# Controllability of Kinematic Control Systems on Stratified Configuration Spaces

Bill Goodwine and Joel W. Burdick

***Abstract*—This paper considers nonlinear kinematic controllability of a class of systems called *stratified*. Roughly speaking, such stratified systems have a configuration space which can be decomposed into submanifolds upon which the system has different sets of equations of motion. For such systems, considering controllability is difficult because of the discontinuous form of the equations of motion. The main result in this paper is a controllability test, analogous to Chow's theorem, is based upon a construction involving distributions, and the extension thereof to robotic gaits.**

***Index Terms*—Controllability, legged locomotion, robot control, stratified systems.**

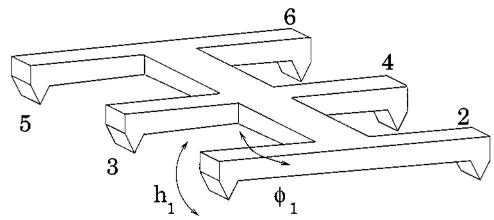


Fig. 1. Schematic of a simple hexapod robot.

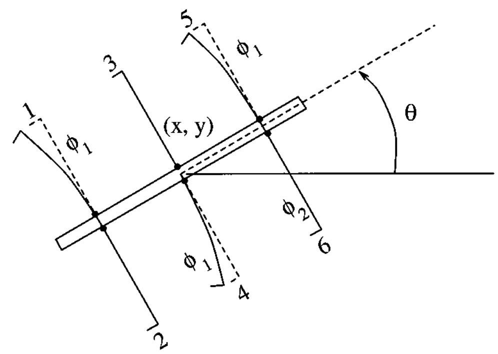


Fig. 2. Definition of kinematic variables.

## I. INTRODUCTION

MANY interesting and important control systems evolve on *stratified* configuration spaces. Roughly speaking, we will call a configuration manifold stratified if it contains submanifolds upon which the system has different equations of motion. Certain robotic systems, in particular, are of this nature. For such systems, the equations of motion on each submanifold may change in a nonsmooth, or even discontinuous manner, when the system moves from one submanifold to another. A legged robot has discontinuous equations of motion near points in the configuration space where each of its "feet" come into contact with the ground, but it is precisely the ability of the robot to lift its feet off of the ground that enables it to move about. Similarly, a robotic hand grasping an object often cannot reorient the object without lifting its fingers off of the object. Despite the obvious utility of such systems, however, a comprehensive means to analyze their controllability properties, to our knowledge, has not appeared in the literature.

For example, consider the six-legged hexapod robot illustrated in Figs. 1 and 2. This model will be fully explored in complete detail in Section V. Note that each leg has only two degrees of freedom. In particular, the robot can only lift its legs up and down and move them forward and backward. In contrast to most mechanical designs in the robotics literature, such a leg can not be extended outward from its body. Such limited control authority may be desirable in practical situations because it decreases the mechanical complexity of the robot; however, such decreased complexity comes at the cost of requiring more sophisticated control theory. Note that for this model, it is not immediately clear whether the robot can move "sideways," and if it cannot move sideways, then it is not controllable. In this, and other stratified cases, traditional nonlinear controllability analyzes are inapplicable because they rely upon differentiation in one form or another. Yet it is precisely the discontinuous nature of such systems that is often their most important characteristic because the system must cycle through different contact states to effectively be controlled. Therefore, it is necessary to incorporate explicitly into a controllability analysis the nonsmooth or discontinuous nature of these systems.

This paper first considers some basic issues regarding the appropriate definition of controllability for stratified systems and then extends standard controllability tests for smooth driftless nonlinear systems to the case where the configuration manifold is stratified. Although Brockett [5] illustrated some of the aspects of the problem of discontinuous or impacting systems, and there is quite a bit written concerning so-called "hybrid systems," (see, e.g., [3], [4], and [6]–[11]) none of these has exploited the particular geometry of stratified systems to develop a controllability test. Additionally, there is a vast literature on the particular problem of legged robotic locomotion. However, prior efforts have focused either on a particular morphology, (e.g., biped [17], quadruped [20], or hexapod [27]), or a particular locomotion assumption, (e.g., hopping [26] or quasistatic

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[27]), and the issue of controllability is an implicitly assumed property. Less effort has been devoted to uncovering principles that span all morphologies and assumptions. Our goal is to generate *general* results, i.e., results formulated in sufficient generality so that they are independent, in robotics applications, of a particular morphology.

Some recent works have attempted to uncover some of the fundamental structure underlying locomotion mechanics. Kelly and Murray [18] showed that a number of kinematic locomotive systems can be modeled using connections on principal fiber bundles. They also provide results on controllability, as well as an interpretation of movement in terms of geometric phases. Ostrowski [24], [25] developed analogous results for a class of dynamic nonholonomic locomotion systems. Also, Tsakiris and Krishnaprasad [19] have used methods from nonlinear control theory to develop motion planning schemes for "G"-snakes, a class of kinematic undulatory mechanisms. These approaches are limited to *smooth* systems, however, and thus are not directly applicable to stratified systems.

Section II defines a stratified configuration space and discusses its generic geometric structure. Section III reviews controllability for smooth systems and the associated mathematical concepts. Section IV first motivates and develops our controllability theory in detail for a simplified subset of stratified problems, illustrates its application with a simple example and then presents the controllability test. Section V presents the hexapod robot example problem in full detail. Section VI presents controllability results for a more general stratified case, and again illustrates its application with a simple example. Additionally, Section VI defines and presents a test for *gait controllability*, particularly useful for and motivated by legged robotic systems, and also returns to the hexapod example to analyze its gait controllability.

## II. Stratified Configuration Spaces

We will motivate our definition of a stratified configuration space with a simple example.

*Example 2.1:* Consider a biped robot. The configuration manifold for the robot describes the spatial position and orientation of a reference frame rigidly attached to the body of the robot mechanism as well as variables such as joint angles which describe the mechanism's internal geometry. The set of configurations corresponding to one of the feet in contact with the ground is a codimension one regular submanifold of the configuration space. The same is true when the other foot contacts the ground. Similarly, when both feet are in contact with the ground, the system is on a codimension two regular submanifold of the configuration space formed by the intersection of the single contact submanifolds. The structure of the configuration manifold for such a biped is schematically illustrated in Fig. 3. The goal in this paper is to exploit the geometric structure of such configuration spaces. Note that when a foot contacts the ground, because the robot is subjected to additional constraints, it will have a different set of equations of motion than when the foot is not in contact with the ground. Also, except for when the robot transitions from a state where a foot is off of the ground to one where a foot contacts the ground, the equations of motion for the system are smooth. In other words, restricted to each stratum, the equations of motion are smooth.

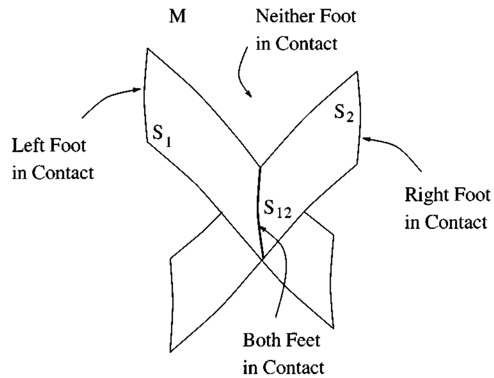


Fig. 3. Schematic view of the configuration manifold structure of a biped robot.

We will refer to the configuration space for the biped robot in Example 2.1 as *stratified*, which will be specifically defined subsequently. To avoid confusion regarding the terminology, we note that the terms "stratification" and "strata" have been used previously for different mathematical structures. In particular, the term *regularly stratified* set has been used to describe a finite union of disjoint smooth manifolds, also called strata, which satisfy the Whitney condition (see [14] for details on such stratifications). Additionally, the terms "strata" and "stratification" have been used in yet another different context; namely, describing the topology of orbit spaces of Lie group actions, and are a slight generalization of the notion of a foliation [1].

By considering legged robot systems that are more general than the biped in Example 2.1, we can develop a general definition of stratified configuration spaces. Let $M$ denote the legged robot's entire configuration manifold (it will often be convenient to denote this space as $S_0$). Let $S_i \subset M$ denote the codimension one submanifold of $M$ that corresponds to all configurations where only the $i$th foot contacts the terrain. Denote, the intersection of $S_i$ and $S_j$, by $S_{ij} = S_i \cap S_j$. The set $S_{ij}$ physically corresponds to states where both the $i$th and $j$th feet are on the ground. Further intersections can be similarly defined in a recursive fashion: $S_{ijk} = S_i \cap S_j \cap S_k = S_i \cap S_{jk}$, etc. Note that the ordering of the indices is irrelevant, i.e., $S_{ij} = S_{ji}$. We will term the lowest dimension stratum containing the point $x$ as the *bottom stratum*, and any other submanifolds containing $x$ as *higher strata*. When making relative comparisons among different strata, we will refer to lower dimensional strata as *lower strata*, and higher dimensional strata as *higher strata*.

*Definition 2.2 (Stratified Configuration Manifold):* Let $M$ be a manifold (possibly with boundary), and $n$ functions $\Phi_i$: $M \mapsto \mathbb{R}$, $i = 1, \ldots, n$ be such that the level sets $S_i = \Phi_i^{-1}(0) \subset M$ are regular submanifolds of $M$, for each $i$, and the intersection of any number of the level sets, $S_{i_1 i_2 \ldots i_m} = \Phi_{i_1}^{-1}(0) \cap \Phi_{i_2}^{-1}(0) \cap \cdots \cap \Phi_{i_m}^{-1}(0)$, $m \leq n$, is also a regular submanifold of $M$. Then $M$ and the functions $\Phi_i$, $i = 1, \ldots, n$ define a *stratified configuration space*.

We consider driftless nonlinear systems defined on stratified configuration manifolds and will write the equations of motion for the system at $x \in M = S_0$ as

$$\dot{x} = g_{0,1}(x)u_{0,1} + \cdots g_{0,n_0}(x)u_{0,n_0}$$

and the equations of motion for the system in one of the strata at $x \in S_i$ as

$$\dot{x} = g_{i,1}(x)u_{i,1} + \cdots g_{i,n_i}(x)u_{i,n_i}$$

where $n_i$ may be different than $n_0$. For an arbitrary stratum, $S_I = S_{i_1 i_2 \ldots i_m}$, we have

$$\dot{x} = g_{I,1}(x)u_{I,1} + \cdots + g_{I,n_I}(x)u_{I,n_I}$$

where the vector fields, $g_{I,j}$ are smooth and defined for all points in $S_I$. In the context of legged locomotion, the assumption of driftless mechanisms on each strata limits us to quasistatic locomotion.

Finally, we assume that the only discontinuities present in the equations of motion are due to transitions on and off of the strata $S_i$ or their intersections. We also assume the control vector fields restricted to any stratum are smooth away from points contained in intersections with other strata. When a system has the above properties, we will refer to it as a *stratified control system.*

## III. Mathematical Preliminaries

This section addresses some basic topological properties of stratified configuration spaces and the implications thereof with respect to the definition of controllability. First, we must define the term "controllable" as it is usually considered for smooth systems. Given an open set $V \subseteq M$, define $R^V(x_0, T)$ to be the set of states $x_f$ such that there exists $u$: $[0, T] \to \mathcal{U}$ that steers the control system from $x(0) = x_0$ to $x(T) = x_f$ and satisfies $x(t) \in V$ for $0 \le t \le T$, where $\mathcal{U}$ is the set of admissible controls. Define

$$R^V(x_0, \le T) = \bigcup_{0<\tau\le T} R^V(x_0, \tau). \tag{1}$$

We will refer to $R^V(x_0, \le T)$ as the set of states reachable up to time $T$.

*Definition 3.1 (Small Time Local Controllability):* A smooth analytic system is *small time locally controllable* (STLC), or simply "controllable", if $R^V(x_0, \le T)$ contains a neighborhood of $x_0$ for all neighborhoods $V$ of $x_0$ and $T > 0$.

For *smooth* analytic kinematic system, Chow's Theorem [15], [22], [23], [28], and related results from exterior differential system [1], [2], [12], and[21] provide necessary and sufficient conditions for controllability. For stratified systems, Definition 3.1 must be modified for two topological reasons. First, in terms of controllability, it may not be desirable or possible to reach an open neighborhood in the entire configuration space, but rather an open set on a collection of the strata within the whole configuration space. For example, for the biped, it may be desirable that the robot always has at least one foot in contact with the ground, i.e., it is walking, as opposed to running. In such a case, it is most natural to consider controllability in terms of reaching

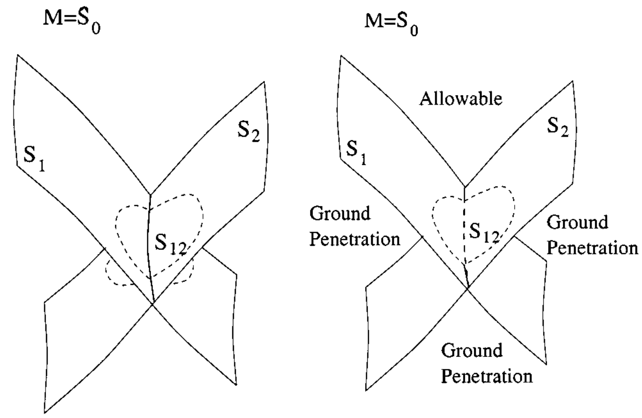


Fig. 4. Stratified open neighborhoods.

an open neighborhood defined in the union of the two strata $S_1$ and $S_2$ (corresponding to each foot in contact with the ground) as illustrated in Fig. 4.

From basic topology [1], we have that if $A$ is a subset of a topological space $S$ with topology $O$ , the *relative topology* on $A$ is defined by $O_A = \{U \cap A | U \in O\}$. Thus, in the biped example, as illustrated on the left in Fig. 4, the dotted regions illustrate an open set in the union $S_1 \cup S_2$. The dotted regions represent the intersection of an open ball in $S_0$ with $S_1 \cup S_2$. Stratified controllability now is defined as the reachable set containing a neighborhood of the starting point, where the neighborhood is open in $S_1 \cup S_2$. The second modification is a result of the fact that, until now, we have considered a stratum to be a submanifold of the configuration space for a stratified system. In fact, it may often be the case that the strata defining the stratification are *boundaries* of the configuration space. In such a case, we have to redefine a neighborhood of a point $x_0$ contained in the boundary to be the union of the portion of the standard neighborhood in the interior of the manifold with the intersection of the standard neighborhood with the boundary. For the biped example, only one of the four "quadrants" defined by the intersecting strata is "allowable" (the other three correspond to one or both feet penetrating the ground). The right figure in Fig. 4 illustrates an open set for such a stratified configuration space with boundary. The open sets illustrated by the dotted lines on each stratum only exist on the "top" half of each stratum. Again, stratified controllability amounts to reaching an open neighborhood of the starting point, where an open set is determined by the natural topology of the problem.

Thus, for a stratified system, the stratum (or union of strata) with respect to which controllability is desired to be determined, must be specified. Additionally, since there are multiple equations of motion corresponding to different strata, the set of admissible controls, $\mathcal{U}$ depends upon the state of the system since, in general, the allowable inputs will be different for different strata. Furthermore, problem-specific constraints such as which switches between strata are allowable are accounted for by $\mathcal{U}$.

*Definition 3.2 (Stratified Controllability):* A stratified system is *stratified controllable in* $S_I$ from $x_0 \in S_I$ if $R^V(x_0, \le T)$ contains a neighborhood of $x_0$ in $S_I$ for all neighborhoods $V \subseteq S_I$ of $x_0$ and $T > 0$, where $R^V(x_0, \le T)$ is defined by Equation (1) with $V \subseteq S_I$.

## IV. Stratified Controllability

In order to clarify the presentation and provide an intuitive understanding of our approach, we first consider an extremely simple example. In particular, we focus on the case where the configuration manifold contains only one submanifold (or stratum). By focusing on this situation, several basic controllability results are rather straightforward to motivate and obtain. Also, as will become clear, these simple results are easily generalized.

*Example 4.1:* Consider the kinematic leg illustrated in Fig. 5. The configuration space, $M$, for the leg is parameterized by the variables $q = (x, l, \theta)$, corresponding to the lateral position of the body, the length of the leg and the angular displacement of the leg, respectively. We assume (unrealistically, but for the purpose of a clear presentation) that the height of the body off of the ground remains fixed, so when the leg is lifted off of the ground, the body does not fall down. The two inputs for the system are the joint velocities $u_1 = \dot{l}$ and $u_2 = \dot{\theta}$.

In this case, the bottom stratum (or boundary) is the set of points

$$S = \{q \in M : l \cos\theta = h\}$$

where $h$ is some fixed height. The equations of motion are given by

$$\frac{d}{dt}\begin{pmatrix} x \\ l \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 \tag{2}$$

when the foot is off the ground, and

$$\frac{d}{dt}\begin{pmatrix} x \\ l \\ \theta \end{pmatrix} = \begin{pmatrix} -\dfrac{l}{\cos\theta} \\ l \tan\theta \\ 1 \end{pmatrix} u_2 \tag{3}$$

when the foot is in contact with the ground (on the bottom stratum, i.e., $q \in S$).

### A. The Distribution Approach

It is clear that if the leg needs to move laterally (in the $x$-direction) while still retaining control over the joint variables, it must cyclically move the leg in and out of contact with the ground. Fig. 6 schematically illustrates the configuration space for the simple kinematic leg example. It consists of the "ambient" space, $M$, where the leg is off of the ground, and the submanifold (or boundary), $S$, which represents the set of points where the leg is in contact with the ground. Since we know the equations of motion in each strata, we can calculate the associated involutive closures of the distributions associated with $M$ and also with $S$, denoted $\overline{\Delta}_M$ and $\overline{\Delta}_S$, respectively. Note that in Fig. 6, the symbols for the involutive distributions are pointing to the manifolds to which they are the tangent space.

If the system starts at a point in $S$, then the set of points it can reach in $S$ is the leaf of the foliation of $S$ defined by $\overline{\Delta}_S$ which contains that point. In Fig. 6, such a leaf is represented

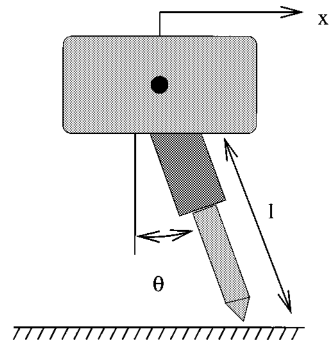


Fig. 5. Kinematic leg.

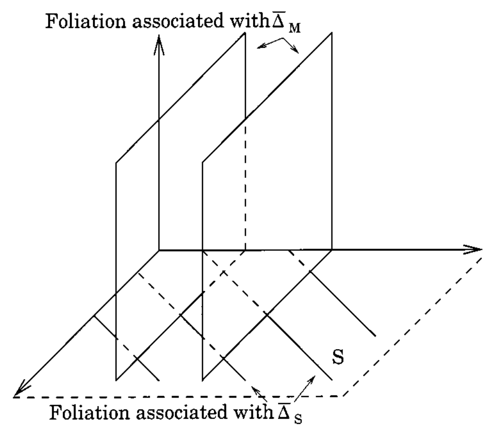


Fig. 6. Controllability of a stratified system.

by the lines in $S$. Similarly, if the system starts at a point in $M$, then the set of points that can be reached in $M$ is represented in Fig. 6 by the vertical sheets in $M$, which represent the foliation defined by $\overline{\Delta}_M$. Any arbitrary point in $S$ is contained in one leaf of the foliation of $M$ defined by $\overline{\Delta}_M$ and one leaf of the foliation of $S$ defined by $\overline{\Delta}_S$. By Chow's Theorem, $\overline{\Delta}_M$ and $\overline{\Delta}_S$ are the directions in which the system can flow on $M$ and $S$, respectively. Since any point in $S$ is also contained in $M$, and the system can move from $S$ to $M$ arbitrarily, then the vector space sum of $\overline{\Delta}_M|_{x_0}$ and $\overline{\Delta}_S|_{x_0}$ represents all the directions in which the system can flow. Thus, if $\overline{\Delta}_M$ and $\overline{\Delta}_S$ intersect *transversely*, i.e.,

$$\overline{\Delta}_M|_{x_0} + \overline{\Delta}_S|_{x_0} = T_{x_0}M$$

the system can then flow in any direction in $M$. This argument suggests the following Proposition.

*Proposition 4.2:* If $x \in S$ and

$$\overline{\Delta}_M|_{x_0} + \overline{\Delta}_S|_{x_0} = T_{x_0}M$$

then the system is stratified controllable from $x_0$ in $M$.

Since this proposition will follow trivially as a corollary of a following more general result (Proposition 4.4), we will not provide the proof.

*Example 4.3 (Kinematic Leg—Continued):* To show that the kinematic leg is controllable, we must show that its equations of motion satisfy the requirement of Proposition 4.2. Since the vector fields in the equations of motion when the foot is not in contact with the ground are constant, then

$$\overline{\Delta}_M = \text{span}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

When the foot is in contact with the ground, there is only one vector field, so

$$\overline{\Delta}_S = \text{span}\begin{pmatrix} -\dfrac{l}{\cos\theta} \\ l\tan\theta \\ 1 \end{pmatrix}.$$

Clearly, away from singularities at $\pm(\pi/2)$

$$\overline{\Delta}_M + \overline{\Delta}_S = TM.$$

Thus, the kinematic leg is controllable.

### *B. Nested Sequence of Strata*

The central aspect of the above controllability discussion was the transversal relationship between the foliations defined by the control vector fields on $M$ and $S$. This notion is easy to generalize to a nested sequence of submanifolds

$$S_p \subset S_{(p-1)} \subset \cdots \subset S_1 \subset S_0 = M.$$

In the nested sequence of submanifolds, the subscript is the codimension of the submanifold. Note that there may be multiple submanifolds with the same codimension at a point. If there are multiple submanifolds with the same codimension, this sequence contains only *one* of them. Also, denote the distribution defined by the control vector fields defined on a stratum $S_i$ by $\Delta_i$, and its associated involutive closure by $\overline{\Delta}_i$. The result for such nested sequences is as follows.

*Proposition 4.4:* If there exists a nested sequence of submanifolds

$$x_0 \in S_p \subset S_{(p-1)} \subset \cdots \subset S_1 \subset S_0$$

such that the associated involutive distributions satisfy

$$\sum_{j=0}^{p} \overline{\Delta}_{S_j}|_{x_0} = T_{x_0}M$$

and each $\overline{\Delta}_{S_j}$ has constant rank for some neighborhood, $V_j \subset S_j$ of $x_0$, then the system is stratified controllable from $x_0$ in $M$.

*Proof:* Let $V_p$ be a neighborhood of the point $x_0$ in the submanifold $S_p$, which is the bottom stratum, i.e., the manifold of smallest dimension in the nested sequence at $x_0$. Choose $X_1 \in \Delta_{S_p}$. For $\epsilon_1$ sufficiently small

$$N_p^1 = \left\{\phi_{t_1}^{X_1}(x_0)\colon 0 < t_1 < \epsilon_1\right\}$$

is a smooth manifold of dimension one. This follows from, for example, the Flow-Box theorem ([23, Th. 2.26]), the Straightening Out Theorem ([1, Th. 4.1.14]) or the Orbit Theorem ([16, Th. 1, Ch. 2]).

Now, construct $N_p^j$ by induction. Assume that the collection of vector fields, $\{X_1, \ldots, X_{j-1}\}$, $X_i \in \Delta_{S_p}$ is such that the mapping

$$\begin{aligned}(t_1, \ldots, t_{j-1}) \mapsto \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)& \\ X_i \in \Delta_{S_p} \quad 0 < t_i < \epsilon_i&\end{aligned} \tag{4}$$

has rank $j-1$. Thus, by the immersion theorem (see, for example, [1, Th. 3.5.7], [23, Th. 2.19]), the set

$$N_p^{j-1} = \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0) \qquad 0 < t_i < \epsilon_i$$

is a $(j-1)$-dimensional manifold. Also, for the $\epsilon_i$ sufficiently small, $N_p^{j-1} \subset V_p$.

If $(j-1) < \dim(\overline{\Delta}_{S_p})$, then there exists $x \in N_p^{j-1}$ and $X_j \in \Delta_{S_p}$ such that $X_j(x) \notin T_x N_p^{j-1}$. If this were not so, then $\overline{\Delta}_{S_p} \subset T_x N_p^{j-1}$ for any $x$ in some open set $W \subset V_p$. This cannot be true since $\dim(\overline{\Delta}_{S_p}) > \dim(N_p^{j-1})$. Thus, for $\epsilon_j$ sufficiently small, the mapping

$$\begin{aligned}(t_1, \ldots, t_j) \mapsto \phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0),& \\ X_i \in \Delta_{S_p}, \quad 0 < t_i < \epsilon_i&\end{aligned} \tag{5}$$

has rank $j$. To see this, consider the tangent mapping

$$\begin{aligned}&T\left(\phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)\right) \\ &\quad = \Big[X_j\Phi_j(x_0) \quad \left(\phi_{t_j}^{X_j}\right)_* X_{j-1}\Phi_j(x_0) \quad \cdots \\ &\qquad \left(\phi_{t_j}^{X_j} \circ \cdots \circ \phi_{t_2}^{X_2}\right)_* X_1\Phi_j(x_0)\Big]\end{aligned}$$

where $\Phi_j(x_0) = \phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)$. We use the notation that for a diffeomorphism, $\phi$, and a vector field, $X$, the notation $\phi_* X = T\phi \circ X \circ \phi^{-1}$. If the rank of this matrix is not $j$, then we can write

$$X_j(\Phi_j(x_0)) = \sum_{i=1}^{j-1} \alpha_i \left(\phi_{t_j}^{X_j} \circ \cdots \circ \phi_{t_{i+1}}^{X_{i+1}}\right)_* X_i(\Phi_j(x_0))$$

for some coefficients, $\alpha_i$. However, if we pull this back along the flow of $X_j$, we have

$$\begin{aligned}&\left(\phi_{t_j}^{-X_j}\right)_* X_j(\Phi_j(x_0)) \\ &\quad = \sum_{i=1}^{j-1} \alpha_i \left(\phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_{i+1}}^{X_{i+1}}\right)_* X_i(\Phi_{j-1}(x_0)) \\ &\Longrightarrow X_j\left(\Phi_{j-1}(x_o)\right) \\ &\quad = \sum_{i=1}^{j-1} \alpha_i \left(\phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_{i+1}}^{X_{i+1}}\right)_* X_i\Phi_{j-1}(x_0)\end{aligned}$$

which contradicts the fact that $X_j \notin TN_p^{j-1}$. Thus

$$N_p^j = \{\phi_{t_j}^{X_j} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)\colon 0 < t_i < \epsilon_i, \, i = 1, \ldots, j\}$$

is a $j$ dimensional manifold. Since $\epsilon$ can be made arbitrarily small, we can assume $N_p^j \subset V_p$. If $k = n_p = \dim(\overline{\Delta}_{S_p})$, $N_p^k \subset V_p$ is an $n_p$-dimensional manifold.

Now, if we let $(s_1, \ldots, s_n)$ satisfy $0 < s_1 < \epsilon_i$ and consider the map

$$(t_1, \ldots, t_{n_p}) \mapsto \phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}, \quad 0 < t_i < \epsilon_i \tag{6}$$

since $\phi_s^{-X} = \phi_{-s}^X$, it follows that the image of this map is an open set of $N_n$ containing the point $x_0$. Hence, $\mathcal{R}^V(x_0, \epsilon)$ contains $x_0$ and an open set in the manifold whose tangent space is $\overline{\Delta}_{s_p}$. By restricting each $\epsilon \leq T/(2n_p)$, we can then find such an open set for any $T > 0$.

Thus far, we have constructed the reachable set for the system restricted to the bottom stratum, $S_p$. The process is to extend the reachable set by using vector fields defined on the next higher stratum, $S_{p-1}$, and then proceed to each higher stratum in order. We will proceed by induction. Assume that we have constructed the reachable set up to and including stratum $S_{k+1}$, and denote this reachable set by $N_{k+1}$. Without loss of generality, assume that

$$\dim\left(\sum_{j=k}^{p} \overline{\Delta}_{S_j}|_{x_0}\right) > \dim\left(\sum_{j=(k+1)}^{p} \overline{\Delta}_{S_j}|_{x_0}\right)$$

(otherwise, the control distribution, $\overline{\Delta}_{S_k}$ would not contribute any "new directions" to the reachable set, in which case we can proceed to the next higher stratum, $S_{k-1}$).

Now, let $n_i = \sum_{j=i}^{p} \dim\left(\overline{\Delta}_{S_i}\right)$, let the vector fields $X_1, \ldots, X_{n_p}$ be defined on $S_p$, let the vector fields $X_{n_p+1}, \ldots X_{n_{p-1}}$ be defined on $S_{p-1}$, etc. We will be considering compositions of flows of the following type:

$$\underbrace{\phi_{t_{n_k}}^{X_{n_k}} \circ \cdots \circ \phi_{t_{n_{k+1}+1}}^{X_{n_{k+1}+1}}}_{\text{on } S_k} \circ \underbrace{\cdots}_{\text{on } S_{k+1}, \ldots, S_{p-1}}$$
$$\underbrace{\circ\, \phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)}_{\text{on } S_p}$$

where the construction starts on the bottom stratum, $S_p$, using vector fields defined there, and proceeds to the higher strata in order.

We also assume (as part of the induction hypothesis) that the mapping

$$\begin{aligned}&(t_1, \ldots, t_{n_p}, \ldots, t_{n_{k+1}})\\ &\quad \mapsto \phi_{t_{n_{k+1}}}^{X_{n_{k+1}}} \circ \cdots \circ \phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}}\\ &\quad\quad \circ\, \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)\end{aligned}$$

has rank $n_{k+1}$, so the set

$$\begin{aligned}N_{k+1}^{n_{k+1}} = &\phi_{t_{n_{k+1}}}^{X_{n_{k+1}}} \circ \cdots \circ \phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}}\\ &\circ\, \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)\end{aligned}$$

is a $(n_{k+1})$-dimensional manifold.

By continuity, there exists a neighborhood, $V_k \subset S_k$ in which $\dim(\overline{\Delta}_{S_k})$ is constant. Since $\dim(N_{k+1}) < \dim(\sum_{j=k}^{p} \overline{\Delta}_{S_j}|_{x_0})$, there exits a vector field, $X \in \Delta_{S_k}$, and a point, $x \in V_k$, such that $X(x) \notin T_x N_k$. If this were not possible, then $X(x) \in T_x N_{k+1} \quad \forall X \in \Delta_{S_k}$ and $x \in V_k$. However, this implies that $\overline{\Delta}_{S_k} \subset TN_{k+1}$. Since, by construction, $\overline{\Delta}_{S_i} \subset TN_i \subset TN_{k+1}$ for $i = (k+1), \ldots, p$

$$\left(\sum_{j=k}^{p} \overline{\Delta}_{S_j}|_{x_0}\right) \subset T_{x_0} N_{k+1}$$

which implies that

$$\dim\left(\sum_{j=k}^{p} \overline{\Delta}_{S_j}|_{x_0}\right) \leq \dim(N_{k+1})$$

which is a contradiction.

By exactly the same argument as before, then, the set

$$\begin{aligned}N_k^{n_k} = &\phi_{s_{n_{k+1}+1}}^{-X_{n_{k+1}+1}} \circ \cdots \circ \phi_{s_{n_k}}^{-X_{n_k}} \circ \phi_{t_{n_k}}^{X_{n_k}} \circ \cdots \circ \phi_{s_1}^{-X_1}\\ &\circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)\end{aligned}$$

is an $n_k$-dimensional manifold containing the point $x_0$, and by construction $N_p^k \subset \mathcal{R}^V(x_0, \leq \epsilon_1 + \cdots + \epsilon_{n_p})$. Hence, $\mathcal{R}^V(x_0, \epsilon)$ contains $x_0$ and an open set in the manifold whose tangent space is $\overline{\Delta}_{s_p}$. By restricting each $\epsilon \leq T/(2n_p)$, we can then find such an open set for any $T > 0$. ■

Note that it is not necessary that the nested sequence actually include the full configuration space $M$. It may, in fact, terminate at some stratum, $S_p$. In such a case, however, controllability amounts to reaching an open neighborhood of the starting point in the highest stratum, $S_p$. Also, note that if the configuration space has a boundary, Proposition 4.4 still applies with a simple modification of the proof. In a manner similar to that in the proof, when extending the reachable set from the submanifold boundary into the manifold in which it is contained, we can always choose the first vector field along which the system flows to be the one that violates the constraint $d\Phi_i(x)\dot{x} = 0$, in the "allowable" direction. However, in the constructed "reversed" flow (6), we do not include this reversed flow corresponding to this vector field which moves the system off of the boundary. In this manner, the final constructed manifold contains $x_0$ and will be an open neighborhood of $x_0$ defined in the appropriate relative topology, i.e., the topology of a manifold with boundary.

Proposition 4.4 only directly applies to a single nested sequence of strata; however, repeatedly applying the test to multiple sequences is possible. The usefulness of this approach is that if the top stratum in each sequence is different, then the test determines controllability for the *union* of the top strata. For example, for the configuration space shown in Fig. 4, Proposition 4.4 applied to the sequence $S_{12} \subset S_1$ will tell if the system can reach an open set in $S_1$ and applied to $S_{12} \subset S_2$ will tell if it can reach an open set in $S_2$, and taken together, gives controllability in the relative topology of the union $S_1 \cup S_2$. This is useful because, for problems like the biped from Example 2.1, reaching open sets in the relative topology of the union of strata is often the most natural way to define controllability.

## V. An Example

Because the kinematic leg example was so simple, it is instructive to include a more complicated example. The following is adapted from [18]. Consider the six-legged robot shown in Figs. 1 and 2. It will be clear from the equations of motion for the system that each leg has only two degrees of freedom. In particular, the leg can move "up and down" and "forward and backward," but *not* "side to side" (in a direction outward from the body). In such a case it is not obvious how the robot can move in any direction.

Assume that the robot walks with a *tripod gait*, alternating movements of legs 1-4-5 with movements of legs 2-3-6. Hence, we are considering motions in only a subset of all possible strata. Suppose that

$$\begin{aligned}
\dot{x} &= \cos\theta(\alpha(h_1)u_1 + \beta(h_2)u_2)\\
\dot{y} &= \sin\theta(\alpha(h_1)u_1 + \beta(h_2)u_2)\\
\dot{\theta} &= l\alpha(h_1)u_1 - l\beta(h_2)u_2\\
\dot{\phi}_1 &= u_1\\
\dot{\phi}_2 &= u_2\\
\dot{h}_1 &= u_3\\
\dot{h}_2 &= u_4
\end{aligned}$$

where

| | |
|---|---|
| $(x, y, \theta)$ | planar position of the robot's center; |
| $\phi_i$ | front to back angular deflection of the legs; |
| $h_i$ | height of the legs off the ground. |

The tripod gait assumption requires that all the legs in a tripod move with the same angle $\dot{\phi}_i$. The inputs $u_1$ and $u_2$ control the leg swing velocities, while the inputs $u_3$ and $u_4$ control the leg lifting velocities.

The functions $\alpha(h_1)$ and $\beta(h_2)$ are defined by

$$\alpha(h_1) = \begin{cases} 1 & \text{if } h_1 = 0 \\ 0 & \text{if } h_1 > 0 \end{cases} \qquad \beta(h_2) = \begin{cases} 1 & \text{if } h_2 = 0 \\ 0 & \text{if } h_2 > 0. \end{cases}$$

Denote the stratum when all the feet are in contact ($\alpha = \beta = 1$) by $S_{12}$ (short for $S_{123456}$), the stratum when leg one is in contact ($\alpha = 1, \beta = 0$), by $S_1$ (short for $S_{145}$), the stratum when leg two is in contact ($\alpha = 0, \beta = 1$), by $S_2$ (short for $S_{236}$). and the stratum when no legs are in contact ($\alpha = \beta = 0$), by $S_0$.

If all legs are in contact with the ground, the equations of motion are

$$\begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}_1\\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta & 0 & 0\\ \sin\theta & \sin\theta & 0 & 0\\ l & -l & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{pmatrix} \tag{7}$$

where $u_3$ and $u_4$ are constrained to be 0. Note that if $f$ represents the first column, and $g$ represents the second column, then

$$[f, g] = \begin{pmatrix} -2l\sin\theta\\ 2l\cos\theta\\ 0\\ 0\\ 0 \end{pmatrix} \quad \text{and} \quad [[f, g], f] = \begin{pmatrix} 2l^2\cos\theta\\ 2l^2\sin\theta\\ 0\\ 0\\ 0 \end{pmatrix}. \tag{8}$$

Clearly, with all the legs in contact with the ground, these vector fields span the $(x, y, \theta)$ directions. However, at this point we have not generated enough directions to simultaneously control the *shape* of the robot as well.

If legs 1, 4 and 5 are in contact with the ground, but legs 2, 3 and 6 are not in contact, the equations of motion are

$$\begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}_1\\ \dot{\phi}_2\\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 & 0\\ \sin\theta & 0 & 0 & 0\\ l & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{pmatrix} \tag{9}$$

where $u_3$ is constrained to be 0.

If legs 2, 3 and 6 are in contact with the ground and legs 1, 4 and 5 are not, then the equations of motion are

$$\begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}_1\\ \dot{\phi}_2\\ \dot{h}_1 \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta & 0 & 0\\ 0 & \sin\theta & 0 & 0\\ 0 & -l & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{pmatrix} \tag{10}$$

where $u_4$ is constrained to be 0.

If none of the legs are in contact with the ground

$$\begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}_1\\ \dot{\phi}_2\\ \dot{h}_1\\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{pmatrix}. \tag{11}$$

If we consider either the distributions associated with the sequence $S_{12} \subset S_1 \subset S_0$ or $S_{12} \subset S_2 \subset S_0$, the distributions spanned by the vector fields comprising (7)–(9) and (11), or the distributions spanned by the vector fields comprising (7), (8), (10), and (11), respectively, the hypotheses of Proposition 4.4 are satisfied. Note that this example has the somewhat unrealistic requirement of considering the equations of motion when none of the feet are in contact with the ground. In fact, this is *required* for controllability in the entire configuration space since

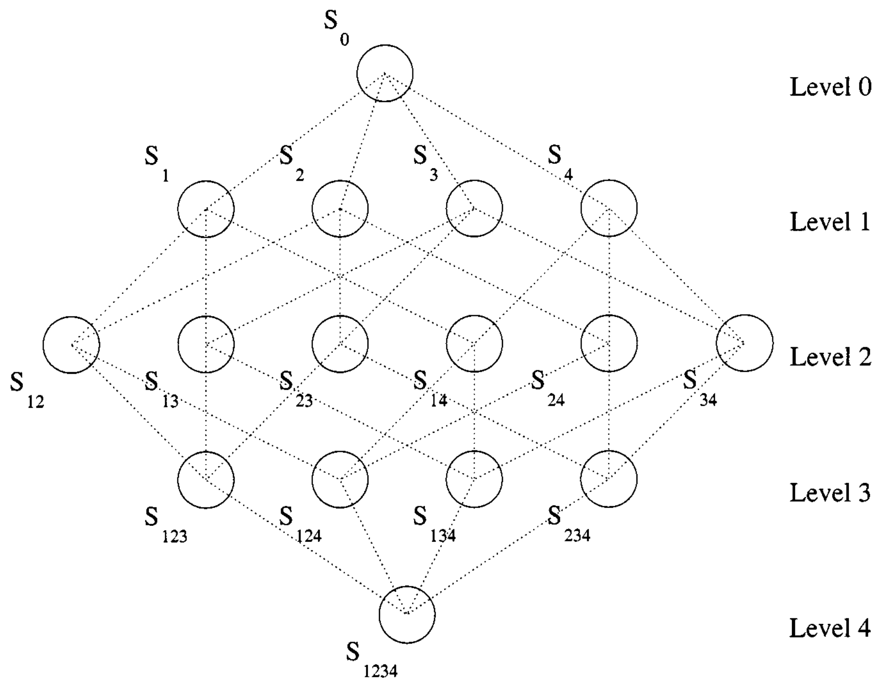


Fig. 7. Four-level stratification.

both leg heights are variables. Since it is undesirable to lift all the feet of the robot out of contact with the ground at once, a better notion of controllability may be to ask that the system reach an open set in the union $S_1 \cup S_2$. Thus, we need to consider the nested sequences $S_{12} \subset S_1$ and $S_{12} \subset S_2$ simultaneously. From (7)–(9), the sum of the associated distributions is six dimensional, as is the sum from (7), (8), and (10). Thus, the system is controllable because it can reach an open neighborhood of a starting point in the bottom strata defined in the relative topology of the union $S_1 \cup S_2$.

## VI. Stratified Systems and Gait Locomotion

This section extends the previous results to overcome the limitation in Proposition 4.4 which considered only the geometry of a nested sequence of submanifolds, thus, possibly excluding the effect of cycling through multiple submanifolds with the same codimension. Assume that at point, $x_0$, the stratum $S_B = S_{i_1 i_2 \ldots i_n}$ is the bottom stratum. We will refer to the *level* of the stratum as its codimension. Thus, the bottom stratum is on the $n$th level, the $(n-1)$th level contains all the strata with codimension $(n-1)$, and so forth. Fig. 7 illustrates the combinatorial structure of a stratification with four levels. In Fig. 7, the nodes of the graph correspond to the different strata. The edges connecting the nodes indicate whether it is possible for the system to move from one stratum to another, i.e., if the nodes are connected by an edge, then the system can move between the strata, if there is no edge, then the system cannot move between the strata. Note that, while the figure simply illustrates edges between nodes only one level apart, it may be the case that multilevel jumps are possible, in which case there would be an edge connecting strata of two levels that are more than one level apart.

If there are $n$ codimension one strata, then the total number of strata is

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$$

which clearly increases quickly with $n$. The corresponding graph structure also grows similarly in complexity. Even with this simplistic pictorial view, it is evident that the a general stratified configuration space is characterized by an interesting algebraic structure. Specifically, as illustrated by the dotted lines connecting the strata, there is an naturally defined graph structure in which to consider the problem. Note that one way to consider a *gait* is simply a choice of a cyclic path through this graph structure, denoted

$$\mathcal{G} = \{S_{I_1}, S_{I_2}, \ldots, S_{I_n}, S_{I_{n+1}} = S_{I_1}\}. \qquad (12)$$

In this ordered sequence, the first and last element are identical, indicating that the gait is a closed loop. Clearly, in order for the gait to be meaningful, it must be possible for the system to switch from stratum $S_{I_i}$ to $S_{I_{i+1}}$ for each $i$. In Fig. 7, this corresponds to each stratum $S_{I_i}$ in the sequence being connected to $S_{I_{i+1}}$ and $S_{I_n}$ being connected to $S_{I_1}$. Limitations on gaits, such as stability requirements, could be expressed as limitations (possibly as a function of configuration) on the cyclic gait paths.

Assume that we know the physical constraints on the system and the manner by which these constraints are manifested as constraints in its graph representation. In other words, assume that there is a collection of strata (or nodes), $\mathcal{S} = \{S_{I_1}, S_{I_2}, \ldots, S_{I_n}\}$ which are deemed "permissible," and similarly a collection of "permissible" edges connecting the nodes, denoted by

$$\mathcal{C} = \{(S_{I_1}, S_{J_1}), (S_{I_2}, S_{J_2}), \ldots, (S_{I_n}, S_{J_n})\}.$$

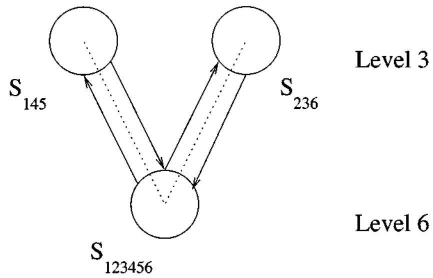


Fig. 8. The simplified hexapod graph.

Which strata and edges are permissible may, of course, be a function of the configuration of the system. Whether a stratum is permissible depends upon whether the equations of motion for the system can be expressed in kinematic form in a neighborhood of the point of interest. For example, for a biped robot, clearly if it lifts both feet off of the ground, it is not a kinematic system because the fact that gravity will make it fall back to the ground.

*Example 6.1 (Hexapod—Revisited):* Recall that the hexapod example in Section V, assumed that the hexapod walked with a tripod gait. That assumption reduced the high dimensional and complex graph structure of the system to a very low dimensional and simple one, as illustrated in Fig. 8. The arrows in the figure represent the tripod gait. Note that, for this problem, it will always be possible for the system to move from a higher stratum onto the bottom stratum. This is manifested in the fact that the robot can always put its feet on the ground regardless of its configuration.

The gait controllability result in the next subsection makes particular assumptions regarding transversality of manifolds. Recall that two submanifolds intersect *transversally* if

$$T_x S_1 + T_x S_2 = T_x M \tag{13}$$

where $S_1$ and $S_2$ are submanifolds of $M$ and $x \in S_1 \cap S_2$. If $S_1$ and $S_2$ are transversal, then following theorem ( [1, Corollary 3.5.13]) is useful.

*Lemma 6.2:* If $S_1$ and $S_2$ are transversal and have finite codimension in $M$, then

$$\operatorname{codim}(S_1 \cap S_2) = \operatorname{codim}(S_1) + \operatorname{codim}(S_2).$$

### *A. Gait Controllability*

This section considers the problem of whether a particular gait is controllable. In this section, we will limit our attention to a particular form of controllability; namely, *gait controllability*. Assume that if $S_{I_{i+1}} \subset S_{I_i}$, then the reachable set of points in $S_{I_i}$ is transversal to the substratum, $S_{I_{i+1}}$. This will always be the case if $\dim(S_{I_{i+1}}) = \dim(S_{I_i}) - 1$ because, since the gait must be allowable, then there must be at least one vector field in $\overline{\Delta}_{S_i}$ along which the systems can flow on or off of $S_{I_{i+1}}$. Switches between strata with dimensions which vary by more than one are allowable as long as this transversality assumption is satisfied.

In the complete stratified structure, there is one bottom stratum, defined by the intersection of all the codimension-1 strata in the configuration space. In Fig. 7, this corresponds to stratum $S_{1234}$. For a locomotion system, such as a legged robot, this bottom stratum corresponds to the set of points in the configuration space where all the feet are in contact with the ground. Now, for gait controllability, the reachable set, $R^V(x_0, \leq T)$, is defined as before, but is restricted to control inputs consistent with the gait, i.e., the reachable set must be consistent with the ordering of the strata that define the gait.

*Definition 6.3 (Gait Controllability):* A gait, $\mathcal{G} = \{S_{I_1}, S_{I_2}, \ldots, S_{I_n}, S_{I_1}\}$ is *gait controllable* from the point $x_0$ if the reachable set $R^V(x_0, \leq T)$ (defined in (1) and consistent with the gait) contains a neighborhood of $x_0$ for all neighborhoods $V$ of $x_0$ and $T > 0$, where the neighborhood is defined by the topology of the lowest stratum, $S_B$.

*Example 6.4 (Kinematic Leg—Revisited):* In the simple kinematic leg example, Example 4.1, illustrated in Fig. 5, the bottom stratum is the set of points $q = (x, l, \theta)$ such that

$$l \cos\theta = h \tag{14}$$

for some fixed height, $h$. This is most naturally parameterized by the variables $x$ and $\theta$, and so an open set in $S$ corresponds to reaching an open neighborhood of $x$ and $\theta$, where $l$ is subject to the constraint expressed by (14).

Let $\overline{\Delta}_I$ denote the involutive closure of the control distribution on $S_I$, where the subscripted index for $\overline{\Delta}_I$ corresponds to the subscripted index for the stratum $S_I$ to which it is associated. Given a gait, $\mathcal{G}$, the *gait distribution* is constructed as follows. First, let $\mathcal{D}_1 = \overline{\Delta}_{S_{I_1}}$. If $S_{I_1} \subset S_{I_2}$, then let $\mathcal{D}_2 = \mathcal{D}_1 + \overline{\Delta}_{S_2}$ (implicitly assuming the appropriate inclusion of $\mathcal{D}_1$ into $S_2$); else, if $S_{I_2} \subset S_{I_1}$, then let $\mathcal{D}_2 = (\mathcal{D}_1 \cap TS_2) + \overline{\Delta}_{S_2}$. In general, then, $\mathcal{D}_i = \mathcal{D}_{i-1} + \overline{\Delta}_{S_i}$ if $S_{I_{i-1}} \subset S_{I_i}$, and $\mathcal{D}_i = (\mathcal{D}_{i-1} \cap TS_i) + \overline{\Delta}_{S_i}$ if $S_{I_i} \subset S_{I_{i-1}}$. Following this procedure for each stratum in the gait (in the order of the gait)gives the gait distribution.

*Proposition 6.5:* If

$$\dim(\mathcal{D}_{n+1}) = \dim T_{x_0} S_{I_1}$$

then the system is gait controllable from $x_0$.

*Proof:* The proof relies on one corollary to Proposition 4.4 and one lemma.

*Corollary 6.6:* In the construction of the gait distribution, if $S_{I_i} \subset S_{I_{i+1}}$, then the dimension of the reachable set increases by the same amount as the increase in dimension between $\mathcal{D}_{I_i}$ and $\mathcal{D}_{I_{i+1}}$ and contains the point $x_0$.

*Lemma 6.7:* In the construction of the gait distribution, if $S_{I_{i+1}} \subset S_{I_i}$, then the dimension of the reachable set increases by the same amount as the increase in dimension between $\mathcal{D}_{I_i}$ and $\mathcal{D}_{I_{i+1}}$ minus the difference between the dimensions of $S_{I_{i+1}}$ and $S_{I_i}$.

*Proof:* This follows from the transversality assumption and the codimension result of Lemma 6.2. ▼

It follows that in the construction of the gait distribution that the dimension of the reachable set will be the dimension of $\mathcal{D}_{n+1}$. If the first and last strata in the gait $\mathcal{G}$ is the bottom stratum, then the result follows since the reachable set it contained in $S_{I_1}$ and has dimension equal to the dimension of $S_{I_1}$. ■

### B. Gait Controllability of the Hexapod Robot Example

This section returns to the hexapod robot example considered in Section V, but considers gait controllability, as opposed to regular controllability. The first step is to construct the gait distribution. Take as the gait the following sequence of strata:

$$\mathcal{G} = \{S_{123456},\, S_{145},\, S_{123456},\, S_{236},\, S_{123456}\}$$

as illustrated in Fig. 8. To simplify notation, let $S_{12} = S_{123456}$, $S_1 = S_{145}$ and $S_2 = S_{236}$. The equations of motion for the system restricted to the bottom stratum, $S_{12}$ are given in (7). Also, a Lie bracket is necessary to construct $\overline{\Delta}_{12}$, as given in (8). By inspection, $\overline{\Delta}_{12} = \mathcal{D}_1$ has a dimension of three. Next extend the construction to $S_1$. Since $S_{12} \subset S_1$, $\mathcal{D}_2 = \mathcal{D}_1 + \overline{\Delta}_1$, where $\overline{\Delta}_1$ is determined from (9). By inspection, then, $\dim(\mathcal{D}_2) = 5$.

The construction next returns to the bottom stratum, $S_{12}$. Note that $S_{12}$ is a codimension-1 submanifold of $S_1$. Also, since $\mathcal{D}_2$ contains the basis vector $\partial/\partial h_1$, it is clear that the transversality assumption is satisfied. Therefore, $\dim(\mathcal{D}_3) = \dim(\mathcal{D}_2) - 1 = 4$. The construction is next extended to stratum $S_2$. As with $S_1$, $S_2$ increases the dimension of $\mathcal{D}_4$ by two, so that $\dim(\mathcal{D}_4) = 6$. "Projecting" this back down to $S_{12}$ as before gives the dimension of the reachable set to be five, which is the dimension of $S_{12}$. Therefore, the hexapod example is gait controllable.

## VII. Conclusion

In this paper, we have formulated a controllability test for systems with stratified configuration spaces. Such a stratified structure provides a means to model many physical systems with governing equations which are discontinuous across subsets of the state space. The general philosophy underlying these extensions was to exploit the particular structure of stratified configuration spaces, which, loosely speaking, allowed us to simultaneously consider the equations of motion for the system on each strata. The examples contained herein illustrated both the steps involved in applying the tests as well as their ease of use.

Several avenues of potentially fruitful further work could be based upon the results in this paper. First, throughout this paper, we have restricted our attention to driftless control systems. Although a much harder problem, controllability tests for smooth systems with drift exist [29], and could potentially be extended to stratified configuration spaces. One difficulty with simply extending the test in [29] is that the test only provides a sufficient condition for controllability. In the case where there is a large number of strata, one is faced with the prospect of the need to satisfy a sufficient condition a large number of times. This is problematic to the extent that sufficient conditions are, generally, too restrictive, in which case, if the test needs to be satisfied multiple times, the restrictive nature of the sufficient conditions are similarly multiplied.

Another possible extension of this work is the trajectory generation, or motion planning problem. While controllability is an important issue from the point of view of establishing a logical framework in which to consider problems in nonlinear control theory, from a practical point of view it is of limited usefulness. At least in the robotics context, the motion planning problem is of paramount importance. In fact, the authors have some preliminary results in this area [13].

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