Stability of High-Frequency Pressure Oscillations in Rocket Combustion Chambers

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The problem of determining the stability of high-frequency pressure oscillations in rocket combustion chambers is treated explicitly as a perturbation of the classical acoustics problem. On the basis of previous experimental results, the energy addition by combustion is emphasized, and an analysis is developed for the stability of stationary pressure waves in chambers using either liquid or gaseous propellants. The formulation is for the three-dimensional case; computations are carried out in detail for a cylindrical chamber in which the mean flow velocity is parallel to the axis and varies only with axial position. The principal result is a formula for the imaginary part of the complex frequency associated with each of the natural modes of the chamber. This yields a single dimensionless group as a measure of the stability of individual chamber modes. Because of the present lack of sufficiently detailed experimental evidence, quantitative interpretation seems impossible, but qualitative agreement with observations can be demonstrated.

Introduction

THE observed instabilities that are grouped under the general label "combustion instability" in rocket motors have been found usually to be of several different kinds. When the frequency is several hundred cycles per second or less, a low-frequency instability, the cyclic variations appear in both the chamber and propellant feed system. The pressure is, to good approximation, in-phase at all points of the chamber; the cause seems to be related to coupling between oscillations in the injection rate, chamber pressure, and burning.[1] A second kind, of much less importance, occasionally has been observed in a somewhat higher range, although below the high frequency oscillations. It has been proposed that oscillations in the mixture ratio may cause this instability by generating entropy and pressure disturbances that are carried by the mean flow.[1,2]

The subject of the present investigation is the high-frequency instability; useful summaries may be found in Refs. 1, 3, and 4. It was recognized from early measurements of frequencies that the problem may be connected with the excitation of acoustic modes by the combustion process. This conclusion is only a beginning step, for it yields no information concerning possible remedies or (what is almost equivalent) how the oscillations are excited and sustained. Significantly, however, it has been found in both laboratory and large-scale devices that the injection system has a great deal of influence on the appearance, or absence, of oscillations in liquid propellant motors. In addition to the presence of combustion within the chamber, a characteristic feature of a rocket motor is the discharge nozzle, usually a converging-diverging design. One would expect that this should have some noticeable effect on whatever waves exist in the chamber. This has been found, at least qualitatively, to be the case.

Recently, rather extensive laboratory tests[5–8] have been conducted using premixed gaseous propellants. These results afford interesting comparisons with those obtained with liquid propellants. The processes prerequisite to the burning of unmixed liquid propellants (atomization, evaporation, etc.) are eliminated. Furthermore, combustion can contribute no net mass to the gaseous phase when the reactants and products are gases at all times. Yet high-frequency instability has been observed with gaseous propell nts to have much the same character as when the propellants are initially in liquid form. One then may draw, at least tentatively, two important conclusions: 1) the presence of liquid drops per se in the chamber is, in some sense, of distinctly minor importance; and 2) it is largely the energy addition, rather than possible mass addition during combustion, which sustains oscillations.

It is partly with these conclusions as a basis that the work covered here differs from the studies by Crocco and Cheng[1] on the one-dimensional problem and by Scala[2] on the axisymmetric problem. By neglecting the effects of the liquid phase in the momentum equation written for the gas phase (a similar simplification also has been used in more recent work by Crocco[9]) and by emphasizing the energy additions, one may deduce a single equation for pressure fluctuations. This differential equation then can be converted to an integral equation that may be solved approximately by a known perturbation-iteration technique. The mechanism for instability is supposed, as in the references cited, to reside in pressure-combustion interaction. Since the representation of this coupling is crucial in the stability problem, it is a fortunate circumstance that in the final results the influence of combustion is separated from other effects. Consequently, different representations may be studied with equal ease.

There are necessarily many approximations, assumptions, and suppositions associated with the description of such a complicated problem. Certain of the choices made here, therefore, may be less valid in some particular cases. It appears that the formal structure is sufficiently flexible to accommodate different special cases and related problems. For example, no mention will be made of ramjet combustion chambers or turbojet afterburners, both of which often exhibit a high-frequency instability. It is very probable that by using the proper boundary conditions one may treat those problems in much the same manner as a rocket chamber.

Governing Equations and Boundary Conditions

No attempt is made here to treat any of the nonlinear aspects of the problem. Thus, one really is investigating a question of incipient instability, and the results should not

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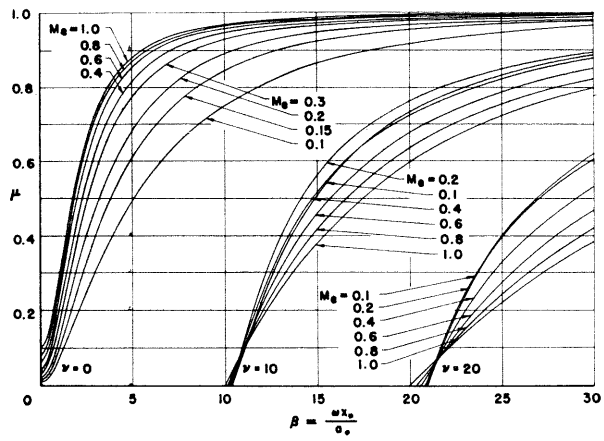

Fig. 1 Real part of nozzle admittance as a function of β for several values of ν ($\gamma = 1.2$)

be expected to apply to situations in which the pressure fluctuations are "large." This approach is justified by the usual argument that at least some of the observed large-amplitude motions may originate as unstable small disturbances suitably described by linear equations. The acoustic equations therefore must result, with additional terms, depending linearly on the fluctuation quantities, and with source terms representing the combustion process.

Experiments surely indicate that there are standing† or travelling pressure waves in the chamber, rather like acoustic or sound waves. Because of this "wavelike" character, one should seek a set of equations of a predominantly hyperbolic type, appropriate to a continuous medium. The medium, of course, comprises the gases in the chamber, whether or not there are liquid drops of propellant as well. Further, it is appropriate to smooth out whatever disorganized motions may be present in the chamber into a "mean flow" on which are superposed the fluctuations associated with the wave motion; a similar view has been taken previously.[1] The chamber gases are lumped together as a single average species whose specific heats and gas constant will be assumed constant always. The equation of state is taken to be the perfect gas law. After neglecting the effects of liquid drops in the momentum equation,‡ thus invoking the first of the two conclusions previously mentioned, one eventually can write as the three conservation equations for the gaseous phase:

Conservation of Mass

$$(\partial\rho/\partial t) + \nabla\cdot(\rho\mathbf{u}) = W_m \tag{1}$$

Conservation of Momentum

$$(\partial\mathbf{u}/\partial t) + (\mathbf{u}\cdot\nabla)\mathbf{u} + (1/\rho)\nabla p = 0 \tag{2}$$

Conservation of Energy

$$\rho(\partial e/\partial t) + \rho(\mathbf{u}\cdot\nabla)e + p\nabla\cdot\mathbf{u} = W_c - eW_m \tag{3}$$

It has been assumed that in the coefficient of W_m in (3) the kinetic energy $u^2/2$ (per unit mass) of the chamber gases is negligibly small compared with the thermodynamic energy. The rate of mass addition by combustion per unit volume is W_m, and W_c is the net heat addition, also per unit volume and time, due to all chemical reactions. The thermodynamic internal energy of the gas phase is $e = \int C_v dT$. Since W_c represents all the energy released in combustion, it includes the thermodynamic energy of the liquid in the case of liquid propellants. Thus the term $-eW_m$ represents (approximately) the thermodynamic energy "absorbed" by the product gases associated with combustion of the liquid.

These equations are applicable, by proper interpretation, to both gas and liquid propellant chambers. To deduce the equations governing the wave motion, the flow as usual is treated as a mean steady flow plus a perturbation flow. For simplicity, it is convenient also to assume that $\bar{\rho},\bar{e},\bar{p}$ are approximately uniform§ in the chamber and that the mean flow Mach number is "small" though variable. These appear to be realistic assumptions and have been invoked previously.[1, 2] They can be relaxed within the present analysis, but only at the expense of added complications that do not seem necessary to determine the essential features of the problem. By setting $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$, etc., and assuming that all products of perturbation quantities are negligible, one finds within the assumptions just mentioned that the equations describing the disturbance flow are

$$(\partial\rho'/\partial t) + \bar{\rho}\nabla\cdot\mathbf{u}' = -\nabla\cdot(\rho'\bar{\mathbf{u}}) + W_m' \tag{4}$$

$$\bar{\rho}(\partial\mathbf{u}'/\partial t) + \nabla p' = -\bar{\rho}[(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}'] \tag{5}$$

$$\bar{\rho}(\partial e'/\partial t) + \bar{p}\nabla\cdot\mathbf{u}' = -\nabla\cdot(\bar{\rho}\bar{\mathbf{u}}e') - p'\nabla\cdot\bar{\mathbf{u}} + W_c' - \bar{e}W_m' \tag{6}$$

$$\frac{p'}{\bar{p}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}} = \frac{\rho'}{\bar{\rho}} + \frac{e'}{\bar{e}} \tag{7}$$

Equation (7) comes from the perfect gas law. After dividing Eq. (4) by $\bar{\rho}$, Eq. (6) by $\bar{\rho}\bar{e}$, adding the results, and taking account of (7), one has an equation for $\eta = p'/\bar{p}$:

$$\frac{\partial\eta}{\partial t} + \gamma\nabla\cdot\mathbf{u}' = \frac{W_c'}{\bar{\rho}\bar{e}} - \gamma\eta(\nabla\cdot\bar{\mathbf{u}}') - (\bar{\mathbf{u}}\cdot\nabla)\eta \tag{8}$$

Equation (5) can be rewritten as

$$\frac{\partial\mathbf{u}'}{\partial t} + \frac{a^2}{\gamma}\nabla\eta = -[(\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}} + (\mathbf{u}\cdot\nabla)\mathbf{u}'] \tag{9}$$

where $a^2 = \gamma\bar{p}/\bar{\rho}$ is the usual sound propagation speed.

The result is that Eqs. (8) and (9) are two equations in the unknowns η and $\mathbf{u}'$. The left-hand sides of these equations

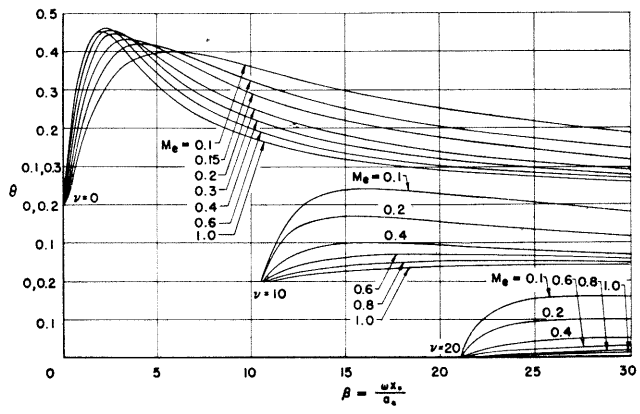

Fig. 2 Imaginary part of nozzle admittance as a function for several values of ν ($\gamma = 1.2$)

† Because, in part, of the energy losses through the chamber exit plane, the modes of oscillation to be treated here are not truly standing waves in the strictest sense. Thus, although "standing wave" is a convenient term, its use in the present work should be accepted with some reservation.

‡ This is an assumption that is not essential and is not always correct, but it is adopted here to simplify the calculations. The term dropped from Eq. (2) has the form $(W_m + c\rho_l)(u - u_l)$, with c a constant, and represents two effects: deceleration of the liquid phase and deceleration of the vaporized gases. If this term is retained (it vanishes strictly only if $\rho_l = 0$) the right-hand sides of Eqs. (5) and (9) are altered, and the net result is an added stabilizing effect [Secs. 3.07 and 3.12 of Ref. 1].

§ If combustion in the gas phase takes place immediately after vaporization, this assumption is satisfied; it is more nearly correct the shorter the time of energy release compared with the wave propagation time.[19]

define the usual acoustics problem; the inhomogeneous terms represent effects of the mean flow $\bar{\mathbf{u}}$ (largely gross convection) and energy generated by combustion. It appears, therefore, that the mathematical formulation presents the qualitative aspects that one might consider to be important. Unless a single equation (in this instance, for η) can be deduced, very cumbersome difficulties are encountered. The coupling between these equations, in the sense that one is prevented from eliminating $\mathbf{u}'$ easily, arises because $\bar{\mathbf{u}}$ varies with position. For $M = |\bar{\mathbf{u}}|/a$ small, however, this coupling is weak, and a single equation for η can be obtained to some approximation.

Since only motions having exponential time-dependence will be treated here, $\partial/\partial t$ can be replaced by iak and the momentum equation (9) solved for $\mathbf{u}'$:

$$\mathbf{u}' \cong -\frac{a}{i\gamma k}\nabla\eta - \frac{1}{\gamma k^2}[(\nabla\eta.\nabla)\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\nabla\eta] \quad (10)$$

in which, since M hereafter will be assumed "small," the usual acoustics formula for particle velocity has been inserted in those terms involving $\bar{\mathbf{u}}$. The complex wave number is $k = (\omega + i\lambda)/a$. Then to the same approximation, the equation for η alone can be obtained by differentiating (8) with respect to time and inserting (10):

$$(\nabla^2 + k^2)\eta = h(\mathbf{r},\eta) \quad (11)$$

$$h(\mathbf{r},\eta) = -i\frac{k}{a}\frac{W_c'}{\bar{\rho}\bar{e}} + i\frac{k}{a}[(\bar{\mathbf{u}}\cdot\nabla)\eta + \gamma\eta(\nabla.\bar{\mathbf{u}})] - \frac{i}{ak}\nabla\cdot[(\bar{\mathbf{u}}\cdot\nabla)\nabla\eta + (\nabla\eta.\nabla)\bar{\mathbf{u}}] \quad (12)$$

Some higher order terms have been neglected in Eq. (12). Note that $\mathbf{u}'$ and η must be treated as complex functions of k and position in the chamber. Equation (11) is an "inhomogeneous" wave equation in the sense that the function h contains perturbations of the usual acoustics problem; the first term is proportional to the fluctuations in energy released and available for driving the pressure waves η, whereas the remaining terms represent the effects of the mean gas motion. Boundary conditions must be set in accordance with Eq. (10).

The closed end of the chamber will, for simplicity, be taken to be flat and rigid to wave reflection. Only a circular chamber of radius R with rigid walls will be considered; the equations just deduced, of course, are not restricted to a particular shape of chamber. Thus, with the axial coordinate measured from the closed end, two boundary conditions follow from (10) when the velocity normal to both the closed end and the side wall must vanish:

$$\begin{aligned} \partial\eta/\partial z &= 0 \qquad z = 0 \\ \partial\eta/\partial r &= 0 \qquad r = R \end{aligned} \quad (13)$$

It is supposed further that the nozzle entrance area is equal to the chamber cross-sectional area (see remark below). It will be assumed that at the chamber exit plane, $z = L$, the mean flow velocity is uniform and axial only, with magnitude u_e; then if combustion is completed within $z < L$, $du_e/dz \cong 0$ at $z = L$. Generally, $\mathbf{u}'$, η, $\partial\eta/\partial z$, etc., all have nonvanishing values at $z = L$. It happens that the most convenient way to specify the boundary condition is to relate the pressure disturbance to the axial component of the velocity fluctuation through an admittance function A:

$$A = \gamma u_e'/\eta_e \quad (14)$$

where u_e' is the axial component of $\mathbf{u}'$ at $z = L$. In view of the foregoing remarks concerning the mean flow velocity, substitution of u_e' from (14) into the axial part of (10) evaluated at $z = L$ gives

$$\frac{1}{\eta}\frac{\partial\eta}{\partial z} = -ik\left[\frac{A}{a} + \frac{M_e}{a}\frac{1}{\eta}\frac{\partial^2\eta}{\partial z^2}\right] \qquad z = L \quad (15)$$

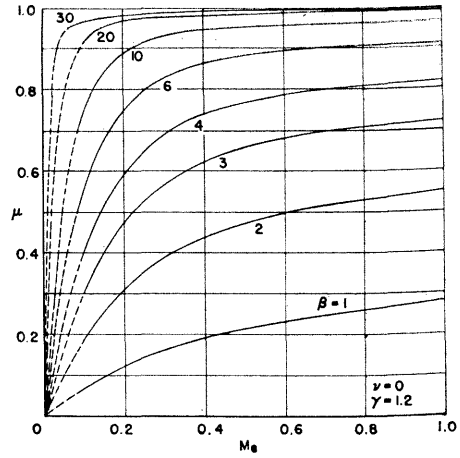

Fig. 3 Real part of nozzle admittance as a function of chamber exit Mach number for several values of β ($\nu = 0$)

The quantity A depends upon, among other things, the frequency of oscillations and the geometry of the nozzle. Calculation of the admittance function is a difficult problem and can be achieved only by considering the behavior of small amplitude waves in a nozzle with variable mean flow temperature, density, and velocity up to a sonic throat. An approximate solution has been discussed[10] for slender nozzles in which the mean flow varies linearly; the analysis is based partly on earlier treatments of the one-dimensional problem by Tsien and Crocco, but it leads to results particularly suited to the present treatment of the axisymmetric chamber problem. The proper boundary condition along the nozzle wall can be satisfied by using nonorthogonal coordinates defined in such a way that one coordinate surface coincides with the nozzle. Details of this procedure will not be presented here; some typical numerical results are shown in Figs. 1–4 taken from Ref. 10. The real and imaginary parts of A/a are μ and θ, respectively. The parameter $\beta = x_*\omega/a_*$; a_* is the speed of sound at the throat, and x_* is defined by $z/x_* = u/a_*$, where $u(z)$ is the (axial) mean flow speed in the nozzle. The mode considered is identified by $\nu = x_*\kappa_{mn}$, with κ_{mn} defined by Eq. (24).

The curves for $\nu = 0$ in the figures shown agree with the calculations carried out by Crocco[1] for the one-dimensional case. It should be noted that $\beta/\nu = \omega/\kappa_{mn}a_*$, and, by using

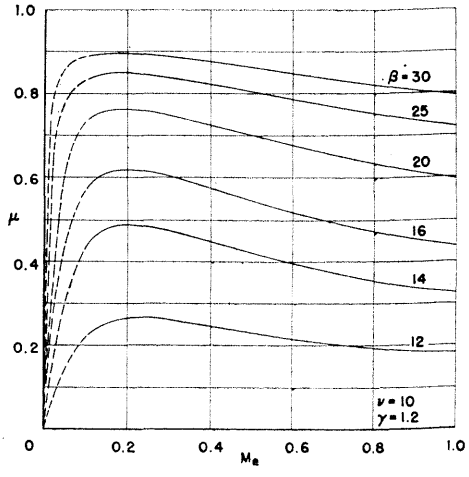

Fig. 4 Real part of nozzle admittance as a function of chamber exit Mach number for several values of β ($\nu = 10$)

later results for ω, it can be shown that, in any practical case, $\beta > \nu$ by an amount sufficiently large that μ and θ always must be positive.

Approximate Solution for Mode Shapes and Eigenvalues

Exact solution to Eq. (11) for η does not appear possible, but a rather simple iterative procedure can be constructed by first defining Green's function $G(r,\phi,z|r_0,\phi_0,z_0)$ satisfying the equation

$$(\nabla^2 + k^2)G(\mathbf{r}|\mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0) \tag{16}$$

$\delta(\mathbf{r} - \mathbf{r}_0)$ is the delta function in three dimensions. Many of the following operations used to gain approximate solution are discussed at length in Ref. 11. An integral equation for η can be found by multiplying Eq. (11) by G, Eq. (16) by η, subtracting the two equations, and integrating over the volume of the chamber. After application of Green's theorem, one finds the desired formal result:

$$\eta(\mathbf{r}) = \int\int\int G(\mathbf{r}|\mathbf{r}_0)h(\mathbf{r}_0,\eta)dV_0 \tag{17}$$

where G must satisfy the same boundary conditions as does η, and the region of integration includes no part of the nozzle. Because of the term involving $\eta^{-1}\partial^2\eta/\partial z^2$ in (15), there is some difficulty with the boundary condition at $z = L$. However, to first order, η has the form of (19) (see subsequent discussion) so that the noted term is then independent of $\mathbf{r}$. Thus the difficulty can be overcome approximately, and $\eta^{-1}\partial\eta/\partial z$ and $G^{-1}\partial G/\partial z$ may be set equal to the same function, namely, the right side of (23).

After some labor (see Ref. 10), the function G (which is the solution for a unit harmonic source placed at $\mathbf{r}_0$ and subject to the proper boundary conditions) can be expanded in a series of eigensolutions for Eq. (11) with $h = 0$:

$$G(\mathbf{r}|\mathbf{r}_0) = \sum_{\substack{q,s=0\\ p=-\infty}}^{\infty} \frac{\epsilon_p}{E_{pqs}^2} \frac{\psi_{pqs}(\mathbf{r})\psi_{pqs}(\mathbf{r}_0)}{(k^2 - k_{pqs}^2)} \tag{18}$$

where

$$\psi_{pqs}(\mathbf{r}) = \cos(\beta_s z)J_p(\kappa_{qs}r) \begin{cases}\sin(p\phi) & p > 0\\ \cos(p\phi) & p \leqslant 0\end{cases} \tag{19}$$

$$\int_0^{2\pi}\int_0^R\int_0^L \psi_{pqs}^2\, dzrdrd\phi = \frac{E_{pqs}^2}{\epsilon_p} \qquad \epsilon_p = \begin{cases}1 & p = 0\\ 2 & p \neq 0\end{cases} \tag{20}$$

$$E_{pqs}^2 = \frac{\pi R^2 L}{2}\left(1 - \frac{p^2}{R^2\kappa_{pq}^2}\right)[J_p(\kappa_{pq}R)]^2\left[1 + \frac{\sin(2\beta_s L)}{2\beta_s L}\right] \tag{21}$$

$$k_{pqs}^2 = \kappa_{pq}^2 + \beta_s^2 \tag{22}$$

The β_s are the roots, for fixed k and K_s, of

$$\beta_s \tanh(2\beta_s L) = -k[(A/a) - M_e(K_s/k)^2] \tag{23}$$

and the K_s are the roots of the same equation but with k^2 related to K_s^2 by $k^2 = \kappa_{pq}^2 + K_s^2$ and with K_s in place of β_s. In all cases A/a must be treated as a complex function of the complex variable k. Finally, the κ_{pq} are the roots of

$$(d/dr)\{J_p(\kappa_{pq}r)\} = 0 \qquad r = R \tag{24}$$

These complications with β_s, K_s, etc., arise because the boundary condition (15) prevents easy construction of a set of orthogonal eigensolutions of the homogeneous problem for Eq. (11) independently of k_{pqs}. Only certain results have been outlined here; further details may be found in Chap. 11 of Ref. 11 and in Ref. 10.

Since $A/a, k, k_{pqs}, K_s$, and β_s are in general complex, it is extremely tedious to extract numerical results according to this procedure. Of these quantities, k^2, given in Eq. (25) for the complete problem, is required ultimately for determining the stability characteristics; in the next section, a special approximate expression for k_{pqs}^2 will be given.

The important point is that it is a simple matter to solve Eq. (17) approximately by successive substitutions, and, by using the expansion (18), one obtains simultaneously an expression for the complex wave numbers of the standing wave modes. It should be noted that, no matter what (reasonable) function is substituted for η in h under the integral, the result given by (17) for η necessarily satisfies the correct boundary conditions. A sufficiently good approximation within the assumptions outlined previously is to set $\eta = \psi_{mnl}$ under the integral; the indices mnl will be used to denote a particular mode.

It is presumed that the problem to be solved when h is nonzero is truly a perturbation of the exactly solvable problem for $h = 0$ in the sense that it is possible to proceed continuously from one to the other by appropriately varying h. The solutions η in both the perturbed and unperturbed problems satisfy the same boundary conditions (13) and (15). Useful information therefore may be gained by separating off the mode ψ_N to which η tends when h tends to zero; ψ_N may, of course, be any one of the "natural" modes. Substitution of (18) into (17) then gives

$$\eta_N(\mathbf{r}) = \frac{\epsilon_N\psi_N(\mathbf{r})}{E_N^2(k^2 - k_N^2)}\int\int\int \psi_N(\mathbf{r}_0)h(\mathbf{r}_0)dV_0 + \sum_{\alpha\neq N}\frac{\epsilon_\alpha\psi_\alpha(\mathbf{r})}{E\alpha^2(k^2 - k\alpha^2)}\int\int\int \psi_\alpha(\mathbf{r}_0)h(\mathbf{r}_0)dV_0$$

The indices $\alpha \equiv pqs$ and $N \equiv mnl$ are abbreviations for the three-tuple identification just used. For $h \to 0$, the coefficient of ψ_N is finite ($k \to k_N$), and the normalization can be chosen at will. It is convenient to perturb from ψ_N:

$$\eta_N = \psi_N + \sum_{\alpha\neq N}\frac{\epsilon_\alpha\,\psi_\alpha}{E_\alpha^2(k^2 - k_\alpha^2)}\int\int\int \psi_\alpha h dV_0$$

so that the normalization leads to the important requirement

$$k^2 \cong k_N^2 + (\epsilon_N/E_N^2)\int\int\int \psi_N(\mathbf{r}_0)h(\mathbf{r}_0,\psi_N)dV_0 \tag{25}$$

Note that the approximation $\eta \approx \psi_N$ has been made in $h(\mathbf{r},\eta)$. Higher approximations may be found by successive iterations in the foregoing expression for η, but formula (25) is sufficiently accurate, providing M and all fluctuations are small. Here, k is the complex wave number for the complete problem; k_N is the complex wave number for $h = 0$ but with the proper boundary conditions so that it contains all effects of the discharge nozzle. Equation (25) contains all the information one can gain from this analysis of standing pressure waves. It remains only to evaluate the integral and to find an approximate expression for $k_N = k_{mnl}$.

If a small quantity, say, ϵ, is introduced as a measure of the amplitude of the pressure oscillations, the argument leading to Eq. (25) can be made somewhat more satisfying in a formal sense. For then, if one assumes that W_e' is of order‖ ϵ (i.e., W_e' proportional to η as later remarks indicate), it is a straightforward matter to determine (25) as the first-order (in ϵ) approximation to k^2.

The preceding discussion has been based in part on the assumption that the area of the nozzle entrance is equal to the cross-sectional area of the chamber. This, however, is not always the case in large scale devices or in laboratory experiments.[5–8] The difference obviously will have some influence on the frequencies and damping of oscillations in the chamber. Such an effect can be incorporated largely in k_N and can be treated approximately by simple modifications of the forementioned procedure.[10] As one would expect, when the nozzle area is smaller than the chamber cross section, the

‖ The assumption is implicit in the steps leading to Eqs. (10) and (11).

frequencies of oscillation are increased and the damping effect of the nozzle is decreased, relative to the case treated here.

Calculation of k and a Stability Criterion

As the problem has been formulated here, there are three main features. The cause of any instability must be related to the energy released by combustion; this may be compensated by the damping effects of the discharge nozzle and by convection of energy from the chamber by the mean flow. Since the formal representation of the pressure oscillations contains a factor e^{iakt} stability of a particular mode is attained when the imaginary part of the corresponding k is greater than zero. Often in previous studies the imaginary part has been set equal to zero, thus restricting subsequent calculations to the "stability boundary." The principal result obtained here is a formula for this quantity, valid near as well as at the stability boundary. Furthermore, as one would expect from a linear approximation, the main effects of energy release, mean gas motion, and the discharge nozzle are separated.

The integral in (25) can be evaluated most easily if the mean flow velocity is assumed to be axial and a function of z only. Departures from this approximation mainly distort the waves in the chamber, an effect that should not influence materially the conclusions reached below. Then the separation just noted may be exhibited explicitly in k^2:

$$k^2 = k_N^2 + \frac{\epsilon_N}{E_N^2} L_N - \frac{\epsilon_N}{a^2 E_N^2} \chi_N \tag{26}$$

$$L_N = ik \iiint \psi_N \left\{ \left(2M - \frac{1}{k^2} \frac{d^2M}{dz^2} \right) \frac{\partial \psi_N}{\partial z} + \frac{dM}{dz} \left(\gamma \psi_N - \frac{2}{k^2} \frac{\partial^2 \psi_N}{\partial z^2} \right) \right\} dV_0 \tag{27}$$

$$\chi_N = \iiint \psi_N \frac{D}{Dt} \left(\frac{W_c'}{\bar{\rho}\bar{e}} \right) dV_0 \tag{28}$$

where $D/Dt = iak + \bar{u}\partial/\partial z$, and $\bar{u}(z)$ is now the (axial) mean flow speed.

Consider first the contribution by the nozzle, contained in k_N. Since the κ_{mn} are tabulated, it is necessary only to compute K_l, of which there are an infinite number for each κ_{mn}. It follows from the definitions of β_l and K_l that, in the Nth term of the expansion (18), $\beta_l = K_l$; thus k_N^2 is computed from (22) with β_l^2 replaced by K_l^2. Then K_l must be computed from

$$\tanh(iK_lL) = M_e \frac{K_l}{(K_l^2 + \kappa_{mn}^2)^{1/2}} - \frac{A}{a} \frac{(K_l^2 + \kappa_{mn}^2)^{1/2}}{K_l} \tag{29}$$

in which $A/a = \mu + i\theta$ is a function of both κ_{mn} and the angular frequency, the real part of $a(K_l^2 + \kappa_{mn}^2)^{1/2}$. Equation (29) eventually will yield values for Ω_l, Λ_l in $K_lL = \Omega_l + i\Lambda_l$. Generally, this will involve some sort of trial-and-error calculation, since explicit expressions for Ω_l and Λ_l cannot be obtained except when $\kappa_{mn} = 0$. One approach is to split Eq. (29) into its real and imaginary parts, giving two coupled algebraic equations that might be solved numerically. An alternative approach may be to modify the graphical procedure used, for example, in Chap. 3 of Ref. 12. In any case, values first must be assigned to M_e and κ_{mn}, so that calculations of this kind are best carried out in specific instances.

Suppose, on the other hand, that conditions are such that $\Lambda_l/\Omega_l \ll 1$. It is then a simple matter to deduce the approximate expressions from Eq. (29):

$$\Lambda_l = \frac{1}{4} \ln \left\{ \frac{(1 + r)^2 + s^2}{(1 - r)^2 + s^2} \right\}$$

$$\Omega_l = \frac{1}{2} \tan^{-1} \left\{ \frac{-2s}{1 - (r^2 + s^2)} \right\}$$

with

$$r = \mu \left(1 + \frac{\kappa_{mn}^2 L^2}{\Omega_l^2} \right) - \frac{M_e}{1 + (\kappa_{mn}^2 L^2/\Omega_l^2)}$$

$$s = \theta \left(1 + \frac{\kappa_{mn}^2 L^2}{\Omega_l^2} \right)$$

These formulas are correctly obtained from (30) when $\kappa_{mn} = 0$. Evidently, as one would expect, the damping and frequency of "standing" oscillations in the chamber are more dependent, respectively, on the real and imaginary parts of the nozzle admittance function. If, in addition, $(r/s)^2 \ll 1$, then the simple formulas result:

$$\Lambda_l \cong \left[\mu \left(1 + \frac{\kappa_{mn}^2 L^2}{\Omega_l^2} \right)^2 - M_e \right] \times \frac{\{\theta^2[1 + (\kappa_{mn}^2 L^2/\Omega_l^2)] + 1\}^{-1}}{[1 + (\kappa_{mn}^2 L^2/\Omega_l^2)]} \tag{30}$$

$$\Omega_l \cong l\pi - \tan^{-1}\theta \left(1 + \frac{\kappa_{mn}^2 L^2}{\Omega_l^2} \right) \tag{31}$$

Even with these approximations, it is not possible fully to uncouple Λ_l and Ω_l if $\kappa_{mn} \neq 0$.

Corresponding to these results, the real and imaginary parts of a $k_N = \omega_N + i\lambda_N$ are

$$\omega_N \cong \frac{a}{L} \left[(\kappa_{mn}L)^2 + \left\{ l\pi - \tan^{-1}\theta \left(1 + \frac{(\kappa_{mn}L)^2}{\Omega_l^2} \right) \right\}^2 \right]^{1/2} \tag{32}$$#

$$\lambda_N \cong \frac{a\Omega_l/L}{[\Omega_l^2 + (\kappa_{mn}L)^2]^{1/2}} \tag{33}$$

Equations (32) and (33) are quite special cases but do illustrate the important effects.

The influence of a nonzero mean flow appears largely in L_N; an explicit result can be obtained when $M(z)$ is specified. The simplest, and yet not unrealistic, variation for M is

$$M = \begin{cases} (z/L')M_e & 0 \leqslant z \leqslant L \\ M_e & L' \leqslant z \leqslant L \end{cases}$$

and the integrations in (27) lead to the approximate result

$$\frac{\epsilon_N L_N}{E_N^2} \approx 2 \frac{M_e}{aL} (i\omega - \lambda) \left[\frac{\gamma + 1}{2} + \left(1 + \frac{\kappa_{mn}^2}{K_l^2} \right)^{-1} \right] \tag{34}$$

Since by supposition $\lambda/\omega \ll 1$, the most significant part of (34) is the imaginary part, $ibM_e\omega/aL$, where the constant b ranges from about 2 for purely radial modes ($K_l = 0$) to around 4 for purely axial modes; the precise value of b is not important here. The mean flow, therefore, contributes largely to damping of waves by convection of energy out of the chamber. There is a slightly destabilizing effect due to a nonzero mean flow at the exit, proportional to M_e, and appearing in (30) and (29) which arose from the boundary condition set at the chamber exit plane. The reason for this term is that convection causes a portion of the pressure fluctuation to be in phase with the velocity fluctuation at $z = L$, as shown by Eq. (10), so that a net amount of work can be done on the waves in the chamber during each cycle. This is a much smaller effect than that of gross convection of waves out of the chamber.

Thus it appears that one can compute the first two terms in (26) without excessive difficulty. On the other hand,

A formula similar to this has been given by Crocco and Cheng for the one-dimensional problem [Eq. (3.11.20) of Ref. 1].

evaluation of χ_N is substantially more troublesome, since it is necessary to represent the combustion process and, in particular, its dependence on pressure fluctuations. The difficulties in doing so are twofold: combustion, even in a static situation, is not thoroughly understood, and in the present case the propellants follow unknown paths through a varying pressure field. However one may choose to describe the burning of propellants, it seems clear that there must remain one or more quantities that may be given at best qualitative interpretation. It is therefore fortunate that χ_N can be calculated separately, for this affords the opportunity for studying various proposed mechanisms for instability. Two representations will be discussed briefly here.

The first is due to Crocco and has been discussed at length in Ref. 1. The central idea is that the rate of combustion is determined largely by the state of the chamber gases and that, so far as a particular element of propellant is concerned, the burning rate is affected most significantly by the state of the gases during an (undetermined) interval—the "sensitive time lag"—just prior to combustion. With the additional assumption that variations in burning rate can be correlated with fluctuations in pressure, an expression for perturbations in mass addition to the gas phase has been deduced for combustion of liquid propellants. The argument can be applied equally well to a description of fluctuations in energy release.

Let $\dot{m}$ be the total mass flow in the chamber and q the heat of reaction per unit mass for the particular propellant combination used. A "distribution function" $f(\mathbf{r})$ can be defined such that $q\dot{m}f$ is the energy released per unit time and volume in the chamber during steady operation. Clearly f is normalized, if combustion is "complete" within the chamber, to satisfy

$$\int\int\int f dV_0 = 1 \tag{35}$$

the integral being over the chamber volume. A straightforward re-interpretation of the argument given in Ref. 1 leads to an expression for W_c':

$$W_c' = \bar{n}q\dot{m}f(\mathbf{r})\,(1-e^{-i\omega\tau})\,\eta(\mathbf{r}) \tag{36}$$

in which τ is the time lag and $\bar{n}$ is a "pressure interaction index." The latter quantity arises most simply upon assuming the combustion rate proportional to $p^{\bar{n}}$ but appears also, for example, if an Arrhenius rate law is used, subject always to the restriction of small changes in pressure.[1]

The result (36) depends upon the history of an element of propellant along its path to the region where combustion occurs. Alternatively, one might direct attention ab initio to the region of combustion; one formula resulting from such an approach may be deduced in the following manner.[10] It is supposed that W_c' can, perhaps artificially, be split into the product of a pure rate and the associated energy released per unit volume. For small fluctuations, it is sufficiently accurate to assume, as before, that changes in the reaction rate may be correlated with changes in pressure. Thus a small change in the local rate of reaction is proportional to $\eta\bar{w}$, where $\bar{w}$ is a mean reaction rate independent of position in the chamber (since the mean pressure is assumed uniform); hence, in particular, $\eta\bar{w}$ vanishes with η. However, the amount of energy actually released where $\eta \neq 0$ depends in some sense on the local energy release in steady operation. For example, if there is no energy being liberated in some region, then a small change in pressure usually will not alter this situation. If, on the other hand, where combustion is taking place there is a reservoir of "activated" but unreacted particles, then for $\eta > 0$ more energy is released and for $\eta < 0$ less energy is released than in the unperturbed state. It is simplest to suppose that the excess energy released when $\eta > 0$ at a point ($\eta \sim e^{i\omega t}$ here) is some (unknown) fraction of the amount of "available" energy (see Refs. 10 and 13) accumulated when $\eta < 0$; this in turn is proportional to the product of $q\dot{m}f$ and π/ω, the time in which $\eta < 0$ during each cycle. Combination of these arguments gives

$$W_c' = \Delta(\bar{w}/\omega)q\dot{m}fe^{-i\epsilon}\eta \tag{37}$$

in which Δ is an unknown "constant" and $e^{-i\epsilon}$ represents a possible phase difference between oscillations in pressure and energy released. This expression is valid at best for steady oscillations; the unknown quantities Δ and ϵ may depend on ω. One might expect, for example, that ϵ has a value approximated by the ratio of a combustion "relaxation time" to the period of the oscillations.

Since (36) and (37) arise from distinct qualitative arguments, there is really no basis for expecting strict correspondence between the two relations. It is possible, of course, to obtain two equations between $\bar{n},\tau,\Delta,\epsilon$ by separating the real and imaginary parts of (36) and (37). Some further remarks concerning these expressions for W_c' will be offered later.

In any event, χ_N will be a complex quantity. The real part affects the angular frequency of oscillations and is of secondary interest. It is, in fact, a small quantity, and ω is nearly equal to ω_N, as experiments[1, 3, 4] have shown. With $dM/dz = 0$ at $z = L, \lambda \ll \omega$, and (36) for W_c', one finds

$$\mathrm{Im}(\chi_N) \cong \frac{\bar{n}q\dot{m}\omega}{\bar{\rho}\bar{e}}\left\{\int\int\int [(1-\cos\omega\tau)\psi_N^2 + \frac{a}{\omega}\sin(\omega\tau)\psi_N\frac{\partial}{\partial z}(M\psi_N)]fdV_0\right\} \tag{38}$$

and, if (37) is used for W_c',

$$\mathrm{Im}(\chi_N) = \frac{\Delta q\dot{m}\bar{w}}{\bar{\rho}\bar{e}}\left\{\int\int\int [\cos(\epsilon)\psi_N^2 + \frac{a}{\omega}\sin(\epsilon)\psi_N\frac{\partial}{\partial z}(M\psi_N)]fdV_0\right\} \tag{39}$$

Since M has been assumed small and $\omega L/a > 1$, it appears that, under conditions most favorable for unstable oscillations, $\cos\epsilon \sim 1$, $\cos\omega\tau \sim 1$, and the first terms in (38) and (39) dominate. After invoking various assumptions previously discussed, the real and imaginary parts of Eq. (26) finally yield

$$\left(\frac{\omega L}{a}\right)^2 \cong \left(\frac{\omega_N L}{a}\right)^2 - \left(\frac{L}{a}\right)^2\frac{M_e b}{2}\left(\frac{a\lambda}{L}\right) + \frac{\epsilon_N}{2E_N^2}\mathrm{Re}(\chi_N) \tag{40}$$

$$\lambda \cong \lambda_N\left(\frac{\omega_N}{\omega}\right) + \frac{bM_e}{2}\left(\frac{a}{L}\right) - \frac{\epsilon_N}{2E_N^2\omega}\mathrm{Im}(\chi_N) \tag{41}$$

As already noted, $\omega L/a$ is given principally by the first term in (40) which is closely equal to the usual acoustics formula for a closed chamber. The simplified formulas (32) and (40) show that the frequencies of oscillations should be somewhat less than those for a closed chamber; this is in accord with observations (e.g., Refs. 1, 3, and 4). It is then sufficiently accurate to set $\omega_N/\omega \approx 1$ in λ.

Interpretation of these results is simplified by defining the dimensionless quantity I_c:

$$I_c = \frac{aL^{-1}D_N}{(\epsilon_N/2E_N^2\omega_N)\mathrm{Im}(\chi_N)} \tag{42}$$

where

$$D_N = \lambda_N(L/a) + (bM_e/2) \tag{43}$$

There is a value of I_c for each mode; according to Eq. (41), $I_c > 1$ if the mode is stable. The definition is merely the ratio of damping to destabilizing influences as they appear

in the exponential time factor of the expression for the mode shape.

Discussion of Results

Because of the simplicity of (42), it is quite easy to determine the effects of the various variables in the problem. Although in practice it is not always possible to change only one quantity while holding all others constant, it is convenient for the sake of discussion to consider each factor separately. In principle, it should be possible to use (42), or equivalent formulas, in quantitative interpretation of experimental data. Unfortunately, published reports contain insufficient information [especially, $f(\mathbf{r})$ is not known], and the best that can be done at present is to seek qualitative verification.

Before the important question of scaling is treated, several significant conclusions based on (42) should be noted, bearing in mind that any effect tending to decrease I_c is a "destabilizing" influence. The numerator contains the features (accounted for here) that may be called "inherently" damping, namely, the discharge nozzle and the mean flow, which, roughly, carries energy out of the chamber. Both of these appear essentially as chamber exit conditions, since the constant b is relatively insensitive to the mean velocity distribution within the chamber. The quantity λ_N, representing the damping effect of the nozzle, is closely proportional to a/L [cf., Eq. (33)], so that D_N is independent of a/L. Thus, the effect of any damping that takes place at the chamber exit must be inversely proportional to L as it appears in λ and I_c. That is, for a particular value of damping D_N, a smaller fraction of energy in the chamber can be removed in unit time the longer the chamber. In addition, D_N itself decreases as L (or R) is increased, for then ω_N decreases, and hence, according to Fig. 1, so also does μ. This holds for a linear variation of the mean flow velocity in the nozzle but may not always be true.[9] It should be noted that for $l = 0$ (modes not involving fluctuations in the axial direction) the nozzle contributes no stabilizing influence; experiments[3] indicate at least that changes in the nozzle have no effect on those modes. Calculations such as those shown in Figs. 1–4 can be used to deduce the consequences of modifications in the nozzle.

Additional terms will appear in D_N if other sources of damping are included. For example, the suggestion has sometimes been made that the Reynolds number is a significant dimensionless group in scaling procedures, since the viscous stresses in the chamber tend to stabilize oscillations. That this effect is in fact very small compared to the two effects represented in (43) can be shown quite easily. The simplest estimate follows from a calculation similar to one that may be found in Lamb's *Hydrodynamics*.[20] If the viscous stresses (for constant viscosity coefficient) are included in the ordinary acoustic equations, the wave equation becomes

$$\partial^2\eta/\partial t^2 = [a^2 + \tfrac{4}{3}\nu(\partial/\partial t)]\nabla^2\eta$$

where ν is the kinematic viscosity based on mean chamber conditions. When $\eta \sim \exp(ikat)$, one finds $(\nabla^2 + K^2)\eta = 0$, with $K^2 = k^2[1 + \frac{4}{3}ik(\nu/a)]^{-1}$. Now if the nozzle and mean flow are ignored, K^2 is a real number for the modes associated with a closed chamber: $K^2 = (\pi l/L)^2 + \kappa_{mn}^2$. With $k = (\omega + i\lambda)/a$ as before, one finds the approximate relation, $\lambda \cong \nu K^2$ or, since $K \cong \omega_N/a$, $\lambda \cong \frac{2}{3}\nu(\omega_N/a)^2$. This may be compared with, say, the damping by convection of energy out of the chamber by the mean flow; the ratio of these two quantities is $(3bM_ea^3)/(4\omega_N^2L\nu)$. At the temperatures and pressures appropriate to rocket chambers, $a \approx 3 \times 10^3$ fps and $\nu \approx 10^{-4}$ ft²/sec. Hence, for typical values of b and ω_N, even if L is as small as 1 ft and M_e as low as $\frac{1}{10}$, the effect of damping due to convection by the mean flow is roughly 10^5 times as large as that due to viscous stresses in the central part of the chamber. Although it is true that the stresses within boundary layers at the walls may be much larger, these act only in a small part of the chamber and cannot affect the acoustic waves present in the remaining volume.

The most interesting part of I_c is the denominator; suppose, to simplify writing, that the first terms of (38) or (39) are dominant, and one can write approximately

$$\mathrm{Im}(\chi_N) = (q\dot{m}/\bar{p})\beta\int\psi_N^2 f dV_0 \tag{44}$$

in which the form of β depends on the expression chosen for W_c'. The group $q\dot{m}\beta/\bar{p}$ contains all the characteristics of the propellants and the average thermodynamic conditions in the chamber. One can deduce very easily the influence of these variables when a particular form for W_c' has been selected. In Ref. 10, for example, qualitative agreement with certain experimental results has been found when $\mathrm{Im}(\chi_N)$ is given by (37). The changes examined include both propellant properties and geometry.

It is more appropriate here to consider the question of scaling. Particularly, one is interested in the consequences of increasing size; a large part of practical concern with the instability problem originates with difficulties encountered in constructing large rocket motors based on experience gained from smaller motors. It appears that a sufficient increase in size inevitably leads to an instability of the kind discussed here.[3, 4] Although there are some discussions (e.g., Refs. 13–16) of scaling from the point of view of dimensional analysis and physical similarity, the large number of possible parameters severely limits the value of such an approach. But with a more detailed study, one has some means of assessing the relative importance of certain variables, as just illustrated in connection with the Reynolds number. A useful definition arising from (42) and (44) is

$$F_N = (\epsilon_N V_c/2E_N^2)\int\psi_N^2 f dV_0 \tag{45}$$

where V_c is the chamber volume; I_c becomes

$$I_c = \frac{aL^{-1}D_N}{(A_t/c^*A_c)(\beta/\omega_N)(A_cqF_N/V_c)} \tag{46}$$

with $c^* = \bar{p}\dot{m}/A_t$ the characteristic velocity commonly used in analyses of rocket motors. The throat and chamber cross-sectional areas are A_t and A_c. In order to emphasize changes of scale, it will be supposed here that the nozzle shape, chamber exit Mach number M_e, and the average temperature and pressure within the chamber are not altered. Consequently, the combination A_t/c^*A_c remains sensibly constant, and D_N changes mainly in response to changes in the frequency of oscillations.

Because of the normalizations of ψ_N [Eq. (20)] and f [Eq. (35)], F_N remains constant if geometric similarity is preserved and if the shape $f(\mathbf{r})$ of the distribution of energy release does not vary. The combination qF_N/V_c is proportional to the average amount of energy released per unit volume and mass flow and "consumed" in maintaining the pressure oscillations. Since c^* is closely a function of chamber temperature only, the mass flow $\dot{m}$ must be proportional to the cross-sectional area of the chamber when $\bar{p}$ and $\bar{T}$ remain constant. Hence, if strict geometric similarity is maintained, the average energy released per unit volume is proportional to $A_c(qF_N/V_c)$, which varies as L^{-1}. This stabilizing tendency, for increased L, precisely compensates the corresponding effective decrease in damping just noted.

Under the conditions supposed, the effect of changing size therefore appears entirely in $D_N(\omega_N/\beta)$. If the nozzle admittance function behaves as shown earlier, then an increase in size is unfavorable for all modes for which $l \neq 0$ since the frequency, and hence D_N, decreases. If (36) is used for W_c', then β contains $\omega \approx \omega_N$ as a factor, and the changes in D_N and $\cos(\omega\tau)$ are the only consequences of changing scale. On the other hand, if (37) is used, β depends on ω_N only to the extent that Δ does. In either case, $D_N(\omega_N/\beta)$

may vary rather strongly with ω_N. That I_c should, in fact, decrease with decreasing frequency for all chamber modes is suggested by two experimental results: instabilities arise when a particular design is scaled up, and only the lowest modes have been observed in any motor of a fixed size. It thus appears that the present results may contain an explanation for these observations, although a convincing quantitative demonstration is yet to be carried out.

Furthermore, the only way to correct such a tendency for all modes under the conditions supposed here is to effect a change in F_N by changing $f(\mathbf{r})$. It is obvious on physical grounds, for example, that, if combustion occurs only at pressure nodes of a particular mode, then that mode cannot be sustained; this has been pointed out in Ref. 1. It is an immediate result from (46), for if $f \neq 0$ only where $\psi_N^2 \rightarrow 0$ then $F_N \rightarrow 0$ and $I_c \rightarrow \infty$; the reason for this may be traced back to the assumption that fluctuations in energy release are proportional to fluctuations in pressure. Generally, it is desirable to have F_N as small as possible for the lowest modes.

One then concludes that, within the assumptions adopted here, the most effective means to influence the stability of standing pressure oscillations is to change the distribution of combustion in steady operation. This is accomplished most obviously by modifying the injection system, although the distribution of burning is coupled also to the chamber geometry and the gas motions (swirling, etc.). The conclusion is at least consistent with experience, for instabilities have been cured in full-scale devices by just such a procedure, and the importance of the distribution of combustion also has been demonstrated in laboratory experiments dealing with radial oscillations[8] and treating purely axial modes.[17] The "transverse" modes also have been studied experimentally in Ref. 18, using a sector motor.

Similar arguments can be carried through for other scaling procedures,[10] but in any case the distribution of combustion, and hence the injection system, must be of primary importance in correcting an unstable mode. Changes in other variables, as for instance the chamber pressure, very well may be useful, but the precise behavior of I_c depends on the expression chosen for W_c'.

This discussion is in no sense complete, chiefly because of the uncertainties associated with W_c' and $f(\mathbf{r})$. Application of the stability criterion deduced here rests on a priori choice of an expression for W_c'. The proper values of $f(\mathbf{r})$ probably should be estimated from experience, since accurate prediction by analysis seems out of the question. Laboratory experiments then may serve both to test the stability criterion and to clarify the form of $f(\mathbf{r})$.

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