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SLOWLY MODULATED OSCILLATIONS IN NONLINEAR DIFFUSION PROCESSES*

DONALD S. COHEN,† FRANK C. HOPPENSTEADT‡ AND ROBERT M. MIURA§

Abstract. It is shown here that certain systems of nonlinear (parabolic) reaction-diffusion equations have solutions which are approximated by oscillatory functions in the form $R(\xi-c\tau)P(t^*)$ where $P(t^*)$ represents a sinusoidal oscillation on a fast time scale t^* and $R(\xi-c\tau)$ represents a slowly-varying modulating amplitude on slow space (ξ) and slow time (τ) scales. Such solutions describe phenomena in chemical reactors, chemical and biological reactions, and in other media where a stable oscillation at each point (or site) undergoes a slow amplitude change due to diffusion.

1. Introduction. Nonlinear diffusion-reaction systems arise as models of chemical reactors, chemical and biological reactions and in several areas of population biology, notably genetics and ecology. Parameters occurring in the models for these diverse phenomena describe reaction rates, diffusivities, selection intensities, intrinsic growth rates, etc. In the absence of diffusion, these systems frequently are characterized by a stable static state for certain parameter values, by a stable limit cycle for others, or by more complicated behavior for still other parameter values. It often happens that as parameter values pass through a certain critical set, a stable static state will split into an unstable static state and a finite amplitude stable limit cycle which grows out of the static state. Parameter values for which such a bifurcation occurs are referred to here as *bifurcation points*. In this paper, we investigate some effects which diffusion can have on reacting systems whose parameters are near a bifurcation point.

The class of nonlinear diffusion equations which we will study here is:

$$\begin{aligned} u_t &= \theta_1 u_{xx} + F(u, v; \lambda), \\ v_t &= \theta_2 v_{xx} + G(u, v; \lambda), \end{aligned} \tag{1.1}$$

where $u_t = \partial u/\partial t$, $u_{xx} = \partial^2 u/\partial x^2$, etc. u and v usually represent concentrations of chemical reactants or biological species, and they are to be determined by this system and some initial and boundary conditions appropriate to the particular physical or biological system modeled. However, we do not state these auxiliary conditions since our interest is restricted here to studying the form of solutions away from regions where initial and boundary effects are dominant.

The constants θ_1 and θ_2 measure diffusivities of u and v, respectively, and the functions F and G describe interactions between u and v. We suppose that the

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‡ Courant Institute of Mathematical Sciences, New York University, New York, New York 10012. Supported in part by the U. S. Air Force Scientific Research Office under Grant AFOSR-71-2107 and by the U.S. Army Research Office under Grant DAAG29-74-G-0219.

§ Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada V6T 1W5. Supported in part by the National Science Foundation under Grant GP-34319.

reaction system contains a nondimensional parameter λ which characterizes certain aspects of the phenomenon, and we focus on the dependence of solutions on the parameter λ.

Since our attention is to be directed at behavior near a bifurcation point for the reacting system, we suppose that $\lambda = \lambda_0$ corresponds to the reaction system being at a bifurcation point, and then we analyze the diffusion-reaction system from λ near λ_0.

With no loss of generality, we may assume that $u = 0$, $v = 0$, corresponds to the static state of interest since a simple transformation of variables can always accomplish this. Therefore, u and v should be interpreted as deviations of concentrations from a static state of the reaction system. By distinguishing linear terms, we can rewrite the system as

$$\begin{aligned} u_t &= \theta_1 u_{xx} + \alpha(\lambda)u - \beta(\lambda)v + f(u, v, \lambda), \\ v_t &= \theta_2 v_{xx} + \beta(\lambda)u + \gamma(\lambda)v + g(u, v, \lambda), \end{aligned} \tag{1.2}$$

where f and g are second order in u and v. The coefficients α, β and γ, and the functions f and g are assumed to satisfy the following conditions:

H1: $\alpha(\lambda)$, $\beta(\lambda)$ and $\gamma(\lambda)$ are smooth functions of λ satisfying $\alpha(\lambda_0) = \gamma(\lambda_0) = 0$, $\alpha(\lambda) > 0$ and $\gamma(\lambda) \geqq 0$ for $\lambda > \lambda_0$, and $\beta(\lambda) \geqq 0$ for all λ.

H2: $f(u, v, \lambda)$ and $g(u, v, \lambda)$ are smooth functions of u, v and λ satisfying $f(u, v, \lambda)$, $g(u, v, \lambda) = o(|u| + |v|)$ as $|u| + |v| \to 0$.

Condition H1 ensures that the linear reaction matrix is purely oscillatory for $\lambda = \lambda_0$ and unstable for $\lambda > \lambda_0$. These conditions are similar to those arising in the Hopf theory for bifurcation of limit cycles. Systems in this form are familiar in various applications. For example, in the chemical reactor literature of problems involving oscillatory phenomena the appropriate transformations which reduce the basic equations of motion to form (1.2) are well known [1]. Also, a model of this form has been proposed for outbreaks of spruce budworms in Eastern Canada.[1]

Condition H2 guarantees the smoothness needed for our analysis and that the entire linear part of the system is accounted for by α, β and γ.

In various chemical and biochemical reactions, an oscillatory or periodic variation of reactants at each point in space will undergo a slow change or drift [2]–[6]. This takes place in the form of a slowly evolving envelope modulating the amplitudes of rapid oscillations. Similar phenomena are observed in certain biological systems. For example, plankton patches [7] correspond to a stationary envelope having a fixed wave length.

These observations motivate the study of (1.2) for solutions of special form. We will show that for $\lambda = \lambda_0 + \varepsilon$, where $0 < \varepsilon \ll 1$, the reaction-diffusion equations (1.2) possess small amplitude solutions that are approximated by

$$R(\xi - c\tau)P(t^*) \tag{1.3}$$

where P is a periodic (sinusoidal) function of $t^* = t(1 + O(\varepsilon^2))$ and the amplitude modulation R is a slowly varying steady progressing wave on slow space and slow

[1] Donald A. Ludwig, University of British Columbia, private communication.

time scales. Rather than (1.3), a general multi-scale analysis of (1.2) can be based on the leading-order approximation

$$R(x, \xi, \tau)P(t^*)$$

where $P(t^*)$ is a periodic function of t^*. Methods analogous to those developed below would lead to a diffusion-like equation for R. Instead of this general form, we specify the form of solutions we seek as in (1.3) and (2.2) thereby eliminating calculations which are not necessary for our limited goals. Our ultimate goal is to devise a method for obtaining approximations to finite amplitude locally periodic but nonsinusoidal solutions of reaction-diffusion equations of the general type (1.1).

The method used here is motivated by multi-scale methods for ordinary differential equations and can be viewed as an extension to dissipative systems of a nonlinear WKB method developed for conservative dispersive systems [8], [9]. Similar analyses using equations describing the slowly-varying envelope of a locally small amplitude nearly sinusoidal solution have been carried out [10], [11], [12]. However, in contrast to these previous analyses, the method developed here is not restricted to small amplitude solutions nor to nearly sinusoidal solutions; although for illustrative purposes, the class of examples given here does have these properties. The details for the more general case are being worked out.

In § 2, we develop the method, and in § 3, the equations governing the amplitude modulation R in different cases are studied in detail.

2. Slowly-varying periodic waves. In order to present the basic method and at the same time avoid lengthy algebraic calculations needed for arbitrary nonlinearities $f(u, v, \lambda)$ and $g(u, v, \lambda)$, we shall perform our investigation first on the special case of the system (1.2) given by

$$\begin{aligned} u_t &= \theta_1 u_{xx} + \alpha(\lambda)u - \beta v - \lambda u^3, \\ v_t &= \theta_2 v_{xx} + \beta u + \gamma(\lambda)v - \mu\lambda u^2 v, \end{aligned} \tag{2.1}$$

where $\alpha(\lambda_0)$, $\alpha'(\lambda_0)$, $\gamma(\lambda_0)$ and $\gamma'(\lambda_0)$ are zero, β and $\alpha''(\lambda_0)$ are positive and $\gamma''(\lambda_0) \geqq 0$. This system is commonly used to model first-order tubular chemical reactions where $\alpha(\lambda) = \gamma(\lambda)$ and μ are physical constants measuring reaction rates in that application [5]. Therefore, our results for this special case are also of physical interest.

As stated earlier, our goal in this paper is limited to finding a small amplitude solution of (2.1) which is locally a sinusoidal oscillation modulated by a slowly-varying wave for λ near the bifurcation point λ_0. With this in mind we let $\varepsilon \equiv \lambda - \lambda_0$, $0 < \varepsilon \ll 1$, and

$$\begin{aligned} u &= u(\eta, t^*; \varepsilon) \equiv \varepsilon F(\eta, t^*; \varepsilon), \\ v &= v(\eta, t^*; \varepsilon) \equiv \varepsilon G(\eta, t^*; \varepsilon), \end{aligned} \tag{2.2}$$

where we define new independent variables

$$\eta \equiv \xi - c\tau, \quad \xi \equiv \varepsilon x, \quad \tau \equiv \varepsilon^2 t, \tag{2.3}$$

$$t^* \equiv (1 - \varepsilon\omega(\varepsilon))^{-1} t. \tag{2.4}$$

The steady progressing wave is described by the coordinate η and it propagates with a velocity c which is not yet specified. Now the differentiations become

$$\frac{\partial}{\partial t}=(1-\varepsilon\omega(\varepsilon))^{-1}\frac{\partial}{\partial t^*}-\varepsilon^2 c\frac{\partial}{\partial\eta},\qquad \frac{\partial}{\partial x}=\varepsilon\frac{\partial}{\partial\eta}. \tag{2.5}$$

Note that ξ and τ correspond to slow space and time scales while t^* accounts for the Poincaré correction to the frequency due to the nonlinearity. Thus, ω is chosen as a smooth function of ε so as to suppress secular terms, i.e. such that the functions F and G are bounded functions of η and t^*. This provides a self-consistent procedure for determining u and v which remain bounded for all x and t.

We further assume that $\alpha(\lambda)$ and $\gamma(\lambda)$ can be written as

$$\begin{aligned}\alpha(\lambda)&=\tfrac{1}{2}\varepsilon^2\alpha''(\lambda_0)+\cdots\equiv\varepsilon^2\alpha_2+\varepsilon^3\hat{\alpha}(\varepsilon), &&\alpha_2>0,\\ \gamma(\lambda)&=\tfrac{1}{2}\varepsilon^2\gamma''(\lambda_0)+\cdots\equiv\varepsilon^2\gamma_2+\varepsilon^3\hat{\gamma}(\varepsilon), &&\gamma_2\geqq 0,\end{aligned} \tag{2.6}$$

where $\hat{\alpha}$ and $\hat{\gamma}$ are both of order unity. Using (2.2)–(2.6) in (2.1) yields a system of equations for F and G which we write in the form

$$\begin{aligned}&F_{t^*}+\beta G=\varepsilon\omega\beta G+\varepsilon^2(1-\varepsilon\omega)[\theta_1F_{\eta\eta}+cF_\eta+(\alpha_2+\varepsilon\hat{\alpha})F-(\lambda_0+\varepsilon)F^3],\\ &G_{t^*}-\beta F=-\varepsilon\omega\beta F+\varepsilon^2(1-\varepsilon\omega)\\ &\qquad\cdot[\theta_2G_{\eta\eta}+cG_\eta+(\gamma_2+\varepsilon\hat{\gamma})G-\mu(\lambda_0+\varepsilon)F^2G].\end{aligned} \tag{2.7}$$

We assume

$$\begin{aligned}\omega(\varepsilon)&=\varepsilon\omega_2+\varepsilon^2\omega_3+\cdots,\\ F(\eta,t^*;\varepsilon)&=F_0(\eta,t^*)+\varepsilon^2F_2(\eta,t^*)+o(\varepsilon^2),\\ G(\eta,t^*;\varepsilon)&=G_0(\eta,t^*)+\varepsilon^2G_2(\eta,t^*)+o(\varepsilon^2).\end{aligned}$$

Substituting into (2.7) and equating to zero the coefficient of each power of ε yields a sequence of systems of equations. The system determining F_0 and G_0 is given by

$$\begin{aligned}F_{0t^*}+\beta G_0&=0,\\ G_{0t^*}-\beta F_0&=0,\end{aligned}$$

and its general solution is given by

$$\begin{aligned}F_0(\eta,t^*)&=R(\eta)\cos(\beta t^*+\phi(\eta)),\\ G_0(\eta,t^*)&=R(\eta)\sin(\beta t^*+\phi(\eta)),\end{aligned} \tag{2.8}$$

where the functions $R(\eta)$ and $\phi(\eta)$ will be determined at a later stage of the perturbation scheme. The system for F_2 and G_2 is given by

$$\begin{aligned}F_{2t^*}+\beta G_2&=\theta_1F_{0\eta\eta}+cF_{0\eta}+\alpha_2F_0+\omega_2\beta G_0-\lambda_0F_0^3\equiv M_2,\\ G_{2t^*}-\beta F_2&=\theta_2G_{0\eta\eta}+cG_{0\eta}+\gamma_2G_0-\omega_2\beta F_0-\mu\lambda_0F_0^2G_0\equiv N_2.\end{aligned} \tag{2.9}$$

We can eliminate either one of the two variables F_2 or G_2 to obtain either of the equations

$$F_{2t^*t^*}+\beta^2F_2=M_{2t^*}-\beta N_2, \tag{2.10}$$

$$G_{2t^*t^*}+\beta^2G_2=N_{2t^*}+\beta M_2. \tag{2.11}$$

For solvability the inhomogeneous terms in (2.10) must satisfy the orthogonality conditions

$$\int_0^{2\pi/\beta}\begin{Bmatrix}\cos(\beta t^*+\phi(\eta))\\ \sin(\beta t^*+\phi(\eta))\end{Bmatrix}(M_{2t^*}-\beta N_2)\,dt^*=0.$$

The corresponding constraint for (2.11) is similar. This constraint simply eliminates the occurrence of secular terms in the solutions and keeps the solutions uniformly bounded. Thus using (2.8) in M_2 and N_2 we obtain

$$\begin{aligned}M_{2t^*}-\beta N_2=-2\beta\{&[\theta(R''-R\phi'^2)+cR'+\Gamma R-\nu R^3]\sin(\beta t^*+\phi)\\ &+[\theta(R\phi''+2R'\phi')+cR\phi'-\omega_2\beta R]\cos(\beta t^*+\phi)\\ &+(\text{third harmonics})\},\end{aligned}$$

where we have set

$$\theta\equiv\frac{1}{2}(\theta_1+\theta_2),\quad \Gamma\equiv\frac{1}{2}(\alpha_2+\gamma_2),\quad \nu\equiv\frac{3+\mu}{8}\lambda_0,\quad\text{and}\quad '=\frac{d}{d\eta}. \tag{2.12}$$

To eliminate secular terms the functions R and ϕ must satisfy the ordinary differential equations

$$\theta(R''-R\phi'^2)+cR'+\Gamma R-\nu R^3=0, \tag{2.13}$$

$$\theta(R\phi''+2R'\phi')+cR\phi'-\omega_2\beta R=0. \tag{2.14}$$

When the solution of this system is substituted in (2.8) the leading-order approximation to u and v is obtained. For general nonlinearities, $f(u, v, \lambda)$ and $g(u, v, \lambda)$, the equations for R and ϕ are derived similarly. If attention had not been restricted to finding steady progressing solutions earlier, then a system similar to (2.13), (2.14) arises, but it is a diffusion-like system for R and ϕ, in ξ and τ. The equations (2.13), (2.14) give the steady progressing wave form of that system. Diffusion equations of that form have arisen in other contexts, such as in studies of the Fisher equation [13] and in various fluid stability problems (see, e.g., [10] where other references to methods developed for continuum mechanics are listed).

2.1. The case $c\neq 0$. We now analyze (2.13), (2.14) in the separate cases $c\neq 0$ and $c=0$. First, we shall show that in the case $c\neq 0$ our requirement of bounded solutions for all x and t forces $\phi'=0$. Let

$$\delta\equiv c/\theta\quad\text{and}\quad\zeta\equiv\omega_2\beta/\theta,$$

then multiply (2.14) by $(R/\theta)\exp(\delta\eta)$ obtaining

$$(R^2e^{\delta\eta}\phi')'=\zeta R^2e^{\delta\eta},$$

which we solve for ϕ' getting

$$\phi'(\eta)=\frac{k\,e^{-\delta\eta}}{R^2(\eta)}+\frac{\zeta\,e^{-\delta\eta}}{R^2(\eta)}\int_0^\eta R^2(s)\,e^{\delta s}\,ds, \tag{2.15}$$

where k is an arbitrary constant. Integrate once by parts to obtain

$$\phi'(\eta)=\frac{k\,e^{-\delta\eta}}{R^2(\eta)}+\frac{\zeta}{\delta}\left(1-\frac{R^2(0)}{R^2(\eta)}e^{-\delta\eta}\right)-\frac{2\zeta e^{-\delta\eta}}{\delta R^2(\eta)}\int_0^\eta R(s)R'(s)\,e^{\delta s}\,ds.$$

With the requirement that $R(\eta)$ be bounded for all $\eta\in(-\infty,\infty)$, this exponential growth for $\phi'(\eta)$ as $\eta\to\pm\infty$ destroys consistency at higher orders in our perturbation procedure. Thus we take $k=\omega_2=0$, resulting in $\phi'(\eta)\equiv 0$. Then (2.13) reduces to

$$\theta R''+cR'+\Gamma R-\nu R^3=0. \tag{2.16}$$

2.2. The case $c=0$. In the case $c=0$ (or equivalently $\delta=0$), which would correspond to standing wave solutions, (2.15) is replaced by

$$\phi'(\eta)=\frac{k}{R^2(\eta)}+\frac{\zeta}{R^2(\eta)}\int_0^\eta R^2(s)\,ds,$$

where k is an arbitrary constant of integration. Thus for $\phi'(\eta)$ to remain bounded as $\eta\to\pm\infty$, we require $R(\eta)\neq 0$ and $\zeta=0$, so we are left with

$$\phi'(\eta)=\frac{k}{R^2(\eta)}. \tag{2.17}$$

If $k=0$, then (2.13) takes the integrable form

$$\theta R''+\Gamma R-\nu R^3=0. \tag{2.18}$$

If $k\neq 0$, it is convenient to introduce a new variable

$$\rho\equiv R^2>0.$$

Then (2.13) becomes

$$\theta\left(\rho''-\frac{1}{2}\frac{\rho'^2}{\rho}-2\frac{k^2}{\rho}\right)+2\Gamma\rho-2\nu\rho^2=0. \tag{2.19}$$

We give qualitative analyses of (2.16), (2.18), and (2.19) in the next section.

3. The slowly-varying modulating amplitude.

3.1. The case $c\neq 0$. We study the qualitative behavior of the solutions of (2.16) in the (R,R')-plane. Setting $T=R'$, we rewrite (2.16) as the first-order system

$$\begin{aligned} R'&=T,\\ T'&=-\frac{c}{\theta}T-\frac{\Gamma}{\theta}R+\frac{\nu}{\theta}R^3. \end{aligned} \tag{3.1}$$

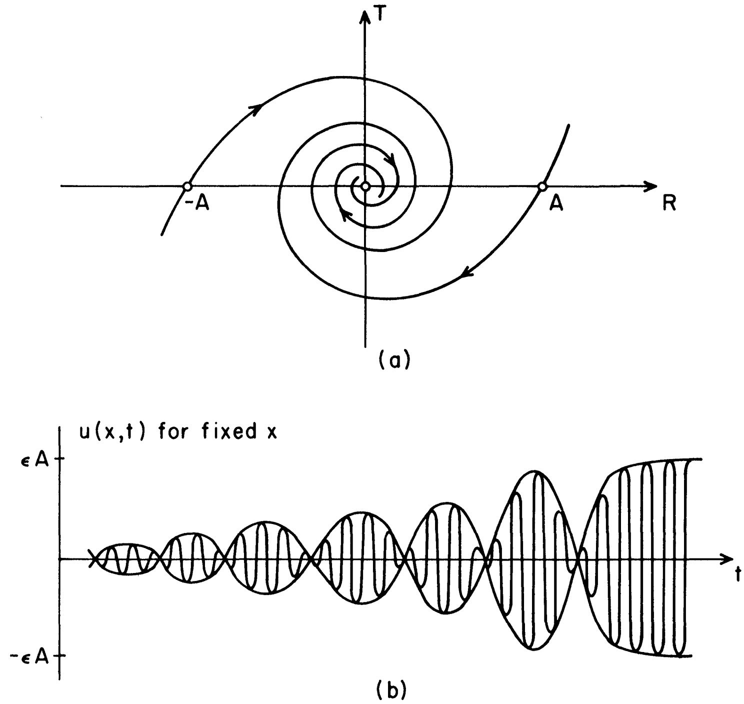

FIG. 1. $0<|c|<\sqrt{4\theta\Gamma}$

Note that the result of the transformation $R\to -R$ and $T\to -T$ leaves the equations invariant, and thus, the trajectories in the phase plane are symmetric with respect to the origin.

The equations (3.1) have three critical points given by $(R, T)=(0, 0)$ and $(R, T)=(\pm A, 0)$ where

$$A \equiv \sqrt{\Gamma/\nu}.$$

Upon linearizing about each critical point, we easily find that the points $(R, T)=(\pm A, 0)$ are saddle points, and that the origin is a spiral point if $0<|c|<\sqrt{4\theta\Gamma}$ and an improper node if $\sqrt{4\theta\Gamma}\leqq|c|$. In Figs. 1(a), 2(a), and 3(a) we have sketched the phase plane portraits corresponding to the three separate cases (the arrows denote the direction of increasing η). In Figs. 1(b), 2(b), and 3(b) we have illustrated the form of the solutions $u(x, t)=\varepsilon R(\eta)\cos(\beta t^*+\phi)$ for fixed x corresponding to each case, where $c>0$. Our perturbation procedure requires that functions $R(\eta)$ be defined and bounded for all real η, and the separatrices in the phase plane from $(\pm A, 0)$ to $(0, 0)$ are the only trajectories satisfying this requirement. Therefore the solutions of (2.16) correspond to separatrices and the amplitude of the oscillations in Figs. 1(b), 2(b), and 3(b) asymptotically approach the value εA. Furthermore, we see that although the location of each critical point does not depend on c, the type does and c is not determined by this analysis.

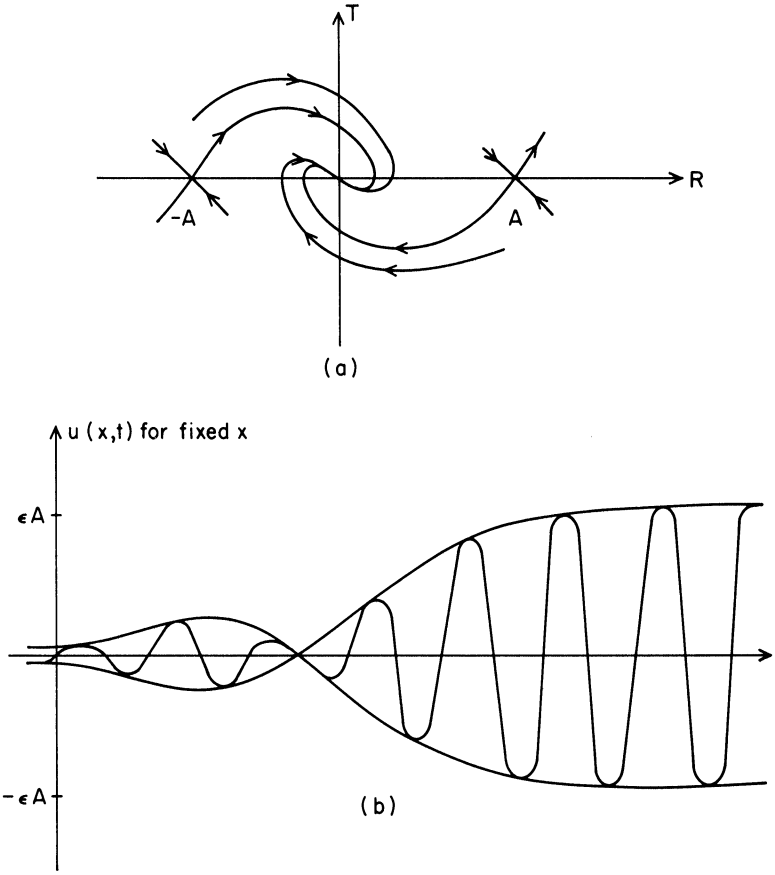

FIG. 2. $|c| = \sqrt{4\theta\Gamma}$

We now show that if $c \neq 0$, then necessarily c must be nonnegative. To demonstrate this, we multiply (2.16) by R' and integrate from $-\infty$ to $+\infty$. Then using the facts that $(R(-\infty), T(-\infty)) = (\pm A, 0)$ and $(R(\infty), T(\infty)) = (0, 0)$ we obtain

$$c = \frac{1}{4}\nu A^2 \Big/ \int_{-\infty}^{\infty} T^2 \, d\eta \tag{3.2}$$

Therefore the arrows in Figs. 1(a), 2(a) and 3(a) correspond to increasing x or decreasing t. However, since (2.13) and (2.14) are invariant under the combined sign changes $\eta \to -\eta$ and $c \to -c$, waves can propagate in either direction.

In summary, our solutions u and v are given to first order in ε by

$$u \sim \varepsilon R(\varepsilon x - c\varepsilon^2 t) \cos(\beta t + \phi),$$

$$v \sim \varepsilon R(\varepsilon x - c\varepsilon^2 t) \sin(\beta t + \phi),$$

where R is determined by (2.16) and ϕ is a constant phase shift (since $\phi' = 0$). Thus, the solutions are small amplitude sinusoidal oscillations modulated by a

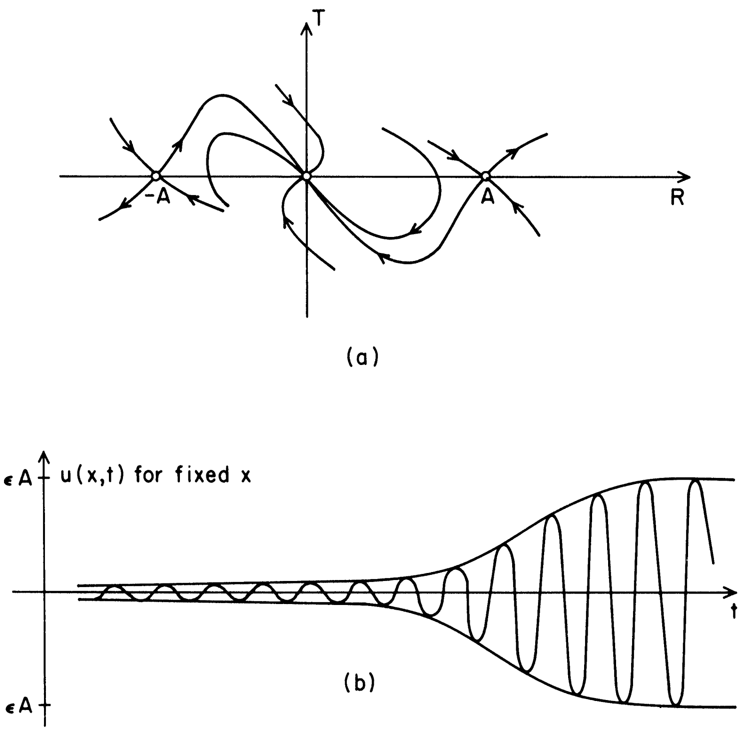

FIG. 3. $\sqrt{4\theta\Gamma}<|c|$

slowly-varying steady progressing wave with given speed c as long as (2.16) possesses bounded solutions for that value of c. Furthermore, the frequency of the oscillations is fixed through order ε^2.

3.2. The case $c=0$ and $k=0$. If $\int_{-\infty}^{\infty} T^2\, d\eta$ is infinite in (3.2), so that $c=0$, then we must treat either (2.18) or (2.19). For $k=0$, we treat (2.18) which can be integrated once (after multiplying by $2R'$) to give

$$\theta R'^2+\Gamma R^2-\tfrac{1}{2}\nu R^4=\text{const.}\geqq 0.$$

The critical points in the (R, T)-plane remain the same but now the origin is a center and the trajectories within the separatices are periodic (see Figs. 4(a) and 4(b); note that the abscissa in Fig. 4(b) is the x-axis and the vertical lines correspond to the temporal oscillations). The amplitude of R is less than A except when the trajectory is along the separatrix from $R=A$ to $R=-A$ in which case the solution $u(x, t)$ appears as in Fig. 4(c).

3.3. The case $c=0$ and $k\neq 0$. When $k\neq 0$, we study (2.19). Although not immediately apparent, (2.19) can be integrated once (this is apparent from (2.13) after using (2.17)) giving

$$\theta(\rho')^2+4\theta k^2+B\rho+4\Gamma\rho^2-2\nu\rho^3=0,$$

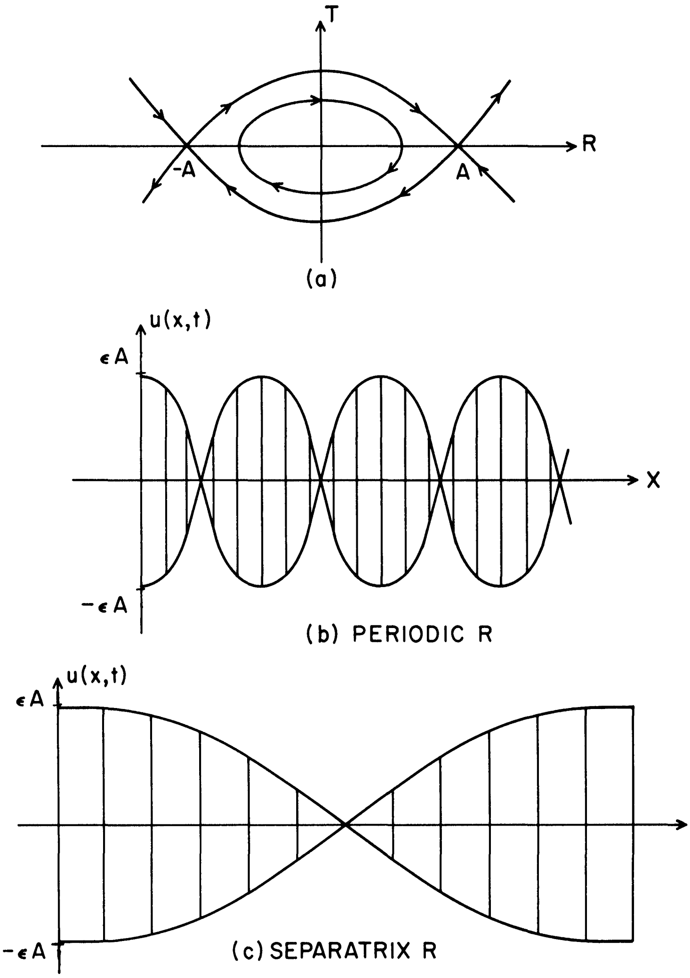

FIG. 4. $c=0$

where B is an unknown integration constant. Because B is unknown, the qualitative behavior of the solution is obtained directly from (2.19). With $r=\phi'$, (2.19) becomes the first-order system

$$\rho'=r,$$

$$r'=\frac{1}{2}\theta\frac{r^2}{\rho}+2\theta\frac{k^2}{\rho}+2\Gamma\rho-2\nu\rho^2.$$

The critical points in the (ρ, r)-plane occur at $r=0$ and the real roots of the cubic polynomial

$$f(\rho)\equiv\nu\rho^3-\Gamma\rho^2-\theta k^2,$$

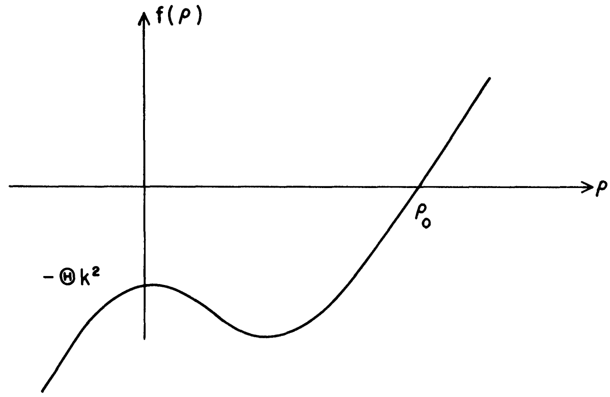

FIG. 5

which has the graph shown in Fig. 5. Clearly there is only one real root at $\rho = \rho_0$ and the other two roots are complex conjugates. Upon linearizing about this critical point, we obtain the real eigenvalues

$$\pm\sqrt{6\nu\rho_0 - 4\Gamma},$$

where the quantity under the square root is simply the slope of $f(\rho)$ at $\rho = \rho_0$ which is positive. Thus $(\rho, r) = (\rho_0, 0)$ is a saddle point and there are no bounded solutions for ρ other than the constant value ρ_0. From the point of view of diffusion effects, this is an uninteresting case.

4. Discussion. When diffusion is absent in the model equations (2.1), the amplitude equation for the bifurcation of a limit cycle from an equilibrium point is precisely (2.16) with $\theta_1 = \theta_2 = 0$, $c = 1$ and $\eta = c\tau$. This case has been studied in [1], [14]–[16]. We can see now that when diffusion is added, the frequency of the fast time oscillation remains unchanged to leading order, but a significant effect is the modulation of this oscillation by a slowly-varying wave. It seems reasonable to suppose that the wave speed c is determined by the initial data, and for any given speed c there probably exists a class of initial data, perhaps small, which evolves to the wave with the given c. This has been demonstrated for Fisher's equation ($\beta = 0$, $f = \lambda u(1-u)$) in [13], but has not yet been shown for the present case. Results of a linear stability analysis would be of interest, and are presently being investigated.

The formal perturbation scheme derived here gives oscillatory solutions which are slowly modulated in space and time. Such phenomena are observed in chemical reactors. Three such solutions have been described here—one shows the existence of precursor waves (Fig. 1(b)), a second shows a propagating front of oscillatory behavior (Figs. 2(b) and 3(b)), and the third shows no propagation at all, but a spatially periodic modulation of the temporal oscillation (Fig. 4(b)).

In the chemical engineering literature concentration and temperature profiles which undergo a slow change or drift are called *creeping profiles*. Such behavior has only relatively recently been observed [6], [17] in experiments on fixed-bed catalytic tubular chemical reactors. In particular, temperature profiles with the amplitude changing just like that illustrated in our Fig. 2 are observed. These profiles can propagate in either direction, as we found, with various speeds which are very slow compared to other changes. In the context of the theory of such chemical reactors it has been shown in [1] that equations (2.1) are a reliable model for a reactor in which there is a simple first-order exothermic reaction. In this case u corresponds to the temperature and v to the chemical concentration of one of the two reacting species. The other concentration immediately follows from the stochiometric balance.

An equation which is included in the class given by (2.1) is the regular oscillation limit of the Nagumo equation

$$U_t = U_{xx} + \varepsilon^2(U - \tfrac{1}{3}U^3) - V,$$
$$V_t = U.$$

The relaxation oscillation case ($\varepsilon \gg 1$) has been studied extensively in connection with nerve conduction problems. On the other hand, in the regular oscillation case ($\varepsilon \ll 1$), our analysis can be applied directly by making the identifications

$$U \equiv \frac{\sqrt{3\lambda}}{\varepsilon} u, \quad V \equiv \frac{\sqrt{3\lambda}}{\varepsilon} v, \quad \varepsilon \equiv \lambda - 1, \quad \alpha(\lambda) \equiv (\lambda - 1)^2,$$
$$\beta(\lambda) \equiv 1, \quad \gamma(\lambda) \equiv 0, \quad \theta_1 = \lambda_0 = 1, \quad \theta_2 = \mu = 0.$$

The result is that the leading-order solutions are given by

$$U \sim \sqrt{3} R(\varepsilon x - c\varepsilon^2 t) \cos(t + \phi),$$
$$V \sim \sqrt{3} R(\varepsilon x - c\varepsilon^2 t) \sin(t + \phi).$$

It is evident from the equation for R which results in this case, that the qualitative behavior of R is the same as described in § 3.

The system (1.2) without diffusion is typical of models for plankton populations (see, e.g., [18]) where for certain environmental parameters, stable oscillations between phytoplankton and zooplankton can be established when there is a limiting nutrient. In this case, our results describe certain dynamic behavior which is frequently observed, and they suggest other behavior on a finer scale which has been speculated in the literature. For example, in the case described in § 3.2 (Fig. 4(b)), a stationary, spatially periodic envelope occurs. This configuration corresponds to patches of plankton blooms within which oscillations occur, and contiguous patches are out of phase. Plankton patches are frequently observed [7], and quite recently oscillations (e.g., of the blue-green algae Oscillatoria (Trichodesmium) Sp. [19]) within patches have been detected.

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