Detecting Non-Abelian Statistics in the $\nu = 5/2$ Fractional Quantum Hall State

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In this Letter we propose an interferometric experiment to detect non-Abelian quasiparticle statistics—one of the hallmark characteristics of the Moore-Read state expected to describe the observed fractional quantum Hall effect plateau at $\nu = 5/2$. The implications for using this state for constructing a topologically protected qubit as has been recently proposed by Das Sarma *et al.* are also addressed.

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Introduction.—One of the most interesting aspects of the fractional quantum Hall effect (FQHE) is the fractionalized nature of its quasiparticle excitations. In addition to carrying a fraction of the electron charge, these excitations are generally expected to have exotic exchange statistics which are neither bosonic nor fermionic. These exotic statistics, generally allowed in $2 + 1$ dimensions [1], are given by representations of the braid group (as opposed to higher dimensions where statistics is represented by the permutation group), and particles that transform as such have been dubbed anyons [2]. The fractional charge of quasiparticles in the $\nu = 1/3$ Laughlin state was first measured a decade ago [3], but confirmation of their statistics remained elusive until very recently [4]. Aside from the experimental difficulties associated with measuring quasiparticle interference patterns, there are also conceptual issues regarding how to isolate the contribution of braiding statistics from that of the Aharonov-Bohm phases that arise due to the quasiparticle charge encircling a region of magnetic flux. For a careful discussion of this subject, see [5]. Curiously, isolating these pieces may prove easier in a more exotic state with non-Abelian statistics. In such a system, the Hilbert space is multidimensional and exchange transformations may rotate different states into one another. This notion, along with a topological protection inherent in these systems make them attractive candidates for fault-tolerant quantum computation [6–8]. A concrete proposal for creating a topologically protected qubit has been recently put forward in [9].

While the existence of Abelian anyons has been well established in the context of FQHE, the more exciting prospect that non-Abelian anyons exist has not been experimentally confirmed. The prime candidate for finding non-Abelian statistics seems to be the FQH state observed at the $\nu = 5/2$ plateau [10]. While its first Landau level counterpart, the $\nu = 1/2$ state, is widely believed to be a Fermi liquid of composite fermions [11], it is most likely that the $\nu = 5/2$ system is the p-wave (spin-polarized) superconducting condensate described by the Moore-Read (MR) state [12,13]. Experimental evidence of spin polarization [14], together with careful numerical studies [15], indicate a preference for the MR state over other potential candidates, notably the Abelian (3,3,1) Halperin state [16], the non-Abelian (albeit critical) Haldane-Rezayi state [17], and the compressible striped phase [18].

Proposed experimental setup.—The experimental device we would like to consider is a two-point-contact interferometer composed of a quantum Hall bar with two front gates on either side of an antidot (see Fig. 1). Biasing the front gates can be used to create constrictions in the Hall bar, adjusting the tunneling amplitudes t_1 and t_2. The relative amplitudes can be compared by individually switching them on. The tunneling between the opposite edge currents leads to the deviation of σ_{xy} from its quantized value, or equivalently, to the appearance of σ_{xx}. The goal of the experiment is to observe the interference between the two tunneling paths that the quasihole current may traverse. For this experiment, we are interested in the *weak* backscattering regime, i.e., the case where the tunneling amplitudes t_1 and t_2 are small. The main reason for this is to ensure that the tunneling current is entirely due to charge $e/4$ quasiholes (with essentially no contribution from the higher charge composites), which is a crucial component of our predictions. In this regime, such tunneling is indeed the most relevant perturbation [19,20], but this need not be true in the strong tunneling regime, where the constrictions are effectively pinched off. We should

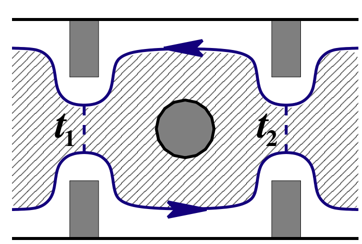


FIG. 1 (color online). A two-point-contact interferometer for measuring the quasiparticle statistics. The hatched region contains an incompressible FQH liquid. The front gates (gray rectangles) are used to bring the opposite edge currents (indicated by arrows) close to each other to form two tunneling junctions. Applying voltage to the central gate creates an antidot in the middle and controls the number of quasiparticles contained there.

also mention that interpreting the interference pattern is simplified when t_1 and t_2 are small.

For the purposes of this experiment, we envision three main experimentally variable parameters: (i) the central gate voltage allowing one to control the number n of quasiholes on the antidot, (ii) the magnetic field B, and (iii) the back gate voltage controlling the uniform electron density. This setup is essentially identical to that proposed for measuring statistics in the Abelian FQHE [5], later adopted for the non-Abelian case in [21], and not dissimilar to the one experimentally realized in [4].

To lowest order in t_1 and t_2, the tunneling current and, hence, longitudinal conductivity σ_{xx} in this system will be proportional to the probability that current entering the bottom edge leaves through the top edge:

$$\begin{aligned}\sigma_{xx} &\propto |(t_1 U_1 + t_2 U_2)|\Psi\rangle|^2 \\ &= |t_1|^2 + |t_2|^2 + 2\,\mathrm{Re}\{t_1^* t_2 \langle\Psi|U_1^{-1}U_2|\Psi\rangle\} \\ &= |t_1|^2 + |t_2|^2 + 2\,\mathrm{Re}\{t_1^* t_2 e^{i\alpha}\langle\Psi|M_n|\Psi\rangle\}. \qquad (1)\end{aligned}$$

In this expression, U_1 and U_2 are the unitary evolution operators for a quasihole taking the two respective paths, and $|\Psi\rangle$ is the initial state of the system. In the third line, $e^{i\alpha}$ is the phase acquired from the dynamics of traveling along the edge around the center region together with the Aharonov-Bohm phase from taking the quasihole charge around the magnetic flux through the center region. The operator M_n is the transformation due solely to the braiding statistics of winding a single quasiparticle around n quasiparticles. Its value for the MR state was related to the Jones polynomial in [21] using the Chern-Simons effective theory. We shall extend their analysis and show explicitly how to detect the non-Abelian statistics.

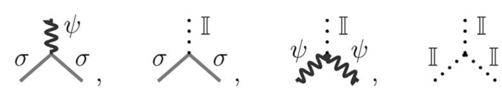
If we keep the filling factor fixed by simultaneously adjusting B and the electron concentration, so as to keep the quasihole number constant, the Aharonov-Bohm phase as a function of Φ will have a periodicity of $(e/e^*)\Phi_0$ where $\Phi_0 = 2\pi/e$ (in units $\hbar = c = 1$) and e^* is the electric charge of the quasiholes [5]. Thus, varying the flux Φ allows one to determine the quasiparticle charge. Note that for $\nu = 5/2$, a quasihole charge of $e^* = e/4$ rather than $e^* = e/2$, would be indicative of a paired state.

The Moore-Read state.—The braiding statistics of particles in a $2+1$ dimensional quantum system may be described by a general model of anyons (see [8,22] and references therein). Such a model is defined by a set of particle types, fusion rules, and braiding rules, all of which are required to satisfy certain consistency conditions. The particle types and their fusion rules can be, respectively, thought of as generalizations of group representations and their tensor products, specifying values of conserved charges and the possible values that may be obtained when forming composite objects (composite in this context need not necessarily mean that the constituents are bound together, but simply that their local properties are not being individually probed, as in the case when they are being viewed from far away).

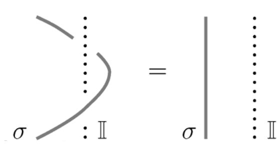
The anyon model that describes the MR state can be denoted as U(1) × Ising. (The term "Ising" is used here in reference to the anyon model obtained from the holomorphic part of the conformal field theory that describes the Ising model at criticality.) In this notation, U(1) refers to the familiar Abelian charge-flux sector, for which particle type is specified by the amount of charge and flux carried, the fusion rules are simply addition of these quantities (i.e., the conservation of charge and flux), and the braiding rules are specified by the usual phases acquired from winding charge-flux composites, i.e., winding one (q, ϕ) charge-flux composite around another produces a phase of $e^{iq\phi}$ [23]. Though less familiar, the Ising anyon model (which contributes all of the non-Abelian statistics to the MR state), is fairly simple. It has three particle types, conventionally denoted as: $\mathbb{I}$ (vacuum), σ (spin/vortex), and ψ (Majorana fermion) [24], and the following fusion rules:

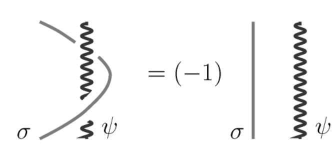
$$\begin{aligned}&\mathbb{I}\times\mathbb{I}=\mathbb{I}, \qquad \mathbb{I}\times\sigma=\sigma, \qquad \mathbb{I}\times\psi=\psi, \\ &\sigma\times\sigma=\mathbb{I}+\psi, \qquad \sigma\times\psi=\sigma, \qquad \psi\times\psi=\mathbb{I}.\end{aligned} \qquad (2)$$

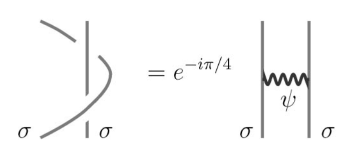
In words, combining the two particle types on the left-hand side of the equal sign gives some superposition of states carrying the labels on the right-hand side (as mentioned earlier, one may think of the symbols × and +, respectively, as generalizations of the tensor product, ⊗, and the direct sum, ⊕). Graphically, these rules can be represented by locally permitting only the following set of trivalent vertices (in any desired orientation):



From the anyon model rules and consistency conditions (which we will not present here, but instead refer the reader to [22] for details [26]), we can distill the following braiding rules:


(3a)

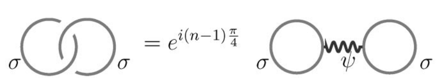

(3b)


(3c)

We emphasize that these diagrams merely keep track of particle fusion and braiding statistics. There are no additional propagators or interactions associated with these diagrams that need to be calculated, and these relations are unchanged by any smooth deformations in which worldlines do not cross. The signature of non-Abelian statistics is apparent in Eq. 3(c), where winding two σ particles around each other is seen to be equivalent (up to a phase) to exchanging a ψ particle between them.

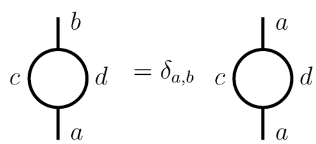
Each quasihole in the MR state carries a U(1) charge flux of $(e/4, \Phi_0/2)$ as well as the Ising label σ. A straightforward application of the fusion rules determines that a composite of n quasiholes will have U(1) charge flux $(ne/4, n\Phi_0/2)$ and Ising label Q_n, where Q_n must equal σ when n is odd, but can equal either $\mathbb{I}$ or ψ when n is even. We can combine the braiding rules of the two sectors to get the rules for winding a single quasihole counterclockwise around n quasiholes by making the following modifications to the diagram equations of the Ising sector Eqs. 3(a)–(c): assign $(e/4, \Phi_0/2)$ to the leftmost and $(ne/4, n\Phi_0/2)$ to the rightmost worldlines on each side, assign $(0, 0)$ to the ψ world line in 3(c), and multiply the right-hand side of each equation by $\exp(\mathrm{i}n\pi/4)$. These rules agree with those obtained by explicitly manipulating quasihole wave functions in the MR state [27].

The inner product for the interference term $\langle\Psi|M_n|\Psi\rangle$ is represented diagrammatically by the standard closure, where each world line is looped back onto itself in a manner that introduces no additional braiding. From Eq. 3(c), we find that if there is an odd number of quasiholes on the antidot, $\langle\Psi|M_n|\Psi\rangle$ is proportional to the following diagram [leaving U(1) labels implicit]:



(4)

But this diagram has vanishing amplitude as a result of the following general consistency condition in anyon models:



(5)

where the labels indicate particle types permitted by the fusion rules. Thus, with no interference, we have

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2, \qquad n\ \text{odd}. \tag{6}$$

When there is an even number n of quasiholes on the antidot, the environment will effectively measure the state, forcing it into either an overall $\mathbb{I}$ or ψ (not a superposition of the two). It is easy to see from 3(a),(b) that the interference term $\langle\Psi|M_n|\Psi\rangle = (-1)^{N_\psi} e^{\mathrm{i}n\pi/4}$, and thus,

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + (-1)^{N_\psi} 2|t_1\|t_2|\cos\left(\beta + n\frac{\pi}{4}\right), \qquad n\ \text{even}, \tag{7}$$

where $\beta = \alpha + \arg(t_2/t_1)$ can be varied by changing B or the relative tunneling phase. Here, $N_\psi = 1$ when the n quasiholes are in the ψ state and 0 otherwise. We note, that for two well-separated quasiholes, the energies of the two possible combined states ($\mathbb{I}$ or ψ) are *equal*. This, however, is not going to be the case for two quasiholes on the same antidot. In particular, one can write down the two corresponding wave functions for the case of a "small antidot" with two quasiholes located at the origin [28]:

$$\Psi_{2\mathrm{qh},\mathbb{I}} = \prod_j z_j \Psi_{\mathrm{GS}}, \tag{8}$$

where

$$\Psi_{\mathrm{GS}} = \mathcal{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\right)\prod_{j<k}(z_j - z_k)^2\prod_j e^{-|z_j|^2/4} \tag{9}$$

is the ground state wave function for the MR state with $\mathcal{A}(\ldots)$ denoting the antisymmetrized sum over all possible pairings of electron coordinates, and

$$\Psi_{2\mathrm{qh},\psi} = \Pi_j z_j \mathcal{A}\left(\frac{z_1 - z_2}{z_1 z_2}\frac{1}{z_3 - z_4}\frac{1}{z_5 - z_6}\cdots\right) \times \prod_{j<k}(z_j - z_k)^2\prod_j e^{-|z_j|^2/4}. \tag{10}$$

While these wave functions are clearly different, using them as variational functions to estimate the energy difference for the case of realistic electron-electron interactions appears hopeless since even the ground state wave function (9) is not actually a ground state for any such realistic interaction. While this remains an open problem, a very naïve argument would suggest that the energy difference should scale as e^2/R where R is the antidot radius; however, it is entirely possible that such a term will have a small prefactor. If charging the antidot is done adiabatically, one may hope that upon addition of two new quasiholes, the system will remain in the same energetically preferred state (probably $\mathbb{I}$). In such a case, σ_{xx} is expected to cycle through all four possible values given by Eq. (7) as a function of an increasing even number of quasiholes, while it returns to the same value given by Eq. (6) for any odd number of quasiholes. However, if the combined state of an even number of quasiholes is chosen randomly every time, we cannot expect such even number periodicity, though the magnitude of the current will generically change whenever two quasiholes are added. The real test for the non-Abelian nature is done by changing the magnetic field B at fixed filling fraction, for a various number of quasiholes on the antidot. In doing so, Aharonov-Bohm oscillations with period $4\Phi_0$ should be observed in the even n case and no oscillations whatsoever should be seen for the odd n case [29].

Implications for a topological qubit scheme.—We finally turn to the implications of our results to the proposed implementation of a topological qubit [9], which is schematically shown in Fig. 2.

Assuming the qubit is implemented as prescribed, one nevertheless has to address the issue that "stray" quasiparticles may disrupt the ability to both measure and switch the state of the qubit. These stray excitations may be trapped elsewhere in the system by a local disorder poten-

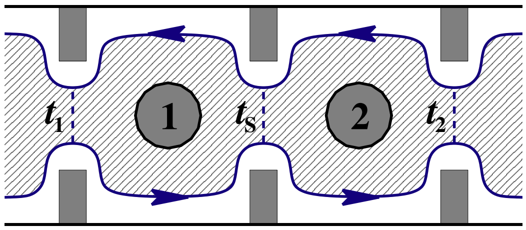


FIG. 2 (color online). The configuration for a topologically protected qubit proposed in [9]. A two-point interferometer is used to measure the combined state of a quasihole pair split onto two separate antidots. A bit flip that switches between the $\mathbb{1}$ and ψ states is performed by tunneling a single quasihole through the switching constriction, whose tunneling amplitude t_S can be turned on and off by controlling the middle set of gates.

tial. From 3(a)–(c) and the related discussion, it is clear that in order to be able to detect the state of the qubit, the *total* number of quasiparticles (and quasiholes), including the strays, in the area between the "measurement" tunneling contacts (t_1 and t_2 in Fig. 2) must be *even*, otherwise the interference necessary to distinguish the states will not be seen. Similarly, in order for switching to work, the total number of quasiparticles in the left partition (i.e., between t_1 and t_S in Fig. 2) must be *odd*, otherwise the state would simply acquire an Abelian phase.

Conclusion.—To summarize, in this Letter we propose an interferometric experiment for detecting non-Abelian quasiparticle statistics in the MR state, the leading candidate for the $\nu = 5/2$ FQHE plateau. Interestingly, while performing this experiment at $\nu = 5/2$ is expected to be more difficult than for well established Laughlin states due to the smaller energy gap, the signature of a non-Abelian state would be much easier to interpret due to the clear separation of non-Abelian statistics from other effects that can only contribute Abelian phases. The experimental setup discussed here, while simpler than that recently proposed for a topological qubit, may be a first step in the implementation of that scheme.

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[24] To avoid confusion, we point out that there is an alternative way of treating these excitations [25]—namely, by identifying σ quasiparticles with the zero-energy Majorana modes inside a vortex core in a p-wave superconductor. In this language, ψ particles (Majorana fermions in our notation) become usual fermionic modes. While we make no recourse to this description, it is worth mentioning that it provides an alternative derivation of the non-Abelian braiding rules, up to a phase factor.
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