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STABILITY OF RIGID BODY MOTION USING THE ENERGY-CASIMIR METHOD

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ABSTRACT. The Energy-Casimir method, due to Mewcomb, Arnold and others is illustrated by application to the motion of a free rigid body and the heavy top.

§1. INTRODUCTION

In the preceding paper of Weinstein, a general framework for calculating stability criteria is reviewed. In this note we illustrate the method in the concrete cases of a rigid body and heavy top. The classical stability results are obtained. The purpose of this note is to illustrate the basic ideas of the method with simple "hands-on" examples that should aid in the understanding of fluid and plasma examples in Holm's lecture that follows.

Let us recall the basic procedures used in the "Energy-Casimir method".

<u>Step A</u>. <u>Equations of Motion and Conserved Quantities</u>

Write the equations as evolution equations

$$\frac{dx}{dt} = X(x) \qquad \text{(EM)}$$

where $x \in P$, the phase space and X is a vector field on P.

Find a conserved energy $H: P \to \mathbb{R}$; i.e.

$$\frac{d}{dt} H(x(t)) = 0 \qquad \text{(H)}$$

for any solution $x(t)$ of (EM), and a family of conserved quantities $F: P \to \mathbb{R}$. (These conserved quantities are typically Casimirs or are generated by symmetry groups -- See Weinstein's lecture for the definitions of these and the definition of Liapunov stability).

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Step B. First Variation

Let x_e be an equilibrium point; i.e. $X(x_e) = 0$, whose (Liapunov) stability we wish to ascertain. Find all F in step A with the property that $H_F = H + F$ has a critical point at x_e:

$$d(H + F)(x_e) = 0 \qquad (VP)$$

Step C. Second Variation

Compute the second derivative $d^2H_F(x_e)$ and see if it is definite, either positive or negative for some F satisfying step B. If P is finite dimensional then x_e is Liapunov stable -- this follows from conservation of H_F.* (If P is infinite dimensional, as for fluids and plasmas, then this second variation test is not sufficient for nonlinear stability; this deficiency can be remedied by convexity estimates.)

In the next two sections we shall go through these three steps for our two examples.

§2. RIGID FREE BODY

A. Equations of Motion and Conserved Quantities

The free rigid body equations are

$$\dot{\underline{m}} = d\underline{m}/dt = \underline{m} \times \underline{\omega} \qquad (2.1)$$

where $\underline{m}, \underline{\omega} \in \mathbb{R}^3$, $\underline{\omega}$ is the angular velocity and $\underline{m}$ the angular momentum both viewed in the body; the relation between $\underline{m}$ and $\underline{\omega}$ is given by $m_i = I_i\omega_i$, $i = 1,2,3$, where $I = (I_1, I_2, I_3)$ is the diagonalized moment of inertia tensor, $I_1, I_2, I_3 > 0$. This system is Hamiltonian in the Lie-Poisson structure of $\mathbb{R}^3$ considered as the dual of the Lie algebra of the rotation group SO(3). Explicitly, for $F, G: \mathbb{R}^3 \to \mathbb{R}$,

$$\{F,G\}(\underline{m}) = -\underline{m}\cdot(\nabla F(\underline{m}) \times \nabla G(\underline{m})) \qquad (2.2)$$

and with respect to this bracket, (2.1) is easily verified to be Hamiltonian in the sense that (2.1) is equivalent to $\dot{f} = \{F,H\}$ where the H is equal to the kinetic energy:

$$H(\underline{m}) = \frac{1}{2}\,\underline{m}\cdot\underline{\omega} = \sum_{i=1}^{3} m_i^2/I_i \qquad (2.3)$$

For any smooth function $\phi: \mathbb{R} \to \mathbb{R}$, the function

$$C_\phi(\underline{m}) = \phi(|\underline{m}|^2/2) \qquad (2.4)$$

*See, e.g., Siegel and Moser [1971], p. 208.

is a Casimir function for (2.2), i.e. its bracket with *any* other function G is identically zero, as an easy computation shows. Thus, for any ϕ, C_ϕ is a conserved function.

B. First Variation

We shall find a Casimir function C_ϕ such that $H_C := H + C_\phi$ has a critical point at a given equilibrium point of (2.1). Such points occur when $\underline{m}$ is parallel to $\underline{\omega}$. We shall assume without loss of generality, that $\underline{m}$ and $\underline{\omega}$ point in the Ox-direction. Then, after normalizing if necessary, we may even assume that the equilibrium solution is $\underline{m}_e = (1,0,0)$. The derivative of

$$H_{C_\phi}(\underline{m}) = \sum_{i=1}^{3} m_i^2/I_i + \phi(\tfrac{1}{2}|\underline{m}|^2)$$

is

$$DH_{C_\phi}(\underline{m})\cdot\delta\underline{m} = (\underline{\omega} + \underline{m}\phi'(|\underline{m}|^2/2))\cdot\delta\underline{m}\,. \qquad (2.5)$$

This equals zero at $\underline{m} = (1,0,0)$, provided that

$$\phi'(1/2) = -1/I_1 \qquad (2.6)$$

C. Second Variation

Using (2.5) and (2.6), the second derivative at the equilibrium $\underline{m}_e = (1,0,0)$ is

$$D^2H_{C_\phi}(\underline{m}_e)\cdot(\delta\underline{m})^2 = \delta\underline{\omega}\cdot\delta\underline{m} + \phi'|\underline{m}_e|^2/2)|\delta\underline{m}|^2 + (\underline{m}_e\cdot\delta\underline{m})^2\phi''(|\underline{m}_e|^2/2)$$

$$= \sum_{i=1}^{3} (\delta m_i)^2/I_i - |\delta\underline{m}|^2/I_1 + \phi''(\tfrac{1}{2})(\delta m_1)^2$$

$$= \left(\frac{1}{I_2} - \frac{1}{I_1}\right)(\delta m_2)^2 + \left(\frac{1}{I_3} - \frac{1}{I_1}\right)(\delta m_3)^2 + \phi''(\tfrac{1}{2})(\delta m_1)^2 \qquad (2.7)$$

This quadratic form is positive definite if and only if

$$\phi''(1/2) > 0 \qquad (2.8)$$

and

$$I_1 > I_2, \quad I_1 > I_3. \qquad (2.9)$$

Consequently, $\phi(x) = (-2/I_1)x + (x - \frac{1}{2})^2$ makes the second derivative of H_{C_ϕ} at $(1,0,0)$ positive definite, so stationary rotation around the longest axis is (Liapunov) stable.

The quadratic form (2.7) is indefinite if

$$I_1 > I_2, \quad I_3 > I_1 \tag{2.10}$$

or the other way around. Consequently, we cannot show by this method that rotation around the middle axis is stable. (In fact, it is unstable.)

Finally, the quadratic form is negative definite, provided

$$\phi''(\tfrac{1}{2}) < 0 \tag{2.11}$$

and

$$I_1 < I_2, \quad I_1 < I_3. \tag{2.12}$$

It is obvious that we may find a function ϕ satisfying the requirements (2.6) and (2.11); e.g. $\phi(x) = (-2/I_1)x - (x - \frac{1}{2})^2$. This proves that rotation around the short axis is (Liapunov) stable.

We summarize the results in the following well-known theorem.

<u>Rigid Body Stability Theorem</u>. *In the motion of a free rigid body, rotation around the long and short axes is (Liapunov) stable.*

<u>Remarks</u>. 1) It is important to keep the Casimirs as general as possible, because otherwise (2.8) and (2.11) could be contradictory. Had we simply chosen $\phi(x) = -(2/I_1)x$, (2.8) would be verified, but (2.11) not. It is only the choice of *two different* Casimirs that enables us to prove the two stability results, even though the level surfaces of these Casimirs are the same.

2) In this case, rotations about the intermediate axis are unstable. This is true even for the linearized equations as an eigenvalue analysis shows.

3) The same stability theorem can also be proved by working with the second derivative along a coadjoint orbit in $\mathbb{R}^3$; i.e. a two-sphere; see Arnold [1966]. This coadjoint orbit method also suggests instability of rotation around the intermediate axis, but it has the deficiency of being inapplicable where the rank of the Poisson structure jumps. (See Weinstein's lecture in this volume.)

§3. LAGRANGE TOP

A. <u>Equations of Motion and Conserved Quantities</u>

The heavy top equations are

$$d\underline{m}/dt = \underline{m} \times \underline{\omega} + Mgl\underline{\gamma} \times \underline{X} \tag{3.1a}$$

$$d\underline{\gamma}/dt = \underline{\gamma} \times \underline{\omega} \tag{3.1b}$$

where $\underline{m}, \underline{\gamma}, \underline{\omega}, \underline{X} \in \mathbb{R}^3$. Here $\underline{m}$ and $\underline{\omega}$ are the angular momentum and angular velocity in the body, $m_i = I_i\omega_i$, $I_i > 0$, $i = 1, 2, 3$, with $\underline{I} = (I_1, I_2, I_3)$ the moment of inertia tensor. The vector $\underline{\gamma}$ represents the motion of the unit vector along the Oz-axis as seen from the body, and the constant vector $\underline{X}$ is the unit vector along the line segment of length l connecting the fixed point to the center of mass of the body; M is the total mass of the body, and g is the strength of the gravitational acceleration, which is along Oz pointing down.

This system is Hamiltonian in the Lie-Poisson structure of $\mathbb{R}^3 \times \mathbb{R}^3$ regarded as the dual of the Lie algebra of the Euclidean group $E(3) = SO(3) \ltimes \mathbb{R}^3$ ($\ltimes$ denotes semidirect product). The Poisson bracket is given by (see Holmes and Marsden [1983] and references therein):

$$(F,G)(\underline{m},\underline{v}) = -\underline{m}\cdot(\underline{\nabla}_{\underline{m}}F \times \underline{\nabla}_{\underline{m}}G)$$

$$-\underline{\gamma}\cdot(\underline{\nabla}_{\underline{m}}F \times \underline{\nabla}_{\underline{\gamma}}G + \underline{\nabla}_{\underline{\gamma}}F \times \underline{\nabla}_{\underline{m}}G). \tag{3.2}$$

The Hamiltonian of this system is the total energy

$$H(\underline{m},\underline{\gamma}) = \tfrac{1}{2}\underline{m}\cdot\underline{\omega} + Mgl\underline{\gamma}\cdot\underline{X}. \tag{3.3}$$

This can be easily verified directly. For further information, see Ratiu's lecture in this volume. The functions $\underline{m}\cdot\underline{\gamma}$ and $|\underline{\gamma}|^2$ are *Casimir functions* for (3.2), i.e. their brackets with any function $G:\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ vanish. Hence the same is true for

$$C(\underline{m},\underline{\gamma}) = \phi(\underline{m}\cdot\underline{\gamma}, |\underline{\gamma}|^2) \tag{3.4}$$

where ϕ is any function from $\mathbb{R}^2$ to $\mathbb{R}$.

We shall be concerned here only with the Lagrange top. This is a heavy top for which $I_1 = I_2$, i.e. it is symmetric, and the center of mass lies on the axis of symmetry in the body, i.e. $\underline{X} = (0,0,1)$. This assumption simplifies the equations of motion (3.1a) to

$$\dot{m}_1 = (I_2 - I_3)m_2m_3/I_2I_3 - Mgl\gamma_2$$

$$\dot{m}_2 = (I_3 - I_1)m_1m_3/I_1I_3 + Mgl\gamma_1$$

$$\dot{m}_3 = (I_1 - I_2)m_1m_2/I_1I_2.$$

Since $I_1 = I_2$, we have $\dot{m}_3 = 0$; thus m_3 and hence any function $\phi(m_3)$ of m_3 is conserved.

B. First Variation

We shall study the equilibrium solution $\underline{m}_e = (0,0,\bar{m}_3)$, $\gamma_e = (0,0,1)$, which represents the spinning of a symmetric top in its upright position. To begin, we look for conserved quantities of the form $H_{\phi,\phi} = H + \phi(\underline{m}\cdot\underline{\gamma}, |\underline{\gamma}|^2) + \phi(m_3)$ which have a critical point at the equilibrium.

The first derivative of $H_{\phi,\phi}$ is given by

$$DH_{\phi,\phi}(\underline{m},\underline{\gamma})\cdot(\delta\underline{m},\delta\underline{\gamma}) = (\underline{\omega} + \dot{\phi}(\underline{m}\cdot\underline{\gamma}, |\underline{\gamma}|^2)\underline{\gamma})\cdot\delta\underline{m} + [Mgl\underline{k} + \dot{\phi}(\underline{m}\cdot\underline{\gamma}, |\underline{\gamma}|^2)\underline{m}$$

$$+ 2\phi'(\underline{m}\cdot\underline{\gamma}, |\underline{\gamma}|^2)\underline{\gamma}]\cdot\delta\underline{\gamma} + \phi'(m_3)\delta m_3, \tag{3.5}$$

where $\dot{\phi} = \partial\phi/\partial(\underline{m}\cdot\underline{\gamma})$, $\phi' = \partial\phi/\partial(|\underline{\gamma}|^2)$. At the equilibirum solution $\underline{m}_e, \underline{\gamma}_e$, the first derivative of $H_{\phi,\phi}$ vanishes, provided that

$$\bar{\omega}_3 + \dot{\phi}(\bar{m}_3,1) + \phi'(\bar{m}_3) = 0; \; \bar{\omega}_3 = \bar{m}_3/I_3$$

$$Mgl + \dot{\phi}(\bar{m}_3,1)\bar{m}_3 + 2\phi'(\bar{m}_3,1) = 0;$$

(The remaining equations, involving indices 1 and 2 are trivially verified.) Solving for $\dot{\phi}(\bar{m}_3,1)$ and $\phi'(\bar{m}_3,1)$ we get the conditions:

$$\dot{\phi}(\bar{m}_3,1) = -\left(\frac{1}{I_3} + \phi(\bar{m}_3)\right)\bar{m}_3$$

$$\phi'(\bar{m}_3,1) = \tfrac{1}{2}\left(\frac{1}{I_3} + \phi'(\bar{m}_3)\right)\bar{m}_3^2 - \tfrac{1}{2}Mgl \tag{3.6}$$

C. Second Variation

We shall check for definiteness of the second variative of $H_{\phi,\phi}$ at the equilibrium point $\underline{m}_e = (0,0,\bar{m}_3)$, $\underline{\gamma}_e = (0,0,1)$. To simplify notation we shall set

$$a = \phi''(\bar{m}_3)$$

$$b = 4\phi''(\bar{m}_3,1)$$

$$c = \dot{\phi}(\bar{m}_3,1)$$

$$d = 2\dot{\phi}'(\bar{m}_3,1).$$

With this notation, (3.5), and (3.6), we find that the matrix of the second derivative at $\underline{m}_e, \underline{\gamma}_e$ is

$$\begin{bmatrix} 1/I_1 & 0 & 0 & \dot{\phi}(\bar{m}_3,1) & 0 & 0 \\ 0 & 1/I_2 & 0 & 0 & \dot{\phi}(\bar{m}_3,1) & 0 \\ 0 & 0 & (1/I_3)+a+c & 0 & 0 & \dot{\phi}(\bar{m}_3,1)+2\bar{m}_3c+d \\ \dot{\phi}(\bar{m}_3,1) & 0 & 0 & 2\dot{\phi}'(\bar{m}_3,1) & 0 & 0 \\ 0 & \dot{\phi}(\bar{m}_3,1) & 0 & 0 & 2\phi'(\bar{m}_2,1) & 0 \\ 0 & 0 & \dot{\phi}(\bar{m}_3,1)+2\bar{m}_3c+d & 0 & 0 & 2\phi'(\bar{m}_3,1)+b+\bar{m}_3^2c+2\bar{m}_3d \end{bmatrix} \tag{3.7}$$

If this form is definite, it must be positive definite since the (1,1)-entry is positive. The six principal determinants have the following values, (recall that $I_1 = I_2$):

$$1/I_1$$

$$1/I_1^2$$

$$(1/I_3 + a + c)/I_1^2$$

$$\frac{1}{I_1}\left(\frac{1}{I_3} + a + c\right)\left(\frac{2}{I_1}\phi'(\bar{m}_3,1) - \dot{\phi}(\bar{m}_3,1)^2\right)$$

$$\left(\frac{2}{I_1}\phi'(\bar{m}_3,1) - \dot{\phi}(\bar{m}_3,1)^2\right)^2\left(\frac{1}{I_3} + a + c\right)$$

$$\left(\frac{2}{I_1}\phi'(\bar{m}_3,1) - \dot{\phi}(\bar{m}_3,1)^2\right)\left[(2\phi'(\bar{m}_3,1) + b + \bar{m}_3^2c + 2\bar{m}_3d)\left(\frac{1}{I_3} + a + c\right)\right.$$

$$\left. - (\dot{\phi}(\bar{m}_3,1) + 2\bar{m}_3c + d)^2\right].$$

Consequently, the quadratic form given by (3.7) is positive definite, if and only if

$$\frac{1}{I_3} + a + c > 0 \tag{3.8}$$

$$\frac{2}{I_1}\,\phi'(\bar{m}_3,1) - \dot{\phi}(\bar{m}_3,1)^2 > 0 \tag{3.9}$$

$$(2\phi'(\bar{m}_3,1) + b + \bar{m}_3^2 c + 2\bar{m}_3 d)\left(\frac{1}{I_3} + a + c\right) - (\dot{\phi}(\bar{m}_3,1) + 2\bar{m}_3 c + d)^2 > 0 \tag{3.10}$$

Conditions (3.8) and (3.10) can always be satisfied if we choose the numbers a, b, c, and d appropriately; e.g. $a = c = d = 0$ and b sufficiently large and positive. Thus, the determining condition for stability is (3.9). By (3.6), this becomes

$$\frac{1}{I_1}\left[\left(\frac{1}{I_3} + \phi'(\bar{m}_3)\right)\bar{m}_3^2 - Mg\ell\right] - \left(\frac{1}{I_3} + \phi'(\bar{m}_3)\right)^2\bar{m}_3^2 > 0\,. \tag{3.11}$$

We can choose $\phi'(\bar{m}_3)$ so that $\frac{1}{I_3} + \phi'(\bar{m}_3) = e$ has any value we wish. The left side of (3.11) is a quadratic polynomial in e, whose leading coefficient is negative. In order for this to be positive for some e, it is necessary and sufficient for the discriminant

$$(\bar{m}_3^2/I_1)^2 - 4\bar{m}_3^2 Mg\ell/I_1$$

to be positive; that is,

$$\bar{m}_3^2 > 4Mg\ell I_1$$

which is the well-known stability condition for a fast top. We have proved the following.

Heavy Top Stability Theorem. *An upright spinning Lagrange top is stable provided that the angular velocity is strictly larger than* $\sqrt{4Mg\ell/I_1}$.

Remarks. 1) The method suggests but does not prove that one has instability when $\bar{m}_3^2 < 4Mg\ell I_1$. In fact, an eigenvalue analysis shows that the equilibrium is linearly unstable and hence nonlinearly unstable in this case.

2) When $I_2 = I_1 + \epsilon$ for small ϵ, the conserved quantity $\phi(m_3)$ is no longer available. In this case, a sufficiently fast top is still linearly stable, but nonlinear stability can only be established by KAM theory. Other regions of phase space are known to possess chaotic dynamics in this case (Holmes and Marsden [1983]).

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