Inequalities of Schwarz and Hlder type for random operators

Gerhard C. Hegerfeldt
Division of Physics, Mathematics, and Astronomy, California Institute of Technology, Pasadena, California 91125 and Institut für Theoretische Physik, Universität Göttingen, Göttingen, West Germany

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Let A and B be random operators on a Hilbert space, and let \( \langle \cdot \rangle \) denote averages (expectations). We prove the inequality \( \| \langle A^*B \rangle \| < \| \langle A A^* \rangle \|^{1/2} \| \langle B^*B \rangle \|^{1/2} \). A generalized Hölder inequality involving traces is also proved.

I. SCHWARZ INEQUALITY

In this paper we prove two inequalities, one of which was announced and extensively used before.\(^1\) Despite the simple proof, the inequalities seem not to have been published elsewhere; only a special case of the Schwarz-type inequality for commuting operators has appeared.\(^2\)

A random operator \( A \) is an operator-valued function \( A(\cdot) \) on some space \( \Omega \), with a given probability measure \( \mu \) on \( \Omega \). Averages or expectations are denoted interchangeably by \( \langle \cdot \rangle \) or \( E \). Thus

\[
\langle A \rangle = E A : = \int A(\omega) d\mu(\omega).
\]

We have to be a little bit more precise. Let \( \mathcal{H} \) be a separable Hilbert space and \( B(\mathcal{H}) \) the set of all bounded operators on \( \mathcal{H} \). Let \( \Sigma \) be the \( \sigma \) algebra of sets on \( \Omega \) on which \( \mu \) is defined. We assume that the complex-valued function \( \varphi, A(\cdot)\varphi \) is \( \Sigma \)-measurable for all \( \varphi, \psi \in \mathcal{H} \). Thus, a random operator is a weakly measurable \( B(\mathcal{H}) \)-valued function. A typical example is a random matrix.

It may happen that the expectation or average \( E(\varphi, A(\cdot)\varphi) \) exists for all \( \varphi, \psi \in \mathcal{H} \); in addition there is a bounded operator, denoted by \( E A \) or \( \langle A \rangle \), such that

\[
E(\varphi, A(\cdot)\varphi) = E(\varphi, A \varphi).\]

we say that the expectation of \( A \) exists (in the Pettis sense) and is given by \( E A \), or \( \langle A \rangle \). Clearly, \( \langle A \rangle \) exists if \( \| A \| \) does, and then \( \| \langle A \rangle \| < \| A \| \).

Theorem 1: Let \( A \) and \( B \) be random operators on a Hilbert space \( \mathcal{H} \). Then

\[
\| \langle A^*B \rangle \| < \| \langle A A^* \rangle \|^{1/2} \| \langle B^*B \rangle \|^{1/2},
\]

where the existence of the right-hand side (rhs) implies the existence of the left-hand side (lhs) and where \( \| \cdot \| \) is the usual operator norm on \( B(\mathcal{H}) \).

The consequences are analogous to those of Schwarz’s inequality and are proved in a similar way.

Corollary 1: For a random operator \( A \) we define

\[
\| A \|_\mu := \| \langle A A^* \rangle \|^{1/2},
\]

if the rhs exists. Then

\[
\| A + B \|_\mu < \| A \|_\mu + \| B \|_\mu \quad \text{(triangle inequality)},
\]

\[
\| A \|_\mu - \| B \|_\mu < \| A - B \|_\mu.
\]

If \( \| A \|_\mu = 0 \), then \( A = 0 \) with probability 1. Thus \( \| \cdot \|_\mu \) is a norm on (equivalence classes of) random variables.

Proof of Theorem 1: We use the ordinary Schwarz inequality, first for the \( d\mu \) integral (i.e., for expectations) and then for the scalar product in \( \mathcal{H} \). Let the rhs of Eq. (1.1) exist. Then

\[
\| \langle A^\varphi \rangle \| = E(\varphi, A A^\varphi) = \| \langle A A^* \rangle \|^{1/2} \| \varphi \|_\mu^2
\]

and similarly for \( B \). The two Schwarz inequalities now give

\[
\| \langle A^\varphi \rangle \|^{1/2} \| \langle B \psi \rangle \|^{1/2} > \langle E(\varphi, A^* A) \rangle,
\]

with existence implied. We now take the sup over \( \varphi \) and \( \psi \) with \( \| \varphi \| = \| \psi \| = 1 \). This shows that the rhs of Eq. (1.3) defines a bounded operator \( \langle A^*B \rangle \). Since \( \| \langle A A^* \rangle \|^{1/2} = \| \langle A^*A \rangle \|, \) its norm is bounded by the rhs of Eq. (1.1).

Q.E.D.

Using the existence statement of Theorem 1, the following analog of an inequality of Lieb and Ruskai\(^3\) is an easy consequence.

Corollary 2: Let \( \langle A A^* \rangle \) and \( \langle B B^* \rangle \) exist. Then for any \( \epsilon > 0 \)

\[
\langle A A^* \rangle + \epsilon ^{-1} \langle B B^* \rangle \geq \langle A A^* \rangle. \]

As a consequence

\[
\langle A B \rangle \langle B A \rangle \leq \langle B B^* \rangle \langle A A^* \rangle.
\]

Proof: Let \( Q = \{ B B^* \} + \epsilon ^{-1} \{ A A^* \}, \) which is nonrandom. Then one has

\[
0 \leq \langle A - B Q \rangle^* (A - B Q) + \epsilon Q Q^*.
\]

Expanding and taking expectation gives Eq. (1.4). From

\[
\| \langle B B^* \rangle + \epsilon ^{-1} \langle B B^* \rangle \| < 1
\]

one then obtains Eq. (1.5). Both also follow directly from Eq. (1.3).

Q.E.D.

An alternative proof of Theorem 1 was proposed to me, which is based on the observation that

\[
\langle A A^* \rangle \langle A A^* \rangle > 0.\]

Sandwiching this with \( \phi = (\lambda \psi \psi) \) from both sides and then taking expectations and the sup over \( \varphi, \psi \) yields Eq. (1.1). In a similar way Corollary 2 can be derived directly, with \( B B^* \) replaced by \( B B^* + \epsilon \).

We remark in passing that for dim \( \mathcal{H} < \infty \) the normed space \( \{ A : \langle A A^* \rangle_{\| \cdot \|_\mu}^{1/2} = \| A \|_\mu < \infty \} \) is complete, and the norms \( \| A \|_\mu \) and \( \| A^* A \|_{\| \cdot \|_\mu} \) are equivalent. For dim \( \mathcal{H} = \infty \) this is in general not true, and \( \langle A A^* \rangle \| < \infty \) does not imply \( \| A A^* \| < \infty \).

\( ^a \)Address until October 1985.
II. HÖLDER INEQUALITY

Defining $|A|^p := (A^* A)^{p/2}$, Eq. (1.1) can be written as
$$\|A B\| \leq \|A\|^1/2 \|B\|^1/2.$$ 

The corresponding Hölder-type inequality for $p \neq 2$ does not hold in general if $\dim \mathcal{H} > 2$. This can be shown by counterexamples. For trace norms, however, one has the following.

**Theorem 2:** Let $A$ and $B$ be random operators on a Hilbert space $\mathcal{H}$. Then, for $r > 1$ and $1/p + 1/q = 1/r$, $p, q > 0$,
$$\frac{1}{r} \leq \frac{1}{r} \leq \{\text{Tr} (A B)\}^{1/r} \leq \frac{1}{r} \leq \{\text{Tr} (A B)\}^{1/r}.$$ 

where existence of the rhs implies existence of the rest. Here $A^*$ may be replaced by $A$ in the middle and the lhs.

Similarly as before, we use Hölder’s inequality for integrals and then for trace norms. But first we note a simple fact.

**Lemma 1:** On positive random operators, trace and expectations commute,
$$E \text{Tr} |A| = \text{Tr} E |A|,$$ 
and existence of either side implies that of the other. In this case $A$ is trace class almost surely, $E A$ exists and is trace class, and
$$\text{Tr} EA = E \text{Tr} A.$$ 

**Proof:** Let $\{\varphi_n\}$ be an orthonormal basis in $\mathcal{H}$. Then
$$E |A| = \sum_n \text{Tr} |A| |\varphi_n\rangle \langle \varphi_n|,$$
by positivity. Hence, if the rhs is finite then $E |A|$ and $E A$ exist as bounded operators and the rhs equals $\text{Tr} E |A|$. Equation (2.3) then follows from Lebesgue’s bounded convergence.

**Q.E.D.**

**Proof of Theorem:** By Hölder’s inequality, first for integrals and then for trace norms, we have
$$\{E \text{Tr} |A|^p\}^{1/p} \leq \{E \text{Tr} B^q\}^{1/q} \leq \sum \langle \varphi_n| (\text{Tr} B)^{q/p} |\varphi_n\rangle^{1/q} \leq \{E \text{Tr} A |B|^q\}^{1/q} \leq \{E \text{Tr} A^* B |A|^q\}^{1/q}.$$ 

(2.4)

By Lemma 1, this proves the second part of Eq. (2.1), together with existence. The remainder follows from Lemma 2.

**Lemma 2:** Let $A$ be a random operator and let $p > 1$.

Then
$$\text{Tr} |EA|^p \leq \text{Tr} E |A|^p,$$ 

(2.5)

where existence of the rhs implies that of the lhs.

**Proof:** Let the rhs exist. By Lemma 1, $|A|^p$ is trace class almost surely. Since $|A|^p \leq \text{Tr} |A|^p$ one has
$$E |A|^p \leq \{E \text{Tr} |A|^p\}^{1/p} < \infty.$$ 

Hence $EA$ exists as a bounded operator.

Now let $X$ be any nonrandom operator with $|X|^r$ trace class, $1/p + 1/q = 1/r$. Then, by the second half of Eq. (2.1), $E |X A|^r$ exists and is trace class. Thus, by Lemma 1, $EXA = XEA$ is also trace class. By duality one now has

$$\{\text{Tr} |EA|^p\}^{1/p} = \sup_{\text{Tr}|X^r| = 1} \text{Tr} |XE A| \leq E \sup_{\text{Tr}|X^r| = 1} \text{Tr} |X A| \leq \{E \text{Tr} |A|^p\}^{1/p}.$$ 

(2.6)

Q.E.D.

It was pointed out to me that the argument in Eq. (2.4) can be replaced by an equivalent linear version of Hölder’s inequality, i.e.,
$$r^{-1} \text{Tr} A^* B |\lambda|^p \text{Tr} |A|^p + q^{-1} \lambda^{-q} \text{Tr} |B|^q,$$
for all $\lambda > 0$. Taking expectation and using Lemma 1 also gives the second part of Eq. (2.1).

**Remark:** Finiteness of the measure $\mu$ has only entered in the proof of Lemma 2. For nonfinite $\mu$, Theorem 1, Theorem 2 with $r = 1$, and Lemma 2 with $\mu = 1$ still hold, as does the second inequality of Theorem 2 for any $r > 1$.

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5. If $\mu = 1$, we take $X$ bounded and replace $\{\text{Tr} |X|^r\}^{1/q}$ by $\|X\|$.