THE ROLE OF APPROXIMATE ANALYTICAL RESULTS IN THE STUDY OF TWO-PHASE FLOW IN NOZZLES

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Abstract

The small slip approximation to the theory of two-phase flow in rocket nozzles is reviewed to show that the inaccuracies associated with drag and heat transfer laws, and those associated with the fundamental approximation, are independent and that the former may be removed algebraicly. Selected applications of the approximate theory are discussed to indicate that these stress the nature of the dependence of the results upon the relevant physical parameters and the possible consequence of scaling laws, rather than numerical accuracy too often limited by inaccurate initial data.

It is suggested that approximate analytical results may offer much more assistance to the rocket engineer than has yet been used to advantage.

Introduction

The performance losses caused in rocket nozzles by the presence of small solid particles in the exhaust, and other details of this two-phase flow, may be calculated with acceptable accuracy provided that accurate information on particle size, solid and gas properties, and the laws of particle and gas interaction are known. Numerical calculations may be tedious and time consuming and, even more important, sufficiently complex that any deep understanding of the physical processes is lost together with the capability to estimate the effects changes in material properties, nozzle shape, etc.

The approximate method that offers the best combination of simplicity, physical insight, and accuracy of results is the linearized analysis introduced by Rannie [1] and by Marble [2] and now in rather wide use for conventional nozzles of reasonable size. To be sure, the technique has limitations. In particular, the accuracy deteriorates when applied to very small nozzles or to regions of extreme acceleration.

The range of accurate applicability is very wide, however, and wider than appears to be realized at the present time. It is often erroneously supposed that approximations in the drag and heat transfer laws are related to the linearizations. This need not be the case if the problem is properly formulated.

It is the purpose of this paper to review the foundations of the linearized theory of one-dimensional, two-phase flow in nozzles and to reexamine some of the success this analysis has had in securing accurate results for and physical insight into some rather complex problems. Finally, the limitations of the linearized analysis will be investigated to clarify its range of applicability and some means of extending its range of usefulness will be discussed.

The Linearized Theory of One-Dimensional Two-Phase Flow

For the usual mass fractions of solids in rocket exhaust and in view of the fact that the density of the solid is of the order 10^3 times that of the gas, the volume occupied by the solids may be neglected in the continuum equations for the gas. Then the equations of continuity, momentum, and the first law of thermodynamics may be written

$$\rho u A = \dot{m} \tag{1}$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = F_p \tag{2}$$

$$\rho u c_p \frac{dT}{dx} - u \frac{dp}{dx} = (u_p - u) F_p + Q_p \tag{3}$$

where F_p is the effective force per unit volume exerted by the particles upon gas, Q_p is the heat transferred per unit volume from the particles to the gas, and $(u_p - u)F_p$ is the dissipative work associated with the motion of particles relative to gas.

The corresponding set of equations may be written for the solid phase, considering it as a sort of continuum and where ρ_p represents the mass of solid phase per unit volume rather than the density of material constituting the solid particles

$$\rho_p u_p A = \kappa \dot{m} \tag{4}$$

$$\rho_p u_p \frac{du_p}{dx} = -F_p \tag{5}$$

$$\rho_p u_p c \frac{dT_p}{dx} = -Q_p \tag{6}$$

where κ is the constant ratio of solid mass to gas mass flowing through any cross section of the nozzle.

Because of the approximations that we intend to employ, it is convenient to introduce the particle slip velocity

$$u_s = u - u_p \tag{7}$$

having the physical significance of the amount that the particle velocity lags behind the gas velocity in the accelerating flow. Similarly, the temperature and density of the particle cloud are replaced by

$$T_s = T - T_p \tag{8}$$

$$\rho_s = \kappa \rho - \rho_p \tag{9}$$

If the particle cloud were at all times in dynamic and thermal equilibrium with the gas, these three quantities defined in equations (7), (8), and (9) would vanish identically; they are measures of the deviation of the two-phase flow from a state of equilibrium.

The departure from equilibrium between the phases is governed by the exchange of momentum and heat, quantities that are given by F_p and Q_p. It has proven convenient to formulate these as

$$F_p = \frac{\rho p a^2}{\lambda_v} \cdot \frac{u_s}{a} \cdot \alpha\left(\frac{u_s}{a}, k\right) \tag{10}$$

$$Q_p = \frac{\rho_p a c_p T_s}{\lambda_T} \cdot \beta\left(\frac{u_s}{a}, k\right) \quad . \tag{11}$$

The characteristic lengths λ_v and λ_T [2] are distances a particle would cover at the speed of sound while reducing initial velocity and temperature differences respectively to e^{-1} of their original values. When the spherical particles obey Stokes law and have a Nusselt number of unity, the functions α and β are unity. For the flow regimes ordinarily encountered in in nozzles, α and β are still of order unity [1] but when the slip Mach number u_s/a or the Knudsen number k are very large, the values of α and β may be seriously altered.

It is convenient to construct a new set of equations [3] from (1) - (6), utilizing these new variables, the first three of which

$$\rho u A = m \tag{12}$$

$$\bar{c}_p(T-T_c)+\tfrac{1}{2}u^2 = \frac{\varkappa}{1+\varkappa}\{cT_s + uu_s - \tfrac{1}{2}u_s^{\,2}\} \tag{13}$$

$$\left(\frac{T}{T_c}\right)\left(\frac{p_c}{p}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} = \exp\left\{\frac{\varkappa}{1+\varkappa}\int_0^x \frac{1}{\bar{c}_p T}\left[c\,\frac{dT_s}{dx} + u_s\,\frac{d}{dx}(u-u_s)\right]dx\right\} \tag{14}$$

resemble those that arise in conventional nozzle theory, the remaining three of which

$$\rho_s u + \varkappa \rho u_s = \rho_s u_s \tag{15}$$

$$\left(1-\frac{\rho_s}{\varkappa\rho}\right)\frac{a^2}{\lambda_v}\cdot\frac{u_s}{a}\cdot\alpha\left(\frac{u_s}{a}, k\right)+u\,\frac{du}{dx} = u\,\frac{du_s}{dx} \tag{16}$$

$$\left(1-\frac{\rho_s}{\varkappa\rho}\right)\frac{a}{\lambda_T}\left(\frac{c_p}{c}\right)T_s\cdot\beta\left(\frac{u_s}{a}, k\right)+u\,\frac{dT}{dx} = u\,\frac{dT_s}{dx} \tag{17}$$

emphasize the slip quantities and involve the drag and heat transfer laws explicitly. The quantities $\bar{c}_p$, $\bar{\gamma}$, reference 2, are the effective specific heat and isentropic exponent for the gas-particle mixture when the gas and solid are in complete dynamic and thermal equilibrium. When this state of equilibrium does hold, the slip quantities vanish identically, equations (15 (17) are redundant, and the right hand sides of equations (13) and (14) become 0 and 1, respectively. The nozzle flow described by these simplified forms of equations (12)-(14) is identified to that for conventional nozzle flow but with the gas properties modified by the mass and thermal capacity of the condensed phase. The nozzle performance under these conditions of equilibrium represents the maximum that can be obtained for the two-phase flow under fixed chamber and discharge conditions.

Under conditions which are appropriate to most rocket motors, the actual performance of the nozzle with suspended solids is rather close to the ideal because the slip quantities are not large. To be somewhat more precise about it, if the nozzle length L is the significant length for acceleration, then the distance x should be measured in terms of L; accordingly, introduce the dimensionless distance along the nozzle as $\xi \equiv x/L$. Equation (15), for example, then may be written

$$\left(1-\frac{\rho_s}{\varkappa\rho}\right)\frac{u_s}{a}\,\alpha\left(\frac{u_s}{a}, k\right)+\frac{\lambda_v}{L}\,\frac{u}{a^2}\,\frac{du}{d\xi} = \frac{\lambda_v}{L}\,\frac{u}{a^2}\,\frac{du_s}{d\xi} \tag{18}$$

where for nozzles of reasonable length and for particles of micron size, the ratio λ_v/L is small. But since $(u/a^2)(du/d\xi)$ is of order unity, it follows that u_s/a is small. Physically, this means that the acceleration experienced by the particle in passing through the nozzle may be approximated by the accelerations which would occur if the solid and gas were in complete equilibrium, the particle slipping at just such a velocity that the resulting drag balances this approximate inertial reaction.

From an analytical viewpoint, this approximation corresponds to an asymptotic expansion [2, 3] of the solution for small λ_v/L and is a singular one in that the highest derivatives of the slip quantities are suppressed in the process. Because use is almost never made of more than the first approximation, we shall not discuss the formal expansions here but simply note some properties of the equilibrium and first approximations to slip quantities, and the first order correction to the gas quantities.

As was first noted by Rannie [1] and subsequently employed to advantage by the present author [2, 3], it is most convenient to utilize as independent variable a thermodynamic quantity that varies monotonically along the nozzle axis. The gas pressure is such a quantity. This change in independent variable avoids difficulty with the perturbation quantities near the nozzle throat, which naturally occur when the distance x is used, and still permit flow field calculations for nozzles of fixed geometry.

The equilibrium solution is readily written down in terms of the pressure along the nozzle, where a superscript zero is employed to denote the equilibrium approximations

$$\frac{\rho^{(0)}}{\rho_c} = \left(\frac{p}{p_c}\right)^{1/\bar{\gamma}} \tag{19}$$

$$\frac{T^{(0)}}{T_c} = \left(\frac{p}{p_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} \tag{20}$$

$$\left(u^{(0)}\right)^2 = 2\bar{c}_p T_c \left\{1 - \left(\frac{p}{p_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right\} \quad . \tag{21}$$

Similarly, the equilibrium relationship between nozzle area and pressure is

$$\frac{\rho_c a_c A^{(0)}}{\dot{m}^{(0)}} = \left(\frac{p}{p_c}\right)^{\frac{1}{\bar{\gamma}}} \left[\frac{2}{\bar{\gamma}-1}\left\{1 - \left(\frac{p}{p_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right\}\right]^{-\frac{1}{2}} \tag{22}$$

which, since the area is prescribed along the nozzle axis, corresponds to the equilibrium approximation to pressure along the axis. Note that, if the nozzle shape is prescribed, the mass flow is not known, the equilibrium approximation to the mass flow being denoted $\dot{m}^{(0)}$ in equation (22).

Without reviewing the specific details, the first approximation to the slip quantities, denoted by a superscript unity, may be obtained quite simply. For the moment, let us assume that the functions

$$\alpha\left(\frac{u_s}{a}\,,\,k\right) \qquad \text{and} \qquad \beta\left(\frac{u_s}{a}\,,\,k\right)$$

are unity; then the particle drag and heat transfer are given respectively by Stokes law and by a Nusselt number of unity. Then the approximate slip quantities are

$$u_s^{(1)} = -a^{(0)}\frac{\lambda_v}{L}\cdot\frac{1}{\bar{\gamma}}\frac{1}{p}\frac{dp}{d\xi} \tag{23}$$

$$T_s^{(1)} = T^{(0)}\frac{\lambda_T}{L}\cdot\frac{\bar{\gamma}-1}{\bar{\gamma}}\cdot\frac{c}{\bar{c}_p}M^0\frac{1}{p}\frac{dp}{d\xi} \tag{24}$$

$$\rho_s^{(1)} = \kappa\rho^{(0)}\frac{\lambda_v}{L}\cdot\frac{1}{\bar{\gamma}}\frac{1}{p}\frac{dp}{d\xi} \tag{25}$$

that is, they are given algebraicly in terms of quantities that are either prescribed or are obtained from the conventional equilibrium solutions.

The first corrections to the solutions for the gas flow are less obvious since they involve approximations to the integral on the right hand side of equation (14). Explicitly, the correction to the gas temperature or the gas density is

$$\frac{T^{(1)}}{T^{(0)}} = \frac{\lambda_v}{L}\frac{\bar{\gamma}-1}{\bar{\gamma}}\frac{\kappa}{1+\kappa}\left\{\eta M^0\frac{1}{p}\frac{dp}{d\xi}+\frac{1}{\bar{\gamma}}\int_{p_c}^{p}\frac{1+(\bar{\gamma}-1)\eta {M^0}^2}{M^0}\frac{1}{p^2}\frac{dp}{d\xi}dp\right\} = -\frac{\rho^{(1)}}{\rho^{(0)}} \tag{26}$$

and the correction to the velocity is

$$\frac{u^{(1)}}{u^{(0)}} = \frac{-1}{(\bar{\gamma}-1)}{M^0}^2\frac{\lambda_v}{L}\frac{\kappa}{1+\kappa}\left\{\left(\frac{\bar{\gamma}-1}{\bar{\gamma}}\right)M^0\frac{1}{p}\frac{dp}{d\xi}+\frac{\bar{\gamma}-1}{\bar{\gamma}^2}\int_{p_c}^{p}\frac{1+(\bar{\gamma}-1)\eta {M^0}^2}{M^0}\frac{1}{p^2}\frac{dp}{d\xi}dp\right\} \tag{27}$$

where

$$\eta \equiv \left(\frac{c}{\bar{c}_p}\right)^2\frac{\lambda_T}{\lambda_v} \quad . \tag{28}$$

For rocket applications, the item of especial significance is the loss of specific impulse and, for the ideally expanded nozzle, this may be written in the form of the fractional impulse loss, where $I^{(0)}$ is the specific impulse for equilibrium flow

$$\frac{I^{(0)}-I}{I^{(0)}} = \left.\frac{(1+\kappa)u^{(0)}-[u^{(0)}+u^{(1)}+\kappa(u^{(0)}+u^{(1)}-u_s^{(1)})]}{(1+\kappa)u^{(0)}}\right]_{p=p_e}$$

$$= \left.-\frac{u^{(1)}}{u^{(0)}}+\frac{\kappa}{1+\kappa}\frac{u_s^{(1)}}{u^{(0)}}\right]_{p=p_e} \quad . \tag{29}$$

Upon substitution from equations (23) and (26), after some algebraic reduction,

$$\frac{I^{(0)}-I}{I^{(0)}} = \frac{\lambda_v}{L} \frac{\varkappa}{1+\varkappa} \left(\frac{1}{\bar{\gamma}} M^{(0)}(p_e)\right)^2 \int_{p_c}^{p_e} \frac{1+(\bar{\gamma}-1)\eta M^{(0)^2}}{M^{(0)}} \frac{1}{p^2} \frac{dp}{d\xi} dp \,. \tag{30}$$

There is no difficulty in evaluating this integral, since it involves no singularity, and the value of $dp/d\xi$ can be evaluated from the equilibrium pressure and the known relationship between the cross-sectional area and x, the distance measured along the nozzle axis from the chamber exit. Since the equilibrium Mach number may be given explicitly in terms of the pressure ratio

$$M^{(0)^2} = \frac{2}{\bar{\gamma}-1} \left[\left(\frac{p_c}{p}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} - 1 \right] \tag{31}$$

the integral in equation (30) may be expressed in terms of either $M^{(0)}$ or p, the shape of the nozzle entering only through the term $dp/d\xi$.

The impulse loss resulting from the non-equilibrium flow in the nozzle is estimated in equation (30) for the case where expansion is carried out to a prescribed exit pressure, p_e. The complementary example, when the expansion is carried out to a prescribed outlet area, is evaluated in reference 2.

The Nozzle Shape for Minimum Impulse Loss

Aside from the fact that equation (30) provides an estimate of the particle slip impulse loss that may be determined by an elementary quadrature, it provides a form appropriate for variational calculations. Because the nozzle shape enters in the integral only through the term $dp/d\xi$, a significant extremum problem may be posed to ask what shape the nozzle should have, when the nozzle length and pressure ratio are fixed, in order that the fractional loss of impulse shall be a minimum. It transpires that this calculation is a straightforward one and yields [3] the optimum shape

$$\xi\left(\frac{p}{p_c}\right) = \frac{H(p/p_c)}{H(p_e/p_c)} \tag{32}$$

where the function $H(p/p_c)$ is defined by the integral

$$H(p/p_c) = \int_{p/p_c}^{1} \sqrt{\frac{1+(\bar{\gamma}-1)\eta M^{(0)^2}}{M^{(0)}}} \frac{dp}{p} \,. \tag{33}$$

This simple formula may be evaluated directly since the relationship between the local pressure p and the equilibrium flow Mach number $M^{(0)}$ is given by equation (31). This function $H(p/p_c)$ is given in Figure 1 for two ratios of the isentropic exponent $\bar{\gamma}$ for equilibrium flow. It is of particular interest that the value of $H(p/p_c)$ varies very nearly linearly with the logarithm of the expansion pressure ratio. As a result, it is easy to calculate the distribution of pressure along the nozzle, and this is given, through use of equation (32), in Figure 2 for a nozzle with a pressure ratio of 100. The near linearity of the $\log(p_c/p)$ with distance along the nozzle is again to be noted.

The crudest approximation to the shape of the nozzle for minimum impulse loss is obtained by substituting the results of equation (32) into the pressure-area relation given by equation (22). The resulting shape of the optimum nozzle is shown in Figure 3 for comparison with a more conventional nozzle of the same expansion ratio.

The characteristics of the optimum shape then become quite clear. The contraction from the chamber is initially quite rapid; the throat region is greatly prolonged to reduce the usually high accelerations, and a reasonable degree of acceleration persists to the end of the nozzle. The fact that the optimum nozzle does not show small accelerations at the nozzle exit suggests that losses due to high slip within the nozzle are as important as particle slip at the nozzle exit. Rather, the acceleration is distributed so that both internal losses and particle slip losses at the exit are moderately small. This fact demonstrates the fallacy of judging impulse losses on the basis of particle lag alone. The fact that the particle velocity lag at the outlet of the optimum nozzle may be several times that of a typical nozzle demonstrates that a considerable portion of the impulse loss may be attributed to dissipation within the nozzle.

One interest in the optimum nozzle is the determination of how much of the particle lag loss in a conventional nozzle can be regained by modified contouring of the nozzle. The only straightforward manner of investigating this is to compare the specific impulse losses in conventional and optimum nozzles having the same length and the same prescribed pressure ratios.

The distribution of losses within the two nozzles is most easily shown by calculating the cumulative impulse loss at various stages along the expansion. The result gives the contribution to the impulse loss of all processes taking place to any point of the nozzle. The results of the cumulative loss calculations are shown in Figure 4 for both conventional and optimum contours. It is to be noted that the slope of the cumulative loss curve corresponds to values of particle slip velocity, confirming the conjecture that dissipation associated with particle slip is a principal contributing factor. The increment in loss which causes the conventional nozzle to have a lower impulse than the optimum occurs in the throat region of the conventional nozzle where the particle slip velocities are very high. Correspondingly, toward the end of the conventional nozzle, impulse losses are being accumulated very slowly, reflecting the low acceleration rate at the conventional nozzle discharge.

In terms of overall performance, the losses induced by the particle lag are not an exceedingly sensitive function of the nozzle shape, at least within the family of contours considered reasonable for the gas alone. As a consequence, one may conclude that, for practical nozzle shapes and reasonable sizes, rather small gains in impulse may be expected from alterations to the nozzle contour. In terms of current practice, this gain would be a fraction of a per cent of the motor impulse.

On the other hand, a reasonable first estimate of the particle loss in a conventional rocket nozzle may be taken as the loss in the optimum nozzle of the same pressure ratio. This approximation may be made even more useful by noting [3] that the function $H(p/p_c)$ may be taken very closely to be

$$H(p/p_c) \cong [4\eta(\bar{\gamma}-1)]^{1/4} \ln(p_c/p) \tag{34}$$

and consequently an adequate approximation to the impulse loss is

$$\frac{I^{(0)}-I}{I^{(0)}} \cong 2\frac{\lambda_v}{L}\left(\frac{\kappa}{1+\kappa}\right) \quad \eta(\bar{\gamma}-1)\left\{\frac{\ln(p_c/p_e)}{\bar{\gamma}\, M_e^{(0)}}\right\}^2 . \tag{35}$$

This result contains all of the elements that are of first importance in the problem of performance loss due to a solid phase in the exhaust. Furthermore, in spite of the fact that compromises have been made in absolute accuracy, the effect of change in parameters based upon this simple formula is quite accurate.

To summarize, the first interesting result of the first order analysis is that the difference between the particle lag loss in a conventional nozzle and that in the corresponding optimum nozzle is not very large. Consequently, it is usually unwarranted to construct exotic nozzle shapes, since this effort, carried out at the expense of many other features of the nozzle, will not reduce the particle loss by more than one third. This is not to imply that improvement in existing nozzles cannot be made, but rather that the gains will be small unless the original nozzle was very short, of very small scale, or possessed some unusual entrance condition or throat contour.

As a consequence, the performance of a conventional nozzle may be approximated by the performance of the optimum nozzle having the same pressure ratio or area ratio. The error made in this approximation is usually less than that resulting from inadequate knowledge of the particle size, gas properties, and detailed flow field.

Droplet Agglomeration in Nozzles

The growth of liquid droplets in rocket nozzles, resulting from particle slip and collision, is another type of problem that demonstrates physical insight and simplicity that can be achieved through the approximate analytical treatment of two-phase flow problems. We wish to estimate the amount of growth in size that droplets may experience when the enter the nozzle as liquid and freeze subsequently as the gas and liquid temperatures fall along the flow path.

The agglomeration of small liquid droplets to form larger ones usually takes place through three separate mechanisms: (i) collision due to Brownian motion of very small droplets, (ii) collisions arising from turbulent diffusion, and (iii) collisions resulting from differential mean velocity of droplets of different sizes. The relative importance of these mechanisms depends upon the magnitude of the acceleration field that produces the relative motion of different size droplets. In the rocket chamber, the Brownian or turbulent diffusion is dominant, but in the rocket nozzle, where accelerations of 10^4 g are common, the collisions from differential particle velocity probably control the agglomeration rate.

These differential slip velocities for particles of different radii, and the resulting collision processes, have been examined in reference 4 and these results may be used in estimating the rate of droplet growth during acceleration of the mixture through the nozzle. Although our scanty knowledge of physical details does not allow a reliable absolute calculation of droplet growth, it is possible to estimate the order of magnitude, the dependence of the process upon chamber pressure, nozzle geometry, and other factors of influence in experiments.

If m denotes the mass of a liquid droplet having a radius σ, we consider the droplet mass spectrum $f(m)$ as continuous, and consequently

$$f(m)dm \tag{36}$$

is the number of droplets per unit volume having masses in the range m to $m+dm$. Now when the droplet sizes and flow conditions in the nozzle permit the approximate analytical treatment described earlier, the local velocity of the droplet of radius σ and mass m is just

$$u(m) = u^{(0)} - \lambda_v(m)M^{(0)}\frac{du^{(0)}}{dx} \tag{37}$$

where

$$\lambda_v(m) \equiv \frac{ma_c}{6\pi\sigma\mu_c} \tag{38}$$

is the velocity equilibration length for a droplet of mass m and radius σ. The quantities a_c and μ_c are the sound speed and viscosity coefficient at the chamber conditions. The mass flow of droplets in the range (m, dm) is

$$m\, f(m)dm \cdot u(m)A \tag{39}$$

where A is the local nozzle area.

Droplets in the range (m, dm) are produced and lost by collision; it will be assumed that each collision [4] results in the formation of a new droplet having a mass equal to the colliding masses. The number of collisions per unit volume between droplets of mass m_1 and mass m_2 is then

$$f(m_1)dm_1 \cdot f(m_2)dm_2 \cdot \pi(\sigma_1+\sigma_2)^2|u(m_1)-u(m_2)| \;, \tag{40}$$

and employing the approximate relation, equation (37), for particle velocity, the collision number is

$$M\frac{du}{dx} \cdot f(m_1)dm_1 \cdot f(m_2)dm_2 \cdot \pi(\sigma_1+\sigma_2)^2|\lambda(m_1)-\lambda(m_2)| \;, \tag{41}$$

where subscripts and superscripts have been dropped that are not needed. It is to be noted, in particular, that the factor $M(du/dx)$ is the only term relating to the nozzle flow process, and this is separable from those terms that pertain to droplet spectrum and droplet geometry.

A droplet is removed from the range (m, dm) with each collision involving a member of this set. The rate of droplet mass loss for a length dx of the nozzle is then

$$(AM \frac{du}{dx})dx \cdot mf(m)dm \int_0^{\infty} f(m')dm' \, \pi(\sigma+\sigma')^2 |\lambda(m)-\lambda(m')| \quad . \tag{42}$$

On the other hand, a droplet in the range (m, dm) is produced by a collision between droplets of mass m' and $m'' = (m-m')$. The production of these droplets in a length dx of the nozzle is

$$\frac{1}{2} (AM \frac{du}{dx})dx \cdot m \int_{m'+m''=m} f(m')dm' \cdot f(m'')dm'' \cdot \pi(\sigma'+\sigma'')^2 |\lambda(m')-\lambda(m'')| \tag{43}$$

where the factor of $\frac{1}{2}$ accounts for counting each collision twice when summing over all values of m', m''. The rate at which the mass flow of droplets in the range (m, dm) changes along the nozzle

$$\frac{d}{dx} \{mf(m)dmudA\} \tag{44}$$

is then given by the difference between the production integral, equation (43), and the loss integral, equation (42).

This spectral equation may be written most conveniently by introducing a new independent variable

$$\xi = m/m_c \tag{45}$$

and a new spectral function

$$\varphi = \frac{\rho_c}{\rho} \cdot \frac{m_c}{n_c} f(m) \tag{46}$$

where m_c, n_c are the average mass and number density of particles in the chamber; ρ_c is the gas density in the chamber. Moreover, if we define a thermodynamic variable

$$\eta = \int_0^M \frac{dM}{[1 + \frac{\bar{\gamma}-1}{2} M^2]^{\bar{\gamma}/(\bar{\gamma}-1)}} \tag{47}$$

and a new characteristic length

$$\ell = \frac{1}{\pi n_c \sigma_c^2} \quad , \tag{48}$$

the spectral equation may be written [5]

$$\frac{d}{d\eta}(\varphi d\xi) = \frac{1}{2}\frac{\lambda(m_c)}{\ell}\int_{\xi'=0}^{\xi} \varphi(\xi')d\xi' \cdot \varphi(\xi-\xi')d\xi \cdot g(\xi', \xi-\xi')$$

$$-\frac{\lambda(m_c)}{\ell}\int_{\xi'=0}^{\infty} \varphi(\xi)d\xi \cdot \varphi(\xi')d\xi' g(\xi, \xi') \ . \tag{49}$$

The function

$$g(\alpha, \beta) = (\alpha^{1/3}+\beta^{1/3})^2|\alpha^{2/3}-\beta^{2/3}| \tag{50}$$

arises from the term $(\sigma+\sigma')^2|\lambda(m)-\lambda(m')|$ after the appropriate change of variables. Accompanying this spectral equation is a statement that the total mass flow of liquid through the nozzle remains constant which, in terms of the new variables, is

$$\int_{\xi=0}^{\infty} \xi\varphi(\xi)d\xi = 1 \ . \tag{51}$$

Now the independent variable of the problem, η, is a thermodynamic quantity defined by the equilibrium flow of gas and droplets through the nozzle, and consequently, the change in the spectrum of droplet sizes from rocket chamber to any point in the nozzle depends only upon the change in thermodynamic state and not upon the configuration of the nozzle. Therefore, even without detailed calculations, it appears that the droplet growth will not depend upon the scale or shape of the nozzle but only upon the pressure or temperature ratio from the chamber. This result becomes obvious as a result of the approximate analysis and is valid so long as the small-slip approximation may be made.

Since it is our purpose to trace the general growth of particle size, it is sufficient to deal with the spectral mean droplet mass

$$\bar{\xi} \equiv \frac{\int_0^{\infty} \xi\varphi(\xi)d\xi}{\int_0^{\infty} \varphi(\xi)d\xi} \tag{52}$$

rather than with the details of the spectrum. Integrating the spectral equation (49) over the droplet mass and using the condition given by equation (51), we find that

$$\frac{d}{d\eta}\left(\frac{1}{\bar{\xi}}\right) = -\frac{1}{2}\frac{\lambda(m_c)}{\ell}\int_0^{\infty}\varphi(\xi)d\xi\int_0^{\infty}\varphi(\xi')d\xi' g(\xi, \xi') \ . \tag{53}$$

Since the result we desire, namely $\overline{\xi}$, is given in terms of an integral, this form is convenient for approximation and one has confidence that, because of the integrations involved, the result will not be sensitive to reasonable approximations in the dimensionless spectral function φ. To this end, a similarity solution has been investigated where the similarity variable is

$$\omega \equiv \xi/\overline{\xi} \tag{54}$$

and the spectral function is of the form

$$\varphi(\xi,\eta) = k(\overline{\xi})\cdot\psi(\omega) \ . \tag{55}$$

It transpires, reference 5, that

$$k(\overline{\xi}) = (\overline{\xi})^{-2} \tag{56}$$

and

$$\overline{\xi} = \left(1 - C\frac{\lambda(m_c)}{\ell}\eta\right)^{-3} \tag{57}$$

where $C = 0.131$ as a reasonable estimate.

In particular, we find that the ratio of the mean droplet radius at any point to the mean droplet radius in the chamber is

$$\frac{\overline{\sigma}}{\sigma_c} \equiv \overline{\xi}^{1/3} = \frac{1}{1 - 0.131\dfrac{\lambda(m_c)}{\ell}\eta} \ . \tag{58}$$

Now the variable η is shown in Figure 5 as a function of the local Mach number or of the local gas temperature. The ratio $\lambda(m_c)/\ell$ is

$$\frac{\lambda(m_c)}{\ell} = \frac{1}{6}\frac{\rho_c a_c \sigma_c}{\mu_c}\cdot \varkappa \ . \tag{59}$$

For a fixed chamber temperature, $\lambda(m_c)/\ell$ increases linearly with the chamber pressure and with the initial droplet radius.

The mean particle radius that is to be observed in experiments is determined by the value of η at which the droplets solidify and the value of $\lambda(m_c)/\ell$ determined from the initial state of the mixture in the chamber. In the case of aluminum oxide, solidification occurs at a value $T/T_c = 0.667$, corresponding to the value of $\eta \equiv \eta_s \cong 1.0$ at solidification. The value of η_s is insensitive to moderate variations of solidification temperature and chamber temperature. Moreover, it appears that η_s is insensitive to a particle temperature lag of several hundred degrees. Hence, for nozzles carrying aluminized propellants, $\eta_s \cong 1.0$ is a rather good approximation.

As a consequence, the details of the observed particle sizes must depend only upon the ratio $\lambda(m_c)/\ell$. In particular, this quantity is directly proportional to the chamber pressure p_c, so that it appears that, as the chamber pressure rises, the final droplet radius increases also, in some cases quite rapidly. Since there is no adequate knowledge of initial droplet radii, equation (58) cannot be treated quantitatively. If we accept the assumption that the collision accommodation coefficient is unity, then the pressure dependence of $\bar{\sigma}$ may be employed to estimate the value of σ_c that is appropriate. Based on the data from reference 6 at 100 psi and 500 psi, equation (58) describes the variation adequately for $\sigma_c \sim 0.05$ microns. It may or may not be relevant that experimental results often show a background of residual particles having a radius less than 0.1 microns. It is probable that, when equation (58) leads to very large droplet radii, as it easily may for appropriate values of the parameters, the final droplet radius may be determined by hydrodynamic instability of the liquid, a value that will not be independent of nozzle geometry.

Concluding Remarks

In reviewing the foundations and some applications of the approximate calculation of two-phase flow in rocket nozzles, it is shown that the aim has been to obtain simple, useful analytic results, that contain the significant quantities of the problem and provide a physical insight into the true mechanism. The emphasis is upon the manner in which the desired results vary with the physical parameters of the problem rather than striving for detail that may not be warranted by the accuracy of data necessary for its solution. It appears that the areas where approximate analytical results provide convenient performance trends and scaling laws have yet to be used to maximum advantage.

Much of the inaccuracy attributed to these approximate calculations may be avoided by inverting the algebraic equations

$$\frac{u_s^{(1)}}{a^{(0)}}\,\alpha\left(\frac{u_s^{(1)}}{a^{(0)}},\ k\right) = -\frac{\lambda_v}{L}\,\frac{1}{\bar{\gamma}}\,\frac{1}{p}\,\frac{dp}{d\xi}$$

$$\frac{T_s^{(1)}}{T^{(0)}}\,\beta\left(\frac{u_s^{(1)}}{a^{(0)}},\ k\right) = \frac{\lambda_T}{L}\,\frac{\bar{\gamma}-1}{\bar{\gamma}}\,\frac{c}{\bar{c}_p}\,\frac{1}{p}\,\frac{dp}{d\xi}$$

exactly for $u_s^{(1)}$ and $T_s^{(1)}$ rather than employing their low Reynolds number counterparts, equations (23) and (24). We emphasize that this algebraic step does not alter the ease with which subsequent results may be obtained; the approximations are valid under the conditions stated regardless of the drag and heat transfer laws. Although this fact was first indicated by Rannie [1] more than five years ago, it does not seem to have been employed by the practitioners in the field.

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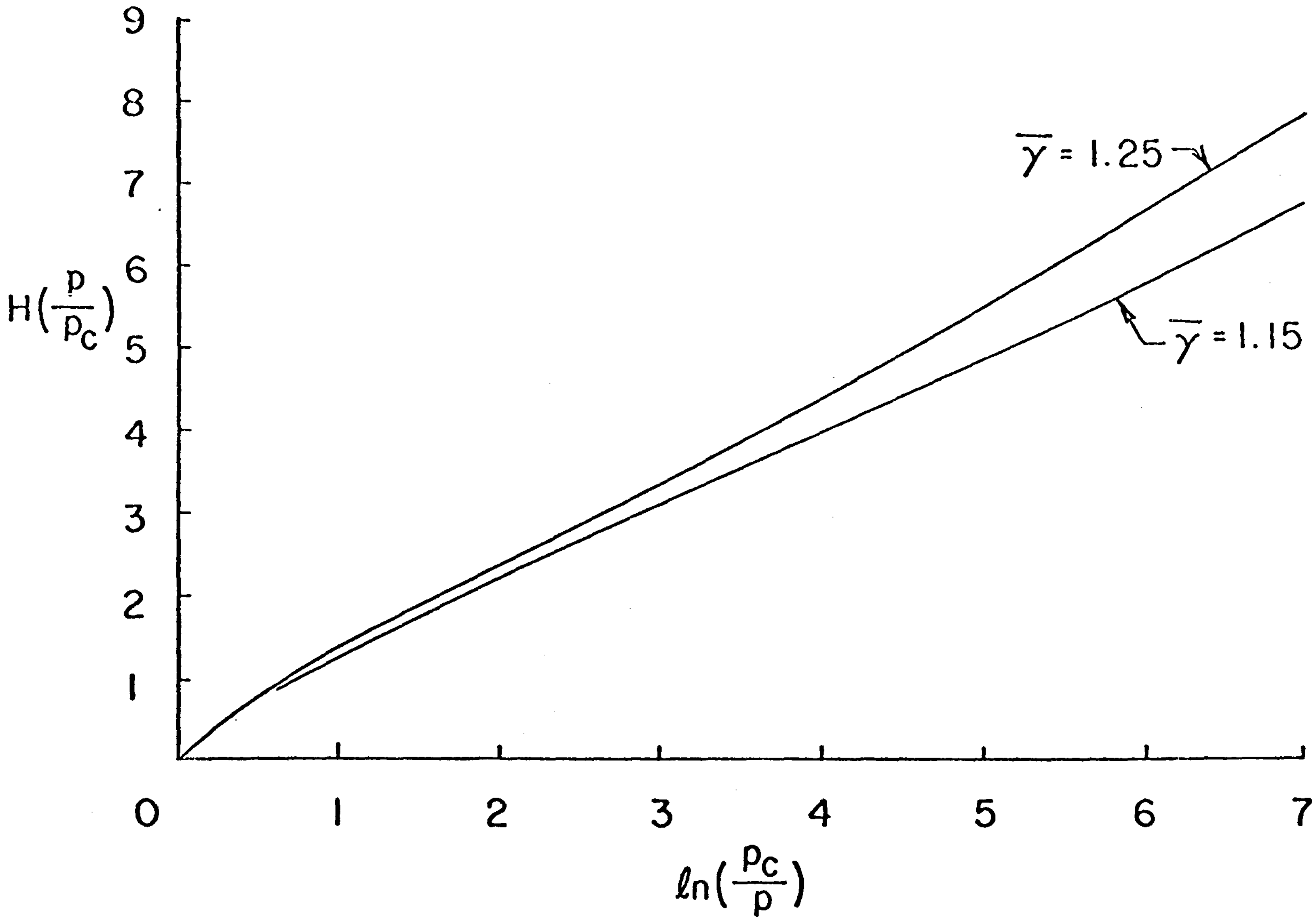

Figure 1. Values of $H(p/p_c)$ as a Function of Pressure Ratio.

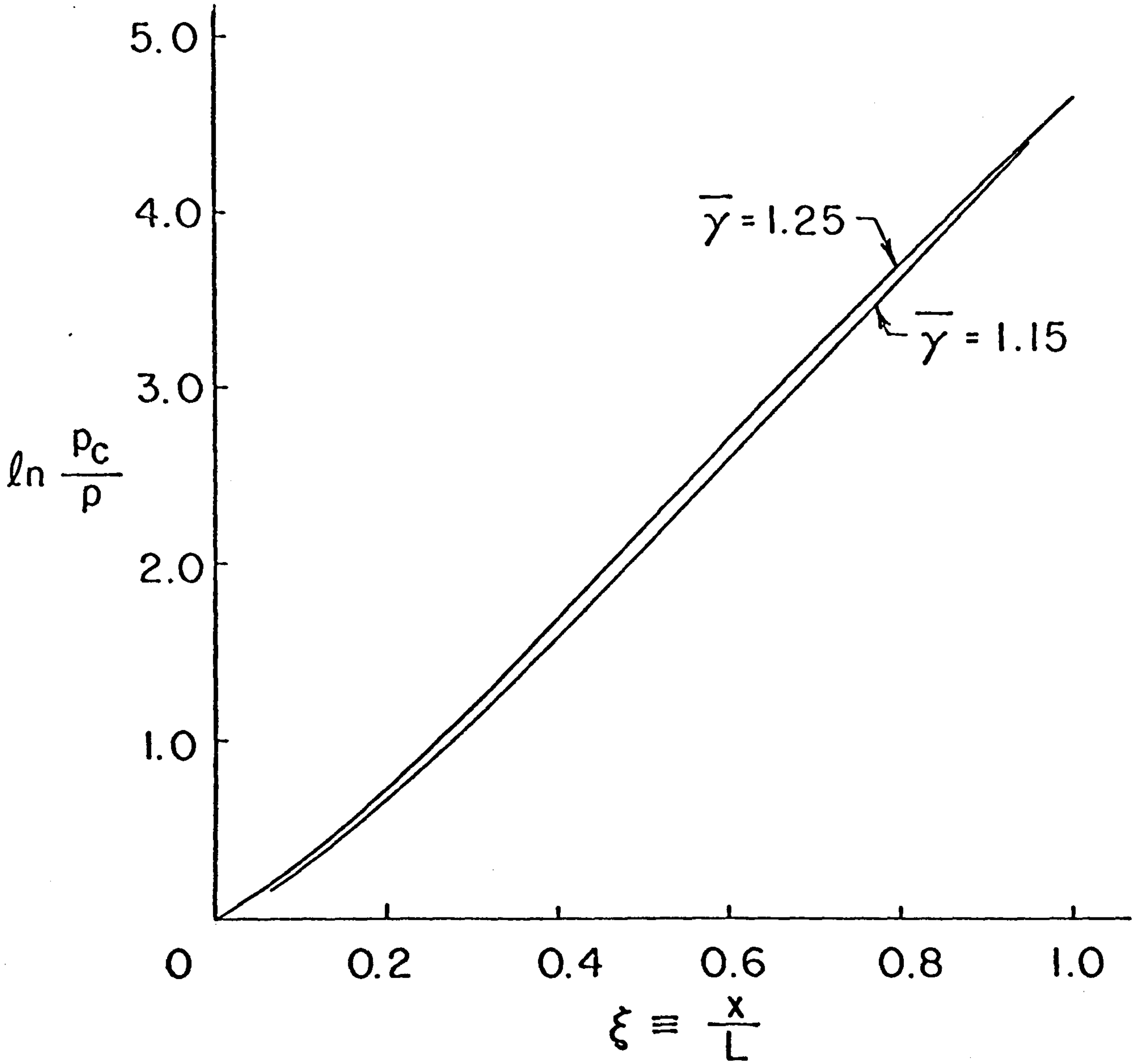

Figure 2. Pressure Distribution Along Nozzle of Optimum Contour.

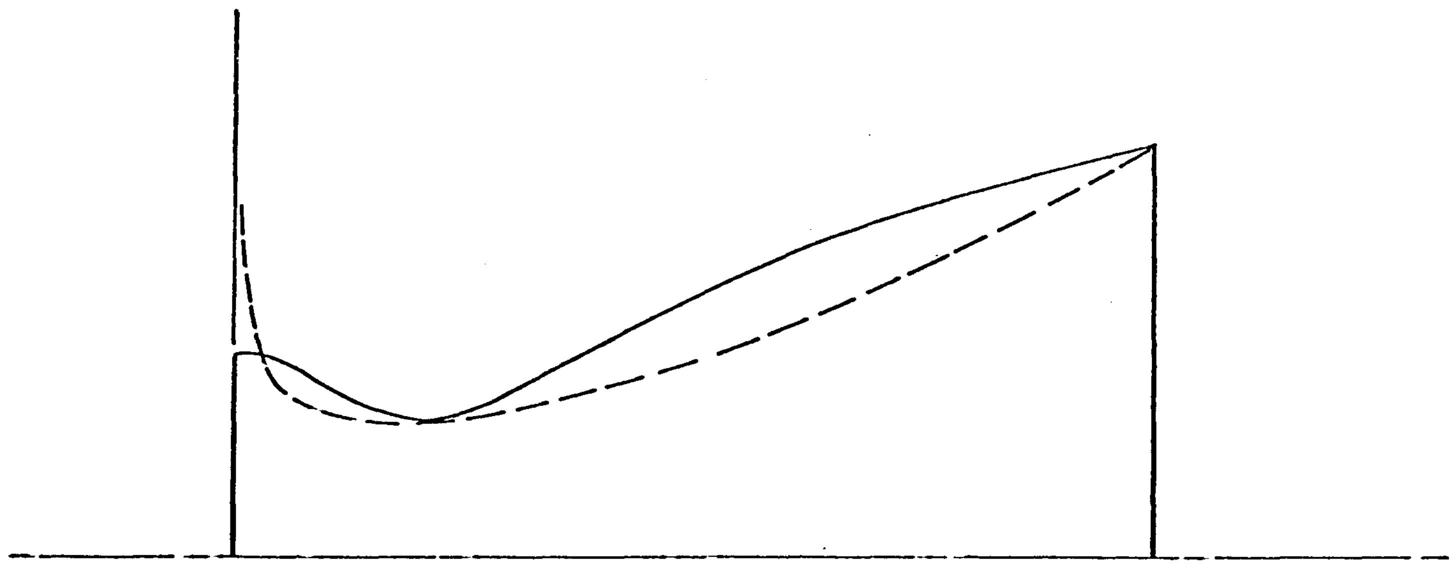

Figure 3. Comparison of Optimum Nozzle Contour with That of Conventional Nozzle.

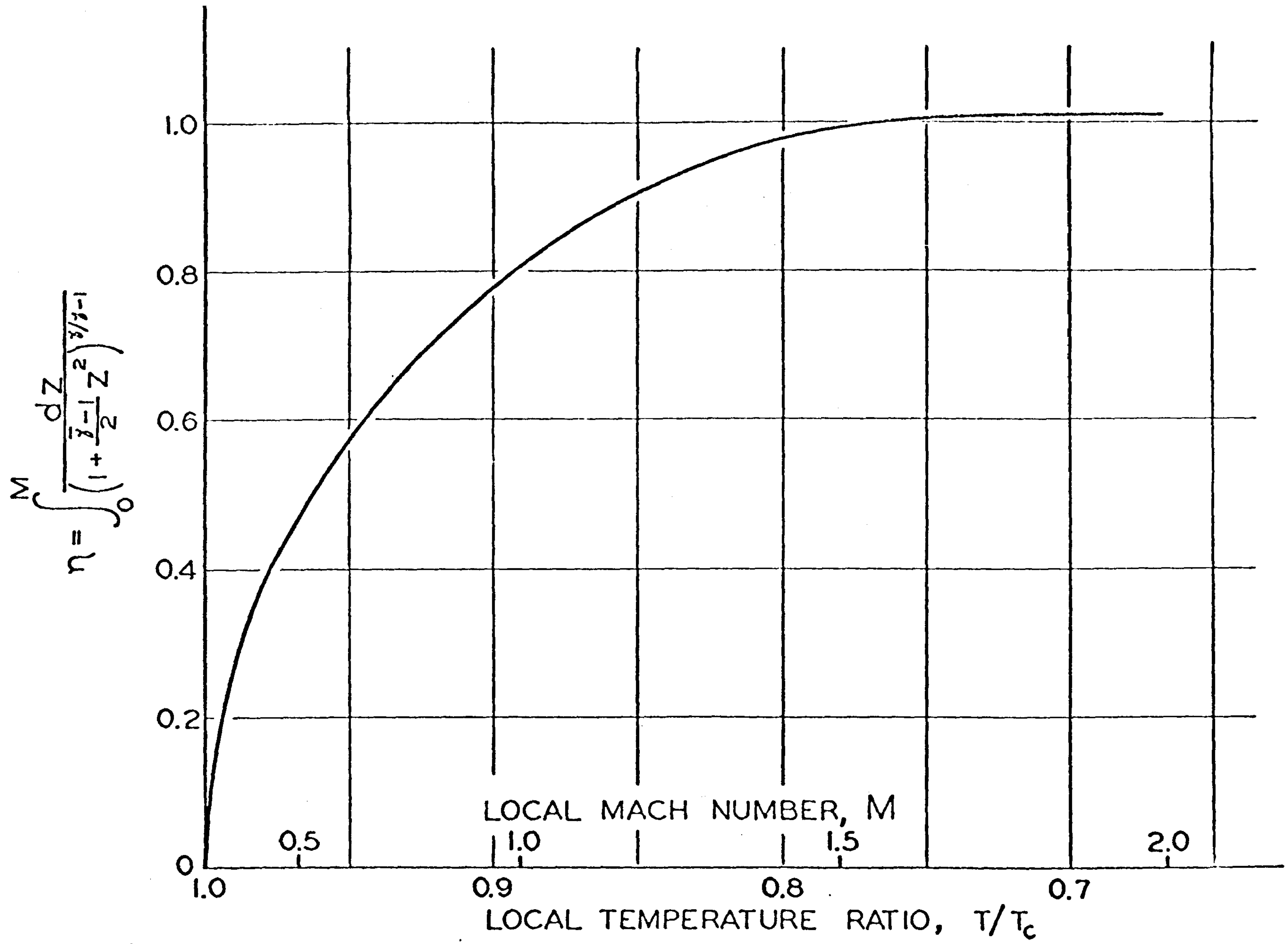

Figure 5. Values of State Variable η as a Function of Local Nozzle Temperature Ratio or of Local Nozzle Mach Number.