HAMILTONIAN MECHANICS ON LIE GROUPS AND HYDRODYNAMICS

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Introduction. Some of the most classical and important examples in mechanics are systems whose configuration space is a Lie group. The particular examples we have in mind are the rigid body (on the Lie group SO(3)) and the perfect fluid (on the Lie group of volume preserving diffeomorphisms).

Most of what we have to say is classical and well known. What we do is to put it in the language of global analysis with perhaps some simplification. Our sources are mainly the papers of Arnold and Blancheton [3], [4].

The paper is divided into two parts. In the first we present the general theory. In the second we describe the case of hydrodynamics. Some connections will be made with the calculus of variations in the future. In addition, a more complete exposition of the present work will appear in lecture note forms shortly [11].

1. Abstract theory. Let $G$ be a Lie group. By this we mean that $G$ is a smooth manifold modelled on a locally convex topological vector space (locally convex since, amongst other reasons, the Hahn-Banach theorem is needed) and is also a group such that group multiplication and inversion are $C^\infty$ mappings. The tangent bundle of $G$ is denoted $TG$ and the fiber over $x \in G$ is written $T_xG$.

Let $g$ be a (weak) Riemannian metric on $G$. This means that the tensor $g$ is an inner product on each $T_xG$, but inducing a different topology in general. For each $x \in G$, $e_x \in T_xG$ and $f_x \in T_xG$ we write $\langle e_x, f_x \rangle = g(x) \cdot (e_x, f_x)$.

A weak symplectic form on a manifold $M$ is a closed two form $\omega$ such that the mapping $\omega_\flat : TM \to T^*M$ defined by $\omega_\flat(x)(f_x) = g(x) \cdot (e_x, f_x)$ is injective on each fiber. (If $\omega_\flat$ is an isomorphism on each fiber, we call $\omega$ a symplectic form.)

For infinite dimensional mechanics (continuum mechanics and quantum mechanics for example), if one wishes to work with a symplectic form it is necessary to use domains for vector fields in the same sense as occurs in semigroup theory; see [8]. If, however, one wishes to exploit differentiability directly and have the vector field defined and smooth in the usual sense, then it is necessary to use weak symplectic forms instead. The reason will become evident shortly. We shall use weak symplectic forms here.

Except in unusual and artificial circumstances (these can be obtained for the wave equation using the spaces in [10]), the manifolds one needs to get a smooth

1 See also [14].

2 The term "Lie Group" may be misleading since, in the infinite dimensional case, the usual Lie theorems do not hold. The term ILH Lie Group, or Fréchet Lie Group may be better.

3 $g$ is assumed smooth.
vector field are usually Fréchet and not Banach, the $C^\infty$ functions for example. In that case the standard flow theorem for vector fields is false. Instead one must use techniques of Browder, Kato and others. For the case of hydrodynamics with viscosity a good part of a classical book [6] is needed. Also, the flow is possibly only local, that is, cannot be extended for all time. For the nonviscous case we are concerned with, see Kato [5].

Let $M$ be a manifold and $\omega$ a weak symplectic form. A one form $\alpha$ can be lifted when there exists a vector field (unique) $X$ on $M$ such that $X^b = \alpha$, where $X^a(m) = 2\omega_x(m) \cdot X(m)$, $m \in M$. We write $X = \alpha^a$.

For a smooth function $f: M \to \mathbb{R}$, such that $df$ can be lifted, we write $X_f = (df)^a$, and we call $X_f$ a Hamiltonian vector field.

It will be necessary to recall a few theorems about Hamiltonian systems.

**Theorem 1.** Let $(M, \omega)$ be a weak symplectic manifold and $X_H$ a Hamiltonian vector field with a local smooth flow $F_t$. Then

(i) $H \circ F_t = H$ (conservation of energy), and

(ii) $F_t^* \omega = \omega$, or $F_t$ is symplectic (preserves the form $\omega$).


Recall that if $M$ is a manifold then $T^*M$ admits a natural weak symplectic structure given by (locally)

$$\omega(\langle \xi_1, \alpha_1 \rangle, \langle \xi_2, \alpha_2 \rangle) = [\alpha_2(e_1) - \alpha_1(e_2)]/2$$

for $e_i \in T_{\xi_i}M$ and $\alpha_i \in T^*_{\xi_i}M$. [$\omega$ is symplectic iff $M$ is modelled on a semireflexive space.] See [8, Theorem 2.4].

**Theorem 2.** Let $G$ be a Lie group and $M$ a manifold with $\Phi: G \times M \to M$ a smooth action of $G$ on $M$, which extends naturally to an action $\Phi^*$ of $G$ on $T^*M$. Suppose $X_H$ is a Hamiltonian vector field on $T^*M$ and $H$ is invariant under the action $\Phi^*$. Then the following functions $P_x$ are invariant under the flow of $X_H$.

Let $X$ be an infinitesimal generator of $\Phi$, so $X$ is a vector field on $M$ and define $P_X: T^*M \to \mathbb{R}$ by $P_x(\alpha_m) = \alpha_m(X(m))$. (We call $P_X$ the momentum of $X$.)

For the proof see [8, Theorem 5.3]. This is the basic conservation law of mechanics.

In case $g$ is a weak Riemannian metric on $M$ inducing a map $g_h: TM \to T^*M$, then we deduce (using [8, Theorem 5.2]) that if $X_H$ is a Hamiltonian vector field on $TM$ (with respect to the form $\omega_g(e_x) \cdot ((e_1, f_1), (e_2, f_2)) = [\langle f_2, e_1 \rangle - \langle f_1, e_2 \rangle]/2$) and $H$ is invariant under the induced (adjoint) action on $TM$ then the function $P_X(e_m) = \langle e_m, X(m) \rangle$ is invariant under the flow of $X_H$.

The latter situation is the one which arises naturally in the case of a Lagrangian system and will be of concern to us below.

Let us now return to the setting of Lie groups. We shall require our group and metric $g$ to have certain regularity properties introduced as follows.

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4 The general nonviscous existence problem is settled in [14].
DEFINITION.\footnote{In the rest of section one, “left” and “right” can be interchanged, and this introduces a minus sign in Theorem 5 and Lemma 6.} Let $G$ be a Lie group and $g$ a weak Riemannian metric on $G$. We say that $g$ is compatible with $G$ iff

(i) $g$ is left invariant; that is, for each $x \in G$, $L_x^*g = g$ where $L_x$ is the diffeomorphism defined by $L_x(y) = xy$;

(ii) the kinetic energy function $T: TG \to \mathbb{R}$ defined by $T(e_x) = \langle e_x, e_x \rangle / 2$ can be lifted to a (smooth) vectorfield $X_T$ using the weak symplectic form

$$\omega_g((e_1, f_1), (e_2, f_2)) = \langle f_2, e_1 \rangle - \langle f_1, e_2 \rangle / 2;$$

and

(iii) $X_T$ possesses a local smooth flow (called the geodesic flow of $g$).

Also, we say that $G$ is a regular Lie group iff every (smooth) left invariant vectorfield on $G$ has a flow.

In the finite dimensional case these conditions are of course redundant. In the infinite dimensional case it is (iii) above which is difficult to verify.

The main conservation theorem is the following (the cases of hydrodynamics and a rigid body are due to Euler):\footnote{In the rest of section one, “left” and “right” can be interchanged, and this introduces a minus sign in Theorem 5 and Lemma 6.}

**THEOREM 3.** Let $G$ be a regular Lie group and $X_H$ a Hamiltonian vectorfield on $TG$ with $H$ invariant by left translations ($H \circ TL_x = H$ for each $x \in G$, $TL_x: TG \to TG$ being the tangent map). Then for each $v \in T_eG$ ($e = identity$ element of $G$) the function $P_v$ is invariant under the flow of $X_H$, where

$$P_v: TG \to \mathbb{R}$$

is defined by

$$P_v(u_x) = \langle T_eR_x \cdot v, u_x \rangle$$

where $u_x \in T_xG$, $R_x$ is right translation by $x$ and $T_eR_x$ is the tangent of $R_x$ evaluated at $e \in G$.

**COROLLARY 4.** Let $G$ be a regular Lie group and $g$ a compatible weak Riemannian metric. Then the functions $P_v$ are invariant under the geodesic flow of $g$. Further, this flow is a symplectic (local) diffeomorphism and conserves (kinetic) energy.

**PROOF.** Let $G$ act on itself by left translations. Each $v \in T_eG$ determines an exponential flow on $G$ by assumption, and its derivative is the infinitesimal generator $X$. Let $E_v$ be the exponential map of $v$. Then

$$X(y) = d(E_v(y))/dt = d(E_v(e) \cdot y)/dt \text{ at } t = 0.$$ 

Now $E_v(e) \cdot y = R_y \circ F_x(e)$ and so by the composite mapping theorem, $d(E_v(e) \cdot y)/dt = T_eR_y \cdot d(E_v(e))/dt = T_eR_y \cdot v$. The result is now an immediate consequence of the conservation theorem.

In practice it is usually most convenient to work in the Lie algebra $T_eG$ by pulling back the flow to $T_eG$ by left translation (in the so called “body coordinates”). The pull back of the vectorfield $X_T$ to $T_eG$ is the Euler equations. They are determined as follows:
THEOREM 5. Let $G$ be a (regular) Lie group, $g$ a compatible metric and $X_T$ the corresponding Hamiltonian vectorfield with flow $F_i: T_G \to T_G$ (which may be just local). Define $H_i$ on $T_eG$ by

$$H_i(v) = T_{x(t)} L_{x(t)^{-1}} \cdot F_i v$$

(where defined), where $F_i v \in T_{x(t)} G$. Then $H_i$ is a smooth flow on $T_eG$ and has vector-field $Y$ uniquely determined by: $Y: T_eG \to T_eG$, $\langle Y(u), v \rangle = \langle [u, v], u \rangle$ where $[u, v]$ is the Lie bracket in $T_eG$.

PROOF. It is easily checked that $H_i$ is a flow. To compute $Y$ we must compute $dH_i(v)/dt$ at $t = 0$. For this we use the following:

LEMMA 6. Let $x(t)$ be a smooth curve in $G$, $v \in T_eG$ and $v(t) = \text{Ad}_{x(t)}^{-1} \cdot v$ where

$$\text{Ad}_x = T_x R_{x^{-1}} \circ T_x L_x = T_e (R_{x^{-1}} \circ L_x): T_eG \to T_eG.$$  

Then we have

$$dv/dt = [v(t), T_{x(t)} L_{x(t)^{-1}} \cdot (dx/dt)].$$

We omit the proof, as it is more or less standard; see [11] or [13].

To prove the theorem, we start with

$$d \langle F_i u, T_e R_{x(t)} v \rangle /dt = 0$$

by the conservation law, for each $u, v \in T_eG$. By left invariance of $g$,

$$\langle F_i u, T_e R_{x(t)} v \rangle = \langle T_{x(t)} L_{x(t)^{-1}} \cdot F_i u, \text{Ad}_{x(t)}^{-1} \cdot v \rangle = \langle H_i u, \text{Ad}_{x(t)}^{-1} \cdot v \rangle.$$  

Differentiating, using Leibnitz's rule and the lemma, gives at $t = 0$,

$$\langle Y(u), v \rangle + \langle u, [v, u] \rangle = 0$$

where we use the fact that $dx/dt = F_i u$ (which is because we have a second order equation). [11]

If one finds the flow of $Y$ on $T_eG$, the problem of finding $F_i$ is solved using the equation

$$dx/dt = T_e L_{x(t)} \cdot H_i u = F_i u$$

and the fact that $F_i$ is left invariant. For groups of diffeomorphisms and in particular for hydrodynamics, this is just a problem in ordinary differential equations.

COROLLARY 7. If $v_0$ is a critical point of $Y$ on $T_eG$ then $x(t) = \exp(tv_0) \in G$ is the geodesic with initial value $v_0$.

PROOF. We have $H_i v_0 = v_0$ and so

$$F_i v_0 = T_e L_{x(t)} v_0 = dx/dt.$$  

This is the equation for $x(t) = \exp(tv_0)$. [11]
By left invariance, the conservation law and conservation of energy pull back to the Lie algebra $T_eG$.

For many circumstances it is important to work in a Hilbert space. Therefore we could complete $T_eG$ with respect to $\langle \cdot, \cdot \rangle$, to obtain a Hilbert space $H$. It is absolutely crucial to recognize that on $H$, the vectorfield $Y$ is not smooth and is not everywhere defined. Also, the flow $H_t$ is not defined on all of $H$. (This can be seen for the diffeomorphism group where $T_eG$ corresponds to $C^\infty$ vectorfields, so $H$ would be $L_2$ vectorfields. For flows of nonsmooth vectorfields see [9].)

In the finite dimensional case, or if the energy function $T$ had $D^2T$ continuous on $H$ (so $Y$ would be smooth on $H$ say), then definiteness of the quadratic form $D^2T^6$ at a stationary point $v_0$ would imply its stability (the quadratic form is computed in Arnold [3]). Unfortunately for hydrodynamics this is not the case, so it is unknown whether or not this criterion for stability is valid, as far as we know.

2. Hydrodynamics of perfect fluids. Let $D$ be a compact orientable $n$-manifold with boundary (or without boundary). Let $g_0$ be a Riemannian metric on $D$ and $\Omega$ a volume (orientation) on $D$ (that is, a nonvanishing $n$-form) which is generally one derived from $g_0$. For tangent vectors on $D$ we write $v \cdot u$ or $\langle v, u \rangle$ for the inner product with respect to $g_0$.

Let $G$ be the group of volume preserving diffeomorphisms on $D$. Leslie [6], and Omori [11] show that the group of all diffeomorphisms has a structure modelled on a Frechet space for which the group is a Lie group. The procedure has become a more or less standard one in manifolds of maps. See also [2]. A recent theorem of D. Ebin 7 tells us that $G$ is also a Lie group (in fact a Lie subgroup). Then $T_eG$ may be identified with $C^\infty$ vectorfields $X$ on $D$ such that

(i) $\text{div}_\Omega X = 0$, and
(ii) $i_*i^*\Omega = 0$ where $i: \text{bd}(D) \to D$ is the inclusion map.

For the full diffeomorphism group, (i) is omitted.

Condition (ii) means that $X$ is parallel to the boundary. Also, the Lie bracket in $T_eG$ is the negative of the usual Lie bracket of vectorfields. This requires the observation that, under an action, the map taking the Lie algebra to the infinitesimal generator is an antihomorphism, a standard result [13]. Letting $G$ act naturally on $D$ gives the stated result.

Define a weak Riemannian structure for $G$ or all diffeomorphisms by setting

$$\langle X, Y \rangle = \int_D X \cdot Y \Omega$$

for $X, Y$ vectorfields. Extend by right translation to all of $G$.\footnote{On the foliation described in [3].}

**Conjecture.** 9 On the regular Lie group $G$, $\langle \cdot, \cdot \rangle$ is compatible with $G$.\footnote{See [14] for the proof.}
That $G$ is regular is simple. Namely if $X \in T_x G$, and $X$ has flow $F_t$, then the map $f \mapsto f \circ F_t$ is the exponential map of $X$. This is easily seen. One cannot do this for general $X$ which are just in $L^2$.

For compatibility of $g = \langle \cdot , \cdot \rangle$ one must show that we get a vectorfield $X$ and that it has a local flow. From §1, it is enough to work with the Euler equations. We shall just show how to get the Euler equations $Y$ (Theorem 5). Conjecture 8 is true for the full diffeomorphism group and suitable metrics. Details may be found in [11].

**Theorem 9.** For $G$ described above, the vectorfield $Y$ of Theorem 5 is given by:

$$Y(X) = -i_x d(\hat{X}) + df = -L_x \hat{X} + dg$$

where $\hat{X}$ denotes the one form obtained from $X$ via $g_0$, $i_x$ is the inner product, and $L_x$ is the Lie derivative. Traditionally, $p$, defined by $p = f - \langle X, X \rangle/2 = g + \langle X, X \rangle/2$ is called the pressure.\(^{10}\)

**Proof.** First, we note that for vectorfields $X, X_0, Y$ that

$$\int_D \langle X, [X_0, Y] \rangle \Omega = \int_D i_{[X_0, Y]} \hat{X} \Omega = \int_D (i_{[X_0, Y]} + L_Y i_x) \hat{X} \Omega$$

since $\int_D L_Y (i_{X_0} \hat{X}) \Omega = 0$ by Stoke's theorem and the boundary condition $i_x \Omega = 0$ on $\text{bd}(D)$. But $i_{[X_0, Y]} + L_Y i_{X_0} = i_{X_0} L_Y$ (see [1]), and so we finally obtain

$$\int_D \langle X, [X_0, Y] \rangle \Omega = \int_D i_{X_0} i_Y d(\hat{X}) \Omega = \int_D \langle Y, L_{X_0} \hat{X} \rangle \Omega.$$  

Of course in this the fact that the vectorfields are divergence free is essential. (On the full diffeomorphism group the equations are a little different.)\(^{11}\) If this condition were omitted, the pressure term would be absent.

It is a classical theorem (Hodge theory)\(^{12}\) that a vectorfield $Z$ can be written uniquely

$$Z = Z_0 + \text{grad}(f)$$

where $\text{grad}(f) = df$, and $Z_0$ and $\text{grad}(f)$ are orthogonal and $\text{div} Z_0 = 0$, and $Z_0$ is parallel to the boundary. (The function $f$ is obtained by solving Laplace's equation $\nabla^2 f = \text{div} Z$.) One easily sees the orthogonality directly, as follows:

For any $X$ in $T_x G$, observe that $\langle X, \text{grad}(f) \rangle = \int X \cdot \text{grad}(f) \Omega = 0$ since

$$\int X \cdot \text{grad}(f) \Omega = \int L_x f \Omega = \int L_x (f \Omega) = \int d X(f \Omega) = 0,$$

\(^{10}\)The equations for $Y$ are usually written in the equivalent form $d\psi/dt + \nabla \psi = d\psi$ where $\nabla$ is the covariant derivative and $\psi$ is an integral curve of $Y$.

\(^{11}\)For the right invariant metric on the full diffeomorphism group, the equations are $d\psi/dt + \psi(d\psi) + L_\psi = 0$. The existence problem is unknown for these equations.

\(^{12}\)Cf. Morrey, *Multiple integrals in the calculus of variations*, Chapter 7 and also [11], [14].
using Stoke's theorem and \( i_X \Omega = 0 \) or \( \text{bd}(D) \). Now for the theorem we let \( f \) be such that \( i_X d(\tilde{X}) - df \) is divergence free and then observe, by our remarks, that

\[
- \langle Y(X), X_0 \rangle = \langle [X, X_0], X \rangle = \int_D i_{X_0} i_X d(\tilde{X}) \Omega
\]

\[
= \langle i_X d(\tilde{X}), X_0 \rangle
\]

\[
= \langle i_X d(\tilde{X}) - \text{grad}(f), X_0 \rangle.
\]

Note that \( i_X d(\tilde{X}) \) might not be parallel to the boundary but \( L_X(\tilde{X}) \) is, and these differ by a gradient. By nondegeneracy we conclude the result.

The pressure thus constrains the motion to being divergence free.

Observe that the motion of a stationary point \( X_0 \) is just its own flow. See Corollary 7. This holds for harmonic vectorfields for example.

Finally we give a theorem classically known as Kelvin's circulation theorem. One should note that the theorem as proven in standard hydrodynamics books lacks rigor.

**Theorem 10.** Let \( X(t) \) be the velocity vectorfield in \( T_eG \) above. Let \( l \) be a smooth closed loop in \( D \) (a compact 1-manifold) and \( l_t = F(t) \), where \( F \) is the motion of the fluid (geodesic). Then \( \int_l X(t) \) is independent of \( t \).

**Proof.** Since \( F \) is a diffeomorphism,

\[
\int_{l_t} \tilde{X}(t) = \int_l F_t^* \tilde{X}(t).
\]

Now \( d(F_t^* \tilde{X}(t))/dt = F_t^* L_X \tilde{X} + F_t^* d\tilde{X} / dt \), which by Theorem 9 equals \( F_t^* g = d(F_t^* g) \). Thus by Stoke's theorem,

\[
\frac{d}{dt} \int_{l_t} \tilde{X}(t) = \int_l d(F_t^* g) = 0. \quad \square
\]

Curiously this is not true for the geodesic flow on the full diffeomorphism group.

This theorem is quite analogous to the circulation theorem for mechanics. It also holds for \( TG \), so we give it.

**Theorem 11.** Let \( M \) be a (weak) symplectic manifold with form \( \omega = d\theta \). Let \( F_t \) be a smooth Hamiltonian flow (or local flow) on \( M \). Let \( A \) be a compact two manifold in \( M \) with boundary \( l = \text{bd}(A) \), and \( l_t = F_t(l) \). Then \( \int_{l_t} \theta \) is independent of \( t \).

**Proof.** By Stoke's theorem,

\[
\int_{l_t} \theta = \int_A \omega = \int_A F_t^* \omega = \int_A \omega,
\]

since \( F_t^* \omega = \omega \).

**References**


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