Symmetry, Stability, Geometric Phases, and Mechanical Integrators (Part I)

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New analytical techniques and recent algorithms which numerically compute the time evolution of mechanical systems enable today's scientists, engineers, and mathematicians to predict events more accurately and more rapidly than ever before. Beyond the problems of simulation and prediction, however, lie the problems of *understanding* a dynamical system and choosing a correct dynamical system to *model* a given physical situation. Many systems remain too intricate to fully understand, but modern methods of mathematical analysis can sometimes offer insight. Most of this insight is obtained by viewing dynamics geometrically, and in fact the recent advances in mechanics which we review in this article all share this geometric perspective. Much of the value of these techniques lies in their applications, and although applications exist in a broad range of disciplines, we will focus on examples from space mechanics and robotics because these are simple to visualize.

A key problem in space mechanics is the problem of efficiently and effectively controlling the attitude of satellites in their orbits. Several spacecraft, including the very first U.S. satellite, Explorer I, have been unable to complete their missions because they began to tumble in space and could not be stabilized. Much research has been devoted to prevent current orbiting telescopes from suffering a similar fate. These telescopes must be controlled with high precision, since small errors can seriously degrade observations made of objects thousands of light years away. Several problems have plagued the Hubble Space Telescope, including low-frequency vibrations in the structure's solar-power panels due to unanticipated thermal expansion effects as the telescope passes from night into day. These vibrations were further amplified by the telescope's computer controlled stabilization mechanisms (Wilford, [1990]). Two of the topics we shall discuss—stability and numerical integration—are pertinent to the analysis and control of such vibrations.

Stability and control are also important issues in the field of robotics. This is certainly the case for a team at the MIT Artificial Intelligence Laboratory which is trying to construct a somersaulting robot (Hodgins and Raibert [1989]) as shown in Figure 1. Specifically, the project is to build a robot which will gather a running start, launch itself into the air, complete a forward revolution, and then land firmly on its feet. As might be imagined, the challenges involved in such a venture are formidable.

Recent ideas of Berry [1984, 1985], Hannay [1985], and Montgomery [1990], however, may help to solve this problem as well as provide the means for a way of efficiently controlling mechanical systems such as orbiting telescopes. It is amusing to note that many of these recent ideas are related to a natural curiosity that has fascinated and motivated investigations in physiology as well as dynamics: How does a falling cat often manage to land upright even if released while upside down from a complete rest? (See Figure 2.) The cat cannot violate the conservation of angular momentum, yet somehow it manages to turn itself 180 degrees in mid-air. This process has been investigated many times over the past century (see *Nature* [1894], Crabtree [1909], Kane and Scher [1969] and references therein) and recently has been analyzed by Montgomery [1990] with an emphasis on how the cat (or, more generally, a deformable body) can efficiently readjust its orientation by changing its shape. By "efficiently," we mean that the reorientation minimizes some function—for example the total energy expended. Montgomery's results characterize the deformations which allow a cat to most efficiently reorient itself without violating conservation of angular momentum.

We begin with a review of Hamiltonian systems and canonical formulations. We then introduce noncanonical formulations and the concept of reduction of dynamics. Recent results in determining stability are presented in the next section, and these are followed by a discussion of geometric phases in mechanics. We conclude with a survey of some recent advances in numerical integration algorithms.

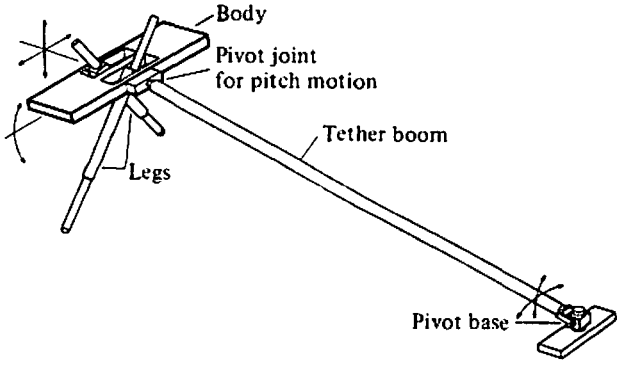

Figure 1: Diagram of the planar biped robot constructed at MIT (from Hodgins and Raibert [1990]). The robot is designed to take a running start, jump into the air, pitch itself forward so that it completes a forward flip, and continue running when it lands. ©1990 MIT Press, used by permission.

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Figure 2: A falling cat manages to land on its feet even if released upside down without initial angular momentum. The explanation of this counter-intuitive feat may lead to new ways of controlling the dynamics of mechanical systems such as robots and space telescopes (drawing from R. Montgomery *Commun. Math. Phys.* 128, 567 [1990].)

Hamiltonian Formulation

The equations of motion for a classical mechanical system consisting of n particles may be written as a set of first order equations in the form established by Hamilton:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \quad \text{(CHE)}$$

The generalized configuration coordinates $(q^1, \ldots, q^n)$ and momenta $(p_1, \ldots, p_n)$ together define the system's instantaneous state, which may also be regarded as the coordinates of a point in a $2n$-dimensional vector space called the *phase space*. We denote such a point by $(\mathbf{q},\mathbf{p})$. The *Hamiltonian function* $H(\mathbf{q},\mathbf{p})$ completely defines the system. In the absence of constraining forces and time dependence, $H(\mathbf{q},\mathbf{p})$ is simply the total energy of the system.

In the modern theory of Hamiltonian systems, this classical setting is generalized in two essential ways. First, the phase space, which identifies the possible states of the system, is allowed to be a differentiable manifold rather than merely a linear vector space. This generalization allows for the simplest and most natural characterization of systems consisting of bodies whose motions are spatially constrained. The set of all possible spatial positions of bodies in the system is known as the *configuration space*. For example, the configuration space for a three dimensional rigid body moving freely in space is $SE(3)$, the six dimensional group of Euclidean (rigid) transformations of three-space, that is, all possible rotations and translations. If translations are ignored and only rotations are considered, then the configuration space is $SO(3)$.

When the constraints defining a system are complicated, the configuration space may be an equally complicated manifold. For example, if two rigid bodies are connected at a point by an idealized ball-in-socket joint, then to specify the position of the bodies. we must specify a single translation (since the bodies are coupled) but we need to specify two rotations (since the bodies are free to rotate in any manner). The configuration space is therefore $SE(3) \times SO(3)$. This is already a fairly complicated object, but remember that one must keep track of both positions *and* momenta of each component body in order to formulate the system's dynamics completely. If Q denotes the configuration space (only positions), then the corresponding phase space P (positions and momenta) is the manifold known as the *cotangent bundle* of Q, which is denoted by T^*Q. Describing dynamics on such a manifold in terms of standard vector calculus can be quite cumbersome and computationally costly, but the modern theory of Hamiltonian systems allows us to take advantage of the powerful *differential calculus on manifolds*.

The second important way in which the modern theory of Hamiltonian systems generalizes the classical theory is by relaxing the requirement of using canonical phase space coordinate systems, i.e., coordinate systems for which the equations of motion take the standard form (CHE). An arbitrary transformation of the coordinates $(\mathbf{q},\mathbf{p})$ does not necessarily result in a system in which the new coordinates obey the canonical equations. As a simple example, the canonical description of the simple harmonic oscillator is defined by the Hamiltonian $H(q,p) = (q^2 + p^2)/2$, but if we change variables according to $q = xy$ and $p = y$, then it is easy to verify that x and y are not canonical coordinates.

Canonical coordinates are sometimes convenient variables through which to study Hamiltonian systems, but rigid body dynamics, celestial mechanics, robotics, and biomechanics provide a rich supply of examples of systems for which canonical coordinates are unwieldy and awkward. The free motion of a rigid body in space is the simplest such example. It was treated by Euler in the 18th century and yet it remains remarkably rich as an illustrative example.

As mentioned earlier, the rigid body problem in its primitive formulation has the six dimensional configuration space $SE(3)$. This means that the phase space, $T^*SE(3)$, is twelve dimensional. Assuming that no external forces act on the body, conservation of linear momentum allows us to solve for the components of the position and momentum vectors of the center of mass. This reduces the problem to finding the body's rotational orientation in space as if its center of mass were fixed. Each possible orientation corresponds to an element of the rotation group $SO(3)$, which we may view as a configuration space for all non-trivial motions of the body.

Euler formulated a description of the body's orientation in space in terms of three angles between axes which are either fixed in space or are attached to symmetry planes of the body's motion, as shown in Figure 3. The three *Euler angles*, ψ, ϕ, and θ, are generalized coordinates for the problem.

It is possible to construct a canonical Hamiltonian of the body's rotational motion in terms of the three Euler angles and their conjugate momenta. This leads to a fairly complicated system of six coupled ordinary differential equations. Euler's formulation, however, is simpler than the canonical Hamiltonian approach. Assuming that no external moments act on the body, the angular momentum vector is conserved. Euler used this fact to write the three associated momentum equations in a coordinate system fixed *within the body* rather than fixed in space. Letting (Π_1, Π_2, Π_3) denote the components of the angular momentum vector $\mathbf{\Pi} = \mathbf{\Pi}(t)$ along the principal inertial axes of the body, the momentum equations are given by the well-known *Euler equations:*

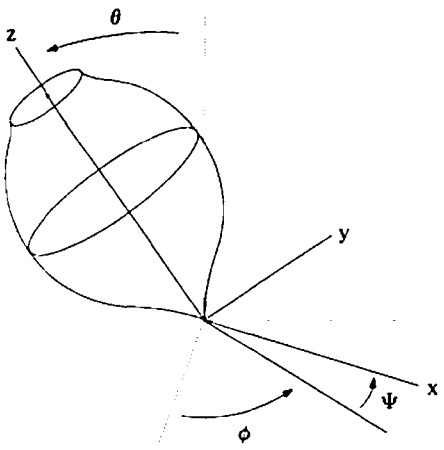

Figure 3: Diagram of Euler angles θ, ϕ, ψ in the case of a symmetric top. (After Goldstein [1980]).

$$\begin{aligned}\dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3,\\ \dot{\Pi}_2 &= \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1,\\ \dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2,\end{aligned} \qquad \text{(EE)}$$

where the constants I_1, I_2 and I_3 are the principal moments of inertia of the body. It was Arnold [1966a] who first clarified in a satisfactory way the relationships between the various representations (body, space, Euler angles) of the equations and showed how the same ideas apply to fluid mechanics as well.

The formulation above is remarkable for the simplicity of its geometrical interpretation. Viewing (Π_1, Π_2, Π_3) as coordinates in a three dimensional vector space, the Euler equations are evolution equations for a point in this space. An integral (constant) of motion for the system is given by the magnitude of the angular momentum vector: $\|\mathbf{\Pi}\|^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2$. This can be verified directly from the Euler equations (EE). Because of this, the evolution in time of any initial point $\mathbf{\Pi}(0)$ is constrained to the sphere $\|\mathbf{\Pi}\| = \|\mathbf{\Pi}(0)\| =$ constant. Thus we may view the Euler equations as describing a two dimensional evolution on an invariant sphere. We call this sphere the *reduced* phase space for the rigid body equations. The constant $\|\mathbf{\Pi}\|$ may be interpreted as a parameter which determines the size of the invariant sphere.

A basic fact about this description is that *this two dimensional system is a Hamiltonian system on the two-sphere* S^2. The Hamiltonian structure is not obvious from Euler's equations because the description in terms of angular momentum is inherently non-canonical. This means that there is no way to choose a pair of coordinates from (Π_1, Π_2, Π_3) to satisfy the canonical Hamilton equations (CHE). As mentioned above, however, Hamiltonian systems may be generalized to include Euler's formulation. The Hamiltonian for the reduced system is

$$H = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right), \qquad \text{(RBH)}$$

and we shall shortly show how this function allows us to recover Euler's equations (EE). Since solutions curves of (EE) are confined to the level sets of H (which are in general ellipsoids) as well as to the invariant spheres $\|\mathbf{\Pi}\| =$ constant, the intersection of these surfaces are precisely the trajectories of the rigid body, shown in Figure 4.

When considering a reduced phase space such as the sphere in the case of the rigid body equations, we call the fixed points *relative equilibria*. The equilibria are "relative" in the sense that they are equilibria only on the reduced phase space. These equilibria correspond to periodic orbits in the unreduced phase space, specifically to steady rotations about a principal inertial axis. The locations and stability types of the relative equilibria for the rigid body are clear from Figure 4. The four points located at the intersections of the invariant sphere with the x and z axes correspond to pure rotational motions of the body about its major and minor principal axes. These motions are stable, whereas the other two relative equilibria corresponding to rotations about the intermediate principal axis are unstable.

We shall shortly see how the stability analysis for a large class of more complicated systems can be greatly simplified through a careful choice of non-canonical coordinates. We managed to visualize the trajectories of the rigid body without really doing any calculations, but this occurrence is rare; the rigid body is a rather special system. Not only is the rigid body problem completely integrable (one can write down the solution in terms of integrals), but the problem reduces in some sense to a two dimensional manifold and allows questions about trajectories to be phrased in terms of level sets of integrals. Many Hamiltonian systems are not completely integrable and trajectories must be studied numerically. However, the fact that we were able to reduce the number of dimensions in the problem (from twelve to two) and the fact that this reduction was accomplished by appealing to non-canonical coordinates turns out to be a general feature of Hamiltonian systems with symmetry. One of the major results of contemporary theoretical mechanics has been the rigorous formalization of a general reduction procedure.

One of the most attractive features of the reduction procedure is that it may be applied to non-integrable or chaotic systems just as easily as to integrable ones. In a Hamiltonian context, non-integrability is generally taken to mean that, once the "ob-

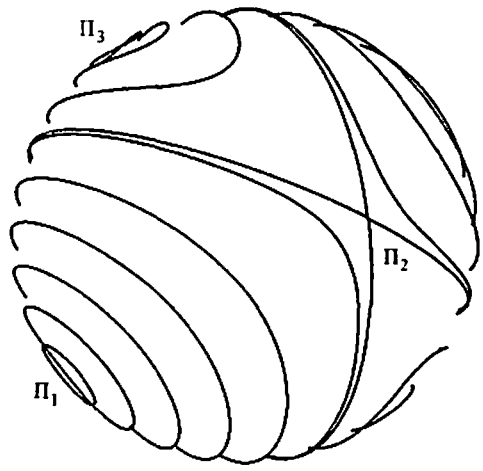

Figure 4: Phase portrait for the rigid body. The magnitude of the angular momentum vector determines a sphere. The intersection of the sphere with the ellipsoids of constant Hamiltonian gives the trajectories of the rigid body. This figure, as well as Figures 12, 13, and 14, were produced using the software package kaos which may be obtained from John Guckenheimer, Cornell University. Figure provided by Mark Meyers.

vious'' integrals are removed any analytic constant of motion is function of the Hamiltonian. We will not attempt to formulate a general definition of chaos, but rather use the term in a loose way to refer to systems whose motion is so extremely complicated that long-term prediction of dynamics is virtually impossible. It can sometimes be very difficult to establish whether a given system is chaotic or non-integrable. Sometimes theoretical tools such as ''Melnikov's method'' (cf. Guckenheimer and Holmes [1983] and Wiggins [1988]) are available. Other times, one resorts to numerics or direct observation. For instance, numerical integration suggests that irregular natural satellites such as Saturn's moon, Hyperion, tumble in their orbits in a highly irregular manner (Wisdom, Peale, and Mignard [1984]). The equations of motion for an irregular body in the presence of a non-uniform gravitational field are similar to the Euler equations except that there is a configuration-dependent gravitational moment term in the equations which may render the system non-integrable.

The evidence that Hyperion tumbles chaotically in space leads to difficulties in numerically modelling this system. It turns out that the manifold $SO(3)$ cannot be covered by a single three dimensional coordinate chart such as the Euler angle chart. Hence, an integration algorithm using canonical variables must employ more than one coordinate system, alternating between coordinates on the basis of the body's current configuration. For a body which tumbles in a complicated fashion, the body's configuration might switch from one chart of $SO(3)$ to another in a very short time interval. In the worst case, this could entail switching coordinate charts at nearly every step of the integration algorithm. The computational cost for such a procedure could be prohibitive. This situation is worse still for odies with internal degrees of freedom like robots and large-cale space structures. Such examples dramatically point out the need to go beyond canonical formulations in the context of practical problems.

Geometry, Symmetry, and Reduction

To motivate the discussion that follows, let us recap the two major elements of our discussion of the problem of the free rigid body: (1) The equations of motion for a system may be simpler in terms of non-canonical coordinates (e.g., Euler's equations) than in canonical coordinates; (2) The essential dynamics of a system may be described in terms of trajectories on a manifold (e.g., the invariant momentum sphere) which has a lower dimension than the dimension of the problem's original phase space. The reduction of dimension involved may be difficult to recognize and cumbersome to formulate within a canonical framework.

We now describe how modern developments in mechanics have led to coordinate-free formulations of equations of motion. These provide a framework for the non-canonical formulation of problems. We then outline a general method for reducing the dimension of the phase space of a Hamiltonian system provided that the system is invariant under an appropriate symmetry group.

We have emphasized the distinction between canonical and non-canonical coordinates by contrasting Hamilton's (canonical) equations with Euler's equations. We may view this distinction from a different perspective by introducing *Poisson bracket* notation. Given two smooth (C^∞) real-valued functions F and H defined on the phase space of a Hamiltonian system, efine the (canonical) Poisson bracket of F and H by

$$\{F,H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i} \frac{\partial F}{\partial p_i} \right),$$

where every (q^i,p_i) is a conjugate pair of canonical coordinates. Now suppose that H is the Hamiltonian function for the system. Then the formula for the Poisson bracket is precisely the directional derivative of F along the flow, that is,

$$\dot{F} = \{F,H\}.$$

In particular, Hamilton's equations themselves are recovered if we let F be each of the canonical coordinates in turn:

$$\dot{q}^i = \{q^i,H\} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = \{p_i,H\} = -\frac{\partial H}{\partial q^i}.$$

Once H is specified, the statement ''$\dot{F} = \{F,H\}$ for all smooth functions F'' is equivalent to Hamilton's equations. In fact, it tells how *any* function F evolves along the flow.

This representation of the canonical equations of motion leads to a generalization of the bracket notation to cover non-canonical formulations. There is an appropriate definition of the binary operation $\{\ ,\ \}$ such that the equations of motion in the given coordinates are equivalent to $\dot{F} = \{F,H\}$ which is valid in any system of coordinates. This holds for Hamiltonian systems on reduced phase spaces, such as the angular momentum sphere of the free rigid body, as well as systems expressed in their unreduced forms.

As an example, we once again consider Euler's equations. The solution to the equations are trajectories given in terms of the coordinates (Π_1, Π_2, Π_3) of the three dimensional ''angular momentum space,'' and the constraint $\|\mathbf{\Pi}\|$ = constant reduces the dynamics to a sphere imbedded in this space. We define the following non-canonical bracket of smooth functions on the angular momentum space

$$\{F,H\} = -\mathbf{\Pi} \cdot (\nabla F \times \nabla H),$$

where the gradients are taken with respect to the (Π_1, Π_2, Π_3) coordinates. The geometry of the scalar triple product operation insures that the induced bracket of functions defined on any invariant sphere is represented by the same formula. If H is the rigid body Hamiltonian (see (RBH)) and F is, in turn, allowed to be each of the three coordinate functions Π_i, then the formula $\dot{F} = \{F,H\}$ yields the three Euler equations.

The non-canonical bracket corresponding to the reduced free rigid body problem is an example of what is known as a *Lie-Poisson bracket* (see Appendix A). Other bracket operations have been developed to handle a wide variety of Hamiltonian problems in non-canonical form, including some problems outside of the framework of traditional Newtonian mechanics (see, for instance, Arnold [1966a] or Marsden et al. [1983]). The generalization of the Poisson bracket exemplifies the geometrical emphasis of modern theoretical mechanics. When studying Hamiltonian dynamics from a geometrical perspective, it is essential to distinguish features of the dynamics which depend on the Hamiltonian function from those which depend only on properties of the phase space. The generalized bracket operation is a geometrical invariant in the sense that it depends only on the structure of the phase space. The phase spaces arising in mechanics often have an additional geometrical structure closely related to the Poisson bracket. Specifically, they may be equipped with a certain differential two-form called the *symplectic form*. The symplectic form defines the geometry of a *symplectic manifold* much as the metric tensor defines the geometry of a Riemannian manifold. Bracket operations can be defined entirely in terms of the symplectic form without reference to a particular coordinate system. (See Marsden et al. [1983].)

The classical concept of a *canonical transformation* can also be given a more geometrical definition within this framework.

A canonical transformation is classically defined as a transformation of phase space which takes one canonical coordinate system to another. The modern analogue of this concept is a *symplectic map*—a smooth map of a symplectic manifold to itself which preserves the symplectic form or, equivalently, the Poisson bracket operation. Symplectic maps of cotangent bundles arise naturally in mechanics since every smooth map on a configuration space induces a symplectic map on the cotangent bundle of that space. This induced map is known as a *cotangent lift*.

The geometry of symplectic manifolds is an essential ingredient in the formulation of the reduction procedure for Hamiltonian systems with symmetry. We now outline some important ingredients of this procedure. Some additional information is contained in Appendix A. In Euler's problem of the free rotation of a rigid body in space (assuming that we have already exploited conservation of linear momentum), the six dimensional phase space is $T^*SO(3)$—the cotangent bundle of the three dimensional rotation group. The reduction from six to two dimensions is classically described as a consequence of two essential features of the problem:

(1) the existence of a coordinate system in which the Hamiltonian can be expressed independently of the body's configuration, and

(2) the existence of a conserved quantity, μ, the angular momentum in space.

Condition (1) is equivalent to rotational invariance of the Hamiltonian, while condition (2) expresses the conservation of the total angular momentum of the rigid body. These two conditions are generalized to arbitrary mechanical systems with symmetry in the general reduction theory of Meyer [1973] and Marsden and Weinstein [1974], which was inspired by the seminal works of Arnold [1966a] and Smale [1970]. In this theory, one begins with a given phase space that we denote by P. We assume there is a group G of symmetry transformations of P that transform P to itself by canonical transformations. Generalizing (1) above, one assumes that the Hamiltonian is invariant under these transformations. Generalizing (2), we use the symmetry group to generate a vector-valued conserved quantity which we denote J; it is called the *momentum map*.

Analogous to the set where the total angular momentum has a given value, we consider the set of all phase space points where J has a given value μ. We call this set the μ-level set for J. The analogue of the two dimensional body angular momentum sphere in Figure 4 is the reduced phase space, denoted P_μ, that is constructed as follows: P_μ is the μ-level set for J with any two points that can be transformed one to the other by a group transformation, identified. This identification procedure is not unlike the procedure one uses to bend an interval into a circle by identifying the two endpoints of the interval—what were two points before become one point in the new system. In the reduction theorem, many points can get identified with one new point, but the idea is the same. The reduction process states that P_μ inherits the symplectic (or Poisson bracket) structure from that of P, so it can be used as a new phase space. Also, dynamical trajectories of the Hamiltonian H on P determine new reduced trajectories on the reduced space. This new dynamical system is, naturally, called the *reduced system*. The trajectories on the sphere in Figure 4 are the reduced trajectories for the rigid body problem.

We saw that steady rotations of the rigid body correspond to fixed points on the reduced manifold, namely, the body angular momentum sphere in Figure 4. In general, fixed points of the reduced dynamics on P_μ are called *relative equilibria*, following terminology first introduced by Poincaré around 1880. The reduction process can be applied to the system which models the motion of the moon Hyperion, to spinning tops, to fluid and plasma systems, and to systems of coupled rigid bodies. For example, if a system of coupled rigid bodies is undergoing steady rotation, with the internal parts not moving relative to each other, this will be a relative equilibrium of the system. An oblate Earth in steady rotation is a relative equilibrium for a fluid-elastic body. In general, the bigger the symmetry group, the richer the supply of relative equilibria.

Stability

Having discussed the reduction procedure, we turn to the stability of the reduced dynamics. There is a standard procedure for finding the stability of equilibria of an ordinary differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{f}$ is smooth. The procedure involves solving for the equilibria (fixed points) of the differential equation. These are the points $\mathbf{x}_e$ such that $\mathbf{f}(\mathbf{x}_e) = 0$; i.e., points that are fixed in time under the dynamics. The goal of this procedure is to determine the stability of the fixed point $\mathbf{x}_e$. By stability here we mean that any solution to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ that starts near $\mathbf{x}_e$ remains close to $\mathbf{x}_e$ for *all* future time.

A traditional method of ascertaining the stability of $\mathbf{x}_e$ is to examine the first variation equation

$$\dot{\xi} = \mathbf{D}_x\mathbf{f}(\mathbf{x}_e)\xi$$

where $\mathbf{D}_x\mathbf{f}(\mathbf{x}_e)$ is the Jacobian of $\mathbf{f}$ at $\mathbf{x}_e$ and is defined to be the matrix of partial derivatives

$$\mathbf{D}_x\mathbf{f}(\mathbf{x}_e) = \left[\frac{\partial f_i}{\partial x_j}\right]_{x=x_e}$$

The eigenvalues of $\mathbf{D}_x\mathbf{f}(\mathbf{x}_e)$ are then examined. If all the eigenvalues lie in the left half plane, then, by a result of Liapunov [1909], the fixed point is stable. If any of the eigenvalues lie in the right half plane, then the fixed point is unstable. However, for Hamiltonian systems the eigenvalues come in pairs or quartets symmetric about the origin and so they cannot all lie in the left half plane. Thus, this standard stability result will never allow us to deduce whether a Hamiltonian system contains a stable fixed point. As the class of Hamiltonian systems includes the equations which are used to model motions of orbiting space stations and space telescopes, it is imperative to develop explicit conditions to ensure the stability of their orbits.

When the Hamiltonian is in canonical form one can use a stability test for fixed points due to Lagrange and Dirichlet. This method uses the fact that for a fixed point $(\mathbf{q}_e,\mathbf{p}_e)$ of such a system,

$$\frac{\partial H}{\partial q^i}(\mathbf{q}_e,\mathbf{p}_e) = \frac{\partial H}{\partial p_i}(\mathbf{q}_e,\mathbf{p}_e) = 0.$$

Hence, the fixed point occurs at a critical point of the Hamiltonian. If the $2n \times 2n$ matrix $\mathbf{D}^2H$ of second partial derivatives is either positive or negative definite at $(\mathbf{q}_e,\mathbf{p}_e)$ then one has a stable fixed point. Consider the positive definite case. Conceptually, the reason for stability is very simple: since H has a minimum at $(\mathbf{q}_e,\mathbf{p}_e)$ and energy is conserved, solutions stay on level surfaces of H, so that a solution starting near the minimum has to *stay* near the minimum. For a Hamiltonian of the form kinetic plus potential (V), critical points occur when $\mathbf{p}_e = 0$ and $\mathbf{q}_e$ is a critical point of the potential of V. This criterion then reduces to asking for a minimum of V.

In fact, this criterion was used to solve one of the classical problems of the 19th century: the problem of rotating gravitating fluid masses. This problem was studied by Newton, Mac-

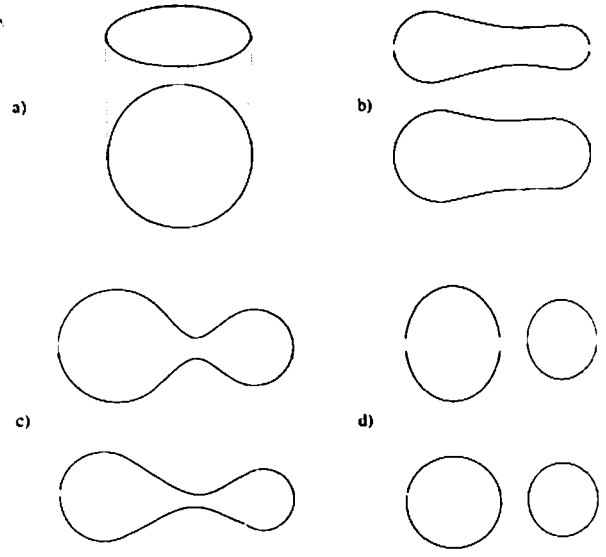

Figure 5: The formation of pear-shaped objects of equilibrium of a rotating fluid mass as it solidifies. Shown are the horizontal and vertical projections of the fluid masses during bifurcation (after Poincaré [1892]).

Laurin, Jacobi, Riemann, Poincaré, and others. The motivation for its study was in the conjectured birth of two planets by the splitting of a large mass of solidifying rotating fluid as shown in Figure 5. This is an example of what has since become known as a *symmetry-breaking bifurcation*. These ideas are important in understanding pattern formation and many of the resulting symmetric objects we see in nature. Poincaré [1892, 1901] was a major contributor to the study of this phenomenon and used the potential energy and angular momentum to deduce the stability and bifurcation of rotating fluids.

The Lagrange-Dirichlet method was generalized by Arnold [1966b] into what has become known as the *energy-Casimir method*. Arnold analyzed the stability of stationary flows of perfect fluids and also developed an explicit stability criterion for the case in which the configuration space for the Hamiltonian of this system is a group which coincides with the symmetry group of the mechanical system. A *Casimir* C is characterized by the fact that it Poisson commutes with *any* function F defined on the phase space of the Hamiltonian system, i.e.,

$$\{C,F\} = 0.$$

(The name Casimir is used in recognition of work by H. B. G. Casimir, who introduced closely related ideas in representation theory.) Large classes of Casimirs usually occur when the reduction procedure is performed, resulting in systems with noncanonical brackets.

For example, in the case of the rigid body discussed previously, if Φ is a function of one variable and Π is the angular momentum vector in the inertial coordinate system, then

$$C(\Pi) = \Phi(\|\Pi\|^2)$$

is a Casimir for the rigid body bracket. The energy-Casimir method involves choosing C such that $H + C$ has a critical point at an equilibrium z_e and then examining $\mathbf{D}^2(H + C)(z_e)$. If this matrix is positive or negative definite then the equilibrium z_e is stable. When the phase space is obtained by reduction, the equilibrium z_e is a relative equilibrium of the original Hamiltonian system.

The energy-Casimir method has been applied to a variety of problems including problems in fluids and plasmas (Holm, Marsden, Ratiu, Weinstein [1985]) and rigid bodies with flexible attachments (Krishnaprasad and Marsden [1987]). If applicable, the energy-Casimir method may permit an explicit determination of the stability of the relative equilibria. It is important to remember, however, that these techniques give stability information only. As such one cannot use them to infer *instability* without further investigation.

The energy-Casimir method is restricted to certain types of systems, since its implementation relies on an abundant supply of Casimir functions. In some important examples, Casimirs have not yet been found and may not even exist. Two methods developed to overcome this difficulty are known as the energy momentum method (EMM) and the reduced energy momentum method (REMM). These two methods are closely linked to the method of reduction. They use conserved quantities, namely the energy and momentum maps, that are usually readily available, rather than Casimirs.

The energy momentum method (Simo, Posbergh and Marsden [1990a,b], Simo, Lewis and Marsden [1990], and Lewis and Simo [1990]) involves the *augmented Hamiltonian* defined by

$$H_\xi = H(q, p) - \xi \cdot J(q, p),$$

where J is the momentum map described in the previous section and ξ may be thought of as a Lagrange multiplier. One then sets the first variation of H_ξ equal to zero to obtain the relative equilibria. To ascertain stability, the second variation $\mathbf{D}^2H_\xi$ is calculated. One is then interested in determining the definiteness of the second variation.

Definiteness in this context has to be properly interpreted to take into account the conservation of the momentum map J and the fact that $\mathbf{D}^2H_\xi$ may have zero eigenvalues due to symmetry. The variations of p and q must satisfy the linearized angular momentum constraint $(\delta q, \delta p) \in \ker[\mathbf{D}J(q_e, p_e)]$, and must not lie in symmetry directions; only these variations are used to calculate the second variation of the augmented Hamiltonian H_ξ. The energy momentum method has been applied to the stability of relative equilibria of among others, coupled rigid bodies and geometrically exact rods (Simo, Posbergh and Marsden [1990a,b] and Patrick [1990]).

Cornerstones in the development of the EMM and REMM were laid by Routh [1877] and Smale [1970], who studied the stability of relative equilibria of simple mechanical systems. Simple mechanical systems are those whose Hamiltonian may be written as the sum of the potential and kinetic energies; the linear harmonic oscillator $\ddot{x} + \omega^2 x = 0$ is an example of such a system. Smale showed that there is a naturally occurring connection that plays an important role in the reduction of a simple mechanical system with symmetry. (A connection can be thought of physically as a generalization of the electromagnetic vector potential, A. See Zwanziger, Koenig, and Pines [1990]). We now call this the *mechanical connection*. Smale also showed that the relative equilibria of these systems are given by the critical points of the *amended potential function* V_{μ_e}, defined below.

The amended potential plays a crucial role in the REMM (see Simo, Lewis and Marsden [1990], and Lewis and Simo [1990]). The REMM exploits properties of the reduction method to put the second variation into a normal form. First one calculates the amended potential V_{μ_e} which is the potential energy of the system plus a generalization of the potential energy of the centrifugal forces in stationary rotation:

$$V_{\mu_e}(q) = V(q) + \frac{1}{2}\mu_e \cdot \mathbb{I}^{-1}(q)\mu_e,$$

where $\mathbb{I}$ is the locked inertia tensor, a generalization of the

inertia tensor of the rigid body obtained by *locking* all the joints in the configuration q. The momentum p_e need not be zero since the system is typically in motion. The second variation directly yields the stability of the relative equilibria. However, an interesting phenomenon occurs if the tangent space $\mathcal{V}$ is split into two specially chosen subspaces $\mathcal{V}_{RIG}$ and $\mathcal{V}_{INT}$ (Simo, Lewis and Marsden [1990]). In this case the second variation block diagonalizes:

$$\mathbf{D}^2V_{\mu_e}|\mathcal{V}\times\mathcal{V} = \begin{bmatrix} \mathbf{D}^2V_{\mu_e}|\,\mathcal{V}_{RIG}\times\mathcal{V}_{RIG} & 0 \\ 0 & \mathbf{D}^2V_{\mu_e}|\,\mathcal{V}_{INT}\times\mathcal{V}_{INT} \end{bmatrix}.$$

The space $\mathcal{V}_{RIG}$ (rotation variations) is generated by the symmetry group, and $\mathcal{V}_{INT}$ are the internal or shape variations. In addition, the whole matrix $\mathbf{D}^2H_\xi$ block diagonalizes in a very efficient manner. This often allows the stability conditions associated with $\mathbf{D}^2V_{\mu_e}|\,\mathcal{V}\times\mathcal{V}$ to recast in terms of a standard eigenvalue problem for the second variation of the amended potential.

This splitting/diagonalization has important computational implications. In the case of pseudo-rigid bodies (Lewis and Simo [1990]), this splitting results in reducing the stability problem to the examination of a single 3×3 matrix instead of a full 18×18 array. (The large matrix becomes diagonal except for a 3×3 subblock on the diagonal.) The block diagonalization approach enabled Lewis and Simo to solve their problem analytically, whereas without it, a substantial numerical computation would have been necessary. The idea of block diagonalization can be taken further. It turns out that $\mathbf{D}^2H_\xi$ and the symplectic structure can be explicitly brought into normal form *simultaneously*. Although investigations are still at an early stage, this result promises to simplify computations in perturbation theory and the study of bifurcation phenomena.

In general, this diagonalization *explicitly separates the rotational and internal modes*, a result which is extremely important not only in rotating and elastic fluid systems, but also in molecular dynamics and robotics. Similar simplifications are expected in the analysis of other problems to be tackled using the reduced energy momentum method.

Appendix A: On the Reduction Construction

In this appendix, we explain a few of the general notions used in the reduction theorem. In the text, we used the example of the free rigid body to illustrate the concept of reduction. The angular momentum space for the rigid body can be interpreted as the dual space of the Lie algebra of $SO(3)$. This is a three dimensional vector space usually identified with $\mathbb{R}^3$. The analogue of the angular momentum space in general reduction theory is $\mathfrak{g}^*$, the dual of the Lie algebra of the symmetry group G.

The momentum map is a map $J: P\to\mathfrak{g}^*$ with the property that, for each $\xi\in\mathfrak{g}$, $\langle J,\xi\rangle$ generates, in the sense of Hamilton's equations, the infinitesimal action in the same way that angular momentum $\mathbf{q}\times\mathbf{p}$ generates rotations. The level set with value $\mu\in\mathfrak{g}^*$ is $J^{-1}(\mu)$, which will be a submanifold of P under certain conditions. The group G_μ is the subgroup of G that maps $J^{-1}(\mu)$ to itself. (It can also be defined as the subgroup that fixes the value μ under the coadjoint action of G on $\mathfrak{g}^*$.) The reduced space is then the quotient $P_\mu = J^{-1}(\mu)/G_\mu$.

Whereas P_μ is symplectic, the manifold P/G is *Poisson* (the bracket of two functions on P/G is defined by regarding them as G-invariant functions on P). If μ is considered as a parameter, one can show that the P_μ are the symplectic leaves in P/G in the same way that the spheres $\|\Pi\| =$ constant are the symplectic leaves in the three dimensional angular momentum space.

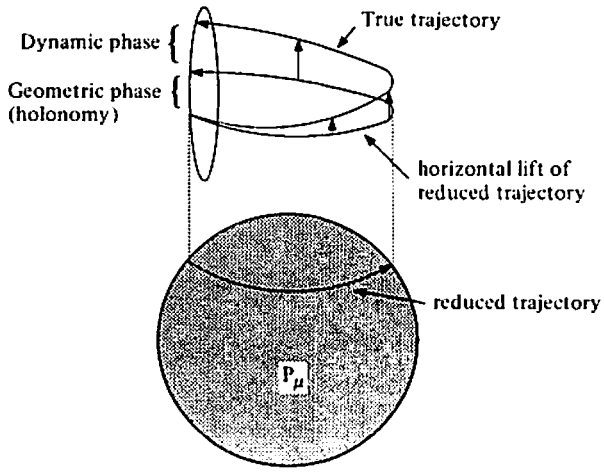

Figure A1: Holonomy for the rigid body (after Marsden, Montgomery, and Ratiu [1990]). As the body completes one period in P_μ, the reduced phase space, the body's true configuration does not return to its original value. The phase difference is equal to the influence of a dynamic phase which takes into account the body's energy, and a geometric phase which depends only on the area of P_μ enclosed by the reduced trajectory.

An important case of the reduction theorem arises when the configuration space is identical to the symmetry group, so that $P = T^*Q = T^*G$. Then the Poisson manifold $P/G = (T^*G)/G$ is identified with the linear space $\mathfrak{g}^*$. This identification induces a special Poisson structure on $\mathfrak{g}^*$ known as the *Lie-Poisson* structure. The Lie-Poisson bracket of two functions F and K on $\mathfrak{g}^*$ is defined by

$$\{F,K\}_\pm(\mu) = \pm\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta K}{\delta\mu}\right]\right\rangle, \qquad \mu\in\mathfrak{g}^*,$$

where the derivative $\delta F/\delta\mu$ is the usual derivative of F regarded as taking values in $\mathfrak{g}$, and $[,]$ is the Lie bracket on $\mathfrak{g}$. The general Lie-Poisson bracket $\{,\}_\pm$ is the bracket obtained from $(T^*G)/G$ using right multiplication for the plus sign and left multiplication for the minus sign. The rigid body bracket discussed in the text is the special case $G = SO(3)$, using the minus sign in the definition above.

Let us indicate how holonomy is linked closely to the reduction process by returning to our rigid body example. Picture the rigid body as tracing out a path in its phase space $T^*SO(3)$. Conservation of angular momentum implies that the path lies in the submanifold consisting of all points which are mapped onto μ by the momentum map. These points are then mapped to a curve in P_μ by the reduction process; i.e., by the quotient map $J^{-1}(\mu)\to P_\mu$. As Figure 4 in the text shows, almost every trajectory on this reduced space is periodic, but this does *not* imply that the original path was periodic, as is shown in Figure A1. The difference between the true trajectory and a periodic trajectory is given by the holonomy plus the dynamic phase. This is given quantitatively by formula (RBP) in the text and the reduction picture presented here is useful in proving it.

We remark that the reduction construction for the rigid body corresponds to the *Hopf fibration* which describes the three-sphere S^3 as a nontrivial circle bundle over S^2. In our example, S^3 is the subset of phase space which is mapped to μ under the reduction process. (More accurately, $J^{-1}(\mu)$ is $SO(3)\approx S^3/\mathbb{Z}_2$.) See Koçak et al. [1986] or Appendix C for more details.

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Part II in next issue