Experimental and Theoretical Studies of Electrothermal Waves

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Experimental and theoretical studies have been made of the electrothermal waves occurring in a nonequilibrium electrical discharge in a potassium-seeded argon plasma. The studies presented in this paper refer to discharges in transverse gas flow and magnetic field. The behavior of these discharges as determined by photographs, photomultiplier measurements, and voltage probes is discussed and the results interpreted in terms of a steady, one-dimensional theory. A single discharge was found to operate in one of three modes—the shorted, transition, or normal mode—depending on the length of the ionization transient. An extension of the one-dimensional theory to the inlet problem predicts the approximate length of this transient and thus provides criteria for the existence of each mode. The normal mode was studied in a duct with a series of circuits discharging in parallel across a gas flow. Here, a regular set of steady, one-dimensional streamers was found in the center of the duct between hot boundary regions along each electrode wall. The properties of the one-dimensional streamers are shown to agree in detail with values predicted from the one-dimensional theory. A scheme is then presented for calculation of the effective conductivity of a duct using the amplitude of conductivity fluctuations predicted by this theory.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>α</td>
<td>load factor of external circuit</td>
</tr>
<tr>
<td>β</td>
<td>Hall parameter</td>
</tr>
<tr>
<td>γ</td>
<td>Hall parameter at which instability occurs</td>
</tr>
<tr>
<td>δ</td>
<td>thickness of hot boundary region</td>
</tr>
<tr>
<td>e</td>
<td>electron temperature in undisturbed case</td>
</tr>
<tr>
<td>x</td>
<td>fraction of Joule heating going into ionization</td>
</tr>
<tr>
<td>λ</td>
<td>wavelength of steady disturbances</td>
</tr>
<tr>
<td>λ₀</td>
<td>wavelength of light at which heat transfer occurs</td>
</tr>
<tr>
<td>ε₀</td>
<td>electron-electron collision frequency</td>
</tr>
<tr>
<td>Δr</td>
<td>width of the resonant line</td>
</tr>
<tr>
<td>σ</td>
<td>electrical conductivity</td>
</tr>
<tr>
<td>∂σ</td>
<td>undisturbed electrical conductivity</td>
</tr>
<tr>
<td>σ₀</td>
<td>initial electrical conductivity</td>
</tr>
<tr>
<td>σₑ</td>
<td>normalized electrical conductivity, σ/σ₀</td>
</tr>
<tr>
<td>σₑ₀</td>
<td>boundary-region electrical conductivity</td>
</tr>
<tr>
<td>σₑ₀</td>
<td>effective electrical conductivity</td>
</tr>
<tr>
<td>r</td>
<td>time since formation of last streamer</td>
</tr>
<tr>
<td>τ₀</td>
<td>time of smooth rise of voltage</td>
</tr>
<tr>
<td>mₑ</td>
<td>mass of the electron</td>
</tr>
<tr>
<td>mₑₚ</td>
<td>reciprocal photon mean free path at line center</td>
</tr>
<tr>
<td>n</td>
<td>exponent of J in e(J) formula</td>
</tr>
<tr>
<td>nₑ</td>
<td>electron density</td>
</tr>
<tr>
<td>nₑ₀</td>
<td>undisturbed electron density</td>
</tr>
<tr>
<td>rₑ</td>
<td>resistance per unit length of a streamer</td>
</tr>
<tr>
<td>lₑ</td>
<td>distance streamers form behind electrodes divided by rₑ</td>
</tr>
<tr>
<td>tₑ</td>
<td>distance streamers form behind electrodes</td>
</tr>
<tr>
<td>vₑ</td>
<td>gas velocity</td>
</tr>
<tr>
<td>w</td>
<td>duct width</td>
</tr>
<tr>
<td>zₑ</td>
<td>distance measured upstream from old streamer</td>
</tr>
<tr>
<td>Eₑ</td>
<td>electric field along a streamer</td>
</tr>
<tr>
<td>Eₑₑ</td>
<td>electric field along a streamer when it forms in inlet region</td>
</tr>
<tr>
<td>Eₑₑₑ</td>
<td>total electric field in undisturbed case</td>
</tr>
</tbody>
</table>

Eₑₑ = electric field along J in undisturbed case
H = heat-transfer term
I = effective conductivity parameter
Iₑ₀ = ionization potential of seed
Iₑ₀ = preionizing current
Jₑ₀ = current density perpendicular to streamer
Jₑ₀ = current density in undisturbed case
K = effective energy diffusion coefficient
Q = electronic heating term
Rₑ₀ = resistance in external circuit
S = mean square conductivity variation
Vₑ₀ = voltage perpendicular to gas flow
Vₑₑ = voltage parallel to gas flow

Introduction

The present paper describes a study of electrothermal instabilities observed in a seeded plasma in which nonequilibrium ionization was important. These disturbances have been observed by a number of investigators, and are due to the strong coupling between electric conductivity and heating of the electron gas. The effect of these instabilities is to reduce the effective electrical conductivity of the plasma. This reduction has been one of the primary means of identifying the presence of the phenomenon, and a large part of previous studies has been devoted to obtaining the influence of the various parameters of the system on the effective conductivity.

A previous paper reported studies of the wavelike disturbances occurring in a steady duct. In this paper, these studies have been extended to include an investigation of disturbances occurring in other geometric configurations, and the reduction of conductivity has been related to the measured characteristics of the disturbances. Voltage and light intensity measurements were used to construct a picture of the pattern of electric field and conductivity variations as a function of magnetic field strength and applied voltage.

Under some circumstances, these patterns were steady, and the characteristics of these disturbances could be compared with the theoretical predictions based on the model discussed in Ref. 1. However, it was necessary to extend the model to

Received June 6, 1968; revision received February 17, 1969.

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include a description of the nonsteady phenomena which occur in the inlet region.

After a brief review of the theoretical work, discussed in detail in Refs. 1 and 15, experimental results are presented and related to the model.

Theory

In Ref. 1, the energy equation for a one-dimensional non-uniformity in electron density was obtained. This equation was of the form

$$I_i \frac{\partial n_i}{\partial t} + \frac{a}{n_e} \frac{\partial n_i}{\partial x} - H = Q(n_i)$$

where $Q$ (the electronic heating term) is

$$Q = \sigma_e E_{Lo}^2 \left[ \gamma \left( \frac{E_o}{E_{Lo}} \right)^2 + 2 \beta \left( \frac{J_o}{\sigma_e E_{Lo}} \right) \left( \frac{E_o}{E_{Lo}} \right) + \frac{1 + \beta^2}{\delta} \left( \frac{J_o}{\sigma_e E_{Lo}} \right)^2 - \sigma^{(2\alpha-1)} \right]$$

and

$$\delta = \sigma / \sigma_e = (J / J_L)^n$$

Note that $Q$ accounts for Joule heating (first three terms) and elastic collision losses (the last term). The term $H$ in Eq. (1) includes both radiative and electronic heat transfer and in Ref. 1 was approximated by the operator $K(dn_i/dx)^2$.

However, most of the important quantitative results obtained from Eq. (1) are dependent primarily on the heating term $Q$ and thus are independent of this approximation. $Q$ has been normalized by the Joule heating $\sigma E_{Lo}^2$ for the uniform solution and is plotted in Fig. 1. Here, the effect of the angle of the $\sigma$ perturbation relative to the vector $J$ for a uniform solution is included in the parameter $E_{Lo}/E_o$. This is the ratio of the electric field along $\sigma$ in the undisturbed case $E_{Lo}$ to the field along the perturbation $E_o$. These parameters are shown schematically in the upper part of Fig. 1.

The Hall parameter $\beta$ also has an important effect on the shape of the curves of $Q$ vs $\sigma$. For a value of $\beta$ less than that at which the instability occurs, $\beta_{crit}$, the slope of $Q$ vs $\sigma$ at $\sigma_{crit}$ is negative; however, for certain values of $E_{Lo}/E_o$, the absolute value of the slope is less than that occurring for $\beta = 0$. This means that the perturbations, such as those introduced by the electrodes, take longer to disperse when $\beta$ is nonzero.

For $\beta > \beta_{crit}$, the slope of $Q$ for certain values of $E_{Lo}/E_o$ becomes positive at $\sigma_{crit}$, and this indicates that $\sigma_{crit}$ is an unstable solution. However, zeros of $Q$ which are stable points do occur for values of $\sigma$ on either side of $\sigma_{crit}$, and these zeros determine the regime of instability plotted in Fig. 8. Thus, if one considers the growth of a streamer from a small disturbance, these zeros determine the maximum possible conductivity variations that can occur, in either the steady or unsteady case (some unsteady cases have been studied by Velikov4 and also by Solbes5). If the streamers are steady, their maximum amplitude is determined by the nearest zero of $Q$ and their shape and half-width are determined by heat transport processes, that is, $H = Q$. However, their amplitude will be somewhat less than the maximum allowed by the instability regime due to the damping effect of this heat transport.

For small-amplitude waves, the heat transport was treated by Lutz,13 from which an expression for the wavelength $\lambda$ of steady sinusoidal disturbances can be obtained

$$\omega = \frac{2(\pi/\lambda)^2}{\gamma + \beta^2 (\pi/\lambda)^2} = \frac{dQ/d\sigma}{\sigma}$$

where the damping due to conduction is

$$\omega = \frac{4}{3} \frac{n_e n_0}{(1 + \beta^2) I_0 m_e m_0 \sigma_e E_{Lo}^2}$$

the damping due to radiation is

$$\omega = \frac{2(2)^{1/2}}{9} \frac{1.24 \times 10^4}{I_0 \lambda_0} \frac{\sigma}{\sigma_e E_{Lo}^2}$$

and the slope of $Q$ vs $\sigma$ at $\sigma_e$ is

$$\frac{dQ}{d\sigma}_{\sigma_e} = \left( \frac{E_o}{E_{Lo}} \right)^2 - \left( 1 + \beta^2 \right) \left( \frac{J_o}{\sigma_e E_{Lo}} \right)^2 \left( \frac{2}{n} - 1 \right)$$

For the conditions occurring in the experiments, it was shown in Ref. 1 that as the amplitude increased, the width of the high-conductivity region, i.e., the streamer, did not increase much from $\lambda/2$; however, the spacing between streamers increased from $\lambda/2$ to $\lambda$ as the amplitude increased. When spacing between the streamers is much greater than $\lambda/2$, considerations other than amplitude are likely to determine this spacing. Such a case occurs with a single discharge transverse to the gas flow, and this problem is discussed in the 'Inlet' section of this paper.

Apparatus

The equipment used to generate the argon-potassium vapor mixture, which is the working fluid in these experiments, has been described in detail in Ref. 3. In the present work, the major change has been the use of a larger plenum chamber, 2.5-cm diameter, upstream of the test section, and of a smoothly contoured entrance between the plenum chamber and the rectangular test section. The test section is 2 cm high by 1 cm wide by 10 cm long. The magnetic field is applied parallel to the 1-cm dimension. The walls are boron nitride, and the electrodes and probes are tungsten rods, 0.075 cm and 0.05 cm, respectively. The working fluid is argon seeded with 0.2 mole % potassium, and at the test section entrance it has a uniform velocity profile, a speed of 100 m/sec, a temperature of $2000^\circ \pm 20^\circ K$, and a pressure of 1 atm. Typical duct-plate configurations and a schematic
The test electrodes were located from 2 to 4 cm downstream; however, it did aid in igniting the charges occurring downstream. This consisted either of a pair of electrodes discharging parallel to the gas flow showed (Fig. 2) that, as the transverse field moved closer to the upstream electrode pair until it appeared between them. This last condition also could be produced by adding a preionizing discharge upstream of the test area. Thus, there were three modes of operation that could be distinguished on the basis of the length of duct required for the ionization transient. In the first mode, where light is emitted only from the gas along the electrode walls, the transient length is larger than the test section length, and the entire duct operates in the shorted mode, as described by Kerrebrock.\textsuperscript{2} In the second mode, light appears in the center of the duct but considerably downstream of the first electrode pair. This mode will be called the “transition” mode. In the third mode, the ionization length is short, and light is emitted the full length of the test section. This mode will be called the “normal” mode; however, it will be shown later that this mode is different than that described by Kerrebrock\textsuperscript{2} in that the current is carried across the duct in a series of transverse streamers.

Qualitatively, the same modes of behavior occurred with a single transverse discharge: therefore, further study of the inlet problem was carried out with a single electrode pair. Also, since the shorted and normal modes are limits of the behavior observed in the transition mode, it was the transition mode that was studied in most detail.

**Inlet**

Photographs of the inlet to a duct with a series of discharges transverse to the gas flow showed (Fig. 2) that, as the magnetic field was increased while holding the current fixed, the discharge appeared to move farther downstream in the duct. For large \( \beta \), the bright region of the discharge (and hence regions of large conductivity and current density) would be several duct widths behind the initial electrode pair. Also it was observed that luminous layers extended back from the electrodes along the electrode wall. In order to hold the current constant as \( \beta \) increased, it was necessary to decrease the resistance in the external circuit. When \( \beta \) was increased without holding the current constant, the light coming from the center of the duct moved back and decreased until only the glow along each electrode wall was seen. Alternatively, if the applied voltage was increased for a constant \( \beta \), the transverse part of the discharge moved closer to the upstream electrode pair until it appeared between them. This last condition also could be produced by adding a preionizing discharge upstream of the test area.

Thus, there were three modes of operation that could be distinguished on the basis of the length of duct required for the ionization transient. In the first mode, where light is emitted only from the gas along the electrode walls, the transient length is larger than the test section length, and the entire duct operates in the shorted mode, as described by Kerrebrock.\textsuperscript{2} In the second mode, light appears in the center of the duct but considerably downstream of the first electrode pair. This mode will be called the “transition” mode. In the third mode, the ionization length is short, and light is emitted the full length of the test section. This mode will be called the “normal” mode; however, it will be shown later that this mode is different than that described by Kerrebrock\textsuperscript{2} in that the current is carried across the duct in a series of transverse streamers.

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**Single Discharge in Cross Flow**

In addition to visual observations, voltage and photomultiplier measurements were made in the region between the electrodes. Typical measurements made while the duct was operating in the transition mode are shown in Fig. 3. In this mode, the voltage between the transverse electrodes \( V_\perp \) showed a sawtooth pattern. From the photomultiplier outputs \( L_1, L_2, \) and \( L_3 \) it can be seen that each sharp drop in the \( V_\perp \) voltage is correlated with an increase in radiant energy which has been interpreted as the formation of a new streamer between the electrodes, and that a gradual increase of the \( V_\perp \) voltage occurs as this streamer is being swept downstream with the gas. Also note that the Hall voltage \( V_\parallel \) is very small during the gradual rise of the applied field (the amplification of \( V_\parallel \) is twice that of \( V_\perp \) and does not begin to rise until just before the formation of a new streamer. Correlation of arrival times from both light and voltage probes showed that streamers were transported by the gas and had negligible speeds relative to the gas.

From measurements of this type over a range of \( \beta \), the following qualitative explanation of the inlet phenomenon was developed. As \( \beta \) increases from zero, nonuniformities introduced at the electrodes take longer to disperse and thus form high-conductivity paths along the electrode walls. These streamerlike paths tend to short out the Hall field, which, if entirely shorted out, would decrease the electronic heating rate for a given transverse field by a factor \( 1/(1 + \beta) \). This decrease in the electronic heating lengthens the ionization transient, thus shifting the discharge farther downstream. For \( \beta > \beta_{crit} \), the transverse part of the discharge forms into a solitary streamer of the type described in Ref. 1. On either side of the streamer, the conductivity is at the lowest zero of \( Q \); thus, there is no net electronic heating there. However,
as this streamer is carried downstream, the electric field between the electrodes rises. Note that in this experiment the maximum increase in the field is the battery source voltage divided by the electrode spacing. In a generator, the same increase in field would occur; however, there the maximum field would be the open-circuit electric field \( U \times B \).

As a result of this increase in electric field, the ionization rate and thus the conductivity between the electrodes increases. When the conductivity has increased to approximately the mean value it had in the old streamer, an increase in Hall field must follow in order to maintain the continuity of current along the duct.

This can be seen by considering regions of high and low conductivity adjacent to each other, as depicted between the electrodes in Fig. 4. Except for small variations of \( \beta \) with \( \sigma \), Ohm's law states that the vectors of the total current and electric field, \( J \) and \( E \), are at a fixed angle to each other. Also, from the current continuity and the fact that the electric field is curl-free, the component of current perpendicular to the conductivity interface, \( J_x \), and the component of field parallel to the interface, \( E_x \), must be the same in both regions. Thus, going from the low- to the high-conductivity region implies a rotation of the vectors \( J \) and \( E \) while the angle between them remains constant, and as well an increase in their magnitudes so as to satisfy the condition that

\[
\nabla \times J = 0.
\]

Thus, going from the low- to the high-conductivity region implies a rotation of the vectors \( J \) and \( E \) while the angle between them remains constant, and as well an increase in their magnitudes so as to satisfy the condition that \( \nabla \times J = 0 \).

In the inlet, the critical time in this process is the time required for the conductivity to be increased from the value entering the test section to the value at which the Hall field starts increasing. This time varies greatly with conditions in the duct and determines in which mode the duct will operate.

**Calculation of Inlet Relaxation Length**

The time required for the first phase of the stream production process can be calculated. Using the conditions existing during this phase, that is, Hall field zero and \( \beta = \sigma / \sigma_1 < 1 \), the energy equation is approximately

\[
\frac{d\sigma}{dt} = k(\sigma / n_0 J)\sigma E_x^2/[1 + \beta^2] \tag{2}
\]

where the factor \( k \) determines the fraction of the Joule heating that goes into ionization (the rest is radiated). For Eq. (2) with \( \beta = 0 \) to give the ionization rates measured by Cool, the value 0.5 was given to \( k \). In Eq. (2), the effect of the magnetic field is that it reduces the Joule heating rate by the factor \( 1/(1 + \beta^2) \).

The electric field \( E_x \) is calculated for the idealized circuit shown in Fig. 4. Here, the old streamer is considered simply as a conducting path of constant resistance \( r \) per unit length, giving

\[
E_x = E_{xi} \left( \frac{1 + 2x/w}{1 + (2/w)\alpha(v_i + x)} \right) \tag{3}
\]

where

\[
\alpha = \frac{r(w + 2v_i)}{R_{ex} + r(w + 2v_i)}
\]

and \( E_{xi} \) is the field along the streamer when it first forms. Since \( \alpha \) is also the ratio of the minimum voltage across the duct to the source voltage, it is approximately equal to the expected loading factor for a generator. Using Eq. (3) in (2) and integrating from the time a given slug of gas enters the electric field (which is assumed to start on a line between the electrodes) until it is at a point \( v = x \) behind the electrodes gives an equation for \( \sigma \) as a function of \( x \) at a time after the formation of the last streamer. \( \sigma(x) \) has a maximum between the electrodes and the old streamer. The value of \( \tau \) at which this maximum reaches \( \sigma_1 \) is taken as the time \( \tau_1 \) for the smooth rise in voltage. Because of the \( E^2 \) dependence of the Joule heating term, the conductivity is rising rapidly when it crosses the value \( \sigma_1 \); thus, \( \tau_1 \) is not strongly dependent on the value of \( \sigma \) chosen. The parameter that determines how the electric field increases as the old streamer is blown downstream is the distance it has been blown \( v \tau_1 \), normalized by the duct width \( w \). This parameter is plotted in Fig. 5a as a function of the reciprocal of the reduced Joule heating \( (1 + \beta^2)/E_x^2 \) and shows good agreement with the observed time for the smooth part of the voltage rise. In Fig. 5b, the total

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**Fig. 3** Light and voltage measurements in inlet test section.

**Fig. 4** Idealized inlet circuit.
The length of the ionization region is \( r_{r, r} \); and if \( (2r_{r, r})/w < 2\lambda/w \), the duct is in the normal mode.

**Preionization**

One way of decreasing \( r_{r, r} \), and thus producing the normal mode is to increase the initial value of the conductivity \( \sigma_i \) by preionization. In Fig. 6, the effect is shown of a preionizing discharge two duct widths upstream of the test electrode pair. As the preionizing current \( I_{pre} \) is increased, the time for the gradual rise of the voltage decreases to the point where the voltage is essentially constant and the streamers are close together. This set of streamers now looks much like those predicted by the periodic solution obtained in Ref. 1. Here, the distance between the streamers depends on their amplitude, and their maximum amplitude is fixed by the lowest zero of \( Q \). A duct with a regular series of transverse streamers of this type has been defined here as operating in the normal mode.

The light intensity seen at the test section when only the preionizer was on is shown at the bottom of Fig. 6. From these measurements, \( \sigma_i \) with preionization was estimated to be 0.1 to 0.2 of \( \sigma_n \), which was 1.5 mho/cm. However, this is an increase of about a factor of 10 over the \( \sigma_i \) without preionization, which is the equilibrium value of 0.03 mho/cm for this plasma. In Fig. 5, the calculated effect on \( r_{r, r} \), of a factor of 10 increase in \( \sigma_i \), is shown to be consistent with the observations.

**Multiple Transverse Discharges**

The normal mode of operation was studied in a duct with a series of circuits discharging in parallel across the test section. The test section for the multiple discharges had electrodes spaced every \( \frac{1}{4} \) cm (see Fig. 7), which is less than the streamer spacing of 1 cm.

The development of the streamer pattern is shown in Fig. 7 as a function of \( \beta \). As \( \beta \) was increased above the critical value of about 2, a set of regular streamers appeared (Fig. 7) of the same size and spacing as seen with a single preionized discharge (Fig. 6). The similarity of the two situations indicates that the upstream discharges act mainly as preionizers.

**Prediction of the Mode of Operation**

The condition \((2r_{r, r})/w < 0.5(1/\alpha)\) for a transverse streamer to form means that the circuit must be operating nearly as a constant current source. In the case \( \alpha = 0 \), that is, a constant current, the period is given explicitly by

\[
\frac{2\tau_{r, r}}{w} = \left[ \frac{27}{\ln \frac{\sigma_i}{\sigma_t}} \left( \frac{\tau_{r, r}}{w} + (1 + \beta)^2 \right) \right]^{1/2} - 1 \quad (4)
\]

where \( \tau_{r, r} \) is the ionization parameter for \( \beta = 0 \), equals \( (I_n \omega)/\left(\sigma_i E_r\right) \) (Ref. 3). This value of \( (2r_{r, r})/w \) then can be used to predict which mode a given discharge will operate in. That is, if \((2r_{r, r})/w > 0.5(1/\alpha)\), the duct is shorted; if \(2\lambda/w < (2r_{r, r})/w < 0.5(1/\alpha)\), the duct is in the transition mode and

\[
\frac{2\tau_{r, r}}{w} = \left[ \frac{27}{\ln \frac{\sigma_i}{\sigma_t}} \left( \frac{\tau_{r, r}}{w} + (1 + \beta)^2 \right) \right]^{1/2} - 1
\]
for the following ones. Detailed light probe measurements, e.g., \( L_1 \) and \( L_2 \) of Fig. 7, showed that the streamers were straight, one-dimensional disturbances which lay perpendicular to the gas velocity and were well defined to within 2 mm of the 0.75-mm-diam electrodes. A similar regular pattern of transverse streamers was observed by Zauderer in high-speed photographs of a magnetohydrodynamic generator with a similar configuration of electrodes but with xenon gas as the working fluid. It is therefore expected that this is a practically important mode of operation.

In the experiments shown in Fig. 7, when \( \beta \) was increased from zero to 1.4, which is below the value at which the instability develops, the electric field increased and the light intensities decreased. This is due to the formation of higher conductivity zones along the electrode walls, as was predicted by Kerrebrock for the shorted mode of a magnetohydrodynamic generator. These zones, also observed by Fischer, are of the same origin as the streamers formed behind the electrodes in the case of a single discharge.

In Fig. 8, the conductivity and field fluctuations measured in the transverse streamers for \( \beta = 4.5 \) are shown to lie within the instability regime on a \( \sigma / \sigma_n \) vs \( E_{\text{tr}}/E_n \) plot. In Fig. 9, the measured values of \( E_{\text{tr}}/E_n \) are plotted as a function of \( \beta \), and this diagram shows that the fluctuations lie in the instability regime over the whole range of \( \beta \). For a constant current, note that \( E_{\text{tr}} \) over \( E_n \) is the ratio of the perpendicular field measured with \( \beta = 0 \) to that when the magnetic field is applied. Conductivity nonuniformities occurring at higher values of \( \beta \) do cause a change in the average conductivity, and thus in \( E_{\text{tr}} \); however, the associated correction to \( E_{\text{tr}} \) would be less than 5%.

From the steady theory, the maximum steady wave amplitude can be calculated as a function of wave angle, that is, of \( E_{\text{tr}}/E_n \); however, the angle, if any, at which steady waves will actually exist cannot be directly determined. To answer this problem analytically, a stability analysis of the steady waves must be done. This is very difficult to carry out in general; however, a simple analysis suggests that the value of the effective Hall parameter defined in terms of average fields and currents should be a minimum to produce the most stable set of steady waves. This criterion is essentially the same as that used by Solbes. In Fig. 9, it can be seen that the measured values of \( E_{\text{tr}}/E_n \) are indeed scattered about the theoretical curve for minimum \( \beta_{\text{eff}} \). Also note that all the values lie below the theoretical curve for the maximum possible amplitude waves. However, it also was found in Ref. 15 that a necessary condition for stability of an infinite plasma was that \( \beta_{\text{eff}} \) must be less than \( \beta_{\text{min}} \), and for theoretical values of \( \beta \) greater than about three, this condition was not produced by the predicted set of steady waves.

The observation of apparently steady waves in this experiment up to theoretical Hall parameters of seven may be due to the stabilizing effect of the wall electrodes. That the walls may have had a stabilizing effect is indicated by the result discussed in Refs. 4 and 15, e.g., that in a test section with the electrodes many wavelengths away from the discharge, the steady streamers did become unstable for Hall parameters greater than three.
However, since the transverse streamers here are one-dimensional and steady, it is possible to use the model of Ref. 1 to calculate the approximate peak-to-peak conductivity variation $\Delta \sigma$ occurring in the duct. This $\Delta \sigma$ then can be used to predict the effective conductivity of the plasma.

Calculations of $\Delta \sigma$

Theoretically, for steady waves, the nearest zero of $Q$ (Fig 8) fixes the maximum amplitude for a given value of $E_{Lm}/E_c$. In the region on this plot from $E_{Lm}/E_c = 0.52$ to 0.2, the lower zero fixes the amplitude and from $E_{Lm}/E_c = 0.61$ to 0.52, the upper zero does. Since the maximum variation occurs when the two zeros are equidistant, it follows that if $E_{Lm}/E_c$ is allowed to vary, the true maximum peak-to-peak amplitude is the difference in conductivity between the upper and lower zeros occurring when $E_{Lm}/E_c = 0.52$. For this case, $\Delta \sigma/\sigma$ = 1.2. This was the criterion used to obtain the upper curve in Fig. 10.

Measured values of $\Delta \sigma/\sigma$ lie below this maximum possible amplitude, and in the range of the value for waves that give minimum $S_{\text{int}}$, the dashed curve in Fig. 10. The values of $\Delta \sigma/\sigma$ for minimum $S_{\text{int}}$ are also very similar to those calculated by Solbes using a different approach.

These results were essentially independent of the direction of the $J \times B$ force relative to the gas velocity, that is, whether the duct was accelerating or braking the gas flow. However, $\Delta \sigma$ is dependent on $n$, the exponent of $J$ in the $\sigma(J)$ formula. The value $n = 0.61$ was chosen to agree with the $\sigma$ vs $J$ curve measured by Cool.$^3$

Effective Conductivity in the Normal Mode

In these experiments, the transverse current was held constant as $\beta$ was increased; therefore, $E_{Lm}/E_c$ is equal to the ratio of the effective conductivity $\sigma_{\text{eff}}$ to the conductivity $\sigma$ with $\beta = 0$. To show that the measured $\sigma_{\text{eff}}$ is consistent with the observed amplitude and pattern of conductivity fluctuations, a calculation of the effective conductivity was made for a duct with hot boundary regions on either side of a region with transverse conducting paths. This was done by calculating average fields and currents in each region from an analysis similar to Ross's and matching them across the division between the two regions. This analysis gives a relationship between $\sigma_{\text{eff}}$ and ($\sigma$). By averaging the energy equation,$^{15}$ an approximate relation between ($\sigma$) and $\sigma$ can then be obtained,

$$ (\sigma)/\sigma = [(\sigma_{\text{eff}})/\sigma][1 + \frac{1}{2}[(2/n - 1)(2/n - 2)]S]^{1/3} $$

where $n$ is the exponent in the $\sigma$ vs $J$ relation. Using these two expressions and the fact that $n$ was approximately $\frac{2}{3}$

these experiments gives

$$ \frac{\sigma_{\text{eff}}}{\sigma} = \left[ \frac{\gamma + 1/(1 - (\sigma)/\sigma_{BL})[S]}{(\gamma + 2)(1 + S)^{1/3}} \right]^{1/3} $$

where $\gamma$ specifies the effective boundary region and is equal to

$$ \gamma = 1 + (w/\delta)[1/(\sigma_{BL}/\sigma - 1)] $$

Here, $w$ is the width of the duct, $\delta$ is the width of the hot boundary region, and $\sigma_{BL}$ is the average conductivity of the boundary. $S$ is the mean square deviation for a sine wave of the normalized peak-to-peak amplitude plotted in Fig. 10. With no transverse streamers, that is, with $\Delta \sigma/\sigma$ = 0, the $\sigma_{\text{eff}}$ would decrease along the curve marked 1-D in Fig. 11. This curve gives the effective conductivity for one-dimensional nonuniformities parallel to the flow and the magnetic field. The actual data lie considerably above this curve, due to the beneficial effect of the transverse streamers. The solid curves in Fig. 11 show the $\sigma_{\text{eff}}$ calculated from Eq. (6) with $S = 0.125$, that is, for $\Delta \sigma/\sigma = 1.0$ and for $\gamma = 3.0$ and 5.0. Most of the data are contained in this range of $\gamma$, which corresponds to two boundary regions, each of thickness 0.5 cm and having a conductivity 1.5 to 2.0 times that in the mainstream. These values are consistent with estimates that could be made from photographs of the duct. However, $\gamma$, generally, cannot be calculated with any degree of accuracy, and therefore only $S$ is known.

Louis$^7$ has suggested that even though the actual conductivity nonuniformities may display distinct regularity, the effective conductivity might be predicted adequately by assuming an isotropic distribution of these nonuniformities. The calculation of $\sigma_{\text{eff}}$ for such disturbances was made using the procedure of Yoshikawa and Rose.$^8$ This procedure accurately gives the effect of superposition of waves only for small-amplitude disturbances$^{16}$; however, it may give an estimate of the effect of larger disturbances.

The procedure was also modified somewhat by using the assumption that the collision frequency was constant rather than the assumption, used earlier, that it was dominated by Coulomb collisions. Although this assumption is closer to the condition of this experiment, it does not change the result significantly. From this calculation, again correcting for the change of ($\sigma$),

$$ \frac{\sigma_{\text{eff}}}{\sigma} = \left[ \frac{1 + (S)^{1/2} - 2S/\beta + (S/\beta^2)}{[1 - (1/\beta)S + \beta S^2](1 + S)^{1/3}} \right] $$

where $I$, as defined in Ref. 15, goes to $\pi/4$ for large $\beta$ and $S$ is the mean square deviation of the conductivity. Equation (7) gives the curve marked 3-D in Fig. 11, where $S = 0.125$. This curve was too high; however, when the calculation of Ref. 8 was reformulated such that only isotropic conductivity fluctuations in the two dimensions perpendicular to the mag-
netic field are allowed, better agreement was obtained. Here, the equation for $\sigma_{el}$ is

$$\frac{\sigma_{el}}{\sigma_n} = \left[ \frac{1 - S + (S^2/4)\beta^4}{1 - S/2 + (S/2)\beta^2(1 + S)^{1/2}} \right]^{3/2} \tag{8}$$

This result, marked 2-D in Fig. 11, agrees with the data at the higher values of $\beta$.

Conclusions

In a magnetic field, a single pair of electrodes discharging across a gas flow was observed to exhibit three qualitatively different modes of behavior: a shorted mode where high conductivity existed only along the electrode walls, a transition mode where transverse streamers were formed with a regular but long period, and a normal mode where the current was carried across the duct in a series of closely spaced streamers. There is a basic reason for the difference in behaviors of such a discharge with and without a magnetic field. With the magnetic field, regions of high conductivity along the walls, produced by current concentrations at the electrodes, short out the Hall field and hence decrease the Joule heating by a factor $1/(1 + \beta^2)$. Based on this concept, a simple theory of the jet was developed from which the ratio of the length of the ionization zone to the duct width [i.e., $(v_\tau g)/u$] was calculated. Qualitatively, this parameter 1) increased with the ratio of final to initial conductivity, 2) increased with the parameter $(v_\tau g)/u$ where $v_\tau$ is the ionization parameter for $B = 0$, $(\sigma_{el0})/(\sigma_n E_0)$, and 3) increased with $(1 + \beta^2)$. The maximum length this ionization zone can have and still permit new streamers to be formed between the electrodes depends on the ratio of available source voltage to the duct voltage. That is, it depends on the load parameter of the circuit. When this limit is exceeded, the duct is in the shorted mode. In the other limit, when the ionization length is shorter than the wavelength of steady waves, the duct is in the normal mode.

For the practically important case of the inlet to a series of circuits discharging in parallel across a gas flow, that is, an accelerator or MHD generator, the situation is more complicated, and the criteria developed here will not be directly applicable. However, the qualitative effect of the various parameters will be the same, and the criterion for occurrence of the normal mode will probably still be valid.

The normal mode of operation was studied in detail in a duct with multiple transverse discharges. Here, it was observed that the light intensity pattern could be divided into two regions. The first is a hot boundary region along each electrode wall and the second is a region of steady, one-dimensional streamers transverse to the gas flow and lying between the hot wall layers.

The streamers in the center of the duct were studied in detail, and it was shown that their peak-to-peak amplitude, about $1.0 \sigma$ in this case, can be predicted by the steady, one-dimensional theory. Using the measured values for the streamers in the center of the duct and reasonable values for the boundary regions, the relative effect that the two regions have on the effective conductivity of the duct is shown. The hot boundary regions alone would have decreased the effective conductivity by a factor of 10 at $\beta = 5$; however, with the transverse streamers, the effective conductivity was decreased by a factor of 2.5 at the same $\beta$. The beneficial effect of operating a generator or accelerator in the normal mode is therefore obvious.

Since the characteristics of the boundary region would be difficult to calculate with any accuracy, a computational scheme was sought from which the effective conductivity could be derived using only the amplitude predicted by the steady theory. A model of a two-dimensional, isotropic distribution of conductivity nonuniformities of the predicted amplitude was found to give reasonable agreement with the measured values over the range of $\beta$ considered.

References