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ON THE GRAVITATIONAL FIELD PRODUCED BY LIGHT

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Abstract

Expressions are obtained, in accordance with Einstein's approximate solution of the equations of general relativity valid in weak fields, for the effect of steady pencils and passing pulses of light on the line element in their neighborhood. The gravitational fields implied by these line elements are then studied by examining the velocity of test rays of light and the acceleration of test particles in such fields. Test rays moving parallel to the pencil or pulse do so with uniform unit velocity the same as that in the pencil or pulse itself. Test rays moving in other directions experience a gravitational action. A test particle placed at a point equally distant from the two ends of a pencil experiences no acceleration parallel to the pencil, but is accelerated towards the pencil by *twice* the amount which would be calculated from a simple application of the Newtonian theory. The result is satisfactory from the point of view of the conservation of momentum. A test particle placed at a point equally distant from the two ends of the track of a pulse experiences no net integrated acceleration parallel to the track, but experiences a net acceleration towards the track which is satisfactory from the point of view of the conservation of momentum.

§1. *Introduction.* The purpose of this article is to investigate, from the standpoint of the general theory of relativity, the gravitational field in the neighborhood of steady beams and isolated pulses of light.[1]

In Part I of the article we shall obtain a general expression for the line element in the presence of a flow of electromagnetic radiation, which will be taken for simplicity as travelling in the X-direction. The calculation will be made on the assumption that the effect of the radiation is small enough so that Einstein's approximate solution of the gravitational equations valid in weak fields can be employed.

In Part II we shall then examine the special form assumed by the line element in the neighborhood of a thin pencil of radiation travelling along the X-axis. We shall find that the pencil must be taken as having only a finite length in order to keep the field weak enough to justify the use of Einstein's approximate method of solution, and hence shall consider the pencil of radiation as travelling between an emitting and an absorbing body which are a finite distance apart.

Abstracting from the gravitational effect of the absorbing and emitting bodies, we shall then investigate the gravitational action of the radiation by determining the effect of its field on the velocity of test rays of light and on the acceleration of test particles in the neighborhood of the pencil. As to the

[1] An investigation of the gravitational field of radiation has also recently been carried out by Rosenfeld, Zeits. f. Physik **65**, 589 (1930). The article deals, however, with questions relating to quantization rather than with the problems which we shall treat in the present paper.

behavior of the test rays, we shall find that a ray of light moving parallel to the pencil and in the same direction would have unit velocity, the same as that in the pencil itself; but in the case of rays moving in other directions we shall find that they would suffer a gravitational disturbance when in the field of the pencil. As to the behavior of test particles, we shall find that a stationary test particle located at a point equally distant from the two ends of the pencil would experience no acceleration parallel to the track of the pencil, but would be accelerated towards the pencil by *twice* the amount, which would be calculated from the Newtonian theory of gravitation by taking the gravitational mass of the radiation as equal to its energy divided by the square of the velocity of light.

The fact that the acceleration of the particle is twice that which would be calculated from a simple application of the Newtonian theory does not seem surprising, when we recall that the bending of a ray of light in passing through a given gravitational field, say that of the sun, is found theoretically and experimentally in the case of weak enough fields to permit the Newtonian approximation, also to be *twice* that which would be calculated from the Newtonian theory taking the gravitational mass of the light as equal to its inertial mass. The occurrence of the factor 2 in both places permits us to retain our familiar ideas as to the conservation of momentum as a first approximation.

In Part III we shall turn to a consideration of the gravitational field in the neighborhood of a passing pulse of radiation, and shall find that this would vary with the time in a manner to be calculated by the method of retarded potentials, the gravitational influence spreading out from the pulse with the velocity of light. As in the case of the pencil of radiation, we shall find also here that we must limit the track of the pulse so as to lie between an emitting and an absorbing body a finite distance apart, since Einstein's approximate method of solving the gravitational equations would fail at the time when the pulse passes abreast of the point of interest, if we should assume an infinite length for the track.

With regard to the behavior of test rays of light in the neighborhood of the track, we shall again find that a ray of light moving parallel and in the same direction as the pulse would have unit velocity, the same as that of the pulse itself, and that rays of light moving in other directions would be subject to a gravitational effect when in the field of the passing pulse. To investigate the behavior of particles in the field of the pulse, we shall consider a stationary test particle, located at a point equally distant from the two ends of the track lying between the source and absorber, and then determine in accordance with the method of retarded potentials what acceleration the particle would receive as a result of the gravitational influences emitted from this portion of the track. Parallel to the track we shall find that the particle would first be accelerated in the direction of motion of the pulse and later in the opposite direction, but that the time integral of this component of the acceleration corresponding to the total influence emitted from the portion of the track considered would be zero. Perpendicular to the motion of the pulse we shall find as a *net* result that the particle would act as though attracted towards the track, and that the time integral of the perpendicular acceleration, corresponding to the total influence emitted from the portion of the track considered, would be just sufficient so that here also as in the case of the pencil of radiation we could maintain in first approximation our older ideas as to the conservation of momentum for the gravitational interaction of a particle with light.

PART I. GENERAL TREATMENT OF THE GRAVITATIONAL FIELD OF RADITION

§2. *Einstein's approximate solution of the field equations.* We may now proceed to a general treatment of the gravitational field due to radiation. We shall assume the total amount, density and distribution of the radiation to be such that the gravitational potentials $g_{\mu\nu}$ occurring in the general theory of relativity will differ but slightly from the values which they have in empty space. Such an assumption would presumably be justified in the case of any ordinary beam or pulse of light that we might encounter in nature or the laboratory, and introduces a great simplification by permitting the use of Einstein's approximate solution of the gravitational equations, valid in the case of weak fields.

In accordance with this approximate treatment we may write the line element in the form

$$ds^2 = g_{\mu\nu}dx_\mu dx_\nu = (\delta_{\mu\nu} + h_{\mu\nu})dx_\mu dx_\nu \tag{1}$$

where the symbols $\delta_{\mu\nu}$ denote the Galilean values of $g_{\mu\nu}$, namely -1, $+1$ and 0, and the $h_{\mu\nu}$ are small quantities of the first order whose square can be neglected. We may also use the symbols $\delta^{\mu\nu}$ and δ^ν_μ to denote the Galilean values of $g^{\mu\nu}$ and g^ν_μ and define the quantities h^ν_μ and h by the equations

$$h_\mu^\nu = \delta^{\nu\alpha}h_{\mu\alpha} \text{ and } h = h_\alpha^\alpha. \tag{2}$$

It can then be shown that an approximate solution of the equations of general relativity which connect the metric with the distribution of matter and energy is given by the equation[2]

$$\left[h_\mu^\nu - \tfrac{1}{2}\delta_\mu^\nu h\right](x, y, z, t) = -4\iiint \frac{[T_\mu^\nu](\bar{x}, \bar{y}, \bar{z}, t-r)}{r} d\bar{x}d\bar{y}d\bar{z} \tag{3}$$

where the integration is to be taken over all elements of spatial volume $d\bar{x}\,d\bar{y}d\bar{z}$, the quantity r is the coordinate distance from the element of volume at $\bar{x}$, $\bar{y}$, $\bar{z}$ to the point of interest at x, y, z, and $[T_\mu^\nu]$ is the value of the energy-momentum tensor in the element of volume at such a time $t-r$, that an influence travelling from $\bar{x}$, $\bar{y}$, $\bar{z}$ with unit velocity would reach the point of interest x, y, z at the time of interest t.

§3. *The energy-momentum tensor for radiation.* To use the above solution in our problem, we must have an expression for the energy-momentum tensor for radiation. In any purely electromagnetic field, however, is it known that

[2] See Eddington, "The Mathematical Theory of Relativity," Cambridge 1923, Eq. (57.6)

the components of the energy-momentum tensor using natural coordinates have the values illustrated by the following typical examples.[3]

$$T_1^1 = \tfrac{1}{2}(X^2 - Y^2 - Z^2) + \tfrac{1}{2}(\alpha^2 - \beta^2 - \gamma^2) \tag{4}$$

$$T_1^2 = XY + \alpha\beta \tag{5}$$

$$T_1^4 = \beta Z - \gamma Y \tag{6}$$

$$T_4^4 = \tfrac{1}{2}(X^2 + Y^2 + Z^2) + \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) \tag{7}$$

where X, Y, Z are the components of the electric field strength and α, β, γ are the components of the magnetic field strength at the point of interest.

Applying these equations now to the case of radiation which we take for simplicity to be moving parallel to the X-axis in the positive direction, and noting the relations of (4) and (5) to the Maxwellian stresses in the field, of (6) to the Poynting vector, and of (7) to the energy density, it can be shown that the only surviving components of the energy-momentum tensor for the case of polarized or unpolarized incoherent radiation will be

$$T_1^1 = -\rho \qquad T_4^4 = \rho \qquad T_1^4 = -\rho \qquad T_4^1 = \rho \tag{8}$$

where ρ is the density of energy, and, since the velocity of light is unity, also the density of energy flow at the point of interest. The results assume the justifiability of neglecting diffraction phenomena, such as would occur at the boundary of a beam of radiation where the hypothesis of a flow of energy solely in the X-direction could not be strictly valid.

§4. *The general form of line element due to radiation.* The above equations give the components of the energy-momentum tensor in coordinates which are Galilean at the point of measurement, but to the order of approximation of the solution we are to use, they may also be taken as the components of that tensor in our general system of coordinates which differs from the Galilean form only by quantities of the first order, and hence may be substituted into equation (3) to determine the form of the line element.

Doing so we obtain

$$\begin{aligned} h_1^1 - \tfrac{1}{2}h &= 4\int\frac{[\rho]dV}{r} \\ h_2^2 - \tfrac{1}{2}h &= 0 \\ h_3^3 - \tfrac{1}{2}h &= 0 \\ h_4^4 - \frac{1}{2}h &= -4\int\frac{[\rho]dV}{r} \\ h_1^4 = -h_4^1 &= 4\int\frac{[\rho]dV}{r} \end{aligned} \tag{9}$$

with all other components of h_μ^ν equal to zero. And remembering the value of h and the method of raising suffixes given by equations (2) we can easily solve these equations for the $h_{\mu\nu}$. We then obtain as the only surviving components

[3] See Eddington, reference 2, Eqs. (77.41–4).

$$h_{11} = h_{44} = -h_{14} = -h_{41} = -4\int \frac{[\rho]dV}{r} \tag{10}$$

and these values of the $h_{\mu\nu}$ can be substituted into Eq. (1) to give the general form of the line element due to radiation travelling in the X-direction.

Part II. The Gravitational Action of a Pencil of Light

§5. *The line element in the neighborhood of a limited pencil of radiation.* As a first application of the expression for the line element due to radiation travelling in the X-direction, it would be natural to try to treat the gravitational field in the neighborhood of a thin pencil of radiation of uniform density, stretching in the X-direction from minus to plus infinity. But this proves to be impossible by the method adopted, since the values of the $h_{\mu\nu}$ come out infinite when the integration indicated in Eq. (10) is performed, which would invalidate the approximate solution of the gravitational equations that has been used.

If we consider, however, a thin pencil of radiation of limited length l and constant linear density ρ, passing steadily along the X-axis between a source at $x=0$ and an absorber at $x=l$, this difficulty does not arise, since we can then evidently write in accordance with Eq. (10) for the contribution of the radiation to the gravitational field at any point of interest x, y, z in the neighborhood of the pencil,

$$\begin{aligned} 4\int \frac{[\rho]dV}{r} &= -h_{11} = -h_{44} = h_{14} = h_{41} \\ &= \int_{u=0}^{u=l} \frac{4\rho du}{[(x-u)^2+y^2+z]^{21/2}} \\ &= 4\rho \log \frac{[(l-x)^2+y^2+z^2]^{1/2}+l-x}{[x^2+y^2+z^2]^{1/2}-x} \end{aligned} \tag{11}$$

which remains finite for finite values of the density ρ and length of pencil l.

It should be noted that Eq. (11) has been derived on the assumption of a *steady* pencil of radiation between the source and absorber, so that no explicit introduction of retarded potentials into the calculation was necessary. Hence of course the expression obtained would not be applicable in the neighborhood of times when the pencil is being started or stopped. It should also be noted that Eq. (11) gives only the contribution of the radiation in the pencil to the gravitational field and neglects the contribution of the bodies which act as source and absorber. This includes a neglect of any effects resulting from changes in the motion or internal condition of these bodies which might themselves be thought of as connected with the flow of radiation.

With these restrictions, however, the values of the surviving components of $h_{\mu\nu}$ given by Eq. (11) may be substituted into Eq. (1) to give the form of the line element in the neighborhood of the pencil of radiation.

§6. *Velocity of a test ray of light in the neighborhood of the pencil.* Having obtained Eq. (11) which gives the effect of the radiation on the form of the line element, we may now investigate the motion of test rays of light and test particles as affected by the presence of the pencil. In accordance with the theory of general relativity, the velocity of light will be given by setting the expression for the line element ds equal to zero. Substituting the values for the components of $h_{\mu\nu}$ given by (11) into the general expression for the line element (1) setting the result equal to zero, and writing

$$h = 4\rho \log \frac{[(l-x)^2 + y^2 + z^2]^{1/2} + l - x}{[x^2 + y^2 + z^2]^{1/2} - x} \tag{12}$$

as an abbreviation, (not the h of §2) we then easily obtain for the velocity of a test ray in the X-direction parallel to the pencil, the two cases

$$\frac{dx}{dt} = +1 \quad \text{and} \quad -\frac{1-h}{1+h} \tag{13}$$

and for the velocity say in the Y-direction in a plane perpendicular to the pencil, the two cases

$$\frac{dy}{dt} = \pm (1-h)^{1/2}. \tag{14}$$

In accordance with these expressions, we note that a test ray of light moving parallel to the pencil and in the same direction would have unit velocity the same as that in the pencil itself, but that rays moving in other directions would have a variable velocity depending as might be expected on their position in the gravitational field of the pencil. Furthermore, it can easily be shown for the case of a test ray moving parallel to the pencil and in the same direction that we should have, not only $dx/dt = 1$, but also $d^2x/dt^2 = d^2y/dt^2 = d^2z/dt^2 = 0$, which is a satisfactory result from the point of view of the stability of the light pencil itself.

§7. *Acceleration of a test particle in the neighborhood of the pencil.* We may also use our knowledge of the line element to investigate the gravitational acceleration which would be experienced by a test particle placed in the neighborhood of the pencil.

In accordance with the theory of general relativity, the acceleration of a particle is determined by the equation for a geodesic

$$\frac{d^2x_\alpha}{ds^2} + \{\mu\nu, \alpha\} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0 \tag{15}$$

and if we apply this equation to a particle which is at rest in our system of coordinates, we can substitute

$$\frac{dx_\mu}{ds} = 0 \text{ for the cases } \mu = 1, 2, 3 \tag{16a}$$

and to our order of approximation

$$\frac{dx_\mu}{ds} = \frac{dt}{ds} = 1 \text{ for the case } \mu = 4 \tag{16b}$$

so that equation (15) will then assume the simple form

$$\frac{d^2x}{dt^2} = -\{44, 1\}, \quad \frac{d^2y}{dt^2} = -\{44, 2\}, \quad \frac{d^2z}{dt^2} = -\{44, 3\} \tag{17}$$

for the cases $\alpha = 1, 2, 3$.

To calculate the values of the Christoffel symbols occurring in equations (17), we have the general equation of definition

$$\{\mu\nu, \sigma\} = \frac{1}{2} g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right) \tag{18}$$

and noting the values of the $g_{\mu\nu}$ which correspond to Eqs. (11), we shall evidently obtain to our order of approximation the simple expressions

$$\frac{d^2x}{dt^2} = -\frac{1}{2}\frac{\partial h_{44}}{\partial x}, \quad \frac{d^2y}{dt^2} = -\frac{1}{2}\frac{\partial h_{44}}{\partial y}, \quad \frac{d^2z}{dt^2} = -\frac{1}{2}\frac{\partial h_{44}}{\partial z}. \tag{19}$$

Substituting then for h_{44} the value given by (11), and performing the indicated differentiations, we can finally obtain, after some simplifications, for the acceleration of a stationary test particle parallel to the line of the pencil the expression

$$\frac{d^2x}{dt^2} = 2\rho\left\{\frac{1}{[x^2 + y^2 + z^2]^{1/2}} - \frac{1}{[(l - x)^2 + y^2 + z^2]^{1/2}}\right\} \tag{20}$$

and for the acceleration in a plane perpendicular to the pencil expressions of the form

$$\frac{d^2y}{dt^2} = -\frac{2\rho y}{y^2 + z^2}\left\{\frac{x}{[x^2 + y^2 + z^2]^{1/2}} + \frac{l - x}{[(l - x)^2 + y^2 + z^2]^{1/2}}\right\}. \tag{21}$$

The first thing of importance to be noted from these rather complicated expressions for the acceleration is the fact, in accordance with Eq. (20), that the acceleration parallel to the pencil becomes zero for a particle placed at a point $x = l/2$ midway between the two ends of the pencil. Furthermore for a particle midway between the two ends of the pencil, with $x = l/2$ and $z = 0$, Eq. (21) gives for the acceleration towards the pencil the simple result

$$-\frac{d^2y}{dt^2} = \frac{2\rho l}{y[(l/2)^2 + y^2]^{1/2}}. \tag{22}$$

The most important characteristic of these expressions for the acceleration, however, is the fact that they can easily be shown to be exactly *twice* as great as would be calculated from the simple Newtonian theory by taking the gravitational mass of the radiation equal to its inertial mass. This is due to the fact that the quantity $h_{44}/2$ occurring in Eqs. (19) is twice the simple Newtonian expression for the gravitational potential of the pencil. As noted in the introduction this is a very satisfactory result, since in the case of weak fields we now see that the factor 2 occurs not only as is known in the expression for the action of a particle on light, but also in the expression for the action of light on a particle, and we are hence able to retain in first approximation our familiar ideas as to the conservation of momentum.

Part III. The Gravitational Action of a Pulse of Light

§8. *The line element in the neighborhood of a pulse of radiation.* As already noted, the considerations of Part II were only applicable to the gravitational field surrounding a steady pencil of light and we shall now turn to a consideration of the gravitational field of a passing pulse of radiation. This will be more complicated to treat since the field will obviously be non-static and we shall have to make specific use of the method of retarded potentials to determine the way in which the gravitational effect spreads out from the moving pulse.

Let us consider a pulse of radiation, of length λ, linear density ρ, and small cross-section, travelling along the X-axis from $x=0$ to $x=l$, which may be taken as the points at which the radiation emerges from the source and enters the absorber, or as giving an arbitrary portion of the track selected for investigation. Furthermore, let us for convenience choose our time scale to make $t=0$ when the front end of the pulse crosses the point $x=0$, so that at any later time t the front end of the pulse will be located at $x=t$ and the rear end at $x=t-\lambda$.

Let us now take some point of interest x, y, z in the neighborhood of the track, and calculate with the help of Eq. (10) the gravitational field produced by the pulse at this point at the time t. Since Eq. (10) has to be applied in accordance with the method of retarded potentials, let us denote by $x=a$ the position of the front end of the pulse and by $x=b$ the position of the rear end of the pulse when they "emit" the gravitational influence which is received at the point x, y, z at the time t. Then we may evidently write in accordance with Eq. (10) for the gravitational potentials at x, y, z and t

$$
\begin{aligned}
4\int\frac{[\rho]dV}{r} &= -h_{11} = -h_{44} = h_{14} = h_{41} \\
&= \int_{u=b}^{u=a}\frac{4\rho du}{[(x-u)^2+y^2+z]^{21/2}} \\
&= 4\rho\log\frac{[(x-a)^2+y^2+z^2]^{1/2}-x+a}{[(x-b)^2+y^2+z^2]^{1/2}-x+b}
\end{aligned}
\qquad (23)
$$

provided we take the YZ-dimensions of the pulse as small compared with $(y^2+z^2)^{1/2}$.

To evaluate this result, however, we must determine a and b as functions of the time. To do this we note that $t-a$ is the distance through which the gravitational influence "travels" in going from the front end of the pulse to reach the point x, y, z at the time t, and hence we can evidently write

$$(t-a)^2 = (x-a)^2 + y^2 + z^2 \tag{24}$$

and solving obtain for a the expression

$$a = \frac{t^2 - x^2 - y^2 - z^2}{2(t-x)} \cdot \tag{25}$$

Similarly we obtain for b the expression

$$b = \frac{(t-\lambda)^2 - x^2 - y^2 - z^2}{2(t-\lambda-x)} \cdot \tag{26}$$

In using these expressions in connection with Eq. (23), however, we shall take the position of the rear end of the pulse to be $b=0$ until the pulse has completely emerged from the source at $x=0$, and take the position of the front end as $a=l$ after the pulse has started to enter the absorber at $x=l$. We do this since our interest lies in the gravitational influences correlated with the track of the pulse between $x=0$ and $x=l$.

Substituting the above values of a and b into Eq. (23), we then easily obtain the three following cases for the time intervals indicated

$$\begin{aligned} 4\int\frac{[\rho]dV}{r} &= -h_{11} = -h_{44} = h_{14} = h_{41} \\ &= 4\rho\log\frac{t-x}{[x^2+y^2+z^2]^{1/2}-x} \quad \begin{cases}\text{from } t = [x^2+y^2+z^2]^{1/2} \\ \text{to} \quad t = [x^2+y^2+z^2]^{1/2}+\lambda\end{cases} \\ &= 4\rho\log\frac{t-x}{t-\lambda-x} \quad \begin{cases}\text{from } t = [x^2+y^2+z^2]^{1/2}+\lambda \\ \text{to} \quad t = l+[(l-x)^2+y^2+z^2]^{1/2}\end{cases} \\ &= 4\rho\log\frac{[(l-x)^2+y^2+z^2]^{1/2}-x+l}{t-\lambda-x} \quad \begin{cases}\text{from } t = l+[(l-x)^2+y^2+z^2]^{1/2} \\ \text{to} \quad t = l+[(l-x)^2+y^2+z^2]^{1/2}+\lambda\end{cases} \end{aligned} \tag{27}$$

and the form of the line element in the different time intervals indicated will be given by substituting these values for the surviving components of $h_{\mu\nu}$ in the general expression for the line element (1). The results show that here also as in the case of the pencil it is necessary to take a limited track for the light pulse, since if the pulse were regarded as coming from an infinitely remote position on the X-axis the field would become infinite at the time $t=x$ when the pulse comes abreast of the point of interest. With our treatment, however, there is no effect from the track of the pulse before the time $t=[x^2+y^2+z^2]^{1/2}$, nor after the time $t=l+[(l-x)^2+y^2+z^2]^{1/2}+\lambda$.

§9. *Velocity of a test ray of light in the neighborhood of the pulse.* Substituting these values for the components of $h_{\mu\nu}$ into the expression for the line element (1) and setting the result equal to zero, we can now obtain information as to the velocity of test rays as affected by the pulse. Writing for simplicity

$$h = 4\int\frac{[\rho]dV}{r} \tag{28}$$

as a general abbreviation for the different expressions given by Eqs. (27), we then easily obtain for the velocity of a test ray in the X-direction parallel to the track, the two cases

$$\frac{dx}{dt} = +1 \text{ and } -\frac{1-h}{1+h} \tag{29}$$

and for the velocity in a plane perpendicular to the track, say in the Y-direction, the two cases

$$\frac{dy}{dt} = \pm (1-h)^{1/2}. \tag{30}$$

As in the case of the pencil, we note that a test ray of light moving parallel to the motion of the pulse and in the same direction would have unit velocity the same as that for the pulse itself, but that rays moving in other directions would have a variable velocity depending on time and position. Here too, it can easily be shown for the case of a test ray moving parallel to the motion of the pulse and in the same direction that we should have, not only $dx/dt=1$, but also $d^2x/dt^2=d^2y/dt^2=d^2z/dt^2=0$, which is a satisfactory result from the point of view of the stability of the pulse itself.

§10. *Acceleration of a test particle due to the pulse.* We must now investigate the gravitational acceleration which would be experienced by a test particle as a result of the passage of the pulse of light. If we take the particle as stationary the accelerations will evidently be determined by the same Eqs. (17)

$$\frac{d^2x}{dt^2} = -\{44,1\}, \quad \frac{d^2y}{dt^2} = -\{44,2\}, \quad \frac{d^2z}{dt^2} = -\{44,3\} \tag{31}$$

as were applicable in the case of the pencil, but the values of the Christoffel symbols will of course be different. To calculate these we may again start with the general equation of definition

$$\{\mu\nu,\sigma\} = \frac{1}{2}g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right) \tag{32}$$

and noting the values of the $g_{\mu\nu}$ which correspond to Eqs. (27), we can evidently write to our order of approximation

$$\begin{aligned}\{44,1\} &= -\frac{1}{2}\left(\frac{\partial h_{41}}{\partial t} + \frac{\partial h_{41}}{\partial t} - \frac{\partial h_{44}}{\partial x}\right)\\ \{44,2\} &= \frac{1}{2}\frac{\partial h_{44}}{\partial y}\\ \{44,3\} &= \frac{1}{2}\frac{\partial h_{44}}{\partial z}.\end{aligned} \tag{33}$$

Substituting the values of h_{44} and h_{41} given by Eqs. (27), and for simplicity considering the particle to be placed at point for which $z=0$, we can obtain for the acceleration parallel to the track of the pulse during the time intervals indicated

$$\frac{d^2x}{dt^2} = 2\rho\left(\frac{1}{t-x} + \frac{1}{(x^2+y^2)^{1/2}}\right) \qquad \begin{cases}\text{from } t = (x^2+y^2)^{1/2} \\ \text{to} \quad t = (x^2+y^2)^{1/2}+\lambda\end{cases}$$

$$= 2\rho\left(\frac{1}{t-x} - \frac{1}{t-\lambda-x}\right) \qquad \begin{cases}\text{from } t = (x^2+y^2)^{1/2}+\lambda \\ \text{to} \quad t = l+[(l-x)^2+y^2]^{1/2}\end{cases} \qquad (34)$$

$$= -2\rho\left(\frac{1}{t-\lambda-x} + \frac{1}{[(l-x)^2+y^2]^{1/2}}\right) \begin{cases}\text{from } t = l+[(l-x)^2+y^2]^{1/2} \\ \text{to} \quad t = l+[(l-x)^2+y^2]^{1/2}+\lambda\end{cases}$$

And for the acceleration perpendicular to the track we obtain for the same time intervals

$$\frac{d^2y}{dt^2} = \frac{-2\rho y}{(x^2+y^2)^{1/2}[(x^2+y^2)^{1/2}-x]} \qquad \begin{cases}\text{from } t = (x^2+y^2)^{1/2} \\ \text{to} \quad t = (x^2+y^2)^{1/2}+\lambda\end{cases}$$

$$= 0 \qquad \begin{cases}\text{from } t = (x^2+y^2)^{1/2}+\lambda \\ \text{to} \quad t = l+[(l-x)^2+y^2]^{1/2}\end{cases} \qquad (35)$$

$$= \frac{2\rho y}{[(l-x)^2+y^2]^{1/2}} \, \frac{1}{[(l-x)^2+y^2]^{1/2}+l-x} \qquad \begin{cases}\text{from } t = l+[(l-x)^2+y^2]^{1/2} \\ \text{to} \quad t = l+[(l-x)^2+y^2]^{1/2}+\lambda.\end{cases}$$

In accordance with these expressions, we see that parallel to the track the particle would first be accelerated in the same direction as the motion of the pulse, and then in the opposite direction. On the other hand perpendicular to the motion of the pulse, the particle would first be accelerated towards the track and later away from it. And we must now investigate the total integrated acceleration corresponding to the whole track of the pulse, and compare the results with those which might be expected in first approximation from our older ideas as to the conservation of momentum.

§11. *Time integral of acceleration of test particle due to the pulse.* Making use of the expressions for the acceleration of the test particle parallel to the track of the pulse given by Eqs. (34), we may now obtain the time integral of the acceleration over the three intervals given, and adding together obtain an expression for the total time integral corresponding to the track of the pulse. Doing so and cancelling out a considerable number of balancing terms, it can easily be shown that we finally obtain for the total integral

$$\int \frac{d^2x}{dt^2}dt = 2\rho\lambda\left\{\frac{1}{(x^2+y^2)^{1/2}} - \frac{1}{[(l-x)^2+y^2]^{1/2}}\right\}. \qquad (36)$$

As might be expected the net acceleration of the test particle parallel to the track depends on the relative magnitudes of the total length of the track l and the distance along the track x at which the particle is placed. When the test particle is midway between the two ends of the track at $x=l/2$, there is no net acceleration parallel to the track. Otherwise the net parallel acceleration is in the direction towards which the longer segment of the track lies.

Turning now to the acceleration perpendicular to the track it can easily be shown from Eqs. (35) that we obtain for the net integrated acceleration

$$\int \frac{d^2y}{dt^2}dt = -\frac{2\rho\lambda}{y}\left\{\frac{x}{(x^2+y^2)^{1/2}} + \frac{l-x}{[(l-x)^2+y^2]^{1/2}}\right\}. \tag{37}$$

And considering the particle to be located midway between the two ends of the track at $x=l/2$ this reduces to

$$\int \frac{d^2y}{dt^2}dt = -\frac{2\rho\lambda l}{y[(l/2)^2+y^2]^{1/2}}\,. \tag{38}$$

This result may now be compared with what would be expected from a simple application of ideas as to the conservation of momentum for the combined interaction of particle and light pulse. If we write

$$m = \rho\lambda \tag{39}$$

as an abbreviation for the mass of the light pulse, i.e., its energy divided by the square of the velocity of light, and write M for the mass of the particle, Eq. (38) will then give us for the total momentum acquired by the (stationary) test particle towards the track

$$\int M\frac{d^2y}{dt^2}dt = -\frac{2mMl}{y[(l/2)^2+y^2]^{1/2}}\,. \tag{40}$$

On the other hand it is well known that the deflection of light in passing through the gravitational field of a particle can be taken in first approximation as twice that which would be calculated from a simple application of the Newtonian theory of gravitation, so that we can evidently write for the momentum acquired by the light pulse in passing over the track

$$\int_0^l \frac{2mMy}{[(l/2-t)^2+y^2]^{3/2}}dt = +\frac{2mMl}{y[(l/2)^2+y^2]^{1/2}} \tag{41}$$

which is equal and opposite to the expression for the momentum acquired by the particle. And since a similar correspondence can be shown for the more general case when the particle is not located midway between the ends of the track, it is evident that we can retain in considerable measure our simple ideas as to the conservation of momentum as a first approximation even in so complicated a process as that of the interaction between a particle and a pulse of light.

Part IV. Conclusion

§*12*. *Some remarks on the problem*. This completes the material which we desired to present in this article. It is particularly hoped that the treatment has clarified the problem of the mutual gravitational interaction of radiation and a particle, since we have given special attention to the behavior of a particle in the gravitational field of light, while the treatment of the converse problem of the behavior of light in the gravitational field of a particle is well known.

This converse problem is of course the one of usual importance in connection with astronomical observations when we are interested in the effect of the gravitational fields of stars or other particles on the behavior of a ray of light coming from a distant object. Its treatment can usually be carried through with less difficulty and higher approximation than was feasible for the problems which we have considered in the present paper. This arises partly from the fact that we have the exact Schwarzschild solution for the gravitational field surrounding a stationary point particle, and partly from the fact that disturbances in the motions of neighboring particles which are produced by the passage of a pulse of light cannot in general communicate their effects back in time to affect the gravitational field through which the pulse itself has to pass.

One of the main results of the present article has been to show that in the case of weak fields the gravitational action of light on a particle can in considerable measure be correctly calculated from the Newtonian theory of gravitation provided we take the gravitational mass of the radiation as twice its inertial mass. And since it was already known in the case of weak fields that the gravitational action of a particle on light could in considerable measure also be correctly calculated from the Newtonian theory by taking the gravitational mass of the radiation as twice its inertial mass, we have thus shown the possibility of extending the domain in which we can use our simple ideas as to the conservation of momentum. Of course in general we should expect to be able to use the more complicated ideas as to the conservation of momentum which can be obtained by the device of introducing the *pseudo* energy-momentum tensor t_μ^ν, but the extent to which the simple ideas would be applicable does not appear to have been certain prior to this investigation.

§*13. Other examples of the gravitational field due to radiation.* The effects of radiation on the gravitational field, studied in this article, may also be compared with two other cases in which radiation has been found to produce a different (greater) effect than ordinary matter on the gravitational field.

The first example is furnished by the case of the static Einstein universe where it has been found[4] that the radius R of a universe *filled with stationary incoherent matter* would be connected with the proper density of matter ρ by the expression

$$\frac{1}{R^2} = 4\pi\rho \tag{42}$$

while for a universe *filled solely with radiation* of density ρ the relation would be given by the expression

[4] See for example, Tolman, Proc. Nat. Acad. 15, 297 (1929), Eqs. (35) and (37).

$$\frac{1}{R^2} = \frac{16}{3}\pi\rho. \tag{43}$$

The second example is furnished by the case of the gravitational field of a sphere of perfect fluid of proper macroscopic density ρ_{00} and pressure p_0. Writing the line element for such a sphere in the form

$$ds^2 = -e^{\mu}(dx^2 + dy^2 + dz^2) + e^{\nu}dt^2 \tag{44}$$

where μ and ν are functions of $r = (x^2+y^2+z^2)^{1/2}$, it is found[5] that the line element degenerates at great distances from the sphere into the approximate Schwarzschild form

$$ds^2 = -\left(1 + \frac{2m}{r}\right)(dx^2 + dy^2 + dz^2) + \left(1 - \frac{2m}{r}\right)dt^2 \tag{45}$$

where the constant m has the value, obtained by integration over the material in the sphere,

$$m = \int (\rho_{00} + 3p_0)e^{\nu/2}dV_0 \tag{46}$$

dV_0 being the element of proper volume. Since for radiation we have $3p_0 = \rho_{00}$ and for incoherent matter at ordinary temperatures have $p_0 \ll \rho_{00}$, it is evident here too that the radiation in the sphere can be regarded as having a greater effect on the gravitational field than matter of the same density.

[5] Tolman, Phys. Rev. **35**, 875 (1930).