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THE HAMILTONIAN FORMULATION OF CLASSICAL FIELD THEORY

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1. INTRODUCTION

I shall present some results from the theory of classical non-relativistic field theory and discuss how they might be useful in the general relativistic context. Some of the Hamiltonian formalism has already been successfully employed in the general relativistic context, but much more remains to be done in the area of dynamic stability, linearization stability, bifurcation, symmetry aking, and covariant reduction.

2. POISSON MANIFOLDS

Let us begin with some terminology. A *Poisson manifold* is a manifold P together with a bracket $\{\ ,\ \}$ on $\mathcal{F}(P) := C^{\infty}(P,\mathbb{R})$ satisfying

(PB1) $\{\ ,\ \}$ makes $\mathcal{F}(P)$ into a Lie algebra

and

(PB2) $\{FG,H\} = F\{G,H\} + G\{F,H\}$.

We call $C \in \mathcal{F}(P)$ a *Casimir* if

$$\{C,F\} = 0$$

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for all $F \in \mathcal{F}(P)$. If $H \in \mathcal{F}(P)$, the *Hamiltonian vector field* X_H associated to H is defined by

$$X_H[F] = \{F,H\}$$

for all $F \in \mathcal{F}(P)$, where $X_H[F] = dF \cdot X_H$ denotes the derivative of F in the direction X_H. Thus, C is a Casimir when $X_C = 0$.

Besides the standard canonical, or symplectic examples, one has the following non-canonical examples:

EXAMPLES. A. Let $P = \mathbf{R}^3$ with elements denoted $\boldsymbol{\ell}$. Let

$$\{F,H\}(\boldsymbol{\ell}) = -\boldsymbol{\ell} \cdot (\nabla F \times \nabla H).$$

One checks that this makes P into a Poisson manifold and that the functions

$$C(\boldsymbol{\ell}) = \Phi(\|\boldsymbol{\ell}\|^2)$$

for any $\Phi \in \mathcal{F}(\mathbf{R})$, are Casimirs. For $H \in \mathcal{F}(P)$, one has

$$X_H(\boldsymbol{\ell}) = \nabla H \times \boldsymbol{\ell}.$$

In particular, if

$$H = \tfrac{1}{2}\langle \mathbf{I}^{-1}\boldsymbol{\ell}, \boldsymbol{\ell}\rangle$$

for a positive definite symmetric matrix $\mathbf{I}$ (the inertia tensor), then

$$X_H(\boldsymbol{\ell}) = \omega \times \boldsymbol{\ell}, \qquad \omega = \mathbf{I}^{-1}\boldsymbol{\ell}$$

gives Euler's rigid body equations.

B. Let P be the space of triples $(\mathbf{M},\rho,\eta)$ where $\mathbf{M}$ = a one form density, ρ = a density, and η = a function on a domain D in $\mathbf{R}^3$, with appropriate boundary conditions. Let

$$\{F,H\} = \int_D \mathbf{M}\left[\left[\frac{\delta H}{\delta\mathbf{M}}\cdot\nabla\right]\frac{\delta F}{\delta\mathbf{M}} - \left[\frac{\delta F}{\delta\mathbf{M}}\cdot\nabla\right]\frac{\delta H}{\Delta\mathbf{M}}\right]$$
$$+ \int_D \rho\left[\left[\frac{\delta H}{\delta\mathbf{M}}\cdot\nabla\right]\frac{\delta F}{\delta\rho} - \left[\frac{\delta F}{\delta\mathbf{M}}\cdot\nabla\right]\frac{\delta H}{\delta\rho}\right]$$
$$+ \int_D \eta\nabla\cdot\left[\frac{\delta H}{\delta\mathbf{M}}\frac{\delta F}{\delta\eta} - \frac{\delta F}{\delta\mathbf{M}}\frac{\delta H}{\delta\eta}\right]$$

which is a Poisson structure on P. Here $\delta F/\delta\mathbf{M}$ is the functional derivative, a vector field on D. The function

$$C(\mathbf{M},\rho,\eta) = \int_D \rho\Phi(\eta,\Omega)$$

where $\Omega = \frac{1}{\rho}[d\eta \wedge d(\mathbf{M}/\rho)]$ (the potential vorticity) and $\Phi \in \mathcal{F}(\mathbf{R}^2)$, is a Casimir.

The equations for adiabatic compressible flow are

$$\rho\frac{D\mathbf{v}}{dt} = -\nabla p, \quad \frac{D\rho}{dt} + p \operatorname{div} \mathbf{v} = 0, \quad \frac{D\eta}{dt} = 0$$

:re $D/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, $\mathbf{v}$ = velocity field, ρ = density, η = entropy, p = pressure. Here $p = \rho^2(\partial w/\partial\rho)$ where $w(\rho,\eta)$ is the energy density per unit mass, a given function of ρ, η, and we assume $c^2 := \partial p/\partial\rho > 0$. With $M = \rho\mathbf{v}$ and

$$H = \int_D \rho[\|\mathbf{v}\|^2/2 + w],$$

the equations are Hamiltonian relative to the above bracket.

C. The two preceding brackets are special cases of Lie-Poisson brackets. These are defined on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ by

$$\{F,H\}_\pm(\mu) = \pm\langle\mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta H}{\delta\mu}\right]\rangle$$

where $\mu \in \mathfrak{g}^*$, $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, $\delta F/\delta\mu \in \mathfrak{g}$ is the functional derivative:

$$DF(\mu)\cdot\nu := \frac{d}{d\epsilon}F(\mu+\epsilon\nu)\Big|_{\epsilon=0} = \langle\nu, \frac{\delta F}{\delta\mu}\rangle$$

and [,] is the Lie algebra bracket. For example A, $\mathfrak{g} = so(3) = (\mathbb{R}^3, \times)$ (with the '−' sign), and for example B, $\mathfrak{g}$ is the semi-direct product (vector fields) s (functions × densities) (with the '+' sign). ▲

These brackets are not conjured out of thin air, but are produced by the methods of reduction (see Marsden, Weinstein, et. al. [1983] for a review). For example, for the rigid body, one starts with phase space $T^*SO(3)$ with the canonical bracket and reduces by SO(3), an intrinsic symmetry group. This produces a map

$$T^*SO(3) \to so(3)^* \quad (6 \text{ dimensions} \to 3)$$

given by left translation to the identity. This map is a *Poisson map*; i.e., is bracket preserving. This map is in fact a momentum map (Noether conserved quantity)

$$J_G: P \to \mathfrak{g}^*$$

which is defined for a Poisson action of a Lie group G with associated Lie algebra $\mathfrak{g}$, on P by

$$\xi_P[F] = \{F, \langle J, \xi \rangle\}$$

where $\xi \in \mathfrak{g}$ and ξ_P is the corresponding infinitesimal generator of the action. (For SO(3) use the action by SO(3) on the *right*.) It is a general fact that equivariant momentum maps are Poisson maps (use $\mathfrak{g}^*_+$ for left actions, $\mathfrak{g}^*_-$ for right actions). For fluids, the relevant group one reduces by is the particle relabelling (or rearrangement) group.

Obviously, the Casimirs are conserved quantities for any Hamiltonian system. If H is G-invariant, a momentum map J is conserved as well. For Lie-Poisson brackets, conservation of the Casimirs corresponds to the fact that any Hamiltonian system leaves the coadjoint orbits invariant. For the rigid body these are the body angular momentum spheres $\|\mathbf{m}\|^2$ = constant and for adiabatic flow, they are the states that are "kinematically connected"; i.e., there is a diffeomorphism (representing a conceivable particle motion) mapping one to another. These coadjoint orbits are also the *symplectic leaves*;

3. THE ENERGY-CASIMIR METHOD

Two important applications of this formalism are to stability and to bifurcation. The first is based on the *energy-Casimir* method, developed and used by Arnold [1969] and Holm et. al. [1985]. To motivate it, first consider the classical canonical case.

For the Hamiltonian systems in canonical form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

there is a classical stability criterion due to Lagrange and Dirichlet. Clearly, an equilibrium point in this case is a critical point of H. If the $2n \times 2n$ matrix D^2H of second partial derivatives evaluated at (q_e,p_e) is positive or negative definite (i.e. all the eigenvalues have the same sign), then (q_e,p_e) is stable. This follows from conservation of energy and the fact, proven in advanced calculus, that the level sets of H near (q_e,p_e) are approximately ellipsoids. Apart from KAM theory, which gives stability of periodic solutions for two degree of freedom systems, the Lagrange-Dirichlet theorem is the only known general stability theorem for canonical systems.

The energy-Casimir method is a generalization of the Lagrange-Dirichlet method. Given an equilibrium u_e for evolution equations $\dot{u} = X(u)$, it proceeds in the following steps:

Energy-Casimir Method

Step A. Write the equations in Hamiltonian form $\dot{F} = \{F,H\}$.

Step B. Find a family of conserved quantities C, such as a family of Casimirs.

Step C. Select C such that $H + C$ has a critical point at u_e.

Step D. Check to see if $D^2(H + C)(u_e)$, the matrix of second partial derivatives of $H + C$ at u_e, is positive or negative definite.

With regard to step C, we point out that an equilibrium solution need not be a critical point of H alone; in general $DH(u_e) \neq 0$. An example where this occurs is a rigid body spinning about one of its principal axes of inertia. In this case, a critical point of H alone would have zero angular velocity; but a critical point of $H + C$ is a (nontrivial) stationary rotation about one of the principal axes.

Formally, the same argument used to establish the Lagrange-Dirichlet test also works here. Unfortunately, for systems with infinitely many degrees of freedom (like fluids and plasmas), there is a snag. The calculus argument used before simply runs into problems; one might think these are just technical and that we just need to improve the methods. In fact there is widespread belief in this "energy criterion" (see, for instance, the discussion and references in Marsden and Hughes [1983], Chapter 6). However, Ball and Marsden [1984] have shown by means of a "realistic" example from elasticity theory that the difficulty is genuine. One way to overcome this difficulty is to modify step D using a convexity argument of Arnold [1969].

Convexity Analysis

Modified Step D

(a) Let $\Delta u = u - u_e$ denote a finite variation in phase space.

(b) Find quadratic functions Q_1 and Q_2 such that

$$Q_1(\Delta u) \leqslant H(u_e + \Delta u) - H(u_e) - DH(u_e)\cdot\Delta u$$

$$Q_2(\Delta u) \leqslant C(u_e + \Delta u) - C(u_e) - DC(u_e)\cdot\Delta u.$$

(c) Require that $Q_1(\Delta u) + Q_2(\Delta u) > 0$ for all $\Delta u \neq 0$.

(d) Introduce the *norm* $\|\Delta u\|$ by

$$\|\Delta u\|^2 = Q_1(\Delta u) + Q_2(\Delta u),$$

so $\|\Delta u\|$ as a measure of the distance from u to u_e; $d(u,u_e) = \|\Delta u\|$.

(e) Reauire that

$$|H(u_e + \Delta u) - H(u_e)| \leqslant C_1\|\Delta u\|$$

and

$$|C(u_e + \Delta u) - C(u_e)| \leqslant C_2\|\Delta u\|$$

for constants C_1 and C_2, and $\|\Delta u\|$ sufficiently small.

These conditions guarantee stability of u_e and provide the distance measure relative to which stability is defined. The key part of the proof is simply the observation that if we add the two inequalities in (b), we get

$$\|\Delta u\|^2 \leqslant H(u_e + \Delta u) + C(u_e + \Delta u) - H(u_e) - C(u_e);$$

here, $DH(u_e)\cdot\Delta u$ and $DC(u_e)\cdot\Delta u$ have added up to zero by step C. But H and C are constant in time so

$$\|(\Delta u)_{\text{time } t}\|^2 \leqslant [H(u_e + \Delta u) + C(u_e + \Delta u) - H(u_e) - C(u_e)]|_{\text{time } 0} \cdot$$

Now employ the inequalities in (e) to get

$$\|(\Delta u)_{\text{time } t}\|^2 \leqslant (C_1 + C_2)\|(\Delta u)_{\text{time } 0}\| \;.$$

This estimate bounds the temporal growth of finite perturbations in terms of initial perturbations, which is exactly what is needed for stability.

In the next section we give an example of how this technique applies in a concrete example. See Holm, Marsden, Ratiu and Weinstein [1985] and Abarbanel, Holm, Marsden and Ratin [1986] for a more extensive analysis.

4. LIQUID DROPS WITH SURFACE TENSION

Because of the historic and sustained interest in the dynamics of gravitating masses, I shall present a somewhat related example, the rotating liquid drop. Consider a planar liquid drop consisting of an incompressible, inviscid fluid with a free boundary and forces of surface tension on the boundary. The dynamic variables are the free boundary Σ and the spatial velocity field $\mathbf{v}$, a divergence free vector field on the region D_Σ bounded by Σ. The surface Σ is an element of the set S of closed curves (respectively surfaces) in $\mathbf{R}^2$ (respectively $\mathbf{R}^3$) diffeomorphic to the boundary of a reference region D and enclosing the same area (respectively volume) as D. We let N denote the space of all such pairs $(\Sigma, \mathbf{v})$. The Hamiltonian approach to hydrodynamic problems was introduced in the fixed boundary case by Arnold [1966] and developed by Marsden and Weinstein [1974,1982,1983]. The free boundary case has also been studied by Sedenko and Iudovich [1978].

The equations of motion for an ideal fluid with a free boundary Σ with surface tension τ are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p,$$

$$\frac{\partial \Sigma}{\partial t} = \langle \mathbf{v}, \nu \rangle,$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{and} \qquad p|\Sigma = \tau\kappa,$$

where ν is the unit normal to the surface Σ, κ is the mean curvature of Σ and τ is the surface tension coefficient, a numerical constant.

The Poisson bracket will be defined for functions $F, G\colon N \to$ which possess *functional derivatives*, defined as follows:

i) $\delta F/\delta \mathbf{v}$ is a divergence free vector field on D_Σ such that

$$D_{\mathbf{v}}F(\Sigma, \mathbf{v}) \cdot \delta\mathbf{v} = \int_{D_\Sigma} \langle \frac{\delta F}{\delta \mathbf{v}}, \delta\mathbf{v} \rangle dA$$

where the partial (Fréchet) derivative $D_{\mathbf{v}}F$ is computed with Σ fixed.

ii) $\delta F/\delta\varphi$ is the function on Σ with zero integral given by

$$\frac{\delta F}{\delta \varphi} = \langle \frac{\delta F}{\delta \mathbf{v}}, \nu \rangle.$$

(The symbol φ represents the potential for the gradient part of $\mathbf{v}$ in the Helmholtz, or Hodge, decomposition.)

iii) $\delta F/\delta\Sigma$ is a function on Σ determined up to an additive constant as follows. A variation $\delta\Sigma$ of Σ is identified with a function on Σ representing the infinitesimal variation of Σ in its normal direction. It follows from the incompressibility assumption that $\delta\Sigma$ has zero integral. Let $\delta F/\delta\Sigma$ be the function determined up to an additive constant by

$$\int_\Sigma \frac{\delta F}{\delta\Sigma}\,\delta\Sigma\, ds = D_\Sigma F(\Sigma,\mathbf{v}) \cdot \delta\Sigma.$$

We now define a Poisson bracket on N as follows. For functions F and G mapping N to $\mathbf{R}$ and possessing functional derivatives as defined above, set

$$\{F,G\} = \int_{D_\Sigma} \langle\omega, \frac{\delta F}{\delta\mathbf{v}} \times \frac{\delta G}{\delta\mathbf{v}}\rangle dA + \int_\Sigma \left[\frac{\delta F}{\delta\Sigma}\frac{\delta G}{\delta\varphi} - \frac{\delta G}{\delta\Sigma}\frac{\delta F}{\delta\varphi}\right] ds,$$

where $\omega = \operatorname{curl} \mathbf{v}$. This Poisson bracket on N is derived from the canonical cotangent bracket on T^* , where, in the two-dimensional case, $C = \mathrm{Emb}_{\mathrm{vol}}(D,\mathbf{R}^2)$ is the manifold of volume-preserving embeddings of a two-dimensional reference manifold D into $\mathbf{R}^2$, by reduction by the group $G = \mathrm{Diff}_{\mathrm{vol}}(D)$, the group of volume-preserving diffeomorphisms of D (i.e. the group of particle relabelling transformations). (See Lewis, Marsden, Montgomery and Ratiu [1986a] for details.)

We take our Hamiltonian to be

$$H(\Sigma,\mathbf{v}) = \int_{D_\Sigma} \tfrac{1}{2}|\mathbf{v}|^2 dA + \tau \int_\Sigma ds.$$

The functional derivatives of H are computed to be

$$\frac{\delta H}{\delta\mathbf{v}} = \mathbf{v},$$

$$\frac{\delta H}{\delta\varphi} = \langle\frac{\delta H}{\delta\mathbf{v}},\nu\rangle = \langle\mathbf{v},\nu\rangle,$$

and $$\frac{\delta H}{\delta\Sigma} = \tfrac{1}{2}|\mathbf{v}|^2 + \tau\kappa,$$

where $\delta H/\delta\Sigma$ is taken modulo constants. For this H and the Poisson bracket defined above, the equations of motion for the free boundary fluid with surface tension are equivalent to the relation $\dot F = \{F,H\}$ for all functions F on N possessing functional derivatives.

This explains the Hamiltonian structure for the free boundary problem. It is used in the following two ways:

1. *Stability.* The circular planar drop is nonlinearly stable provided

$$\frac{3\tau}{r^3} > (\Omega/2)^2$$

where Ω is the rotation rate of the circular drop. Formal stability is proved using the energy-Casimir method, taking $H + C$ to be the energy plus a multiple of the angular momentum. (See Lewis, Marsden and Ratiu [1986b].) To prove nonlinear stability rigorously requires some more work, analogous to using the convexity estimates. Here it is the Weierstrass theory in a version due to Hestenes that does the job. See Lewis [1987] for details.

2. *Bifurcation.* If the parameters τ, r or Ω are varied to violate the stability condition, one can prove that a bifurcation occurs, so in this sense the stability condition is sharp. The result, due to Lewis, Marsden and Ratiu [1986c] uses the bifurcation theory for Hamiltonian systems with symmetry (see Golubitsky and Stewar [1986]).

5. COMMENTS ON GENERAL RELATIVISTIC FLUIDS

The Poisson structure given in Section 2 has been shown to be relevant for general relativistic fluids by BMW (Bao, Marsden and Walton) [1985]. They show that the evolution equations in a general lapse and shift (not necessarily comoving with the fluid) have the form $\dot{F} = \{F,H\}$ where $H = N\mathcal{H} + X\cdot\mathcal{J}$ (N = lapse, X = shift, $\mathcal{H}$ = superhamiltonian, $\mathcal{J}$ = supermomentum) and where $\{\ \}$ is the canonical bracket for the ADM variables (g,π) plus the Lie-Poisson bracket for the fluid variables. This is the Poisson bracket form which corresponds to the adjoint form of the ADM equations (Fischer and Marsden [1979]).

Remarks and Open Questions:

1. Does the free boundary bracket of §4 also carry over to the general relativistic case?

2. The energy-Casimir method has been used for fluids in a fixed background by Holm and Kuperschmidt [1984, 1986]; is it useful in the coupled case as well?

3. Is the bifurcation theory with symmetry useful for studying general relativistic gravitating masses, fission, etc? We point out, as a curiosity, that the bifurcation that occurs in the liquid drop example is to a shape with "galactic symmetry" or "propellor symmetry", i.e. symmetry under rotation by π, but not under reflections.

4. The Hamiltonian structure can surely be generalized to include electromagnetic effects, following Marsden and Weinstein [1982] and Marsden, Weinstein, et. al. [1983], for both fluids and plasmas.

5. The Hamiltonian structure of Bao et. al. is useful for *linearization stability*. Recall that the main result of the linearization stability program is that a spacetime (with a compact Cauchy surface of constant mean curvature) is linearization stable if and only if it has no Killing fields; if it does, then obstructions to perturbation expansions are the second order Taub conditions. (See Fischer, Marsden and Moncrief [1980], Arms, Marsden and Moncrief [1982], Isenberg and Marsden [1982].) For fluids, this result appears to be true as well, provided one imposes constraints on the perturbations corresponding to, for example, preserving baryon number - these constraints are, mathematically, preserving the coadjoint orbit structure and are closely related to "Lin constraints" (see Cendra and Marsden [1986]).

6. COVARIANT REDUCTION

Finally, I wish to point out some interesting open issues in classical relativistic field theory which are suggested by the preceding sections.

First of all, one should note that there is a beautiful covariant version of the multisymplectic formalism adapted especially for the linearization stability program, and which incorporates momentum maps, called the "GIMMSY project"; see Gotay et. al. [1987]. This is a covariant analogue of the formalism linking Lagrangians on TQ (Q = configuration space) with Hamiltonians on T^*Q via the Legendre transformation.

PROBLEM 1. *Develop a covariant theory of reduction, starting with the Gimmsy setup and producing, after* 3 + 1*ing, the* BMW *bracket for* GR *fluids.* The work of Kunzle and Nester [1984] and Holm [1985] suggests this is reasonable.

On the other hand, it seems that there is also a *covariant* version of Poisson brackets (where the bracket does *not* depend on any hypersurface and involves integration over *spacetime*). In this formalism, due to Marsden, Montgomery, Morrison and Thompson [1986] (M^3T), the Euler-Lagrange field equations take the form $\{F,S\} = 0$ where S is an appropriate action integral. It does work for general relativistic fluids and plasmas, including electromagnetic effects. One obtains covariant analogues of Lie Poisson brackets.

A more grandiose project is:

PROBLEM 2. *Make a commutative diagram of the following sort:*

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Covariant Multisymplectic}\\ \text{Formalism for GR fluids}\\ \text{(GIMMSY)}\end{array} \arrow[r, "\begin{array}{c}\text{Covariant}\\ \text{Reduction}\end{array}"] \arrow[d, "\begin{array}{c}3+1\\ \text{procedure}\end{array}"'] & \begin{array}{c}\text{Covariant Lie Poisson}\\ \text{Formalism for GR}\\ \text{Fluids } (M^3T)\end{array} \arrow[d, "\begin{array}{c}3+1\\ \text{procedure}\end{array}"] \\
\begin{array}{c}\text{3+1 GR fluids in material}\\ \text{(Lagrangian) or inverse}\\ \text{material representation}\\ \text{(Kunzle and Nester)}\end{array} \arrow[r, "\begin{array}{c}\text{Standard}\\ \text{material to}\\ \text{spatial reduc-}\end{array}", "\begin{array}{c}\text{tion (Holm}\\ \text{[1985], Holm,}\\ \text{Marsden and}\\ \text{Ratiu [1986]}\end{array}"'] & \begin{array}{c}\text{3+1 GR fluids in}\\ \text{spatial representation}\\ \text{(BMW)}\end{array}
\end{tikzcd}
$$

Bits of this scheme are known (see the quoted references); it would greatly clarify classical field theory if the whole picture were known in detail for fluids in particular and for GR fluids more generally.

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