2} e_{-}^{3} + \cdots \right)
\]

\[
\times \left( |j, k \rangle - |j, k \rangle \right) = - |j, k \rangle.
\]

(2.8)

Here and through the rest of the article we write \( -a \) as \( -a \) in matrixlike symbols. Repeated application of (2.7) gives

\[
R^{2} |j, k \rangle = (-1)^{L} |j, k \rangle = (-1)^{M} |j, k \rangle,
\]

(2.9)

\[
R^{4} |j, k \rangle = |j, k \rangle,
\]

(2.10)

which demonstrates that \( R \) is an element of order 4. Note that CCO of SU(2) could have been defined by

\[
\bar{R} = \exp e_{+} \exp(-e_{-}) \exp e_{+},
\]

(2.10)
which would imply the interchange of $M$ and $-M$ in (2.7). Indeed one can verify directly that
\[
\tilde{R} |_{\mathcal{I}_r} = (-1)^{|L + M|/2} |_{\mathcal{I}_r}.
\] (2.11)
Hence,
\[
\tilde{R} = R^{-1}.
\] (2.12)

The matrix of $R$ relative to the basis $\{ |_{\mathcal{I}_r} \}$ for $L = 0, 1, 2, 3, \ldots$ is
\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\] (2.13)
and so on. The trace of any of the matrices $R$ in (2.13) is the character of the element $R \in SU(2)$ in the corresponding representation of $SU(2)$. Thus $tr R$ takes only three distinct values:
\[
tr R = \begin{cases}
1 & \text{for } L = 0 \pmod{4} \\
0 & \text{for } L = 1 \text{ or } 3 \pmod{4} \\
-1 & \text{for } L = 2 \pmod{4}
\end{cases}
\] (2.14)
It was shown in Ref. 10 that in $SU(2)$ there is only one conjugacy class of elements of order $4$ whose character values on irreducible representations are restricted to $0, \pm 1$. Thus, $\mathfrak{q}$ is simply connected, and, in notations of Kac (cf. Refs. 10, 11, and 12), its conjugacy class is given as $R \sim |_{\mathcal{I}_r}$. 

Subsequently we need the transformation of generators $e_\pm$ by $R$. For that consider the equalities
\[
|_{\mathcal{I}_r} = h |_{\mathcal{I}_r} = -R |_{\mathcal{I}_r}
\]
\[
= R h |_{\mathcal{I}_r} = R h R^{-1} |_{\mathcal{I}_r} = -R h R^{-1} |_{\mathcal{I}_r},
\]
\[
|_{\mathcal{I}_r} = e_+ |_{\mathcal{I}_r} = R |_{\mathcal{I}_r} = -R e_+ |_{\mathcal{I}_r}
\]
\[
= R e_- R^{-1} |_{\mathcal{I}_r} = -R e_- R^{-1} |_{\mathcal{I}_r},
\]
\[
|_{\mathcal{I}_r} = e_- |_{\mathcal{I}_r} = R e_+ |_{\mathcal{I}_r} = R e_- R^{-1} |_{\mathcal{I}_r}
\]
\[
= R e_+ R^{-1} e_- |_{\mathcal{I}_r} = -R e_+ R^{-1} e_- |_{\mathcal{I}_r},
\]
\[
\text{from which it follows immediately that}
\]
\[
R h R^{-1} = -h, \quad R e_\pm R^{-1} = -e_\pm.
\] (2.16)

Finally, let us also point out that
\[
R |_{\mathcal{I}_r} = R^{-1} |_{\mathcal{I}_r} = (-1)^{L/2} |_{\mathcal{I}_r}.
\] (2.17)

III. CHARGE CONJUGATION OPERATORS OF ARBITRARY SIMPLE COMPACT LIE GROUP

All ideas of this section extend naturally to arbitrary simply connected compact Lie groups, but for simplicity we consider here only simple simply connected compact Lie groups. The purpose of this section is to bring together well-known facts relevant to CCO.

Let $k$ be the Lie algebra of a simply simply connected compact Lie group $K$, $g$ its complexification $k_c$, and $G$ the simply connected complex group with Lie algebra $g$ and with maximal compact subgroup $K$. We let $T$ be a maximal torus of $G$ and $h$ be the corresponding Cartan subgroup of $g$. Thus $h = t_c$, where $t$ is the subalgebra of $g$ corresponding to $T$, and $h_k := \sqrt{-1} t$ is a real Euclidean space (under the Killing form) of dimension $l = \text{rank}(G)$. Here the symbol $:=$ indicates that the left side is defined by the right one.

Relative to $h$ we have the root space decomposition
\[
g = h \oplus \bigoplus_{\alpha \in \Delta} g^\alpha
\] (3.1)
of $g$, where $\Delta \subset h_k^*$ (the dual space to $h_k$) is the root system of $g$ relative to $h$, and for each $\alpha \in \Delta$,
\[
g^\alpha = \{ x \in g | [h, x] = \alpha(h)x \} \quad \text{for all } h \in h.
\] (3.2)
Choosing an ordering of $h_k^*$ leads to an ordering on $\Delta$. Let $\Delta^+$ denote the corresponding set of positive roots in $\Delta$ and let $H = \{ \alpha_1, \ldots, \alpha_l \}$ be the corresponding set of simple roots.

For each $\beta \in \Delta$, $\mathfrak{sl}(2) = g^\beta + g^\beta - [ g^\beta, g^{-\beta} ]$ is a subalgebra of $g$ isomorphic to $\mathfrak{sl}(2)$, and we chose $e_\beta \in g^\beta$, $e_- \beta \in g^{-\beta}$, and $h_\beta \in h$ such that
\[
[ h_\beta, e_\pm \beta ] = \pm 2 e_\pm \beta, \quad [ e_\beta, e_- \beta ] = h_\beta.
\] (3.3)
These $h_\beta$ are uniquely determined by (3.3) and satisfy
\[
\lambda(h_\beta) = 2(\lambda, \beta)/(\beta, \beta)
\] (3.4)
for all $\lambda \in h_k^*$. The choice of $e_\beta \in g^\beta$, $e_- \beta \neq 0$, is arbitrary, whereupon $e_- \beta$ is uniquely determined by $[ e_\beta, e_- \beta ] = h_\beta$. At this time we leave this choice free.

Let $G^\mathfrak{r} \subset G$ be the compact subgroup whose Lie algebra is $\mathfrak{sl}(2)$, and let $SU^\mathfrak{r}(2) = G^\beta \cap K = \{ \exp(\mathfrak{sl}(2) \cap \mathfrak{k}) \}$ be the corresponding compact subgroup. Thus $G^\mathfrak{r} \cong SU(2)$ and $K^\mathfrak{r} \cong SU(2)$.

A specific isomorphism of $G^\mathfrak{r}$ and $SL(2)$ is established by identifying $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$:
\[
e_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (3.5)
Consequently,
\[
\exp(-e_\beta) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \exp e_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\] (3.6)
and
\[
R_\beta := \exp e_- \exp(-e_\beta) \exp e_- \beta \in SU(2)
\] (3.7)
as in (2.13).

Now let $\rho : K \to \text{GL}(V)$ be a finite-dimensional (unitary) representation of $K$ on a complex space $V$ and let $d\rho : k \to \text{End}(V)$ be its differential, i.e., a representation of $k$ on $V$. Both $\rho$ and $d\rho$ have complexifications,
\[
\rho_c : G \to \text{GL}(V),
\]
\[
d\rho_c : g \to \text{End}(V).
\]
Relative to $T, V$ decomposes into weight spaces
\[
V = \bigoplus_{\lambda \in \Omega} V(\lambda),
\] (3.8)
where $\Omega \subset h_k^*$ is the weight system and for all $\lambda \in \Omega$
\[
V(\lambda) = \{ v \in V | d\rho_c(h)v = \lambda(h)v \} \quad \text{for all } h \in h_k.
\] (3.9)
For each $\beta \in \Delta$ we have, by restriction, representations
\[
\rho^\beta : SU^\mathfrak{r}(2) \to \text{GL}(V),
\]
\[ p_\alpha^\beta : G^\beta \rightarrow GL(V), \]
\[ dp_\alpha^\beta : sl_\beta(2) \rightarrow \text{End}(V). \]

If we partition \( \Omega \) into \( \beta \)-weight strings, \( (\lambda + Z \beta) \cup \Omega \), i.e.,
\[ \lambda + q \beta, \lambda + (q-1) \beta, \ldots, \lambda - p \beta, \]
then the sums of the corresponding weight spaces
\[ s \oplus V(\lambda + j \beta) \]
are \( SU^2(\beta) \)-submodules. They can further be decomposed into \( SU^2(\beta) \)-irreducible submodules. Each such submodule is a sum
\[ s \oplus \text{C}(A - k \beta), \]
where \( A = \lambda + n \beta \) for some \( r \) and \( v(A - k \beta) \in V(A - k \beta). \)

The identification (3.5) of \( sl^2(2) \) and \( sl(2) \) leads to representations of \( sl(2) \) and \( S_L(2) \) on \( V \) such that
\[ \begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix} \rightarrow dp_c(e_\beta), \]
\[ \begin{pmatrix}
1 & -1 \\
0 & 1 \\
\end{pmatrix} \rightarrow \exp(d \rho_c(-e_\beta)) = \rho_c(\exp(-e_\beta)), \]
\[ R_\beta - \rho_c(\exp e_\beta \exp e_\beta), \]
An appropriate choice of \( v(A - k \beta) \) allows us to identify them with \( L_\lambda^\beta \) of Sec. II, namely,
\[ v(A - k \beta) \rightarrow v_{A-k\beta}^{1(h)} = L_\lambda^\beta. \]

Then also \( e_\beta \) and \( h_\beta \) act on (3.14) according to (2.5).

Although \( R_\beta \) is defined in terms of nonunitary operators, it lies in \( SU(2) \subset K \) and hence appears as a unitary operator on \( V \). From now on we write \( R_\beta \) for \( \rho_c(R_\beta) \). According to (2.7), one has
\[ R_\beta^{1(h)}(A - k \beta) = (-1)^k(\lambda - k \beta) - \lambda, \]
which demonstrates the "charge conjugating role" of the operators \( R_\beta \). Thus the general effect of \( R_\beta \) on \( V \) is the permutation of weight subspaces:
\[ V(\lambda) \rightarrow V(\lambda - \lambda(\beta) / (\beta, \beta) \beta). \]

The Weyl reflection \( r_\beta h_\beta \rightarrow h_\beta \) is defined by
\[ r_\beta \lambda = \lambda - \lambda(\beta) = \beta - (2(\lambda, \beta) / (\beta, \beta) \beta). \]

Therefore,
\[ R_\beta V(\lambda) = V(r_\beta \lambda). \]

The Weyl group is by definition the group \( W \) generated by the \( r_\alpha, \alpha \in \Delta \). For all \( \alpha, \beta \in \Delta, r_\alpha^2 = 1, \) and \( r_\alpha r_\beta = r_\beta r_\alpha \). It follows that \( W \) is generated by the \( r_i = r_\alpha, i = 1,2, \ldots, l \).

Whereas, \( r_0^2 = 1 \), it is obvious from (2.9) that we only have
\[ R_\beta = 1. \]

From (2.9) and (3.15) one has
\[ R_\beta^2 V(\lambda) = (-1)^k(h_\beta). \]

Following Tits\(^{13}\) we write
\[ R_\beta^2 = (-1)^{h_\beta}. \]

Corresponding to the generators \( e_\beta, e_{-\beta} \) of \( sl^2(2) \), we have
\[ e_{-\beta}, e_\beta, h_{-\beta} = -h_\beta \]
as a set of generators of \( sl^2(2) \). The operator \( R_{-\beta} \) is then
\[ R_{-\beta} = \exp e_\beta \exp e_{-\beta} \exp e_\beta \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow R_{\beta}^{-1}. \]

Thus
\[ R_{-\beta} = R_{\beta}^{-1}. \]

IV. THE FINITE GROUP GENERATED BY CHARGE
CONJUGATION OPERATORS

Our primary interest here is the group \( N \) generated by \( R_\alpha, \beta \in \Delta \). As it stands \( N \) depends on the choice of \( e_\alpha \) (hence, \( R_\alpha \)), \( \beta \in \Delta \). The most convenient form of \( N \) arises by the use of a Chevalley basis\(^{5,6} \) of \( g \). According to Ref. 14, there is a choice of the \( e_\alpha, \beta \in \Delta, \) such that the following occurs: For \( \alpha, \beta \in \Delta, \) where \( \alpha \) and \( \beta \) are linearly independent with root string \( \beta - \alpha, \beta, \ldots, \beta + q_\alpha, \)
\[ [e_\alpha, e_\beta] = \pm (p + 1) e_{\alpha + \beta} \text{ if } \alpha + \beta = \Delta. \]

The matter of sign is not essential to us here. With such a choice of basis,
\[ g_z = \sum Z h_i + \sum_{\beta \in \Delta} Z e_\beta \]
is a Lie ring. Most importantly, for all \( \beta \in \Delta \) and for all \( n \in N, \)
\[ (1/n!)(a d e_\beta)^n : g_z \rightarrow g_z \]
More generally, Kostant has shown\(^{6,15}\) that for every representation \( (\rho, V) \) of \( g \) there is a basis \( v_1, \ldots, v_m \) of \( V \) consisting of weight vectors such that if we set
\[ V_z = \oplus Z v_j, \]
then for all \( \beta \in \Delta \)
\[ \frac{1}{n!} dp(e_\beta)^n : V_z \rightarrow V_z. \]

Here \( v_\beta \) are assumed to be a Chevalley basis. Thus in particular,
\[ R_\beta : V_z \rightarrow V_z. \]

Since \( R_{-\beta} = R_{\beta}^{-1} \) we see that \( R_\beta \) is a bijective mapping of \( V_z \).

From the Chevalley basis the operators \( R_\alpha \) and \( R_\alpha^{-1} \) map \( g_z \) into itself (in the adjoint representation) and are automorphisms of \( g_z \) as a Lie ring. It is easy to check [cf. (2.16)] that \( R_\alpha \circ h_\alpha = -h_\alpha \) and \( R_\alpha \) acts trivially on the orthogonal complement of \( h_\alpha \) in \( h_r \). Thus
\[ R_\alpha h_\alpha = h_{-\alpha}. \]

Also since for each \( \gamma \in \Delta \), the generators \( \pm e_\gamma \) are the only ones for \( g_z = Z_e \) (as \( Z \)-modules),
\[ R_\alpha e_\alpha = \pm e_{-\alpha}, \text{ for all } \alpha, \beta \in \Delta. \]
From \( R_\alpha(h_\beta) = R_\alpha [e_\beta, e_\beta] = [R, e_\beta, R, e_\beta] \) it follows that
\[ R_\alpha e_{-\beta} = \pm e_{-\alpha}, \]
where the sign is the same as in (4.8). From this we have
This is no more than the action of $W$ on $h_\alpha$ induced by transposing the action of $W$ on $h_\lambda$: For all $w \in h_\alpha$, $h \in h_\lambda$,

$$w_{\lambda}(h) = \varphi(w^{-1}h).$$

(4.21)

Since $W$ stabilizes $h_\alpha = \Sigma z h_i$, it thus produces a modulo 2 action on $h_\alpha$. Let $\bar{\alpha} \in h_\alpha$ and $(-1)^{\bar{\alpha}}$ be the corresponding element of $A$. Then

$$w(-1)^{\bar{\alpha}} = (-1)^{\bar{\alpha}} h_{\bar{\lambda}}.$$

(4.22)

If $(\rho, V)$ is any unitary representation of $K$ and $V = \oplus V(\lambda)$ is the weight space decomposition of $V$, then we have seen

$$RV(\lambda) = V(\pi(R) \lambda)$$

and

$$(-1)^{\bar{\alpha}}_{V(\lambda)} = (-1)^{\bar{\alpha}} h_{\bar{\lambda}} [\text{cf. (4.18) and (4.15)}].$$

Important subgroups of $N$ are those which stabilize a given weight space $V(\lambda) \in N$. By the following considerations. Each $W$-orbit, $W(\lambda) \lambda \in \Omega$, contains unique dominant element $\lambda^+$ : defined by

$$\lambda^+(h_i) > 0 \quad \text{for all} \quad i = 1, ..., l.$$  

(4.24)

For $\lambda$ dominant, let

$$J_\lambda = \{ i \in \{ 1, ..., l \} | \lambda^+(h_i) = 0 \}.$$  

(4.25)

Then $N_{\lambda^+}$ is the group generated by $A$ and by the $R_i$, $i \in J$. Alternatively, if we define

$$W_{J_\lambda} = \{ r_{i} | i \in J \} ,$$

then $N_{\lambda^+}$ is the full preimage $N_{J_\lambda} = \pi^{-1}(W_{J_\lambda})$ of $W_{J_\lambda}$ in $N$, 

$$1 \rightarrow A \rightarrow N_{J_\lambda} \rightarrow W_{J_\lambda} \rightarrow N.$$  

(4.26)

The cardinality of the set $W_{\lambda^+} = \{ w \lambda^+ | w \in W \}$ is precisely the index

$$|W_{\lambda^+}| = |W| / |W_{J_\lambda}|$$

(4.27)

of $W_{J_\lambda}$ in $W$. This is trivial to compute since $W_{J_\lambda}$ is the Weyl group of the subgroup system of $A$ based on $\{ a_i | i \in J \}$. For $\lambda$ not dominant, choose $w \in W$ such that $\lambda^+ = w^{-1} \lambda$ is dominant, and define $N_{\lambda^+}$ as above. Then

$$N_{\lambda^+} = wN_{\lambda}w^{-1}.$$  

(4.28)

Note that (4.29) makes sense since the choice of representation $R$ of $w$ in $N$ for computing (4.29) is immaterial.

V. EXAMPLES

In order to illustrate the content of Sec. III and IV, we consider here some particular cases.

A. The group SU(3)

Consider the lowest faithful representation with the highest weight $\Lambda = (1,0)$ of the group SU(3). Its representation space $V^{\Lambda}$ is spanned by the weight vectors

$$| \Lambda^0 \rangle, \quad | \Lambda^1 \rangle, \quad \text{and} \quad | \Lambda^2 \rangle.$$  

(5.1)

For simplicity we omit the highest weight in symbols like (5.1) whenever there can be no ambiguity. According to (3.15).

$$r_{\alpha} h_{\lambda} = h_{\lambda} \alpha(h_{\lambda}).$$  

(4.20)


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\[ R_3 | 10 \rangle = | \bar{T} \rangle, \quad R_3 | \bar{T} \rangle = - | 10 \rangle, \quad R_3 | 0 \bar{T} \rangle = | 0 \bar{T} \rangle, \quad (5.2) \]

\[ R_2 | 10 \rangle = | 10 \rangle, \quad R_2 | \bar{T} \rangle = | 0 \bar{T} \rangle, \quad R_2 | 0 \bar{T} \rangle = - | \bar{T} \rangle, \]

and therefore we have in the SU(3)-representation (1,0)

\[ R_i = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ R_1 R_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.3) \]

\[ R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{etc.} \]

By a direct computation one is led to the conclusion that the subgroup \( N \subseteq SU(3) \) generated by \( R_1 \) and \( R_2 \) is isomorphic to the octahedral group \( O \) of order 24 and that the above representation is the three-dimensional irreducible representation with determinant of all elements equal one. Adopting notations of Ref. 16, it is the representation \( \Gamma_4 \).

Since \( N \subseteq SU(3) \), every SU(3) representation \((p,q)\) reduces with respect to \( N \). That is

\[ (p,q) \supset \bigoplus_{i=1}^{r} m_i \Gamma_i, \quad (5.4) \]

where \( m_i \) is the multiplicity of \( \Gamma_i \) in the reduction. The multiplicities are easily found for any \((p,q)\) from the generating function (4.6)-(4.10) of Ref. 17.

The elements \( R_1 \) and \( R_2 \) of \( N \) lie in the same SU(3)-conjugacy class of regular elements of order 4 in SU(3), namely the one denoted by [2 1 1] in Table I of Ref. 10. Their character values on irreducible SU(3)-representations are restricted to 0, \pm 1.

B. The group SU(\text{n})

As in the previous case one finds a faithful matrix representation of \( N \subseteq SU(n) \) by considering the action of \( R_i, i = 1,2,\ldots,n-1 \), in the lowest faithful representation \((1,0,\ldots,0)\) of SU(\text{n}) according to (3.15):

\[ R_i = I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}, \quad i = 1,2,\ldots,n-1, \quad (5.5) \]

where \( I_k \) is the \( k \times k \) identity matrix. From (4.19) we find that the order of \( N \) is \(|N| = 2n - 1,n! \). It is obvious in (5.5) that all \( R_i \) belong to the same SU(\text{n})-conjugacy class of rational elements of order 4, which is identified as [210...01] in Table 6 of Ref. 12. Except for SU(2) and SU(3), the \( R_i \) are not regular in SU(\text{n}) and consequently the set of their character values over all irreducible SU(\text{n}) representations is an unbounded set of integers. The elements \( R_i \) satisfy the following identities

\[ R_i^4 = 1, \]

\[ R_i R_j = R_j R_i, \quad \text{if} \ |i - j| > 1, \quad (5.6) \]

\[ R_i R_j R_i = R_j R_i R_j, \quad \text{if} \ |i - j| = 1. \]

The exact sequence (4.18) can be written as

\[ 1 \rightarrow Z_2 \times \cdots \times Z_2 \rightarrow \mathbb{Z}_n \rightarrow S_n \rightarrow 1. \quad (5.7) \]

Here the Weyl group SU(\text{n}) is isomorphic to the symmetric group \( S_n \) of \( n \) letters.

C. The groups USp(4) and O(5)

In the symplectic four-dimensional representation \((1,0)\) of these groups, we have

\[ R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ (5.8) \]

relative to the basis of weight vectors. Therefore, also

\[ R_1^4 = R_2^4 = 1, \quad (5.9) \]

\[ R_1 R_2 R_1 R_2 = R_2 R_1 R_2 R_1. \]

Similarly in the five-dimensional orthogonal representation \((0,1)\), one has

\[ R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.10) \]

\[ R_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \]

which also satisfy the identities (5.9). The exact sequence (4.18) is in this case

\[ 1 \rightarrow Z_2 \times Z_2 \rightarrow \mathbb{Z}_5 \rightarrow D_4 \rightarrow 1, \quad (5.11) \]

where \( D_4 \) is the dihedral group. The order of \( N \) is 32. The elements \( R_1 \) and \( R_2 \) are not conjugate to each other because they correspond to simple roots of different length. Their conjugacy classes are identified in Table 6 of Ref. 12 as [201] and [210], respectively, for \( R_1 \) and \( R_2 \). Both elements are rational, which implies that their characters take only integer values in any representation of the Lie group, but they are not regular which means that their character values are unlimited. For any given representation \((a,b)\) the characters \( R_1 \) and \( R_2 \) are easily found from the generating function of Table V of Ref. 10.

D. The group \( G_2 \)

In the lowest representation \((0,1)\) of dim = 7 of \( G_2 \), one has \( R_1 \) and \( R_2 \) as

\[ R_1 = 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1, \quad (5.12) \]

\[ R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

relative to the weight vector basis. The group \( N \) generated by (5.12) is of order 48. One has the exact sequence

\[ 1 \rightarrow Z_2 \times Z_2 \rightarrow \mathbb{Z}_5 \rightarrow D_4 \rightarrow 1, \quad (5.13) \]

where \( D_6 \) is the dihedral group, and the identities
\[ R_1^+ = R_2^+ = 1, \quad (5.14) \]
\[ R_1 R_2 R_1 R_2 R_2 R_2 R_2 R_1 R_2 R_1. \]

The elements \( R_1 \) and \( R_2 \) are rational and nonregular in \( G_2 \). Their \( G_2 \)-conjugacy classes are, respectively [201] and [110]. Their characters in any representation of \( G_2 \) are found from the generating functions of Table VI of Ref. 10.

VI. CONSTRUCTING BASES IN REPRESENTATION SPACES

In this section we demonstrate and illustrate the reduction of the problem of constructing a basis and computing the matrix elements of generators to similar problems of much smaller size involving only the dominant weight vectors.

Given the general decomposition (3.8) of a space \( V \) in which a compact simple Lie group \( K \) acts irreducibly as a representation \( \rho \), it is natural to consider the present problem as a construction of bases in every weight subspace \( V(\lambda) \subseteq V \), where \( \lambda \in \Omega \). Let us point out that in almost every application, and certainly in all of them in elementary particle physics, one chooses a basis of weight vectors relative to some Cartan subalgebra of \( g \) whenever an explicit use of a basis is made. Otherwise one could not associate quantum numbers with the basis vectors—physical states. The dimension of \( V(\lambda) \) is the multiplicity of \( \lambda \) in \( \Omega \) so that the basis construction in \( V(\lambda) \) is a nontrivial problem only when \( \dim V(\lambda) \geq 1 \).

Using notations of Sec. III, let us recall the following. For every weight \( \lambda \in \Omega \) there exists a unique dominant weight \( \lambda^+ \in \Omega \) such that \( \lambda \in W \lambda^+ \). If \( \lambda = \omega \lambda^+ \) then also \( \lambda = \omega \lambda^+ \lambda^+ \). There exists a unique canonical \( \omega \) such that
\[ \lambda = \omega \lambda^+ \quad \omega = r_1 r_2 \ldots r_k, \quad (6.1) \]
in which the number \( k \) of reflections \( r_i \) is minimal. We define
\[ \bar{\omega} = r_1 r_2 \ldots r_k. \quad (6.2) \]
Although \( \omega \) is unique, its expression as a word in the reflections \( r_1 \ldots r_k \) is not. Thus \( \bar{\omega} \) depends on the choice of the writing \( \omega \) in (6.1). If \( r_1 \ldots r_k \) is any other expression for \( \omega \) (minimal or not), then \( \bar{\omega}' = \bar{r}_1 \ldots \bar{r}_k \) is some other preimage of \( \omega \) in \( N \) and \( \bar{\omega}' = \bar{\omega}^{-1} \bar{\omega} \) stabilizes the weight space \( V(\lambda^+) \).

Hence it is unavoidable to consider the effect of such elements \( \bar{\omega} \in N \) on the weight spaces they stabilize.

Let us assume that for each dominant weight \( \lambda^+ \in \Omega \) we have an orthogonal basis [problem (i) of Introduction]
\[ \lambda^+_1, \ldots, \lambda^+_m \quad V(\lambda^+), \quad \dim V(\lambda^+) = m. \quad (6.3) \]
Then for \( \lambda \) as in (6.1) we define
\[ \bar{\omega} \lambda^+_1, \bar{\omega} \lambda^+_2, \ldots, \bar{\omega} \lambda^+_m \quad (6.4) \]
as our basis for \( V(\lambda) \), which solves part (a) of problem (ii) of the Introduction. In the following five examples we illustrate our approach to that problem.

Example 1 [SU(2)]: All \( V(\lambda) \) are one dimensional. A basis of \( V(\lambda^+) \) consists of \( |\lambda^+_m \rangle \), \( M > 0 \). As a basis vector in \( V(-\lambda^+) \) take
\[ |\lambda^+_m \rangle = (-1)^{L-M} R \langle \lambda^+_m | \lambda^+_m \rangle, \quad (6.5) \]
where we have used the phase factor in order to keep the convention in complete agreement with Sec. II.

Example 2 [SU(3) representation (1,0)]: There is only one dominant weight \( \lambda^+ = (1,0) \in \Omega \) of multiplicity 1. The basis is given in (5.2).

Example 3 [the adjoint representation (2,0) of Sp(4)]: By assumption we know the basis vectors
\[ |20\rangle, \quad |01\rangle, \quad |00\rangle, \quad |00\rangle. \]
Then the rest is given by
\[ R_1 |20\rangle = |2\rangle, \quad R_2 |01\rangle = |2\rangle, \quad R_1 |20\rangle = |2\rangle, \quad R_2 |01\rangle = |1\rangle, \quad (6.6) \]

Example 4 [SU(3) representation (3,2) of dim = 42]: Properties of dominant weights are shown on Fig. 1. Assuming that we have pairwise orthogonal dominant weight vectors
\[ |21\rangle, \quad |02\rangle, \quad (6.7) \]
for \( j = 1, 2, 3 \), the basis is given by
\[ |32\rangle, \quad R_1 |31\rangle = R_1 R_2 |32\rangle = |2\rangle, \quad R_1 |32\rangle = |3\rangle, \quad R_1 |52\rangle = R_1 R_2 |32\rangle = |2\rangle, \quad R_2 |32\rangle = |3\rangle, \quad R_2 |52\rangle = R_2 R_1 |32\rangle = |2\rangle, \quad (6.8) \]

FIG. 1. Dominant weights of the weight system of the SU(3)-representation (3,2), their multiplicities, and the positive roots by which they differ.
\[ |10\rangle_j, j = 1,2,3, \quad R_2|\bar{1}\rangle_j = R_2R_1|10\rangle_j = |0\rangle_j, \]
\[ R_1|10\rangle_j = |1\rangle_j, \] (6.8)
\[ |21\rangle_i, i = 1,2,3, \quad R_2|\bar{2}\rangle_i = R_2R_1|21\rangle_i = |\bar{1}\rangle_i, \]
\[ R_1|21\rangle_i = |2\rangle_i, \quad R_1|\bar{2}\rangle_i = R_2R_1|21\rangle_i = |\bar{3}\rangle_i, \]
\[ R_2|21\rangle_i = |3\rangle_i. \quad R_1|3\rangle_i = R_2R_1|21\rangle_i = R_2|\bar{3}\rangle_i = R_2R_1|21\rangle_i = |\bar{2}\rangle_i. \]

The last example, although it still refers only to a group of rank 2, makes it obvious that it is impractical to explicitly write all basis vectors in larger spaces. Instead one should construct the basis for dominant weight subspaces and any other ones only when needed for a particular task at hand.

**Example 5 [O(16)-representation (0000000)] of dim = 11440:** The dominant weights and their multiplicities are found in Ref. 18; they are shown on Fig. 2. Assuming that an orthogonal basis in each of the three dominant weight subspaces of dim > 1 has been constructed, the rest of the construction is a mechanical application of CCO similar to (6.8). Thus from |0000000\rangle one gets \(2^8\times 8! = 1024\) other basis vectors, from each of the three |0000010\rangle, \(j = 1,2,3\), one gets \(2^8\times 8! = 1792\) others, from each of |0000100\rangle, \(k = 1,\ldots, 10\), one gets \(2^8\times 3! \times 2^4 = 448\) new ones, and from each |0000100\rangle, \(n = 1,\ldots, 35\), one gets \(2^8\times 6! = 16\) new basis vectors. Here the numbers are calculated using (4.28) and the orders of Weyl groups given for instance in Refs. 15 or 18.

Suppose now that we need the basis in a particular weight subspace, say \(V(\lambda) = V(01\bar{1}0\bar{0}0\bar{1}). \) Applying \(R_1\) with subscripts corresponding to negative entries in the corresponding weight for as long as possible one finds
\[ V(0001000) = R_3R_2R_1R_1R_1R_3\bar{V}(01\bar{1}0\bar{0}0\bar{1}). \] (6.9)

Then applying \(R_1^{-1}\) in the inverse order to that in (6.9) to the basis vectors \(|0000100\rangle, i = 1,2,3,\) one gets the desired basis. Arrived at in this way the element \(\bar{w}\) of \(\bar{N}\) is bound to be of minimal type.

Let us now exemplify the truly difficult part of our problem: construction of bases in dominant weight subspaces. Also here CCO operators provide a valuable tool.

**Example 6 [the adjoint representation (1,1) of SU(3)]:** The dominant weight subspace \(V(00)\) is of dim = 2. Its basis can be constructed as follows: Consider the highest irreducible component in the tensor product \((1,0) \otimes (0,1)\) which is the adjoint representation. Its dominant weight vectors \(|11\rangle, |00\rangle_+\), and \(|00\rangle_-\) can be chosen as
\[ |11\rangle = |10\rangle|01\rangle, \]
\[ |00\rangle_+ = (1/\sqrt{2})(|\bar{1}\rangle|1\rangle + |1\rangle|\bar{1}\rangle), \]
\[ |00\rangle_- = (1/\sqrt{2})(|\bar{1}\rangle|1\rangle - |1\rangle|\bar{1}\rangle) |01\rangle. \] (6.10)

Here the linearly independent but nonorthogonal vectors \(|00\rangle_+\) and \(|00\rangle_-\) span \(V(00).\) Instead of (6.10) one could observe\(^{19}\) that \(\bar{N}\) acts irreducibly on \(V(00).\) Its two-dimensional representation is generated by matrices
\[ m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_2 = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}. \] (6.11)
(cf. Table IX, Ref. 16) and, by definition of \(N,\) also by \(R_1\) and

\[ R_2. \] On \(V(00)\) one can identify \(m_1\) with \(R_1.\) Then the eigenvectors \(R_1\)
\[ R_1|00\rangle_\pm = \pm |00\rangle_\pm \] (6.12)
provide an orthogonal basis of \(V(00)\). Using (6.10) one has explicitly
\[ |00\rangle_- = (1/\sqrt{2})(|\bar{1}\rangle|1\rangle + |1\rangle|\bar{1}\rangle), \]
\[ |00\rangle_+ = (1/\sqrt{2})(|\bar{1}\rangle|1\rangle - |1\rangle|\bar{1}\rangle) + 2|01\rangle|0\rangle. \] (6.13)

**Example 7:** As our next example let us construct the dominant weight basis vectors (6.7). For that consider the representation (3,2) as the highest irreducible component in (3,0) \(\otimes (0,2).\) Since (3,0) and (0,2) have only one-dimensional weight spaces, their weight vectors \(|3\rangle_\pm\) and \(|2\rangle_\pm\) provide a basis for our problem. For \(\lambda^+\) of multiplicity one in \(\Omega\) of (3,2), one has
\[ |3\rangle_\pm = |2\rangle_{\pm 0}, \quad |2\rangle_\pm = |3\rangle_{\pm 0}, \quad |2\rangle_{\pm 0} = |3\rangle_{\pm 0}. \] (6.14)

The two-dimensional subspace \(V(02)\) is stabilized by \(R_1,\)
\[ R_1V(02) = V(02). \] (6.15)

Hence, its basis can be taken as eigenvectors of \(R_1.\) In order to identify the eigenvalues, it suffices to notice that there are two \(\alpha_1\)-strings passing through \(V(02)\) of \(V\) corresponding to \(SU^a(2)\) representations of dimensions 5 and 3. Since according to (2.17), \(R_1|3\rangle = |5\rangle\) but \(R_1|3\rangle = -|3\rangle,\) the \(R_1\)-eigenvalues are \(\pm 1.\) Consequently, we can choose

FIG. 2. Dominant weights of the representation (000000) of O(16) of dimension 11440, their multiplicities, and the positive roots by which they differ.

\[ R_2. \] On \(V(00)\) one can identify \(m_1\) with \(R_1.\) Then the eigenvectors \(R_1\)
\[ R_1|00\rangle_\pm = \pm |00\rangle_\pm \] (6.12)
provide an orthogonal basis of \(V(00).\) Using (6.10) one has explicitly
\[ |00\rangle_- = (1/\sqrt{2})(|\bar{1}\rangle|1\rangle + |1\rangle|\bar{1}\rangle), \]
\[ |00\rangle_+ = (1/\sqrt{2})(|\bar{1}\rangle|1\rangle - |1\rangle|\bar{1}\rangle) + 2|01\rangle|0\rangle. \] (6.13)

**Example 7:** As our next example let us construct the dominant weight basis vectors (6.7). For that consider the representation (3,2) as the highest irreducible component in (3,0) \(\otimes (0,2).\) Since (3,0) and (0,2) have only one-dimensional weight spaces, their weight vectors \(|3\rangle_\pm\) and \(|2\rangle_\pm\) provide a basis for our problem. For \(\lambda^+\) of multiplicity one in \(\Omega\) of (3,2), one has
\[ |3\rangle_\pm = |2\rangle_{\pm 0}, \quad |2\rangle_\pm = |3\rangle_{\pm 0}, \quad |2\rangle_{\pm 0} = |3\rangle_{\pm 0}. \] (6.14)

The two-dimensional subspace \(V(02)\) is stabilized by \(R_1,\)
\[ R_1V(02) = V(02). \] (6.15)

Hence, its basis can be taken as eigenvectors of \(R_1.\) In order to identify the eigenvalues, it suffices to notice that there are two \(\alpha_1\)-strings passing through \(V(02)\) of \(V\) corresponding to \(SU^a(2)\) representations of dimensions 5 and 3. Since according to (2.17), \(R_1|3\rangle = |5\rangle\) but \(R_1|3\rangle = -|3\rangle,\) the \(R_1\)-eigenvalues are \(\pm 1.\) Consequently, we can choose
as the basis \( V'(02) \), or explicitly
\[
\begin{align*}
|12\rangle_1 &= (1/\sqrt{2}) (|12\rangle_{00} + |10\rangle_{02}), \\
\nonumber
|12\rangle_2 &= (1/\sqrt{6}) (|10\rangle_{02} - |10\rangle_{00} + 2|00\rangle_{02}) \).
\end{align*}
\] (6.17)

Then it is natural to also choose
\[
\begin{align*}
|12\rangle_1 &= (1/\sqrt{6}) e_1 |12\rangle_{00} + (1/\sqrt{2}) (|12\rangle_{02} + |10\rangle_{02}), \\
|12\rangle_2 &= (1/\sqrt{2}) e_2 |12\rangle_{02} = (1/\sqrt{12}) (|12\rangle_{02} - \\
&- (\sqrt{3}|10\rangle_{02} + 2\sqrt{2}|02\rangle_{02}) \).
\end{align*}
\] (6.18)

The coefficients above (other than the overall normalization) are a result of the application of SU(2) generators along the corresponding \( \beta \)-string [cf. (3.14)] according to (2.5). The three-dimensional subspace \( V'(10) \) is stabilized by the subgroup \( (R, R^*_1) \) of \( N \) generated by \( R \) and \( R^*_1 \). There are three \( \alpha \)-strings passing through it of lengths 5, 2, and 1. Due to (2.17), one can thus require that
\[
\begin{align*}
R \cdot |10\rangle_1 &= |10\rangle_1, \text{ and } e_1^2 |10\rangle_1 &= |10\rangle_1 = R |10\rangle_1, \\
R \cdot |10\rangle_2 &= - |10\rangle_2 \text{ and } e_2 |10\rangle_2 &= (1 - R) e_{-2} |10\rangle_2, \\
R \cdot |10\rangle_3 &= |10\rangle_3 \text{ and } e_{\pm 2} |10\rangle_3 &= 0,
\end{align*}
\] (6.19)

where \( \sim \) indicates that both sides differ only by a constant nonzero factor.

Explicitly (6.19) is
\[
\begin{align*}
|10\rangle_1 &= (1/\sqrt{6}) (|10\rangle_{00} + |10\rangle_{02}) + 2|00\rangle_{02}), \\
|10\rangle_2 &= (1/\sqrt{6}) (|10\rangle_{02} - |10\rangle_{00} - \sqrt{2}|00\rangle_{02}), \\
&- \sqrt{2}|10\rangle_{02}), \\
|10\rangle_3 &= (1/\sqrt{30}) (|10\rangle_{00} + 3|10\rangle_{20}) + 2|00\rangle_{02}), \\
&+ \sqrt{2}|10\rangle_{02} - \sqrt{2}|10\rangle_{00} + \sqrt{2}|00\rangle_{02}).
\end{align*}
\] (6.20)

Let us point out that using the product form (6.20) for each \( |10\rangle \) in (6.20), the conditions (6.19) do not guarantee that \( |10\rangle \) lies in \( V \) because the product space is reducible and contains other SU(3) irreducible representations than (3,2). (In fact there are six linearly independent vectors \( |A\rangle \) in the product space corresponding to several highest weights \( \Lambda \).

In order to assure that \( |10\rangle \subset V \), one can proceed for instance as follows. A third vector of \( V'(10) \) which is linearly independent from \( |10\rangle_1 \) and \( |10\rangle_2 \), is \( (1 + R_2) e_{-2} |10\rangle_{22} \). Then \( |10\rangle_2 \) is the linear combination of the three which is normalized and orthogonal to \( |10\rangle_1 \) and \( |10\rangle_2 \).

Finally, let us continue Example 6. In Fig. 3 one finds the dominant weights of the adjoint representation of SU(3) and two matrices representing the action of the generator \( e_{1+2} = e_1 e_2 - e_2 e_1 \) on the chosen basis \( (6.13) \) of \( V'(00) \) and \( e_{-1-2} = e_{-1} e_{-2} - e_{-2} e_{-1} \) on \( V(11) \). Namely,
\[
\begin{align*}
e_{-1-2} |11\rangle &= (\sqrt{2} |00\rangle_- - \sqrt{3}/2 |00\rangle_+), \\
e_{1+2} |00\rangle_- &= (\sqrt{1/2} |11\rangle, \quad e_{1+2} |00\rangle_+ &= (\sqrt{3/2} |11\rangle. 
\end{align*}
\] (6.21)

Using Fig. 3 many other matrix elements can readily be found, for instance,
\[
e_{-2} |12\rangle = e_{-2} R_1 |11\rangle = R_1 R_{-1}^{-1} e_{-2} R_1 |11\rangle = - R_{-1} e_{-1-2} |11\rangle = - R_1 |\sqrt{1/2} |00\rangle_- - \sqrt{3/2} |00\rangle_+ \).
\] (6.23)

Similarly, using (6.12) and (6.23), one has
\[
e_{-2} |12\rangle = (\sqrt{1/2} |00\rangle_- + \sqrt{3/2} |00\rangle_+.
\] (6.24)

Example 9: Consider again Example 7. On Fig. 4 we have summarized the relevant information, i.e., the basis vectors (6.14), (6.17), and (6.18) together with the matrix elements of generators relating them. Thus for instance, the nonzero matrix elements of \( e_{-1-2} = e_{-1} e_{-2} - e_{-2} e_{-1} \) are read off Fig. 4 as
\[
e_{-1-2} |32\rangle = (1/\sqrt{2}) |21\rangle_1 - (3/\sqrt{2}) |21\rangle_2.
\]
\[ e_{-1 - 2} |21\rangle_1 = -|10\rangle_1 + \sqrt{5/2} |10\rangle_3, \]
\[ e_{-1 - 2} |21\rangle_2 = -|10\rangle_1 - 2|10\rangle_2 - \sqrt{5/2} |10\rangle_3, \]

and
\[ e_{-1 - 2} |13\rangle = |02\rangle_1 = \sqrt{2} |02\rangle_2, \]
while for \( e_{1 + 2} = e_1 e_2 - e_2 e_1 \), one has
\[ e_{1 + 2} |10\rangle_1 = |21\rangle_1 + |21\rangle_2, \]
\[ e_{1 + 2} |10\rangle_2 = 2|21\rangle_1, \]
\[ e_{1 + 2} |10\rangle_3 = -\sqrt{5/2} |21\rangle_1 + \sqrt{5/2} |21\rangle_2, \]
\[ e_{1 + 2} |21\rangle_1 = -(1/2) |32\rangle, \]
\[ e_{1 + 2} |21\rangle_1 = (3/2) |32\rangle, \]
\[ e_{1 + 2} |02\rangle_1 = -|13\rangle, \quad e_{1 + 2} |02\rangle_2 = \sqrt{3} |13\rangle. \]

Proceeding as in the previous example one finds any other matrix elements of \( e_{g, \beta} \in \mathcal{A} \), in terms of those of Fig. 4.

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19. Corollary of Chap. VI. Sec. 1, 2 of Ref. 15.