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Annals of N.Y. Acad. Sci 357 (1980) 313–321. #70

A HORSESHOE IN THE DYNAMICS OF A FORCED BEAM*

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If the doors of perception were cleansed, everything would appear to man as it is: Infinite.

William Blake
The Marriage of Heaven and Hell, 1790.

Introduction

In this brief note, we state two theorems concerning the global behavior of a class of periodically perturbed dynamical systems that are close to Hamiltonian. The methods are applicable to ordinary differential equations ($2n$-dimensional systems) and to certain infinite-dimensional evolution equations arising from partial differential equations. Many more details and proofs will appear in a forthcoming paper.[1]

To fix ideas, consider the following experiment, which is similar to one performed by Moon and Holmes.[2] A slender beam, pinned at each end, is buckled between a pair of rigid supports, so that it possesses two (symmetric) equilibria, $w_1(x)$ and $w_2(x)$; $w = w(x, t)$ here denotes the lateral displacement. The supports are then excited sinusoidally and the beam's inertia causes it to move, also. For low excitation levels, the motion, while not sinusoidal, is still periodic, but as the excitation increases, the beam begins to snap back and forth in an irregular, apparently random manner. Tseng and Dugundji observed similar behavior in an earlier study.[3]

The simplest equation of motion for the beam is the following modified Euler-Bernoulli equation for the deflection $w(z, t)$ of the center line of the beam,

$$\ddot{w} + w'''' + \Gamma w'' - \kappa\left(\int_0^1 |w'|^2 d\zeta\right) w'' = \epsilon(f\cos\omega t - \delta\dot{w}), \tag{1}$$

where $\cdot = \partial/\partial t$, $' = \partial/\partial z$, Γ = external load, κ = stiffness due to "membrane" effects, δ = damping, and ϵ is a small parameter. We take $w = w'' = 0$ at $z = 0, 1$, i.e., simply

*This research was supported, in part, by a National Science Foundation grant, no. MCS 78-06718, a US Army Research Office grant, no. DAAG 29-79C-0086, and a Killam visiting fellowship at the University of Calgary, all to J. M., and by a National Science Foundation grant, no. ENG 78-02891, to P. H.

supported, or hinged, ends. With these boundary conditions, the eigenvalues of the linearized, unforced equations form a countable set,

$$\lambda_j = \pm \pi j \sqrt{\Gamma - \pi^2 j^2}, \qquad j = 1, 2, \ldots.$$

Thus, if $\Gamma < \pi^2$, all eigenvalues are imaginary and the trivial solution $w = 0$ is formally stable; for positive damping, it is Lyapunov stable. We shall henceforth assume that

$$\pi^2 < \Gamma < 4\pi^2,$$

in which case the solution $w = 0$ is unstable with one positive and one negative eigenvalue and the nonlinear equation (1) with $\epsilon = 0$ and $\kappa > 0$ has two nontrivial stable buckled equilibrium states.

In studies of a related model for a magnetically buckled cantilever beam,[2] it was shown that a single mode Galerkin approximation takes the form of a Duffing equation

$$\ddot{x} + \frac{1}{2}(x^2 - 1)x = \epsilon(\gamma \cos \omega t - \delta \dot{x}), \tag{2}$$

where $x(t)$ and $\dot{x}(t)$ represent the modal displacement and velocity and γ and δ are parameters derived from the force amplitude and damping. In an earlier paper,[4] Holmes showed that, for $\epsilon > 0$, small and $\gamma > \gamma_c(\delta)$, equation 2 possesses transverse homoclinic orbits (and hence Smale horseshoes) in its Poincaré map. While this does not fully explain the apparent "strange attractor" motions observed in Reference 2 and studied in Reference 4, it is clearly of considerable importance and interest, since it shows, for example, that (2) possesses a countable infinity of periodic orbits of arbitrarily high periods.

In this work, the methods of Melnikov were used[5] (see Reference 6). In the present note, we outline extensions to these methods that enable us to apply them to a class of partial differential equations of which (1) is a member and, thus, we can prove results analogous to those of References 4–6 for the full PDE.

Abstract Results

We consider an evolution equation in a Banach space X of the form

$$\dot{x} = f_0(x) + \epsilon f_1(x, t), \tag{3}$$

where f_1 is periodic of period T in t. Our assumptions on (3) are as follows.

Assumption 1. a. Assume $f_0(x) = Ax + B(x)$, where A is an (unbounded) linear operator that generates a C^0 one parameter group of transformations on X and where $B: X \to X$ is C^∞ and has bounded derivatives on bounded sets.

b. Assume $f_1: X \times S^1 \to X$ is C^∞ and has bounded derivatives on bounded sets, where $S^1 = \mathbb{R}/(T)$, the circle of length T.

c. Assume that F_t^ϵ is defined for all $t \in \mathbb{R}$ for $\epsilon > 0$ sufficiently small and F_t^ϵ maps bounded sets in $X \times S^1$ to bounded sets in $X \times S^1$ uniformly for small $\epsilon \geq 0$ and t in bounded time-intervals.

(See Segal[7] and Holmes and Marsden[8]).

Assumption 1 implies that the associated suspended autonomous system on $X \times S^1$,

$$\begin{cases} \dot{x} = f_0(x) + \epsilon f_1(x, \theta), \\ \dot{\theta} = 1, \end{cases} \tag{4}$$

has a smooth local flow, F_t^ϵ, which can be extended globally in time, i.e., solutions do not escape to infinity in finite time. Energy estimates suffice to prove the latter for equation 1, (cf. References 1 and 9).

Assumption 2. a. Assume that the system $\dot{x} = f_0(x)$ (the unperturbed system) is Hamiltonian with energy $H_0: X \to \mathbb{R}$.

b. Assume there is a symplectic 2-manifold $\Sigma \subset X$ invariant under the flow F_t^0 and that on Σ there is a fixed point p_0 and a homoclinic orbit $x_0(t)$, i.e.,

$$f_0(p_0) = 0, \qquad \dot{x}_0(t) = f_0(x_0(t))$$

and

$$p_0 = \lim_{t \to +\infty} x_0(t) = \lim_{t \to -\infty} x_0(t).$$

This means that X carries a skew symmetric continuous bilinear map, $\Omega: X \times X \to \mathbb{R}$, that is weakly nondegenerate (i.e., $\Omega(u, v) = 0$ for all v implies $u = 0$) called the *symplectic form* and that there is a smooth function, $H_0: X \to \mathbb{R}$, such that

$$\Omega(f_0(x), u) = dH_0(x) \cdot u$$

for all x in D_A, the domain of A. Consult Abraham and Marsden[10] and Chernoff and Marsden[11] for details about Hamiltonian systems.

Remarks. a. For a non-Hamiltonian two-dimensional version, see Holmes[6] and Chow, Hale, and Mallet-Paret.[12] Non-Hamiltonian infinite-dimensional analogues can be developed using the methods of this paper.

b. The condition that Σ be symplectic means that Ω restricted to vectors tangent to Σ defines a nondegenerate bilinear form.

c. Assumption 2 can be replaced by a similar assumption on the existence of heteroclinic orbits connecting two saddle points; the existence of transverse heteroclinic orbits can then be proven using the methods below. For details in the two-dimensional case, see Holmes.[6]

The next assumption states that the homoclinic orbit through p_0 arises from a hyperbolic saddle.

Assumption 3. Assume that $\sigma(Df_0(p_0))$, the spectrum of $Df_0(p_0)$, consists of two nonzero real eigenvalues $\pm\lambda$, with the remainder of the spectrum on the imaginary axis, strictly bounded away from 0. Assume that $\sigma(\exp[tDf_0(p_0)])$, the spectrum of $\exp[tDf_0(p_0)]$, equals the closure of $\exp[t\sigma(Df_0(p_0))]$ and that if $\pm i\mu \in \sigma(Df_0(p_0))$, then μT is strictly bounded away from $m\pi$, $\forall m \in \mathbb{Z}$.

See Reference 1 for details. Note that if $Df_0(p_0)$ and $\exp[tDf_0(p_0)]$ have only point spectra, then $\sigma(\exp[tDf_0(p_0)]) = \exp[t\sigma(Df_0(p_0))]$. This is the case for equation 1.

Consider the suspended system (4) with its flow $F_t^\epsilon: X \times S^1 \to X \times S^1$. Let $P: X \to$

X be defined by

$$P^\epsilon(x) = \pi_1 \cdot (F_T^\epsilon(x, 0)),$$

where $\pi_1: X \times S^1 \to X$ is the projection onto the first factor. The map P^ϵ is the Poincaré map for the flow F_t^ϵ. Note that $P^0(p_0) = p_0$, and that fixed points of P^ϵ correspond to periodic orbits of F_t^ϵ. We can now prove the following lemma.

LEMMA 1. For $\epsilon > 0$ small, there is a unique fixed point p_ϵ for P^ϵ near p_0; moreover, p_ϵ is a smooth function of ϵ.

For ordinary differential equations, Lemma 1 is a standard fact about persistence of fixed points. For general partial differential equations, its validity can be a delicate matter; see Reference 1 for details. Assumption 3 does not hold for equation 1, since $\exp[T\sigma(Df_0(p_0))]$ is dense on the unit circle. In this case, perturbation arguments involving the positive damping must be used (cf. Assumption 4).

Our final hypothesis means, in effect, that the perturbation $f_1(x, t)$ is Hamiltonian plus damping. Using an assumption like Assumption 3, this condition can be stated either in terms of the spectrum of the linearization of equation 4 or in terms of the Poincaré map.

ASSUMPTION 4. Assume that, for $\epsilon > 0$, the spectrum of $DP^\epsilon(p_\epsilon)$ lies strictly inside the unit circle, with the exception of a single real eigenvalue $e^{\lambda T} > 1$.

Appendix A of Reference 1 gives some techniques for checking this condition.

In Lemma 1, we saw that the fixed point p_0 perturbs to another fixed point p_ϵ for the perturbed system. The same is true for the local invariant manifolds of the map P^ϵ, $W_\epsilon^{ss}(p_\epsilon)$ and $W_\epsilon^u(p_\epsilon)$,[13] which remain C^r close to the unperturbed manifolds $W_0^{cs}(p_0)$ and $W_0^u(p_0)$. Here, $W_\epsilon^{ss}(p_\epsilon) \subset W_\epsilon^s(p_\epsilon)$ and the superscript ss denotes the strong stable manifold. Assumptions 3 and 4 guarantee that the center-stable manifold ($W_0^{cs}(p_0)$ of the unperturbed system and the perturbed stable manifold $W_\epsilon^s(p_\epsilon)$ are codimension one, while the unstable manifolds are one-dimensional. The flow in $X \times S^1$ similarly has a periodic orbit, γ_ϵ, C^r close to $\{p_0\} \times S^1$, with invariant manifolds close to $W_0^{cs}(p_0) \times S^1$, etc. See Reference 1 for a detailed statement.

Equipped with these assumptions and preliminary results, we now proceed to calculate the separation of the perturbed manifolds $W_\epsilon^s(p_\epsilon)$ and $W_\epsilon^u(p_\epsilon)$ by calculating the $0(\epsilon)$ components of perturbed solution curves of equation 3 from the first variation equation of (3):

$$\frac{d}{dt} x_1^s(t, t_0) = Df_0(x_0(t - t_0)) x_1^s(t, t_0) + f_1(x_0(t - t_0), t). \tag{5}$$

Here we have expanded solution curves in W_ϵ^son(γ_ϵ); a similar expression holds for those in $W_\epsilon^u(\gamma_\epsilon)$. Points in $W_\epsilon^s(p_\epsilon)$ are obtained by intersecting $W^s(\gamma_\epsilon)$ with the section $X \times \{0\}$. This can also be done on general sections $X \times \{t_0\}$ and equation 5 contains t_0 as an initial starting time.

In a manner similar to that of References 5 and 6, it is then possible to compute a function $M(t_0)$ that acts as a measure of the separation of the perturbed manifolds $W_\epsilon^s(p_\epsilon)$ and $W_\epsilon^u(p_\epsilon)$ on different Poincaré sections $X \times \{t_0\}$. $M(t_0)$ is periodic of period T in t_0 and, as in Reference 6, we prove the following theorem.

THEOREM 1. Let Assumptions 1–4 hold. Let

$$M(t_0) = \int_{-\infty}^{\infty} \Omega(f_0(x_0(t - t_0)), f_1(x_0(t - t_0), t))dt \tag{6}$$

Suppose that $M(t_0)$ has a simple zero as a function of t_0. Then for $\epsilon > 0$ sufficiently small, the stable manifold $W_\epsilon^s(p_\epsilon(t_0))$ of p_ϵ for $P_{t_0}^\epsilon$ and the unstable manifold $W_\epsilon^u(p_\epsilon(t_0))$ intersect transversally.

The main idea of the extension of the two-dimensional Melnikov result lies in the use of a projected distance function $d_\epsilon^\pm(t_0)$, projected from $X \times \{t_0\}$ into the tangent space $T_{x_0(0)}\Sigma$ to Σ at a specified point, $x_0(0)$, lying on the unperturbed homoclinic loop. The C^r closeness of $W_\epsilon^s(p_\epsilon)$ and $W_0^{cs}(p_0)$ then guarantees that $d_\epsilon^\pm(t_0)$ is a good measure of the actual separation of the manifolds $W_\epsilon^s(p_\epsilon(t_0))$ and $W_\epsilon^u(p_\epsilon(t_0))$ near $x_0(0)$. The function $M(t_0)$ in theorem 1 is the leading $(0(\epsilon))$ nonzero term of $d_\epsilon^\pm(t_0)$. The power of the method rests on the fact that $M(t_0)$ is easily calculated in specific cases.

The second major result is an extension of the Smale-Birkhoff homoclinic theorem to infinite dimensions.[14]

THEOREM 2. If the diffeomorphism $P_{t_0}^\epsilon: X \to X$ possesses a hyperbolic saddle point p_ϵ and an associated transverse homoclinic point $q \in W_\epsilon^u(p_\epsilon) \cap W_\epsilon^s(p_\epsilon)$, with $W_\epsilon^u(p_\epsilon)$ of dimension 1 and $W_\epsilon^s(p_\epsilon)$ of codimension 1, then some power of $P_{t_0}^\epsilon$ possesses an invariant zero-dimensional hyperbolic set, Λ, homeomorphic to a Cantor set, on which a power of $P_{t_0}^\epsilon$ is conjugate to a shift on two symbols.

As in the finite-dimensional case, this implies the following corollary.

COROLLARY 1. A power of $P_{t_0}^\epsilon$ restricted to Λ possesses a dense set of periodic points; there are points of arbitrarily high period and there is a nonperiodic orbit dense in Λ.

The hyperbolicity of Λ under a power of $P_{t_0}^\epsilon$ and the structural stability theorem of Robbin (see Reference 14) implies that the situation of Theorem 2 persists under perturbations so that the complex dynamics cannot be removed by making small (lower order, bounded) changes in equation 3.

COROLLARY 2. If $\bar{P}: X \to X$ is a diffeomorphism that is sufficiently close to $P_{t_0}^\epsilon$ in C^1 norm, then a power of $\bar{P}$ has an invariant set $\bar{\Lambda}$ and there is a homeomorphism, $h: \bar{\Lambda} \to \Lambda$, such that $(P_{t_0}^\epsilon)^N \circ h = h \circ \bar{P}^N$ for a suitable power N.

The Chaotic Beam

Using Theorems 1 and 2, we now show that the beam equation (1) possesses horseshoes if the force γ exceeds a certain critical level, dependent upon the damping δ. Verification of the abstract Assumptions 1–4 is carried out in detail in the Appendices of Reference 1.

The partial differential equation of the beam is

$$\ddot{w} + w'''' + \Gamma w'' - \kappa\left(\int_0^1 [w']^2 d\zeta\right) w'' = \epsilon(f \cos \omega t - \delta \dot{w}), \tag{7}$$

with boundary conditions

$$w = w'' = 0 \text{ at } z = 0, 1.$$

The basic space is $X = H_0^2 \times L^2$, where H_0^2 denotes the set of all H^2 functions on $[0, 1]$ satisfying the boundary conditions $w = w'' = 0$ at $z = 0, 1$. For $x = (w, \dot{w}) \in X$, the X-norm is the "energy" norm $\|x\|^2 = |w''|^2 + |\dot{w}|^2$, where $|\cdot|$ denotes the L_2 norm. We write (7) in the form (3),

$$\frac{dx}{dt} = f_0(x) + \epsilon f_1(x, t), \tag{8}$$

where

$$f_0(x) = Ax + B(x) \quad \text{and} \quad f_1(x, t) = \begin{pmatrix} 0 \\ f \cos \omega t - \delta \dot{w} \end{pmatrix}.$$

Here A is the linear operator

$$A \begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \dot{w} \\ -w'''' - \Gamma w'' \end{pmatrix},$$

with domain

$$D(A) = \{(w, \dot{w}) \in H^4 \times H^2 \,|\, w = w'' = 0 \quad \text{and} \quad \dot{w} = 0 \text{ at } z = 0, 1\}$$

and B is the nonlinear mapping of X to X given by

$$B(x) = \begin{pmatrix} 0 \\ \kappa \left(\int_0^1 |w'|^2 \, d\zeta \right) w'' \end{pmatrix}.$$

In the forcing term, f is generally a spatially distributed load. Let $\bar{f}$ denote the mean and expand f in a Fourier series with a period twice the beam length,

$$f(z) = \bar{f} + \sum_{n=1}^{\infty} (\alpha_n \sin(n\pi z) + \beta_n \cos(n\pi z)).$$

The theorems of Holmes and Marsden show that A is a generator and that B and f_1 are smooth maps.[8] This, together with energy estimates (see References 1 and 9), shows that the equations generate a global flow, $F_t^\epsilon: X \times S^1 \to X \times S^1$, consisting of C^∞ maps for each ϵ and t. If x_0 lies in the domain of the (unbounded) operator A, then $F_t^\epsilon(x_0, s)$ is t-differentiable and equation 8 is literally satisfied. Thus, Assumption 1 holds.

For $\epsilon = 0$, the equation is readily verified to be Hamiltonian by using the symplectic form

$$\Omega((w_1, \dot{w}_1), (w_2, \dot{w}_2)) = \int_0^1 [\dot{w}_2(z) w_1(z) - \dot{w}_1(z) w_2(z)] dz$$

and

$$H(w, \dot{w}) = \frac{1}{2}|\dot{w}|^2 - \frac{\Gamma}{2}|w'|^2 + \frac{1}{2}|w''|^2 + \frac{\kappa}{4}|w'|^4.$$

The invariant symplectic 2 manifold Σ is the plane in X spanned by the functions $(a \sin \pi z, b \sin \pi z)$ and the homoclinic loop is given by

$$w_0(z, t) = \frac{2}{\pi} \sqrt{\frac{\Gamma - \pi^2}{\kappa}} \sin(\pi z) \operatorname{sech}(t\pi \sqrt{\Gamma - \pi^2}).$$

Assumption 2 therefore holds. For $\epsilon = 0$, one finds by direct calculation that the spectrum of $Df_0(p_0)$, where $p_0 = (0, 0)$, is discrete with two real eigenvalues,

$$\pm\lambda = \pm\pi\sqrt{\Gamma - \pi^2},$$

and the remainder pure imaginary (since $\Gamma < 4\pi^2$) at

$$\lambda_n = \pm n\pi\sqrt{\Gamma - n^2\pi^2}, \qquad n = 2, 3, \ldots$$

(see Holmes[9]). Assumption 3 is not satisfied, since $\lambda_n T$ is arbitrarily close to $m\pi$ for m, n large. However, the positive damping, $\delta > 0$, comes to the rescue (Assumption 4), and Lemma 1 holds for the beam equation. See Reference 1.

The Melnikov function (6) is given by

$$M(t_0) = \int_{-\infty}^{\infty} \Omega \begin{pmatrix} \dot{w} & 0 \\ w'''' + \kappa|w'|^2 w'' - \Gamma w' & f \cos \omega t - \delta \dot{w} \end{pmatrix} dt$$

$$= \int_{-\infty}^{\infty} \left(\int_0^1 f \cos \omega t \, \dot{w}(z, t - t_0) - \delta \dot{w}(z, t - t_0) \dot{w}(z, t - t_0) dz \right) dt.$$

Substituting the expressions for w and $\dot{w}$ along the homoclinic orbit, the integral can be evaluated by standard methods to give

$$M(t_0) = \frac{2\omega}{\pi} \sqrt{\frac{\Gamma - \pi^2}{\kappa}} \left(\frac{\alpha_1}{2} + \frac{2\bar{f}}{\pi} \right) \frac{\sin(\omega t_0)}{\cosh\left(\dfrac{\omega}{2\sqrt{\Gamma - \pi^2}} \right)} + \frac{4\delta(\Gamma - \pi^2)^{3/2}}{3\pi\kappa}.$$

Thus, if

$$\left| \frac{\alpha_1}{2} + \frac{2\bar{f}}{\pi} \right| > \frac{2\delta(\Gamma - \pi^2)}{3\omega\sqrt{\kappa}} \left[\cosh\left(\frac{\omega}{2\sqrt{\Gamma - \pi^2}} \right) \right],$$

then the hypotheses of Theorem 1 hold and the stable and unstable manifolds intersect transversally. Note that in the spatial integral evaluated in the expression for Ω, only the components $\bar{f}$ and α_1 of $f(z)$ survive, due to the orthogonality of the other Fourier components with the solution

$$\dot{w}_0(t) = -\frac{2}{\sqrt{\kappa}} (\Gamma - \pi^2) \sin(\pi z) \operatorname{sech}(t\pi\sqrt{\Gamma - \pi^2}) \tanh(t\pi\sqrt{\Gamma - \pi^2}).$$

It should be [illegible]ized that, while the formal calculation of $M(t_0)$ is similar to that in the two-dimensional example given in Reference 4, the full power of Theorem 1 is necessary since, in the infinite-dimensional case, the perturbed manifolds $W^s_\varepsilon(p_\varepsilon(t_0))$ and $W^u_\varepsilon(p_\varepsilon(t_0))$ do not lie in Σ.

We have, therefore, shown that there is a complicated invariant hyperbolic Cantor set Λ embedded in the Poincaré map of equation 1 for a calculable open set of parameter values. Although the dynamics near Λ are complex, we do not assert that Λ is a strange attractor. In fact, Λ is unstable in the sense that its generalized unstable manifold (or outset), $W^u(\Lambda)$ is nonempty (it is one-dimensional) and thus points starting near Λ may wander, remaining near Λ for a relatively long time, but eventually leaving a neighborhood of Λ and approaching an attractor. This kind of behavior has been referred to as *transient chaos* (or preturbulence). In two dimensions, Λ can coexist with two simple sinks of period one or with a strange attractor, depending on the parameter values (see Holmes[4]). There is experimental evidence for transient chaos in the magnetic cantilever problem (Holmes and Moon[2]).

We close with a comment on the bifurcations in which the transversal intersections are created. Since the Melnikov function $M(t_0)$ has nondegenerate maxima and minima, it can be shown that, *near* the parameter values at which $M(t_0) = M'(t_0) = 0$ but $M''(t_0) \neq 0$, the stable and unstable manifolds $W^s_\varepsilon(p_\varepsilon(t_0))$ and $W^u_\varepsilon(p_\varepsilon(t_0))$ have quadratic tangencies. This mechanism, described by Newhouse,[15] then implies that $P^\varepsilon_{t_0}$ can have infinitely many stable periodic orbits of arbitrarily high periods near the bifurcation point, at least in the finite-dimensional examples. In practice, it may be difficult to distinguish these long period stable periodic points from transient chaos and from "true" chaos itself. In fact, it is not yet understood what role the Newhouse sinks play in experimental and computer-generated chaotic motions.

Acknowledgments

We are grateful to the following people for their interest in this work and for their comments: Paul Chernoff, Shui-Nee Chow, John Guckenheimer, Jack Hale, Mark Levi, Sheldon Newhouse, David Rand, and Jeffrey Rauch.

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