Oscillatory and monotonic modes of long-wave Marangoni convection in a thin film

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We study long-wave Marangoni convection in a layer heated from below. Using the scaling $k = O(\sqrt{Bi})$, where $k$ is the wave number and $Bi$ is the Biot number, we derive a set of amplitude equations. Analysis of this set shows presence of monotonic and oscillatory modes of instability. Oscillatory mode has not been previously found for such direction of heating. Studies of weakly nonlinear dynamics demonstrate that stable steady and oscillatory patterns can be found near the stability threshold.

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I. INTRODUCTION

Marangoni convection in a liquid layer with upper free boundary is a classical problem in the dynamics of thin films and in the pattern formation [1,2]. In the pioneer theoretical paper, Pearson [3] analyzed the linear stability of the layer with a nondeformable free surface. He considered two cases of thermal boundary conditions at the substrate: the ideal and poor heat conductivity, when either the temperature or the heat flux are specified. In the latter case he found a monotonic long-wave instability mode for heating from below and zero Biot number $Bi$. For $Bi \ll 1$ the critical wave number $k$ is proportional to $Bi^{1/4}$ [1]. Many authors extended the analysis in order to include the deformation of the free surface. Review of analytical and numerical works can be found in [1]. In particular, several oscillatory modes were revealed; these modes were reported only for heating from above.

In the case of heating from below, a nonlinear analysis for ideally conductive substrate was performed in Ref. [4]: it was shown that the subcritical bifurcation occurs and instability with necessity results in film rupture. The behavior of perturbations near the stability threshold was studied in [5] for the case of a poorly conductive substrate. Under assumption of large gravity, and, hence, small surface deflection, the amplitude equation was derived and the subcritical bifurcation was found.

In this paper, we demonstrate the existence of a new oscillatory mode of long-wave instability for the film heated from below. Using the scaling $k = O(\sqrt{Bi})$, which was first suggested in Ref. [6], we derive a set of amplitude equations. Linear stability analysis gives both the monotonic and the oscillatory modes. Pattern selection near the stability threshold clearly demonstrates that instability does not necessarily lead to rupture and that both steady and oscillatory regimes can be found experimentally within certain domains of parameters.

II. PROBLEM FORMULATION

We consider a three-dimensional thin liquid film of the unperturbed height $H_0$ on a planar horizontal substrate heated from below. The heat conductivity of the solid is assumed small in comparison with the one of the liquid, thus the constant vertical temperature gradient $-\Delta A$ is prescribed at the substrate. (The Cartesian reference frame is chosen such that the $x$ and $y$ axes are in the substrate plane and the $z$ axis is normal to the substrate.)

The dimensionless boundary-value problem governing the fluid dynamics reads:

\begin{equation}
\frac{1}{\rho}(\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \nabla^2 \mathbf{v} - Ge, \tag{1a}
\end{equation}

\begin{equation}
T_i + \mathbf{v} \cdot \nabla T = \nabla^2 T, \quad \nabla \cdot \mathbf{v} = 0, \tag{1b}
\end{equation}

\begin{equation}
\mathbf{v} = 0, \quad T_z = -1 \text{ at } z = 0, \tag{2a}
\end{equation}

\begin{equation}
\mathbf{u} \cdot \mathbf{n} = (\rho - CaK)\mathbf{n} - M\nabla_z (T|_{z=0}), \quad \nabla_n T = -BiT, \tag{2b}
\end{equation}

Here, $\mathbf{v} = (u, w)$ is the fluid velocity (where $\mathbf{u}$ is the velocity in the substrate plane and $w$ is the $z$ component), $T$ is the temperature, $p$ is the pressure in the liquid, $\mathbf{u}$ is the viscous stress tensor, $h$ is the dimensionless height of the film, $\mathbf{e}_z$ is the unit vector directed along the $z$ axis, $\mathbf{n}$ and $\tau$ are the normal and tangent unit vectors to the free surface, respectively, $K$ is the mean curvature of the free surface. The dimensionless parameters entering the above set of equations are the capillary number, the Marangoni number, the Galileo number, the Biot number, and the Prandtl number:

\begin{equation}
Ca = \frac{\sigma H_0}{\eta \chi}, \quad M = -\frac{\sigma T A H_0^2}{\eta \chi}, \quad G = \frac{gH_0^3}{\nu \chi}, \quad Bi = \frac{qH_0}{k}, \quad P = \nu / \chi.
\end{equation}

and $P = \nu / \chi$. Here, $\sigma_T = d\sigma / dt; \sigma, g, q, \kappa, \chi, \nu$ and $\eta$ are the surface tension, acceleration of gravity, heat transfer rate, thermal conductivity, thermal diffusivity, kinematic and dynamics viscosity, respectively.

Below we study a large-scale convection using the set of Eqs. (1) and (2).

III. AMPLITUDE EQUATIONS

We rescale the coordinates and the time as follows:
\[ X = \varepsilon x, \quad Y = \varepsilon y, \quad \tau = \varepsilon^2 t, \quad (3) \]

where \( \varepsilon \ll 1 \) is the ratio of \( H_0 \) to a typical horizontal length scale. The temperature field is represented as \( T = -z + \text{Bi}^{-1} + \theta(X, Y, \tau) + O(\varepsilon^2) \).

We assume large values of \( \text{Ca} \) and small values of \( \text{Bi} \),

\[ \text{Ca} = \varepsilon^2 C, \quad \text{Bi} = \varepsilon^2 \beta. \quad (4) \]

Thus we deal with the intermediate asymptotics between the conventional long-wave mode, \( \text{Bi} = O(\varepsilon^4) \) [5], and the case of finite \( \text{Bi} \) [3]. These cases correspond to \( \beta = 0 \) and \( \beta \to \infty \), respectively.

Substituting the rescaled fields into Eqs. (1) and (2) and applying the conventional technique of the lubrication approximation (see [2]), we arrive at

\[ h_\tau = \nabla \cdot \left[ \frac{h^3}{3} \nabla \Pi + \frac{M h^2}{2} \nabla (\theta - h) \right] = \nabla \cdot j, \quad (5) \]

\[ h \theta_\tau = \nabla \cdot (h \nabla \theta) - \frac{1}{2} (\nabla h)^2 - \beta (\theta - h) + j \cdot \nabla (\theta - h) \]

\[ + \nabla \cdot \left[ \frac{h^4}{8} \nabla \Pi + \frac{M h^3}{6} \nabla (\theta - h) \right]. \quad (6) \]

Here \( \Pi = G h - C \nabla^2 h \) and \( \nabla \) is a two-dimensional gradient with respect to \( X \) and \( Y \).

Equations (5) and (6) form a closed set of the amplitude equations governing the nonlinear interaction of two well-known long-wave modes: the Pearson’s mode (\( h = 1 \)) [3] and the surface deformation-induced mode. (Note that the latter mode with \( \theta = \text{const} \) emerges only in the case of the conductive substrate [4].) Conductive state obviously corresponds to \( h = \theta = 1 \).

**IV. LINEAR STABILITY ANALYSIS**

Substituting the perturbed fields \( h = 1 + \xi \) and \( \theta = 1 + \Theta \) into Eqs. (5) and (6), linearizing the equations for perturbations about the equilibrium, and representing the perturbation fields proportional to \( \exp(\lambda \tau + ikX) \), one arrives at

\[ \lambda^2 + \lambda \left[ \beta + k^2 \left( \frac{\tilde{G} - M}{3} \right) \right] + \frac{k^2}{3} (\beta + k^2) \tilde{G} \]

\[ - \frac{M k^4}{2} \left( \frac{1}{3} + \frac{\tilde{G}}{72} \right) = 0, \quad (7) \]

where \( \tilde{G} = G + Ck^2 \). Equation (7) possesses both real (monotonic instability) and complex (oscillatory instability) solutions.

For the monotonic mode \( \lambda = 0 \) at the stability border, thus the marginal stability curve is given by

\[ M_m = \frac{48(\beta + k^2)\tilde{G}}{k^2(72 + \tilde{G})}. \quad (8) \]

These marginal curves have a minimum at the finite values of \( k \) only if

\[ \beta C < 72, \quad (9) \]

otherwise the minimal value, \( M_m^{(\infty)} \), is achieved in the limit \( k \to \infty \), i.e., the long-wave mode is not critical. Hereafter we assume that the inequality (9) holds; since the limit \( C = 0 \) is well studied [10], for all computations we can set \( C = 1 \) without loss of generality [11]. The critical wave number materializing the minimum of the marginal stability curve, Eq. (8), is

\[ (k_c^{(m)})^2 = \frac{\beta C G + \sqrt{72 \beta C G(G + 72 - \beta C)}}{C(72 - \beta C)}. \]

For the oscillatory mode the marginal stability curve is determined by the expression

\[ M_o = 3 + \tilde{G} + \frac{3\beta}{k^2}. \quad (10) \]

The imaginary part of the growth rate for neutral perturbations is

\[ \lambda = \text{Im}(\lambda) = \frac{k^2}{12} \sqrt{(72 + \tilde{G})(M_m - M_o)}, \quad (11) \]

i.e., the oscillatory mode is present only at \( M_o(k) < M_m(k) \).

Minimization of the Marangoni number with respect to \( k \) gives

\[ M_c^{(o)} = 3 + G + 2\sqrt{3\beta C}, \quad k_c^{(o)} = \left( \frac{3\beta}{C} \right)^{1/4}. \quad (12) \]

Examples of the marginal stability curves for these modes are shown in Fig. 1(a). Domains of monotonic and oscillatory instability are demonstrated in Fig. 1(b). It is clear that the oscillatory mode is critical for \( \beta C > 17.4 \) and \( G < 17.2 \). Take, for instance, a layer of water of thickness \( H_0 = 10^{-3} \text{ cm} \). Then \( G = 0.1 \), \( \text{Ca} \approx 10^4 \) and \( \text{Bi} \) has to be approximately \( 10^{-3} \) in order to provide the required value of \( \beta C \); this value seems achievable in experiments.

Equations (10)–(12) indicate why the oscillatory mode has not been found earlier. As we have emphasized above, all previous studies deal with either \( \tilde{G} > 1 \) [3], or \( \beta = 0 \) [5], or \( C = 0 \) [6]. In these cases the oscillatory mode does not exist.
where $K$ is a lattice.

Amplitudes $c.c.$ denotes complex conjugate terms and see that supercritical branching occurs only in two domains with respect to the slow time, and among the variety of possible patterns [7], three are important. They are Rolls with $A_1 \neq 0$, $A_2 = A_3 = 0$ and two types of Hexagons with $A_1 = A_2 = A_3 = 0$. Finally, at $G > G_2$, $H^* (H^*)$ is stable within the interval of supercriticality, whereas Rolls become stable when $M - M^\infty$ increases.

To finalize the discussion of steady patterns, we briefly discuss the competition of patterns on the square and hexagonal lattices. It is clear that at the finite values of $N$, Hexagons emerge subcritically and no stable patterns can be found near the stability threshold. Therefore, weakly nonlinear analysis provides stable patterns only near the dashed-dotted curve shown in Fig. 2(a), where the competition between hexagons and rolls occurs.

### B. Oscillatory mode

For the oscillatory mode the solution is presented in the form

$$\xi = \sum_{j=1}^{n} (A_j e^{i k_j \cdot \mathbf{R}} + B_j e^{-i k_j \cdot \mathbf{R}}) + c.c.$$  

Note that the pair $(A_j, B_j)$ corresponds to counterpropagating waves, which must be taken into account separately. The wave vectors for the square and hexagonal lattices are given by Eqs. (14) and (15), respectively.

For square lattice, the equation governing the dynamics of the amplitudes $A_j$ reads:

$$\dot{A}_j = \left( \gamma - K_0 |A_j|^2 - K_1 S_A \right) A_j , \quad j = 1, 2, 3 ,$$

where $S_A = \Sigma_{l=1}^{n} |A_l|^2$. Here, the dot denotes the derivatives with respect to the slow time, and $\gamma - M - M^\infty$ is the real growth rate. The Landau constants, $K_0$ and $K_1$, are real; they are cumbersome and, thus, are not presented here. Results of the numerical calculations are shown in Fig. 2. One can readily see that supercritical branching occurs only in two domains of parameters. These domains are situated either at rather small values of $\beta C$, Fig. 2(a), or at sufficiently small $G$, Fig. 2(b). In the former case Rolls are selected everywhere except for a very small region shown in the inset. In the latter case Squares are selected everywhere excluding the small region where Rolls are stable.

For hexagonal lattice, the resonant quadratic interaction results in the following amplitude equation:

$$\dot{A}_1 = \gamma A_1 - N A_2 A_3^* - (K_0 |A_1|^2 + K_1 S_A) A_1 ,$$  

and a similar equations for $A_{2,3}$. (Hereafter the asterisk denotes the complex-conjugate terms.) Generally speaking, the quadratic term prevails over cubic ones, which leads to subcritical excitation of the hexagonal patterns through a transcritical bifurcation [7]. However, $N = 0$ at the dashed-dotted line shown in Fig. 2(a) and in the vicinity of this line Eq. (17) becomes appropriate.

Among the variety of possible patterns [7], three are important. They are Rolls with $A_1 \neq 0$, $A_2 = A_3 = 0$ and two types of Hexagons either $H^*$ below or $H^*$ above the dashed-dotted line. At $G_1 < G < G_2 = 10$ Rolls are still subcritical and unstable; stable Hexagons emerge only within the finite interval of supercriticality. Finally, at $G > G_2$, $H^* (H^*)$ is stable within the interval of supercriticality, whereas Rolls become stable when $M - M^\infty$ increases.

Pattern selection on a hexagonal lattice is shown in Fig. 2. At $G < G_1 \approx 8.20$ there are no stable solutions; the subcritical bifurcation occurs for Rolls and one branch of Hexagons either $H^*$ below or $H^*$ above the dashed-dotted line. At $G_1 < G < G_2 = 10$ Rolls are still subcritical and unstable; stable Hexagons emerge only within the finite interval of supercriticality. Finally, at $G > G_2$, $H^* (H^*)$ is stable within the interval of supercriticality, whereas Rolls become stable when $M - M^\infty$ increases.
Three similar equations are obtained from Eqs. Aj replacement equations was performed in Ref. [9]. Furthermore, the infinite growth of one of the amplitudes \( AR \) or demonstrates the infinite growth of one of the amplitudes in the domain of supercritical excitation. Alternating Rolls are stable within the small area marked by “AR”; here depending on the initial condition the system either approaches AR or demonstrates the infinite growth of one of the amplitudes.

For hexagonal lattice, the amplitude equation governing the dynamics of the complex amplitudes \( A_j \) reads:

\[
A_j = \left[ \gamma - K_0 |A_j|^2 - K_1 |B_j|^2 - K_2 S_A - K_3 S_B \right] A_j - K_i B_j^* S_{AB},
\]

\[
j = 1, 2, 3.
\] (20)

Three similar equations are obtained from Eqs. (20) by a replacement \( A_j \rightleftharpoons B_j \).

Analysis of the Hopf bifurcation for the above set of equations was performed in Ref. [9], where 11 limit cycles were found and studied. Based on that paper, the results on pattern selection are presented in Fig. 3(b). The dashed line again separates direct and inverse Hopf bifurcations for TR, it is obviously the same as in the panel (a). However, for the hexagonal lattice, there appears a competition between TR and Traveling Rectangles 2 (TRA2, \( A_1 = B_1 \neq 0 \), whereas all other amplitudes vanish). The latter pattern is stable in the domain marked by “TRA2.” The entire domain of supercritical bifurcation becomes smaller because TRA2 can bifurcate subcritically, see Fig. 3(b). Studying the competition between patterns on hexagonal and square lattices, we found that the stability boundaries for both TR and TRA2 are the same as shown in Fig. 3(b), whereas stability domain for AR nearly disappears.

VI. CONCLUSIONS

We studied the long-wave Marangoni convection in a liquid layer heated from below; the heat flux at the substrate is specified. In such setup, an interaction of two well-known monotonic modes of long-wave instability, the Pearson’s mode and the surface deformation-induced mode, can result in the emergence of a long-wave oscillatory mode. However, the oscillatory mode has not been detected in spite of extensive numerical, analytical, and experimental studies [1] since the publication of Pearson’s paper. We succeed in such analysis and point out the domain of parameters where the oscillatory mode exists, which can be reached in experiments.

Moreover, we point out the domains of parameters where the convection emerges supercritically and, hence, either stationary or oscillatory terminal state with distorted surface is stable. This result is also very unusual, since only subcritical branching was found in the previous studies [4,5].

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[10] For \( \epsilon = 0 \) (i.e., \( \epsilon = \infty \)) the critical Marangoni number reduces to the conventional value \( 48 \gamma^2/(\gamma + 72) \) [5], which is approached as \( k \rightarrow \infty \). The same \( M_c^{(\infty)} \) holds for \( \beta = 0 \) as well, but with zero critical wave number.

[11] This is equivalent to choosing \( \epsilon = \epsilon_c = \epsilon^{1/2} \).