Summary

We focus on the seemingly complicated dynamics of a four-machine power system which is undergoing a sudden fault. Adopting a Hamiltonian (energy) formulation, we consider the system as an interconnection of (one degree of freedom) subsystems. Under certain configuration (a star network) and parameter values we establish the presence of Arnold diffusion which entails periodic, almost periodic, and complicated nonperiodic dynamics all simultaneously present; and erratic transfer of energies between the subsystems.

In section 1 we introduce the transient stability problem in a mathematical setting and explain what our results mean in the power systems context. Section 2 provides insights into Arnold diffusion and summarizes its mathematical formulation as in [8], [1]. Section 3 gives conditions for which Arnold diffusion arises on certain energy levels of the swing equations. These conditions are verified analytically in the case when all but one subsystem (machine) undergo relatively small oscillations.

1. Introduction

Transient stability of a power system describes the dynamical phenomena caused by a sudden fault (such as short circuit) or a large impact (such as lightning). It is precisely the Lyapunov stability in a state space formulation of a simplified differential equation model (called the swing equations) which possesses multiple equilibria: \( \dot{x} = f(x) \). Let \( x_0 \) be a stable equilibrium point of this model which is presumably "closest" to the prefault equilibrium point (see [4], [11]). The transient stability problem is to determine whether or not a given point in the state space belongs to the region of stability of the stable point \( x_0 \). This translates the transient stability problem to one of investigation of the region of stability of a given stable equilibrium point (see [6, 7, 9] for simulations).

The swing equations: We write the swing equations model of an interconnected power system. We assume zero transfer conductances of the reduced network (or assume a generator connected to each node).

\[
\begin{align*}
\frac{d}{dt} \delta_i &= \omega_i - \omega_R \\
M_i \frac{d}{dt} \omega_i - D_i \omega_i &= P_i - \sum_{j \neq i} Y_{ij} \sin(\delta_i - \delta_j) \\
\end{align*}
\]

where \( \delta_i \) and \( \omega_i \) are respectively, the angle and the velocity of the rotor of the \( i \)-th machine; \( \omega_R \) is a constant reference velocity, usually \( (27.6 \text{ rad/sec}) \); \( M_i \) (\( D_i \)) is the inertia (damping) constant; \( P_i := P_i - G_{ii} \omega_i^2 \) is the exogenous power input; \( Y_{ij} := e_{i}^* e_{j} \) is the maximum real power transferred between nodes \( i \) and \( j \) (For details of derivations and notation, refer to [3]). The damping constants \( D_i \) are known to be very small. As often done in the analysis of transient stability, we set them to zero, i.e., \( D_i = 0 \).

Equations (1.1) (1.2) describe an \( n \)-degree of freedom Hamiltonian system with a known energy function

\[
W = \sum_{i=1}^{n} \left( \frac{1}{2} M_i (\omega_i - \omega_R)^2 - \sum_{j \neq i} Y_{ij} \sin(\delta_i - \delta_j) \right)
\]

In the power systems literature the analysis of the transient stability had focused on utilizing this energy function. Most of these Analyses [3, 6, 7, 9] draw insights from analogies to the 1 machine = \( \infty \) bus case (or equivalently the equal area criterion). Moreover, in [6], [9], an estimate of the stability region is produced in the \( \delta \)-space (i.e. angle space) only.

The essence of our contribution, as those of [10], are to refute in certain cases the analogies to the 1 machine = \( \infty \) bus, and prohibit the use of other than the complete state space (i.e. \( \delta - \omega \) space). This is so since the dynamics of (autonomous) deterministic differential equations of dimensions higher than 2, can exhibit complicated behavior (see [1], for instance). Indeed, Kopell and Washburn [10] showed that (horseshoe) chaos, i.e. unpredictable behavior of trajectories, is present is a 3-machine power system under certain configuration and parameter values. The 3-machine case describes a two-degree of freedom system.

Another form of known complicated behavior is Arnold diffusion which entails complex nonperiodic unpredictable trajectories and, moreover, erratic transfer of energies between interconnected degrees of freedom (subsystems). Here we show that Arnold diffusion arises in the interconnected 4-machine case (For the \( n \)-machine case, \( n \geq 4 \), and technical details, see [2]). This case describes a 3-degree of freedom system. The specific configuration of the power system yielding Arnold diffusion is shown in fig. 1, with the noted relative parameter ranges. In the case of 3 machines, this configuration...
produces chaos. This is analogous to that of [10] except that it focuses on energy levels of the whole system, i.e., a Hamiltonian approach, and it explicitly considers the dynamics of the reference machine (machine 4), see section 3.

In the context of power systems, we summarize what the presence of Arnold diffusion implies: (1) analysis cannot be based on analogies to the one machine-infinite bus case which is an autonomous two-dimensional system and hence does not exhibit complicated behavior. (2) It raises questions about the adequacy of the model. Does a physical power system of the same configuration as figure 1, exhibit complicated dynamics in the form of chaos or Arnold diffusion? (3) If the model is indeed adequate, then the conclusion of our result leads to new design constraint on the parameter ranges and the configuration of the interconnected power system. (4) Transient stability analysis cannot be conducted in the angle space, i.e., $\delta$-space, alone as in [6,9]. The complete dynamics can be understood only in the whole state space, i.e., $\delta$-w-space.

2. Arnold diffusion

Arnold diffusion (see [2] e.g.) is a self-generated "stochastic" motion that can occur in near-integrable (i.e. weakly coupled) n-degree of freedom Hamiltonian systems where $n \geq 3$. It entails an erratic transfer of energies between these degrees of freedom. It, therefore, constitutes a new concept of instability in a specific example of a Hamiltonian system: a weakly coupled (i.e. near integrable) time-periodic two degree of freedom Hamiltonian system; one degree of freedom described by the energy function $F$ and the weakly coupling term is time periodic. There is vast experimental work on Arnold diffusion primarily in the plasma physics literature. For an account of these works, see [2], Holmes and Marsden [8] introduced an adaptation of Arnold's result to n-degrees of freedom, where $n \geq 3$, employing a vector-Melnikov integrals version. We summarize their result (see also [8], [2]): consider the (perturbed) near-integrable Hamiltonian system

$$H'(q,p;\delta) = F(q,p) + \sum_{i=1}^{n} G_i (I_i) + \mu H^1(q,p;\delta)$$

where the integrals are, at least, conditionally convergent [8], [2] and $\delta$ denotes Poisson brackets. We require that the multiply $2\pi$-periodic Melnikov vector $M(\delta') = (M_1(\delta'),...,$ $M_n(\delta'))$ by $M(q_1,\ldots,q_n; h, h_1, \ldots, h_{n-1}): = \int_{-\infty}^{\infty} \{I_k, H^1\}dt$, $k = 1, \ldots, n-1$

$$M_n(q_1,\ldots,q_n; h, h_1, \ldots, h_{n-1}): = \int_{-\infty}^{\infty} \{F, H^1\}dt$$

Theorem (Holmes and Marsden [8])

If conditions (C1)-(C4) hold for the perturbed system (2.1), then, for $\mu$ sufficiently small, Arnold diffusion arises in this system.

Remarks: (1) An extension of this Theorem to include non-Hamiltonian systems of differential equations is given in [2]. Specifically, this extension allows the degree of freedom described by the energy function $F$ above to be non-Hamiltonian.

(2) Intuitively, the phenomenon of Arnold diffusion can be thought of as a weak coupling (H1) between 2-dimensional subsystems (degrees of freedom); where one subsystem (F) has a homoclinic orbit, the other subsystems (G) are nonlinear oscillators. Under the conditions (C2) and (C3) the oscillators survive the perturbation, and remain nonlinear oscillators with the same frequencies. These oscillators then act collectively as a periodic forcing to the subsystem (F).

§3. The application to the swing equations of a 4-machine power system.

Here we consider an interconnected power system with the configuration of a star network (fig. 1 with-
out the dotted lines). We require that machine (or area) 4 to be relatively large, machine (or area) 3 to be relatively small; and machines 1 and 2 to be intermediate. We also choose appropriate parameters $P_i$ and $Y_{ij}$ and define $\zeta$ to be a measure of the ratio between machine $k$ ($N_k$, $k = 1,2,3$) and the large (reference) machine ($N_4$) [2]. The latter machine serves to produce the coupling between the first three machines. One obtains, after appropriate scaling of constants, and expansions in $\zeta$ (see [2]), the following Hamiltonian which describes the dynamics of the interconnection of the star network.

$$H = \sum_{i=1}^{\infty} \left( \frac{\zeta_i}{i^2} \right)^2 - \zeta_i^3 + \frac{\zeta_i}{i^2} \left[ \cos(\zeta_i^2) + \cos(\zeta_i^2) \right] + \frac{1}{i^2} \zeta_i \left[ \cos(\zeta_i^2) + \cos(\zeta_i^2) \right]$$

where $\zeta_i$ is the velocity; $\zeta_1, \zeta_2, \zeta_3$ are functions of $\zeta$; and $\zeta$ is a small constant; and $\zeta$ is the perturbation parameter. The first three terms, each, describes the dynamics of a pendulum with constant forcing; the other terms, which are functions of $\zeta$, represent the coupling function. 

(The phase portrait of a forced pendulum is shown in figure 2.) The first two terms are pertaining to the two degrees of freedom associated with the intermediate machines (1) and (2). The third is associated with the small machine, machine 3. It is scaled by the (fixed) small nonzero parameter $\zeta$, which serves to guarantee that, for certain energy levels of the unperturbed system ($\zeta = 0$), subsystem 3 possesses a homoclinic orbit, while subsystems 1 and 2 act only as nonlinear oscillators.

Our Hamiltonian (3.1) describes a coupling of subsystems each of which is a Hamiltonian of a forced pendulum. From figure 2, and by the known properties of forced pendulums, conditions (C1)-(C3) of the Arnold diffusion theorem (section 2) can easily be satisfied. It only remains to satisfy condition (C4) on the Melnikov vector. After simplifications of the expressions utilizing equation (1.2), one obtains (2).

$$\hat{M}_3(t_1,t_2) = \int_{-\infty}^{\infty} \left[ -\frac{d}{dt} \zeta_3(t) \cdot \zeta_k(t-t_k) \right] dt, \quad i = 1,2$$

where the overbar denotes the variables along a homoclinic orbit before perturbation. We stress that these improper integrals must be shown to exist and condition (C4) must be verified analytically, in order to prove the existence of Arnold diffusion. When $\bar{\zeta}_3$ and $\bar{\zeta}_3$ are merely $t$-periodic, such an analytic verification is not apparent. (We note though that Fourier expansions may be utilized to perform the evaluation computationally, see [2].)

In the case when machines 1 and 2 undergo small oscillations; analytic proofs of the Melnikov integrals, so as to possess transversal zeros can be established. In this case the variables of machines 1 and 2, i.e. $\zeta_1$ and $\zeta_2$ become sinusoidal (plus small error terms). The Melnikov integrals are then evaluated to (we relegate the technical details to [2]).

$$\hat{M}_1(t_1,t_2) = a_{11} \cos(\zeta_1^2) + b_{11} \sin(\zeta_1^2) \quad i = 1, 2 \quad (3.3.a)$$

$$\hat{M}_2(t_1,t_2) = \sum_{k=1}^{\infty} a_{jk} \cos(\zeta_k^2) + b_{jk} \sin(\zeta_k^2) \quad (3.3.b)$$

where $a_{11}, b_{11}, a_{2k}$ and $b_{2k}$ are nonzero constants; and $\zeta_1$ and $\zeta_2$ are commensurate frequencies. Let $\hat{\gamma}(t_1,t_2) = (\hat{\gamma}_{N_1}, \hat{\gamma}_{N_2})$ (note that we dropped $\bar{\zeta}_3$ since we consider machine 2 to be acting as the forcing). Thus,

$$\text{det} [\hat{\gamma}] = \frac{\hat{\gamma}_{N_1}}{t_{1}^1} \cdot \frac{\hat{\gamma}_{N_2}}{t_{2}^2} \quad (3.4)$$

Equation (3.3.a) has transversal zeros $t_1$, i.e. $\hat{\gamma}_{N_1}(t_1, t_2) = 0$ and $\hat{\gamma}_{N_1}(t_1, t_2) \neq 0$, such that $\sin(\zeta_1^2) = -\frac{a_{11}}{b_{11}}$ and $\cos(\zeta_1^2) = \frac{b_{11}}{a_{11} + b_{11}}$. Plug

$$a_{11} = -b_{11} \frac{a_{11}}{b_{11}} \quad \text{and} \quad \cos(\zeta_1^2) = \frac{b_{11}}{a_{11} + b_{11}}$$

one such zeros, $t_1$, into (3.3.b) and obtain a $t_2$ such that $\hat{\gamma}_{N_2}(t_1, t_2) = 0$ and $\hat{\gamma}_{N_2}(t_1, t_2) \neq 0$. Then the pair $(t_1, t_2)$ constitute a transversal zero for the vector $\hat{\gamma}(t_1,t_2)$ as seen from (3.4). Carrying out these steps, we set $\hat{\gamma}(t_1, t_2)$ to zero, i.e.,

$$a_{32} \cos(\zeta_2^2) + b_{32} \sin(\zeta_2^2) + \frac{a_{31} b_{31} - b_{31} b_{31}}{a_{31} b_{31}} = 0$$

or

$$a_{32} \cos(\zeta_2^2) + b_{32} \sin(\zeta_2^2) = \frac{a_{31} b_{31} - b_{31} b_{31}}{a_{31} b_{31}} \quad (3.5)$$

and require that
Arnold diffusion arises on certain levels. These levels are given explicitly in [2] as \( H^0 = h \), for all \( h > 0 \), but 'near' \( h \), see [8].

As a closing remark we note that our result on the presence of Arnold diffusion extends to the configuration of figure 1, which adds weak interconnections between the subsystems, as shown by the dotted lines.

References


