

Distinguishing Separable and Entangled States

A. C. Doherty,¹ Pablo A. Parrilo,^{2,3} and Federico M. Spedalieri¹

¹*Institute for Quantum Information, California Institute of Technology, Pasadena, California 91125*

²*Institut für Automatik, ETH Zürich, CH-8092, Switzerland*

³*Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, California 91125*

(Received 12 December 2001; published 23 April 2002)

We show how to design families of operational criteria that distinguish entangled from separable quantum states. The simplest of these tests corresponds to the well-known Peres-Horodecki positive partial transpose (PPT) criterion, and the more complicated tests are strictly stronger. The new criteria are tractable due to powerful computational and theoretical methods for the class of convex optimization problems known as semidefinite programs. We successfully applied the results to many low-dimensional states from the literature where the PPT test fails. As a by-product of the criteria, we provide an explicit construction of the corresponding entanglement witnesses.

DOI: 10.1103/PhysRevLett.88.187904

PACS numbers: 03.67.-a, 03.65.Ca, 03.65.Ud

Entanglement is one of the most striking features of quantum mechanics. Not only is it at the heart of the violation of Bell inequalities [1], but it has lately been recognized as a very useful resource in the field of quantum information. Entanglement can be used to perform several important tasks such as teleportation, quantum key distribution, and quantum computation [2]. Despite its widespread importance, there is no procedure that can tell us whether a given state is entangled or not, and considerable effort has been dedicated to this problem [3,4]. In this Letter we apply powerful tools of optimization theory for problems known as semidefinite programs to construct a hierarchy of tests for entanglement.

A bipartite mixed state ρ is said to be separable [5] (not entangled) if it can be written as a convex combination of pure product states

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|, \quad (1)$$

where $|\psi_i\rangle$ and $|\phi_i\rangle$ are state vectors on the spaces \mathcal{H}_A and \mathcal{H}_B of subsystems A and B , respectively, and $p_i > 0$, $\sum_i p_i = 1$. If a state admits such a decomposition, it can be created by local operations and classical communication, and hence it cannot be an entangled state.

Several operational criteria have been proposed to identify entangled states. Typically these are based on simple properties obeyed by all separable states and are thus necessary but not sufficient conditions for separability (although some sufficient conditions for separability are known [6]). The most famous of these criteria is based on the partial transposition and was first introduced by Peres [7]. It was shown by Horodecki *et al.* [8] to be both necessary and sufficient for separability in $\mathcal{H}_2 \otimes \mathcal{H}_2$ and $\mathcal{H}_2 \otimes \mathcal{H}_3$. If ρ has matrix elements $\rho_{ik,jl} = \langle i| \otimes \langle k| \rho |j\rangle \otimes |l\rangle$ then the partial transpose ρ^{T_A} is defined by $\rho_{ik,jl}^{T_A} = \rho_{jk,il}$. If a state is separable, then it must have a positive partial transpose (PPT). To see this, consider the decomposition (1) for ρ . Partial transposition takes $|\psi_i\rangle\langle\psi_i|$ to $|\psi_i^*\rangle\langle\psi_i^*|$, so the result of this operation

is another valid density matrix and must be positive. Thus any state for which ρ^{T_A} is not positive semidefinite is necessarily entangled. This criterion has the advantage of being very easy to check, but there are PPT states that are nonetheless entangled as was first demonstrated in [9].

Our separability criteria will also be based on simple computationally checkable properties of separable states. Let $\tilde{\rho}$, defined on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_A$, be given by

$$\tilde{\rho} = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \otimes |\psi_i\rangle\langle\psi_i|. \quad (2)$$

First, $\tilde{\rho}$ is an extension of ρ (that is, the partial trace over the third party C is equal to ρ , $\text{Tr}_C[\tilde{\rho}] = \rho$). Second, the state is symmetric under interchanging the two copies of \mathcal{H}_A . To put this more formally we define the swap operator P such that $P|i\rangle \otimes |k\rangle \otimes |j\rangle = |j\rangle \otimes |k\rangle \otimes |i\rangle$. We have $P^2 = I$, and $\pi = (I + P)/2$ is a projector onto the symmetric subspace. Since $\pi\tilde{\rho}\pi = \tilde{\rho}$, the extension $\tilde{\rho}$ has support only on this subspace. Finally the extension $\tilde{\rho}$ is a tripartite separable state. This means that it will have positive partial transposes with respect to any of the parties, and in particular we have $\tilde{\rho}^{T_A} \geq 0$ and $\tilde{\rho}^{T_B} \geq 0$.

We may now formulate an explicit separability criterion based on the existence of the extension discussed above. *If the state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable then there is an extension $\tilde{\rho}$ on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_A$ such that $\pi\tilde{\rho}\pi = \tilde{\rho}$, $\tilde{\rho}^{T_A} \geq 0$, and $\tilde{\rho}^{T_B} \geq 0$.* Note that the symmetry of the extension means that if $\tilde{\rho}^{T_A} \geq 0$ then $\tilde{\rho}^{T_C} \geq 0$, so including this would not make a stronger test. We may generalize this criterion to an arbitrary number of copies of both \mathcal{H}_A and \mathcal{H}_B . *If the state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable then there is an extension $\tilde{\rho}$ with support only on the symmetric subspace of $\mathcal{H}_A^{\otimes k} \otimes \mathcal{H}_B^{\otimes l}$ such that $\tilde{\rho}$ has a positive partial transpose for all partitions of the $k + l$ parties into two groups.* Since the extensions are required to be symmetric, it is only necessary to test the possible partitions into two groups that are not related by permuting copies of \mathcal{H}_A and \mathcal{H}_B . Including testing for positivity of the extension itself, there are $[(k + 1)(l + 1)/2]$ distinct positivity checks to be satisfied by $\tilde{\rho}$.

These results generate a hierarchy of necessary conditions for separability. The first is the usual PPT test for a bipartite density matrix ρ . If the test fails, the state is entangled; if the test is passed, the state could be separable or entangled. In the latter case we look for an extension $\tilde{\rho}$ of ρ to three parties such that $\pi\tilde{\rho}\pi = \tilde{\rho}$ satisfies the PPT test for all possible partial transposes. If no such extension exists, then ρ must be entangled. If such an extension is possible, the state could be separable or entangled, and we need to consider an extension to four parties, and so on.

Each test in this sequence is at least as powerful as the previous one. We can see this by showing that, if there is a PPT extension $\tilde{\rho}_n$ to n parties, then there must be a PPT extension $\tilde{\rho}_{n-1}$ to $n-1$ parties. Let $\tilde{\rho}_{n-1} = \text{Tr}_X[\tilde{\rho}_n]$, X being one of the copies of A or B . Then $\tilde{\rho}_{n-1}$ will inherit from $\tilde{\rho}_n$ the property of having its support on the symmetric subspace. Let's assume that it is not PPT. Then there is a subset I of the parties such that $\tilde{\rho}_{n-1}^{T_I}$ has a negative eigenvalue, where T_I represents the partial transpose with respect to all the parties in subset I . Let $|e\rangle$ be the corresponding eigenvector and let $\{|i\rangle\}$ be a basis of the system X over which the partial trace was performed. Since $\tilde{\rho}_n$ is PPT, then $\langle e|\langle i|\tilde{\rho}_n^{T_I}|e\rangle|i\rangle \geq 0$, for all i . Then

$$\sum_i \langle e|\langle i|\tilde{\rho}_n^{T_I}|e\rangle|i\rangle = \langle e|\text{Tr}_X[\tilde{\rho}_n^{T_I}]|e\rangle \geq 0. \quad (3)$$

Since $X \notin I$, we can commute the trace and the partial transpose, and, using $\tilde{\rho}_{n-1} = \text{Tr}_X[\tilde{\rho}_n]$, we have $\langle e|\tilde{\rho}_{n-1}^{T_I}|e\rangle \geq 0$, which contradicts the fact that $|e\rangle$ is an eigenvector of $\tilde{\rho}_{n-1}^{T_I}$ with negative eigenvalue.

The problem of searching for the extension can be solved efficiently, since it can be stated as a particular case of the class of convex optimizations known as *semidefinite programs* (SDP) [10]. A SDP corresponds to the optimization of a linear function, subject to a linear matrix inequality (LMI). A typical SDP will be

$$\begin{aligned} & \text{minimize } c^T \mathbf{x}, \\ & \text{subject to } F(\mathbf{x}) \geq 0, \end{aligned} \quad (4)$$

where c is a given vector, $\mathbf{x} = (x_1, \dots, x_m)$, and $F(\mathbf{x}) = F_0 + \sum_i x_i F_i$, for some fixed n -by- n Hermitian matrices

F_j . The inequality in the second line of (4) means that the matrix $F(\mathbf{x})$ is positive semidefinite. The vector \mathbf{x} is the variable over which the minimization is performed. In the particular instance in which $c = 0$, there is no function to minimize and the problem reduces to whether or not it is possible to find \mathbf{x} such that $F(\mathbf{x})$ is positive semidefinite. This is termed a feasibility problem. The convexity of SDPs has made it possible to develop sophisticated and reliable analytical and numerical methods for them [10].

The separability criteria we introduced above may all be formulated as semidefinite programs. For brevity we will explicitly consider only the problem of searching for an extension of ρ to three parties. We will also relax the symmetry requirements on the extension $\tilde{\rho}$, and we will ask only $P\tilde{\rho}P = \tilde{\rho}$. This increases the size of the SDP, but simplifies the setup. Let $\{\sigma_i^A\}_{i=1, \dots, d_A^2}$ and $\{\sigma_j^B\}_{j=1, \dots, d_B^2}$ be bases for the space of Hermitian matrices that operate on \mathcal{H}_A and \mathcal{H}_B , respectively, such that they satisfy

$$\text{Tr}(\sigma_i^X \sigma_j^X) = \alpha \delta_{ij} \quad \text{and} \quad \text{Tr}(\sigma_i^X) = \delta_{i1}, \quad (5)$$

where X stands for A or B , and α is some constant—the generators of $SU(n)$ could be used to form such a basis. We can then expand ρ in the basis $\{\sigma_i^A \otimes \sigma_j^B\}$, and write $\rho = \sum_{ij} \rho_{ij} \sigma_i^A \otimes \sigma_j^B$, with $\rho_{ij} = \alpha^{-2} \text{Tr}[\rho \sigma_i^A \otimes \sigma_j^B]$. We can write the extension $\tilde{\rho}$ in a similar way

$$\begin{aligned} \tilde{\rho} = & \sum_{\substack{ij \\ i < k}} \tilde{\rho}_{kji} \{\sigma_i^A \otimes \sigma_j^B \otimes \sigma_k^A + \sigma_k^A \otimes \sigma_j^B \otimes \sigma_i^A\} \\ & + \sum_{kj} \tilde{\rho}_{kjk} \sigma_k^A \otimes \sigma_j^B \otimes \sigma_k^A, \end{aligned} \quad (6)$$

where we have explicitly used the symmetry between the first and third party. We also need to satisfy $\text{Tr}_C(\tilde{\rho}) = \rho$. Using (5), and the fact that the $\sigma_i^A \otimes \sigma_j^B$ form a basis of the space of Hermitian matrices on $\mathcal{H}_A \otimes \mathcal{H}_B$, we obtain $\tilde{\rho}_{ij1} = \rho_{ij}$. The remaining components of $\tilde{\rho}$ will be the variables in our SDP. The LMIs come from requiring that the state $\tilde{\rho}$ and its partial transposes be positive semidefinite. For example, the condition $\tilde{\rho} \geq 0$ will take the form $F(\mathbf{x}) = F_0 + \sum_i x_i F_i \geq 0$ if we define

$$\begin{aligned} F_0 &= \sum_j \rho_{1j} \sigma_1^A \otimes \sigma_j^B \otimes \sigma_1^A + \sum_{i=2, j=1} \rho_{ij} \{\sigma_i^A \otimes \sigma_j^B \otimes \sigma_1^A + \sigma_1^A \otimes \sigma_j^B \otimes \sigma_i^A\}, \\ F_{iji} &= \sigma_i^A \otimes \sigma_j^B \otimes \sigma_i^A, \quad i \geq 2, \\ F_{ijk} &= (\sigma_i^A \otimes \sigma_j^B \otimes \sigma_k^A + \sigma_k^A \otimes \sigma_j^B \otimes \sigma_i^A), \quad k > i \geq 2. \end{aligned}$$

The coefficients $\tilde{\rho}_{ijk}$ ($k \neq 1, k \geq i$) play the role of the variable \mathbf{x} . There are $m = (d_A^2 d_B^2 - d_A^2 d_B^2)/2$ components of \mathbf{x} , where d_I is the dimension of \mathcal{H}_I . Each F is a square matrix of dimension $n = d_A^2 d_B^2$. Positivity of the partial transposes T_A and T_B leads to two more LMIs, $\tilde{\rho}^{T_A} \geq 0$ and $\tilde{\rho}^{T_B} \geq 0$. The F matrices for these two LMIs are related to the matrices F_{ijk} by the appropriate partial transposition. We can write these three LMIs as one, if we define the matrix $G = \tilde{\rho} \oplus \tilde{\rho}^{T_A} \oplus \tilde{\rho}^{T_B}$, so for example, $G_0 = F_0 \oplus F_0^{T_A} \oplus F_0^{T_B}$ (a block-diagonal matrix $C = A \oplus B$ is positive semidefinite if, and only if, both A and B are positive semidefinite). So the feasibility problem reduces to attempting to find $\tilde{\rho}_{ijk}$ ($k \neq 1, k \geq i$) with $G \geq 0$. In fact, the SDP corresponding to minimizing t subject to $tI_{ABA} + G \geq 0$ is always feasible and performs better numerically. A positive optimum gives a value of p^* such that $(1-p)\rho + pI_{AB}/d_A d_B$ is entangled for all $0 \leq p < p^*$. Looking for an extension on $\mathcal{H}_A^{\otimes k} \otimes \mathcal{H}_B^{\otimes l}$ is a semidefinite program

with $m = \binom{d_A+k-1}{k} \binom{d_B+l-1}{l} - d_A^2 d_B^2$ variables and a matrix G with $[(k+1)(l+1)/2]$ blocks of dimension at most $\binom{d_A+[k/2]-1}{[k/2]} \binom{d_B+[l/2]-1}{[l/2]}$.

Numerical SDP solvers are described in detail in [10]. Typically they involve the solution of a series of least squares problems, each requiring a number of operations scaling with a problem size as $O(m^2 n^2)$. For SDPs with a block structure these break into independent parts, each with a value of n determined by the block size. The number of iterations required is known to scale no worse than $O(n^{1/2})$. Thus for any fixed value of (k, l) the computation involved in checking our criteria scales no worse than $O(d_A^{13k/2} d_B^{13l/2})$ which is polynomial in the system size.

Using the SDP solver SeDuMi [11], we applied the first criterion ($k=2, l=1$) to several examples of PPT entangled states with $d_A=2, d_B=4$ or $d_A=3, d_B=3$. On a 500 MHz desktop computer a single state could be tested in under a second for $d_A=2, d_B=4$ and in about eight seconds for $d_A=3, d_B=3$. For the one and two parameter families of PPT entangled states described in [3,9,12] we performed a systematic search of the parameter space, in each case testing hundreds or thousands of different states. We checked 4000 randomly chosen examples of the seven parameter family of PPT entangled states in [13]. We also checked the PPT entangled states constructed from unextendable product bases in [14]. We did not find any PPT entangled state with an extension of the required form, thus verifying the entanglement of all these states. Very close to the separable states the test was inconclusive due to numerical uncertainties. Uncertainties and one example are discussed more fully below.

A very useful property of a SDP is the existence of the dual problem. If a problem can be stated as a SDP like (4), usually called the primal problem, then the dual problem corresponds to another SDP that can be written as

$$\begin{aligned} & \text{maximize} && -\text{Tr}[F_0 Z] \\ & \text{subject to} && Z \geq 0 \\ & && \text{Tr}[F_i Z] = c_i, \end{aligned} \quad (7)$$

where the matrix Z is Hermitian and is the variable over which the maximization is performed. For any feasible solutions of the primal and dual problems we have

$$c^T \mathbf{x} + \text{Tr}[F_0 Z] = \text{Tr}[F(\mathbf{x})Z] \geq 0, \quad (8)$$

where the last inequality follows from the fact that both $F(\mathbf{x})$ and Z are positive semidefinite. Then, for the particular case of a feasibility problem ($c=0$), Eq. (8) will read $\text{Tr}[F_0 Z] \geq 0$. This result can be used to give a certificate of infeasibility for the primal problem: *if there exists Z such that $Z \geq 0$, $\text{Tr}[F_i Z] = 0$, that satisfies $\text{Tr}[F_0 Z] < 0$, then the primal problem must be infeasible.*

In the context of entanglement the role of the ‘‘certificate’’ is played by observables known as *entanglement witnesses* (EW) [8,15]. An EW for a state ρ satisfies

$$\text{Tr}[\rho_{\text{sep}} W] \geq 0 \quad \text{and} \quad \text{Tr}[\rho W] < 0, \quad (9)$$

where ρ_{sep} is any separable state. If our primal SDP is infeasible (which means that the state ρ must be entangled), the dual problem provides a certificate of that infeasibility that can be used to construct an EW for ρ .

First, we note that, due to the block diagonal structure of the LMI, we can restrict any feasible dual solution Z to have the same structure, i.e., $Z = Z_0 \oplus Z_1^{T_A} \oplus Z_2^{T_B}$ where the Z_i are operators on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_A$. Second, we have $\text{Tr}[G_0 Z] = \text{Tr}[F_0(Z_0 + Z_1 + Z_2)]$. We defined F_0 as a linear function of ρ so that $F_0 = \Lambda(\rho)$, where Λ is a linear map from $\mathcal{H}_A \otimes \mathcal{H}_B$ to $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_A$. We can now define an operator \tilde{Z} on $\mathcal{H}_A \otimes \mathcal{H}_B$ through the adjoint map Λ^* such that $\tilde{Z} = \Lambda^*(Z_0 + Z_1 + Z_2)$ and

$$\text{Tr}[\rho \tilde{Z}] = \text{Tr}[\Lambda(\rho)(Z_0 + Z_1 + Z_2)] = \text{Tr}[G_0 Z]. \quad (10)$$

If ρ_{sep} is any separable state, we know that the primal problem is feasible (the extension $\tilde{\rho}$ exists). Then, using $\text{Tr}[G_0 Z] \geq 0$ and (10), we have $\text{Tr}[\rho_{\text{sep}} \tilde{Z}] \geq 0$ for *any* \tilde{Z} obtained from a dual feasible solution. For this particular problem, if the primal is not feasible (which means ρ is an entangled state), a feasible dual solution Z_{EW} that satisfies $\text{Tr}[G_0 Z_{\text{EW}}] < 0$ always exists. Using (10) we can see that the corresponding operator \tilde{Z}_{EW} satisfies $\text{Tr}[\rho \tilde{Z}_{\text{EW}}] < 0$ which together with $\text{Tr}[\rho_{\text{sep}} \tilde{Z}_{\text{EW}}] \geq 0$ means that \tilde{Z}_{EW} is an entanglement witness for ρ .

In numerical work, if the SDP solver cannot find an extension $\tilde{\rho}$ it constructs the matrices Z_i . Evaluating $\text{Tr}[\rho \tilde{Z}_{\text{EW}}]$ and verifying the three positivity conditions provide an *independent* check of the result. Unless this check is not conclusive—for example, if $\text{Tr}[\rho \tilde{Z}_{\text{EW}}]$ is not significantly different from zero—we are able to definitively conclude that no $\tilde{\rho}$ exists.

If W is an EW, then for any product state $|xy\rangle$ we have

$$E(x, y) = \langle xy | W | xy \rangle = \sum_{ijkl} W_{ijkl} x_i^* y_j^* x_k y_l \geq 0, \quad (11)$$

where $\{x_i, y_i\}$ are the components of $|x\rangle, |y\rangle$ in some basis, and W_{ijkl} are the matrix elements of W in the same basis. Equation (11) states that the biquadratic Hermitian form E associated with W must be positive semidefinite (PSD). It is not hard to show that all of the EWs generated by Eq. (10) satisfy the relation

$$\begin{aligned} \langle xyx | \tilde{Z}_{\text{EW}} \otimes I | xyx \rangle &= \langle xyx | (Z_0 + Z_1 + Z_2) | xyx \rangle \\ &= \langle xyx | Z_0 | xyx \rangle + \langle x^* yx | Z_1^{T_A} | x^* yx \rangle \\ &\quad + \langle xy^* x | Z_2^{T_B} | xy^* x \rangle. \end{aligned} \quad (12)$$

Since $Z_0, Z_1^{T_A}$, and $Z_2^{T_B}$ are positive by construction the biquadratic Hermitian form $E(x, y) \langle x | x \rangle$ has a decomposition as a sum of squared (SOS) magnitudes. This guarantees that $E(x, y)$ is PSD. It can be shown that our first separability criterion detects all entangled states that possess an EW such that E may be written in this form. The dual program to our initial SDP may be interpreted as a search for an entanglement witness of this type. Equally, the Peres-Horodecki criterion detects the entanglement of those states which possess entanglement witnesses

for which (11) may be written directly as a SOS—the decomposable entanglement witnesses [16] such that $W = P + Q^{T_A}$ for some PSD P and Q . In general, if there is no EW W such that (11) is a SOS, we can search over W for which (11) is a SOS when multiplied by $\langle x|x \rangle^{k-1} \langle y|y \rangle^{l-1}$ for some $k, l \geq 1$. By duality, this corresponds to our (k, l) separability criterion.

As an example illustrating the methodology, consider the state described in [3], Section 4.6, given by

$$\rho_\alpha = \frac{2}{7} |\psi_+\rangle\langle\psi_+| + \frac{\alpha}{7} \sigma_+ + \frac{5-\alpha}{7} P\sigma_+P, \quad (13)$$

with $0 \leq \alpha \leq 5$, $|\psi_+\rangle = \frac{1}{\sqrt{3}} \sum_{i=0}^2 |ii\rangle$, and $\sigma_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$. Notice that ρ_α is invariant under the simultaneous change of $\alpha \rightarrow 5 - \alpha$ and interchange of the parties. The state is separable for $2 \leq \alpha \leq 3$ and not PPT for $\alpha > 4$ and $\alpha < 1$. Numerically, entanglement witnesses could be constructed for ρ_α in the range $3 + \epsilon < \alpha \leq 4$ (and $1 \leq \alpha < 2 - \epsilon$) with $\epsilon \geq 10^{-8}$. A witness for $\alpha > 3$ can be extracted from these by inspection

$$\begin{aligned} \tilde{Z}_{EW} = & 2(|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22|) + |02\rangle \\ & \times \langle 02| + |10\rangle\langle 10| + |21\rangle\langle 21| - 3|\psi_+\rangle\langle\psi_+|. \end{aligned}$$

This observable is non-negative on separable states:

$$\begin{aligned} 2\langle xy|\tilde{Z}_{EW}|xy\rangle\langle x|x\rangle = & |2x_0x_1y_2^* - x_2x_0y_1^* - x_1x_2y_0^*|^2 + |2x_0x_0^*y_0 - 2x_1x_0^*y_1 + x_1x_1^*y_0 - x_2x_0^*y_2|^2 \\ & + |2x_0x_0^*y_2 - 2x_1x_2^*y_1 + x_2x_2^*y_2 - x_0x_2^*y_0|^2 + |2x_0x_1^*y_0 - 2x_2x_2^*y_1 + x_2x_1^*y_2 - x_1x_1^*y_1|^2 \\ & + 3|x_2x_0y_1^* - x_1x_2y_0^*|^2 + 3|x_1x_1^*y_0 - x_2x_0^*y_2|^2 + 3|x_2x_2^*y_2 - x_0x_2^*y_0|^2 + 3|x_2x_1^*y_2 - x_1x_1^*y_1|^2 \\ \geq & 0. \end{aligned}$$

The expected value on the original state is $\text{Tr}[\tilde{Z}_{EW}\rho_\alpha] = \frac{1}{7}(3 - \alpha)$, demonstrating entanglement for all $\alpha > 3$.

The reformulation of our separability tests as a search for SOS decompositions of the forms $E(x, y)$ provides connections with existing results in real algebra (see [17] for a discussion of the SDP-based approach in a general setting). By Artin's positive solution to Hilbert's 17th problem, for any real PSD form $f(\mathbf{x})$ there exists a SOS form $h(\mathbf{x})$, such that the product $f(\mathbf{x})h(\mathbf{x})$ is SOS [18]. Finding such an $h(\mathbf{x})$ and SOS decomposition proves that f is PSD. For a fixed SOS form $h(x, y)$, we may write a SDP that attempts to find EWs such that $h(x, y)E(x, y)$ is SOS. In our hierarchy of criteria the form h is restricted to be $\langle x|x \rangle^{k-1} \langle y|y \rangle^{l-1}$. While it is conceivable that every PSD bi-Hermitian form is SOS when multiplied by appropriate factors of this type, currently we do not have a proof. Deciding whether a form is positive is computationally hard and so this connection to positive forms also promises to shed light on the computational complexity of the separability problem.

In this Letter we introduced a hierarchy of separability tests that are computationally tractable and strictly stronger than the PPT criterion. Only the second step in this sequence of tests was required to detect the entanglement of a wide class of known PPT entangled states. The method is based on the application of semidefinite programs. By exploiting the duality property of these problems, we showed how to construct entanglement witnesses for states that fail any separability test in the sequence. Finally, the wide range of applications of semidefinite programming, along with the work reported here and in [19], suggests that it may become a useful tool in quantum information and in quantum theory in general.

It is a pleasure to acknowledge conversations with Hideo Mabuchi, John Doyle, John Preskill, and Patrick Hayden.

This work was supported by the Caltech MURI Center for Quantum Networks, the NSF Institute for Quantum Information, and the Caltech MURI Center for Uncertainty Management for Complex Systems.

- [1] J. S. Bell, *Physics* (Long Island City, N.Y.) **1**, 195 (1964).
- [2] M. N. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [3] M. Horodecki, P. Horodecki, and R. Horodecki, *quant-ph/0109124*.
- [4] M. Lewenstein *et al.*, *J. Mod. Opt.* **47**, 2841 (2000).
- [5] R. F. Werner, *Phys. Rev. A* **40**, 4277 (1989).
- [6] S. L. Braunstein *et al.*, *Phys. Rev. Lett.* **83**, 1054 (1999).
- [7] A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996).
- [8] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Lett. A* **223**, 1 (1996).
- [9] P. Horodecki, *Phys. Lett. A* **232**, 333 (1997).
- [10] L. Vandenberghe and S. Boyd, *SIAM Rev.* **38**, 49 (1996).
- [11] J. Sturm, *SeDuMi version 1.05* 2001, available from <http://fewcal.kub.nl/sturm/software/sedumi.html>.
- [12] P. Horodecki and M. Lewenstein, *Phys. Rev. Lett.* **85**, 2657 (2000).
- [13] D. Bruss and A. Peres, *Phys. Rev. A* **61**, 030301(R) (2000).
- [14] C. H. Bennett *et al.*, *Phys. Rev. Lett.* **82**, 5385 (2000).
- [15] B. M. Terhal, *Phys. Lett. A* **271**, 319 (2000).
- [16] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, *Phys. Rev. A* **62**, 052310 (2000).
- [17] P. A. Parrilo, <http://www.cds.caltech.edu/~pablo/pubs>.
- [18] B. Reznick, *Contemporary Mathematics* (American Mathematical Society, Providence, 2000), Vol. 253, pp. 251–272.
- [19] E. M. Rains, *quant-ph/0008047*; K. Audenaert and B. D. Moor, *quant-ph/0109155*.