Nonsmoothable, locally indicable group actions on the interval

Danny Calegari

By the Thurston Stability Theorem, a group of $C^1$ orientation-preserving diffeomorphisms of the closed unit interval is locally indicable. We show that the local order structure of orbits gives a stronger criterion for nonsmoothability that can be used to produce new examples of locally indicable groups of homeomorphisms of the interval that are not conjugate to groups of $C^1$ diffeomorphisms.

This note was inspired by a comment in a lecture by Andrés Navas. I would like to thank Andrés for his encouragement to write it up. I would also like to thank the referee, whose many excellent comments have been incorporated into this paper.

1 Non-smoothable actions

1.1 Thurston Stability Theorem

A simple, but important case of the Thurston Stability Theorem is usually stated in the following way:

**Theorem 1.1** (Thurston Stability Theorem [8]) Let $G$ be a group of orientation-preserving $C^1$ diffeomorphisms of the closed interval $I$. Then $G$ is locally indicable; i.e. every nontrivial finitely generated subgroup $H$ of $G$ admits a surjective homomorphism to $\mathbb{Z}$.

The proof is nonconstructive, and uses the axiom of choice. The idea is to “blow up” the action of $H$ near one of the endpoints at a sequence of points that are moved a definite distance, but not too far. Some subsequence of blow-ups converges to an action by translations.

Note that it is only finitely generated subgroups that admit surjective homomorphisms to $\mathbb{Z}$, as the following example of Sergeraert shows.
Example 1.2 (Sergeraert [7]) Let $G$ be the group of $C^\infty$ orientation-preserving diffeomorphisms of $I$ that are infinitely tangent to the identity at the endpoints. Then $G$ is perfect.

Another countable example comes from Thompson’s group.

Example 1.3 (Navas [6], Ghys–Sergiescu [3]) Thompson’s group $F$ of dyadic rational piecewise linear homeomorphisms of $I$ is known to be conjugate to a group of $C^\infty$ diffeomorphisms. On the other hand, the commutator subgroup $[F, F]$ is simple; since it is non-Abelian, it is perfect.

Given a group $G \subset \text{Homeo}_+(I)$. Theorem 1.1 gives a criterion to show that the action of $G$ is not conjugate into $\text{Diff}^1_+(I)$. It is natural to ask whether Thurston’s criterion is sharp. That is, suppose $G$ is locally indicable. Is it true that every homomorphism from $G$ into $\text{Homeo}_+(I)$ is conjugate into $\text{Diff}^1_+(I)$? It turns out that the answer to this question is no. However, apart from Thurston’s criterion, very few obstructions to conjugating a subgroup of $\text{Homeo}_+(I)$ into $\text{Diff}^1_+(I)$ are known. Most significant are dynamical obstructions concerning the existence of elements with hyperbolic fixed points when the action has positive topological entropy by Hurder [4], or when there is no invariant probability measure for some sub-pseudogroup by Deroin, Kleptsyn and Navas [2] (also, see Cantwell and Conlon [1]).

In this note we give some new examples of actions of locally indicable groups on $I$ that are not conjugate to $C^1$ actions.

Example 1.4 ($\mathbb{Z}^\mathbb{Z}$) Let $T: I \to I$ act freely on the interior, so that $T$ is conjugate to a translation. Let $I_0 \subset \text{int}(I)$ be a closed fundamental domain for $T$, and let $S: I_0 \to I_0$ act freely on the interior. Extend $S$ by the identity outside $I_0$ to an element of $\text{Homeo}_+(I)$. For each $i \in \mathbb{Z}$ let $I_i = T^i(I_0)$ and let $S_i: I_i \to I_i$ be the conjugate $T^iST^{-i}$. For each $f \in \mathbb{Z}^\mathbb{Z}$ define $Z_f$ to be the product:

$$Z_f = \prod_{i \in \mathbb{Z}} S_i^{f(i)}$$

Let $G$ be the group consisting of all elements of the form $Z_f$. Then $G$ is isomorphic to $\mathbb{Z}^\mathbb{Z}$ and is therefore abelian.

However, $G$ is not conjugate into $\text{Diff}^1_+(I)$. For, suppose otherwise, so that there is some homeomorphism $\varphi: I \to I$ so that the conjugate $G^\varphi \subset \text{Diff}^1_+(I)$. We suppose by abuse of notation that $S_i$ denotes the conjugate $S_i^\varphi$. For each $i$, let $p_i$ be the midpoint of $I_i$. Since for each fixed $i$ the sequence $S_i^\varphi(p_i)$ converges to an endpoint of $I_i$.
n goes to infinity, it follows that for each \( i \) there is some \( n_i \) so that \( dS_i^{n_i}(p_i) < 1/2 \). Let \( F \in \mathbb{Z}^Z \) satisfy \( F(i) = n_i \). Then \( dZ_F(p_i) < 1/2 \) for all \( i \). However, \( Z_F \) fixes the endpoints of \( I_i \) for all \( i \), so \( Z_F \) has a sequence of fixed points converging to 1. It follows that \( dZ_F(1) = 1 \). But \( p_i \rightarrow 1 \), so if \( Z_F \) is \( C^1 \) we must have \( dZ_F(1) \leq 1/2 \). This contradiction shows that no such conjugacy exists.

**Remark 1.5** The group \( \mathbb{Z}^Z \) is locally indicable, but uncountable. Note in fact that this group action is not even conjugate to a bi-Lipschitz action. On the other hand, Theorem D from [2] says that every countable group of homeomorphisms of the circle or interval is conjugate to a group of bi-Lipschitz homeomorphisms.

### 1.2 Order structure of orbits

In this section we describe a new criterion for nonsmoothability, depending on the local order structure of orbits.

**Definition 1.6** Let \( G \) act on \( I \) by \( \rho: G \rightarrow \text{Homeo}_+ (I) \). A point \( p \in I \) determines an order \( \prec_p \) on \( G \) by

\[
a \prec_p b \iff a(p) < b(p) \text{ in } I.
\]

Note that with this definition, \( \prec_p \) is really an order on the left \( G \)-space \( G/G_p \), where \( G_p \) denotes the stabilizer of \( p \).

**Lemma 1.7** Suppose \( \rho: G \rightarrow \text{Diff}^1_+ (I) \) is injective. Let \( H \) be a finitely generated subgroup of \( G \), with generators \( S = \{ h_1, \ldots, h_n \} \). Let \( p \in I \) be in the frontier of \( \text{fix}(H) \) (ie the set of common fixed points of all elements of \( H \)) and let \( p_i \rightarrow p \) be a sequence contained in \( I - \text{fix}(H) \). Then there is a sequence \( k_m \in \{ 1, \ldots, n \} \) and \( e_m \in \{ -1, 1 \} \) such that for any \( h \in [H, H] \), and for all sufficiently large \( m \) (depending on \( h \)), there is an inequality:

\[
h \prec_{p_m} h_{k_m}^{e_m}
\]

**Proof** There is a homomorphism \( \rho: H \rightarrow \mathbb{R} \) defined by the formula \( \rho(h) = \log h'(p) \). Of course this homomorphism vanishes on \( [H, H] \). If \( h_i \) is such that \( \rho(h_i) \neq 0 \) then (after replacing \( h_i \) by \( h_i^{-1} \) if necessary) it is clear that for any \( h \in [H, H] \), there is an inequality \( h \prec_{p_m} h_i \) for all \( p_m \) sufficiently close to \( p \). Therefore in the sequel we assume \( \rho \) is trivial.

For each \( i \), let \( U_i \) be the smallest (closed) interval containing \( p_i \cup Sp_i \). Given a bigger open interval \( V_i \) containing \( U_i \), one can rescale \( V_i \) linearly by \( 1/\text{length}(U_i) \) and move...
Danny Calegari

\( p_i \) to the origin thereby obtaining an interval \( \widetilde{V}_i \) on which \( H \) has a partially defined action as a pseudogroup.

The argument of the Thurston Stability Theorem implies that one can choose a sequence \( V_i \) such that any sequence of indices \( \to \infty \) contains a subsequence for which \( \widetilde{V}_i \to \mathbb{R} \), and the pseudogroup actions converge, in the compact-open topology, to a (nontrivial) action of \( H \) on \( \mathbb{R} \) by translations. In an action by translations, some generator or its inverse moves 0 a positive distance, but every element of \([H, H]\) acts trivially. The proof follows.

\[ \square \]

**Example 1.8** Let \( T \) be a hyperbolic once-punctured torus with a cusp. The hyperbolic structure determines up to conjugacy a faithful homomorphism \( \rho: \pi_1(T) \to \text{PSL}(2, \mathbb{R}) \).

The group \( \text{PSL}(2, \mathbb{R}) \) acts by real analytic homeomorphisms on \( \mathbb{RP}^1 = S^1 \). Since \( \pi_1(T) \) is free on two generators (say \( a, b \)) the homomorphism \( \rho \) lifts to an action \( \tilde{\rho} \) on the universal cover \( \mathbb{R} \). We choose a lift so that both \( a \) and \( b \) have fixed points. If we choose coordinates on \( \mathbb{R} \) so that \( a \) fixes \( x \), then \( a \) also fixes \( x + n \) for every integer \( n \). Similarly, if \( b \) fixes \( y \), then \( b \) fixes \( y + n \) for every \( n \). On the other hand, if \( p \in S^1 \) is the parabolic fixed point of \([a, b]\), and \( \tilde{p} \) is a lift of \( p \) to \( \mathbb{R} \), then the commutator \([a, b]\) takes \( \tilde{p} \) to \( \tilde{p} + 1 \). Since the action of every element on \( \mathbb{R} \) commutes with the generator of the deck group \( x \to x + 1 \), the element \([a, b]\) acts on \( \mathbb{R} \) without fixed points, and moves every point in the positive direction, satisfying \([a, b]^n(z) > z + n - 1\) for every \( z \in \mathbb{R} \) and every positive integer \( n \). See Figure 1.

![Figure 1](attachment:image.png)

Figure 1: In the lifted action, \( a \) and \( b \) have fixed points, but \([a, b]\) takes \( \tilde{p} \) to \( \tilde{p} + 1 \).

This action on \( \mathbb{R} \) can be made into an action on \( I \) by homeomorphisms, by including \( \mathbb{R} \) in \( I \) as the interior. Then the points \( \tilde{p} + n \to \infty \) in \( \mathbb{R} \) map to points \( p_n \to 1 \) in \( I \). Note that for each \( n \), the elements \( a \) and \( b \) have fixed points \( q_n, r_n \) respectively.
Nonsmoothable, locally indicable group actions on the interval 613

satisfying \( p_n < q_n < p_{n+1} \) and \( p_n < r_n < p_{n+1} \). Moreover, \([a, b](p_n) = p_{n+1}\) for all \( n \). It follows that

\[
a, a^{-1} < p_n [a, b]^2, \quad b, b^{-1} < p_n [a, b]^2
\]

for every \( n \), so by Lemma 1.7, this action is not topologically conjugate into \( \text{Diff}_+^1(I) \).

On the other hand, this is a faithful action of the free group on two generators. A free group is locally indicable, since every subgroup of a free group is free.

\textbf{Remark 1.9} The relationship between order structures and dynamics of subgroups of homeomorphisms of the interval is subtle and deep. For an introduction to this subject, see e.g. Navas [5].

\section*{References}


\textit{Department of Mathematics, California Institute of Technology}
\textit{Pasadena CA 91125, USA}
danny@its.caltech.edu
http://www.its.caltech.edu/~danny

Received: 3 December 2007 Revised: 1 March 2008