RESPONSE OF A NOZZLE TO AN ENTROPY DISTURBANCE EXAMPLE OF THERMODYNAMICALLY UNSTEADY AERODYNAMICS

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1. INTRODUCTION

The larger number of problems that qualify as unsteady aerodynamics relate to non-uniform motion of surfaces -- such as pitching of airfoils -- or the correspondingly non-uniform motion of a fluid about a surface -- such as a gust passing over an airfoil. Experiment and analysis concerning these problems aims to determine the non-steady forces or surface stresses on the object. These may be thought of as "kinematically" non-steady problems. Another class of problems presents itself when the undisturbed gas stream temperature (or density) is non-steady although the velocity and pressure are steady; such non-uniformities are associated with entropy variations from point to point of the stream. In a locally adiabatic flow these entropy variations are transported with the stream, and when a fixed boundary -- such as an airfoil -- is encountered, the flow field undergoes a non-steady change because the density variations alter the pressure field -- or the stresses at the boundaries. This happens in spite of the fact that the undisturbed free-stream velocity field and the surface boundaries of the flow are independent of time. A gas turbine blade, for example, will experience a time-dependent load simply because of temperature fluctuations in the combustion products flowing over it, although the angle of attack is independent of time. We shall call these "thermodynamically" unsteady flows in contrast with the more familiar kinematically unsteady flows.

It is our aim to examine one case of thermodynamically unsteady flows related to non-uniform entropy regions passing through a nozzle in a ducted flow. It has a relative simplicity which derives from the fact that interesting aspects may be treated as one-dimensional. In spite of this restriction, it demonstrates the character of the problem and has a direct application to the generation of pressure waves in ducts associated with combustion systems.

2. PERTURBATION THEORY WITH NON-UNIFORM ENTROPY

One-dimensional flow in a channel of variable cross-sectional area A(x) satisfies the familiar equations

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) + \frac{\partial u}{\partial x} = -u \cdot \frac{1}{A}\frac{dA}{dx} \qquad (1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2}$$

$$\left\{\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right\} \log\left[(p^{1/\gamma}/\rho)\exp(-s/c_p)\right] = 0 \tag{3}$$

$$T\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x}\right) = Q \tag{4}$$

where Q is the sensible heat addition rate as one follows a unit volume of fluid. For locally adiabatic flow $Q = 0$, so that when the further restriction is made that fluid of uniform entropy is supplied to the system, equation (3) leads to the familiar isentropic law which holds over the entire fluid medium. Our special interest centers on the circumstances where one or both of these conditions are not met.

If, for a given duct geometry, a solution ρ^o, p^o, u^o, s^o is known, in which these variables may be functions of both x and t, then a solution which departs from this only slightly -- possibly due to a time-dependent perturbation to the condition far upstream -- is conveniently expressed $\rho = \rho^o(1+\rho'/\rho^o)$, $p = p^o(1+p'/p^o)$, $u = u^o(1+u'/u^o)$, and $s = s^o(1+s'/s^o)$. Considering the primed terms as perturbations, the first-order perturbation terms are easily written down:

$$\left\{\frac{\partial}{\partial t} + u^o\frac{\partial}{\partial x}\right\}(\rho'/\rho^o) + u^o\frac{\partial}{\partial x}(u'/u^o) = 0 \tag{5}$$

$$u^o\left\{\frac{\partial}{\partial t} + u^o\frac{\partial}{\partial x}\right\}(u'/u^o) + \frac{\gamma p^o}{\rho^o}\frac{\partial}{\partial x}(p'/\gamma p^o)$$

$$+\left(\frac{\partial u^o}{\partial t} + u^o\frac{\partial u^o}{\partial x}\right)(u'/u^o) + \frac{1}{\rho^o}\frac{\partial p^o}{\partial x}(p'/p^o - \rho'/\rho^o) = 0 \tag{6}$$

$$\left\{\frac{\partial}{\partial t} + u^o\frac{\partial}{\partial x}\right\}\left[\frac{p'}{\gamma p^o} - \frac{\rho'}{\rho^o} - \frac{s'}{c_p}\right] + \left[u^o\frac{\partial}{\partial x}\log\left(\frac{(p^o)^{\frac{1}{\gamma}}}{\rho^o}\cdot\exp\left(-\frac{s^o}{c_p}\right)\right)\right]\frac{u'}{u^o} = 0 \tag{7}$$

And, in particular, if the heat addition is at most a perturbation term $Q'(x,t)$, then

$$\left\{\frac{\partial}{\partial t} + u^o\frac{\partial}{\partial x}\right\}(s'/c_p) + u^o\frac{\partial s^o/c_p}{\partial x}(u'/u^o) = Q'/c_pT^o \tag{8}$$

Consider first the case where the undisturbed flow is uniform and steady, so that p^o, u^o are constant, but that a large temperature variation enters from far upstream, leading to the condition

$$\begin{aligned} s^o &= s^o(t - x/u^o) \\ \rho^o &= \rho^o(t - x/u^o) \end{aligned} \tag{9}$$

and consequently to a varying sound speed

$$c^o = (\gamma p^o/\rho^o)^{\frac{1}{2}} = c^o(t - x/u^o) \ . \qquad (10)$$

Then the pressure perturbation satisfies an acoustic-type equation with variable sound speed

$$\left\{\frac{\partial}{\partial t} + u^o \frac{\partial}{\partial x}\right\}^2 (p'/\gamma p^o) - (c^o)^2 \frac{\partial^2}{\partial x^2}(p'/\gamma p^o) - \frac{\partial (c^o)^2}{\partial x} \cdot \frac{\partial}{\partial x}(p'/\gamma p_o) = 0 \ . \qquad (11)$$

This leads to internal reflection and refraction of disturbances generated at the boundaries or far upstream, but no production of disturbance in the proper sense. Note, however, that the entropy -- and hence density -- variations must be large so that $\partial c^o/\partial x$ is of zeroth order, for otherwise the acoustic velocity variation leads to only second-order contributions.

Consider now the circumstance where the undisturbed flow is a steady isentropic one so that s^o and $(p^o)^{1/\gamma}/\rho^o$ are constant, while p^o, u^o, and ρ^o are functions of x. If, in addition, we restrict ourselves to perturbations that are locally adiabatic -- so that $\dot{Q}' = 0$, equation (8) may be integrated to

$$\frac{s'}{c_p} = \frac{s'}{c_p}\left(t - \int^x \frac{d\zeta}{u^o}\right) \ , \qquad (12)$$

which states physically that the entropy disturbance is transported, unmodified, with the gas stream and is fixed by its prescription far upstream. In addition, this permits writing

$$\frac{p'}{\gamma p^o} - \frac{\rho'}{\rho^o} = \frac{s'}{c_p}\left(t - \int^x \frac{d\zeta}{u^o}\right) \qquad (13)$$

from equation (7), the first law of thermodynamics. Now the density perturbation may be eliminated from equations of continuity (5) and momentum (6) to give a pair of relations,

$$\left\{\frac{\partial}{\partial t} + u^o \frac{\partial}{\partial x}\right\}\left(\frac{p'}{\gamma p_o}\right) + u^o \frac{\partial}{\partial x}\left(\frac{u'}{u^o}\right) = 0 \ , \qquad (14)$$

$$\left\{\frac{\partial}{\partial t} + u^o \frac{\partial}{\partial x}\right\}\left(\frac{u'}{u^o}\right) + \frac{(c^o)^2}{u^o}\frac{\partial}{\partial x}\left(\frac{p'}{\gamma p_o}\right)$$

$$+ \left(2\frac{u'}{u^o} - (\gamma - 1)\frac{p'}{\gamma p_o}\right)\frac{du^o}{dx} = \frac{du^o}{dx} \cdot \frac{s'}{c_p} \qquad (15)$$

These differ from the familiar first-order acoustic equations in two respects, the first of which is the presence (on the left hand side of equation (15)) of terms that represent refraction and reflection of the field by the duct geometry and mean flow gradients. Of more novelty and interest is the dipole source term on the right hand side of the momentum equation. This term arises from the unsteady interaction of the entropy disturbance, which moves into the problem from far upstream, and the strong gradient of the undisturbed flow induced by the geometric boundaries. Note again, if the undisturbed flow field were a perturbation from uniform, the dipole term would be of second order.

It is this mechanism -- represented by the interaction of entropy "spots" with strong state gradients, that we wish to examine in some detail, utilizing the unsteady nozzle flow as an example.

3. THEORY OF THE COMPACT NOZZLE

It is our aim in this section to examine the disturbance that accompanies the motion of a gas with spatially non-uniform entropy as it passes through a supercritical nozzle. This example of the second case discussed in Section 2 will provide an instructive characterization of the time-dependent interaction of thermodynamic non-uniformities with strong state gradients and provides further intuitive insight into the mechanism. For one-dimensional gasdynamics it has been shown that equilibrium mass flow through a choked nozzle varies directly as the stagnation pressure and inversely as the square root of the stagnation temperature of the gas entering the nozzle. Therefore, a variation of the inlet state results in a corresponding variation in quasi-steady mass discharge rate. This behavior is accompanied by variation of momentum flux and hence a force acting parallel to the direction of flow having a dipole-like character. As a consequence, kinematical disturbances are generated both ahead and downstream of the nozzle, and it is these disturbances that interest us here.

In this section we present a theory describing the response of supercritical nozzles to normal entropy waves when the nozzles may be considered compact. By compact we mean specifically that the shortest wave length that appears in the flow field is long in comparison with nozzle dimensions. It is a fortunate consequence of this compactness that while wave motion in constant-area ducts ahead of and following the nozzle must be considered, the details of the nozzle flow may be replaced by matching conditions across its short length and treated, in fact, similar to the conventional manner for shock waves.

Thus, for one-dimensional, small amplitude waves in a duct of uniform cross section and constant gas velocity U and acoustic velocity c , equations (5) - (8) yield

$$\left\{ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right\} \left(\frac{\rho'}{\rho} \right) + c \frac{\partial}{\partial x} \left(\frac{u'}{c} \right) = 0 \tag{16}$$

$$\left\{ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right\} \left(\frac{u'}{c} \right) + c \frac{\partial}{\partial x} \left(\frac{p'}{\gamma p} \right) = 0 \tag{17}$$

$$\left\{ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right\} \left(\frac{p'}{\gamma p} - \frac{\rho'}{\rho} \right) = \left\{ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right\} \left(\frac{s'}{c_p} \right) \tag{18}$$

where the primed variables denote perturbation from the unprimed uniform states and s' is the convected entropy disturbance, presumed known at an upstream station. Then the pressure and velocity satisfy conventional acoustic wave equations in a moving medium and take the form

$$\frac{p'}{\gamma p} = P^{+}\exp\left\{i\omega\left[t-\frac{x}{U+c}\right]\right\}+P^{-}\exp\left\{i\omega\left[t-\frac{x}{U-c}\right]\right\} \tag{19}$$

$$\frac{u'}{c} = U^{+}\exp\left\{i\omega\left[t-\frac{x}{U+c}\right]\right\}+U^{-}\exp\left\{i\omega\left[t-\frac{x}{U-c}\right]\right\} \tag{20}$$

where the plus and minus superscripts denote waves propagating, relative to the gas, in the direction and opposite the direction of flow, respectively. Substitution of (19) and (20) into (16) shows that the various coefficients are not independent but that $U^{+} = P^{+}$ and $U^{-} = -P^{-}$ so that the velocity perturbation is conveniently written

$$\frac{u'}{c} = P^{+}\exp\left\{i\omega\left[t-\frac{x}{U+c}\right]\right\}-P^{-}\exp\left\{i\omega\left[t-\frac{x}{U-c}\right]\right\} . \tag{21}$$

On the other hand, the entropy has a solution of frequency ω convected at the stream velocity

$$\frac{s'}{c_p} = \sigma\exp\left\{i\omega\left[t-\frac{x}{U}\right]\right\} , \tag{22}$$

and as a consequence, the temperature and density perturbations consist of <u>both</u> propagating and convecting parts, e.g.

$$\frac{\rho'}{\rho} = P^{+}\exp\left\{i\omega\left[t-\frac{x}{U+c}\right]\right\}+P^{-}\exp\left\{i\omega\left[t-\frac{x}{U-c}\right]\right\}-\sigma\exp\left\{i\omega\left[t-\frac{x}{U}\right]\right\} . \tag{23}$$

Now the flow in the uniform channel upstream of the nozzle is given by relations (19), (21), (22), and (23) with the appropriate values P_1^{+}, P_1^{-}, σ_1 of the multiplicative constants and the corresponding flow downstream with the corresponding values P_2^{+}, P_2^{-}, and σ_2. Within the nozzle itself, because it is compact, the flow may be considered quasi-steady, in contrast to the processes taking place in the regions upstream and downstream. Statements of the required conservation relations across the nozzle will provide matching conditions between the upstream and downstream wave motions.

First, the (quasi-steady) variation in mass flow is identical at the inlet and outlet and, because the nozzle geometry is fixed, this variation is proportional to the fractional variation in density and velocity. Thus, the quantity

$$\frac{1}{M}\frac{u'}{c}+\frac{\rho'}{\rho} \tag{24}$$

has the same value at the nozzle entrance and exit where, in applying the condition, the appropriate different values of M, c, ρ are employed upstream and downstream. Second, it is assumed that the entropy is constant through the nozzle, implying that negligible heat exchange or losses are taking place -- in particular, we imply an absence of shock waves. Then clearly,

$$s'/c_p \tag{25}$$

has the same value at inlet and outlet.

The same assumptions guarantee that the stagnation temperature is identical upstream and downstream, and thus the fractional variation of stagnation temperature

$$\frac{1}{1+\frac{\gamma-1}{2}M^2}\left\{\gamma\left(\frac{p'}{\gamma p}\right)-\frac{\rho'}{\rho}+(\gamma-1)M\frac{u'}{c}\right\} \tag{26}$$

is identical when computed from either the upstream or downstream wave solutions.

These three conditions described above hold for any adiacatic, loss-free, compact element, whether it be nozzle or diffuser, choked or not. The unique property of the supersonic nozzle is that its mass flow is directly proportional to the stagnation pressure and inversely proportional to the square root of stagnation temperature in the approach flow. This leads to the expression for the fractional mass flow variation

$$\frac{1}{1+\frac{\gamma-1}{2}M^2}\left\{\frac{\gamma}{2}(1-M^2)\frac{p'}{\gamma p}+\frac{\gamma+1}{2}M\frac{u'}{c}+\frac{1}{2}(1+\gamma M^2)\frac{\rho'}{\rho}\right\}$$

which, when element is adiabatic and loss-free, holds both upstream and down. In combination with the corresponding -- and independent -- calculation of fractional variation of mean flow from the continuity equation (24), this provides the equation

$$\frac{u'}{c}-\frac{\gamma}{2}M\left(\frac{p'}{\gamma p}\right)+\frac{1}{2}M\frac{\rho'}{\rho}=0 \tag{27}$$

which holds independently at the nozzle entrance and exit. It is the latter fact, that it becomes a boundary condition on each flow field rather than a boundary condition between them, which leads to the unique character of the results for the supercritical nozzle. In addition, equation (27) permits simplification of the remaining matching conditions to the statements that

$$p'/\gamma p \tag{28}$$

and the entropy disturbance, equation (25), are identical at the nozzle inlet and outlet, completing the matching conditions for the supercritical, adiabatic, loss-free nozzle.

Consider first a plane acoustic wave of angular frequency ω moving toward the nozzle from upstream to the right, where the reference field quantities are denoted by subscript 1. We shall call it of strength ε if the coefficient P_1^+ of the streamwise moving wave is equal to ε ; $P_1^+ = U_1^+ = \varepsilon$ and $\sigma \equiv 0$ because the disturbance is an isentropic (acoustic) wave. Then referring to expression (27) for the choked nozzle, we simply evaluate u'/c_1 and $p'/\gamma p_1$ at $x = 0$ from the solutions (19, 20), with the result that the strength P_1^- of the acoustic wave reflected upstream is

$$P_1^- = \frac{1 - \frac{\gamma-1}{2} M_1}{1 + \frac{\gamma-1}{2} M_1} \, \varepsilon \; . \tag{29}$$

The reflection coefficient ranges between unity for a very low approach Mach number toward $1 - (\gamma-1)/(\gamma+1)$ as $M_1 \to 1$, a value which the Mach number cannot attain under our assumptions. The value of unity for low M_1 simply means that the ratio of approach area to choked nozzle area is very large and to the approaching acoustic wave, the nozzle appears very much like a solid wall. The fractional pressure fluctuation at the nozzle inlet, $\frac{p'}{\gamma p_1}(0-, t)$, is identical by our matching condition (28) to the corresponding value $\frac{p'}{\gamma p_2}(0+, t)$ at the nozzle discharge; and since the former is known,

$$\frac{p'}{\gamma p_2}(0+, t) = \frac{2}{1 + \frac{\gamma-1}{2} M_1} \, \varepsilon \, e^{i\omega t} \; . \tag{30}$$

Moreover, the condition (27) holds independently downstream of the nozzle so that from relations among the coefficients, we find directly

$$P_2^+ = \frac{1 + \frac{\gamma-1}{2} M_2}{1 - \frac{\gamma-1}{2} M_1} \, \varepsilon \tag{31}$$

$$P_2^- = \frac{1 - \frac{\gamma-1}{2} M_2}{1 + \frac{\gamma-1}{2} M_1} \, \varepsilon \tag{32}$$

for the two waves downstream of the nozzle. Both of these waves move downstream (for $U_2 > c_2$) but with very different speeds. The faster, P_2^+, is the stronger, and the difference between the two increases with increasing discharge Mach number until, when M_2 is about 5.0, the slow P_2^- wave changes sign.

The phenomenon exhibits more physical familiarity if we observe it from the stream itself, for then an element of gas that is extending itself in the flow direction sends compression waves both ahead and behind. But when the configuration moves past us with supersonic speed, the first wave we encounter is the forward propagating (P_2^+) wave and appears as a compression wave. The second wave to pass us is the rearward propagating (P_2^-) wave but, because we traverse it from "back to front," it appears as an expansion wave. On the other hand, if we travel with a gas element that moves somewhat faster than its surroundings, it sends a compression wave ahead and an expansion wave behind which, in our fixed coordinate system, appear as two compression waves, a P_2^+ followed by a P_2^-. Results (31, 32) contain contributions from both the dilatational and translational modes, the former dominating at modest values of discharge Mach number.

Now suppose, on the other hand, that a temperature disturbance is convected toward the nozzle in the approach flow and there is no impinging pressure wave; this situation is characterized by $P_1^+ = 0$ but $\sigma \neq 0$. Again, because equation (27) for the choked nozzle applies independently upstream and downstream, the single unknown ahead of the nozzle is the strength of the upstream moving wave. This strength has the value

$$P_1^- = \frac{-M_1}{1 + \frac{\gamma - 1}{2} M_1} \frac{\sigma}{2} \tag{33}$$

because the fractional temperature rise ($\equiv \sigma$) causes a quasi-steady increase of volume flow through the nozzle, and this generates an expansion wave propagating to the rear. It should be noted that the pressure wave amplitude is proportional to the approach Mach number, in agreement with other processes that involve momentum transport by the stream (at velocity U_1) rather than by acoustic waves (at velocity $c_1 + U_1$).

Downstream of the nozzle, the entropy disturbance and the fractional pressure are known (by expressions (28), (25)), and condition (27) completes the information required to determine the strengths of the fast and slow pressure waves as

$$P_2^+ = \frac{\frac{1}{2}(M_2 - M_1)}{1 + \frac{\gamma - 1}{2} M_1} \frac{\sigma}{2} \tag{34}$$

$$P_2^- = \frac{-\frac{1}{2}(M_2 + M_1)}{1 + \frac{\gamma - 1}{2} M_1} \frac{\sigma}{2} \tag{35}$$

Referring to our physical discussion of the previous example, it appears that dilatation of the entropy spot dominates the pressure wave production in the exhaust region; that is, the volume of the region containing the temperature non-uniformity expands at a different volumetric rate than its surroundings. This dilatation part, anti-symmetric when viewed from the nozzle, is directly proportional to the discharge Mach number M_2, so that even for modest values of M_2, a two per cent fluctuation in upstream temperature may cause a pressure disturbance of 0.01 atmospheres at the nozzle discharge.

4. THE SUPERCRITICAL NOZZLE OF FINITE LENGTH

Although the analysis for compact elements is frequently adequate, there are many situations where either or both of the approximations of compactness or of plane waves proves inadequate. We emphasize the issue of compactness because it becomes an issue at lower frequencies than do the duct-like modes and because it forms a natural complement to the foregoing compact analysis. Specifically, we are concerned with the situation where the nozzle is not small in comparison with $2\pi(c_1-U_1)/\omega$. The condition that $2\pi(c_1-U_1)/\omega$ is small compared with the nozzle length, which states that pressure waves reflected from the nozzle should be of nearly constant phase over the nozzle length, is probably a severe estimate of compactness but serves to remind us that in a moving medium the contracted wave length is relevant in judging the compactness of an element. The problem may be approached by low-frequency asymptotic analysis for general nozzle geometry or by a full range analysis for rather particular geometry. We develop the latter here because the significant physical results are those that may be expected from nozzles with any reasonable geometry.

For this problem, ρ, U, p, c describe the one-dimensional steady flow through the nozzle. They are functions of x along the nozzle, and take on the values previously denoted U_1, U_2, etc. at the nozzle inlet and outlet. When this flow is disturbed by an acoustic or entropy wave, the variables take on the local values $\rho+\rho'$, $U+u'$, $p+p'$, so that first-order perturbation equations of continuity and momentum have variable coefficients when the dependent variables are taken as ratios of perturbation to local values, as we have written down in equations (12) - (15). The energy equation (13) takes the form

$$\frac{p'}{\gamma p(x)} - \frac{\rho'}{\rho(x)} = \frac{s'}{c_p} = F\left(t - \int_{x_1}^{x} \frac{dx}{U(x)}\right) , \tag{36}$$

where the variable $t - \int_{x_1}^{x} \frac{dx}{U(x)}$ replaces the previous retarded transport variable $t - \frac{x}{U}$ which was appropriate in a uniform stream. The range of integration extends from the nozzle inlet x_1 to the local value of x. The variable velocity of sound is given by $c^2+[(\gamma-1)/2]U^2 = [(\gamma+1)/2]c_*^2$, where c_* is the characteristic sonic velocity at the throat which will, in turn, be taken at $x = x_*$. The distance from the inlet to throat of the nozzle is then x_*-x_1, but for convenience the dimensionless time and reduced frequency will be defined $\tau = c_*t/x_*$ and $\Omega = x_*\omega/c_*$ in terms of the distance x_*.

Now it is a satisfactory approximation for conventional nozzles that the velocity U(x) increases linearly with distance along the nozzle, and we shall take the slope U'(x) to be a constant equal to $(c_*-U_1)/(x_*-x_1)$ where U_1 is the nozzle inlet velocity at $x = x_1$. Now we are free to choose the origin of x coordinate, Figure 1, at the point where the extrapolated nozzle velocity vanishes, recognizing that this lies outside the range of x-variable for the nozzle velocity distribution. Then we write conveniently $U(x)/c_* = x/x_*$ and the actual nozzle inlet length, x_*-x_1, may be recovered from the analysis by an algebraic calculation.

Now, making the mathematical substitution $\xi = (x/x_*)^2$, equations (14) and (15) become

$$\left(\frac{\partial}{\partial\tau} + 2\xi \frac{\partial}{\partial\xi} \right)\left(\frac{p'}{\gamma p} \right) + 2\xi \frac{\partial}{\partial\xi} \left(\frac{u'}{U} \right) = 0 \ , \tag{37}$$

$$\left(\frac{\partial}{\partial\tau} + 2\xi \frac{\partial}{\partial\xi} \right)\left(\frac{u'}{U} \right) + \left[(\gamma+1)\frac{1}{\xi} - (\gamma-1) \right] \xi \frac{\partial}{\partial\xi} \left(\frac{p'}{\gamma p} \right) + 2\frac{u'}{U} - (\gamma-1)\frac{p'}{\gamma p} = F(\tau - \log\sqrt{\xi/\xi_1}) \tag{38}$$

where F has again been used for s'/c_p expressed in dimensionless variables. Because we shall seek periodic solutions, take

$$\frac{p'}{\gamma p} = P(\xi)\exp\{i\Omega t\} \ , \quad \frac{u'}{U} = U(\xi)\exp\{i\Omega\tau\} \ ,$$
$$F(\tau - \log\sqrt{\xi/\xi_1}) = \sigma\exp\{i\Omega[\tau - \log\sqrt{\xi/\xi_1}]\} \ ; \tag{39}$$

substitution of which in (37), (38) leads to an equation for $P(\xi)$:

$$\xi(1-\xi)\frac{d^2P}{d\xi^2} - \left[2 + \frac{2}{\gamma+1} i\Omega\right]\xi \frac{dP}{d\xi} - \frac{i\Omega(2+i\Omega)}{2(\gamma+1)} P = -\sigma\frac{i\Omega}{2(\gamma+1)}\left(\frac{\xi}{\xi_1}\right)^{-i(\Omega/2)} \ , \tag{40}$$

which is of the hypergeometric form. The velocity perturbation, on the other hand, is given by

$$(2+i\Omega)U(\xi) = -(\gamma+1)(1-\xi)\frac{dP}{d\xi} + (\gamma-1+i\Omega)P + \sigma(\xi/\xi_1)^{-i(\Omega/2)} \tag{41}$$

and is known without further integration.

The linear velocity distribution, frequently denoted Tsien's assumption, has been widely employed in combustion stability analysis. As shown by Tsien [1] and subsequently from another viewpoint by L. Crocco [2], the conditions required to determine $P(\xi)$ are: (1) that it be regular at $\xi = 1$, corresponding to the nozzle throat; and (2) that one condition involving at most a linear combination of P and U, or of P and $dP/d\xi$, be specified at ξ_1, the nozzle inlet. The first condition disposes of the singular hypergeometric solution about $\xi=1$, which behaves like

$$(1-\xi)^{-1-\frac{2}{\gamma+1}i\Omega} \ .$$

The second condition involves coupling the one regular hypergeometric solution to the wave system in the flow approaching the nozzle for which we have seen that $U_1^+ = P_1^+$ and $U_1^- = -P_1^-$. Thus, at the nozzle entrance we deduce that

$$P(\xi_1) + M_1U(\xi_1) = 2P_1^+ \ , \tag{42}$$

where P_1^+ is the known impinging wave strength (which may be zero) and the factor M_1 appears because the upstream and nozzle velocity fluctuations are normalized differently. With the pressure and velocity perturbations in the nozzle completely known, the reflected and transmitted wave strengths follow directly:

$$P_1^- = \tfrac{1}{2}\{P(\xi_1) - M_1 U(\xi_1)\} \quad ,$$
$$P_2^+, P_2^- = \tfrac{1}{2}\{P(\xi_2) \pm M_2 U(\xi_2)\} \quad . \qquad (43)$$

It must be remembered that the strengths $P_{1,2}^{+,-}$ are now complex becuase of the phase angle incurred from nozzle inlet to outlet and because the upstream and downstream waves have different reference origins.

Two examples are shown here in Figures 2 and 3 for the strength of the transmitted wave generated by temperature spots (entropy disturbances in the absence of impinging pressure waves), as it depends upon reduced frequency Ω for several values of the discharge Mach number M_2. In Figure 2, the nozzle inlet Mach number is $M_1 = 0.29$ and the asymptotic value of total transmitted wave strength for low frequencies, $\Omega = 0$, corresponds to the result of compact analysis. Of particular interest is the strong dependence of the fluctuation magnitude upon the reduced frequency for relatively low values of Ω and the peak value reached in the neighborhood of $\Omega = 3$ which, for low inlet Mach number, may be more than three times the values for low reduced frequency. In Figure 3, the similar result for an inlet Mach number of 0.61 shows similar trends. The much larger peaks for higher exit Mach number emphasize that the maxima clearly depend upon the discharge Mach number M_2, while the asymptotic values $\Omega \to 0$ are independent of M_2. Readily available asymptotic expansions for the low reduced frequencies describe the initial change of pressure amplitude quite accurately and, since the expansion is in powers of $i\Omega$, it suggests that the initial effect of reduced frequency is to alter the phase relationship between the two downstream waves as they leave the nozzle exit.

5. FREQUENCY DEPENDENCE OF NOZZLE-ENTROPY INTERACTION

The discharge Mach number dependence of the pressure fluctuation at the nozzle outlet, demonstrated quite strikingly in Figures 2 and 3, is interesting not only because it is very strong, but because it is completely absent for the compact nozzle, $\Omega = 0$, and appears only for $\Omega > 0$. A possible origin of this dependence is suggested by the fact that for the compact nozzle, both P_2^+ and P_2^- depend upon M_2 but in such a manner that at the nozzle outlet the two waves combine so that $p'/\gamma p$ is independent of Mach number. It is clear that this cancellation exists only at the nozzle exit -- and possibly at other discrete points downstream -- since the two waves separate with a velocity $2c_2$ as they move downstream. This is shown quite directly by substituting the wave strengths P_2^+ and P_2^-, equations (34) and (35), into (19) to obtain the pressure fluctuation at an arbitrary distance x behind the nozzle

$$\frac{p'}{\gamma p} = \left[M_1 \cos\left(\frac{\omega}{c}\frac{x}{M_2^2-1}\right) + iM_2 \sin\left(\frac{\omega}{c}\frac{x}{M_2^2-1}\right)\right] \frac{\sigma/2}{1+\frac{\gamma-1}{2}M_1} \exp i\omega\left(t - \frac{x}{c}\frac{M_2}{M_2^2-1}\right) \qquad (44)$$

and thus the M_2-dependent amplitude vanishes for the compact nozzle, where $x = 0$. This indicates that, in a nozzle of finite length where a disturbance generated at one section may suffer phase separation by the time the nozzle outlet is reached, the pressure disturbance may exhibit a dependence upon discharge Mach number. It appears instructive, therefore, to examine the wave structure more carefully within the finite nozzle.

Now a strict separation of the pressure distribution into upstream and downstream propagating components is not possible within a channel of strongly varying area, but we can extract the information of interest by a simple device. At any station beyond the throat, that is, at any value $M(x) > 1$, assume the flow field to be continued by a channel of the appropriate area. Then the pressure disturbance may be decomposed rigorously into its P^+ and P^- components at this point, and the transformation of these may be examined as we repeat this process at various distances down the nozzle.

Figures 4 and 5 show the amplitudes of the P^+ and P^- components as a function of reduced frequency for various values of the exit Mach number, for a nozzle inlet Mach number of 0.29. Hence, they are to be compared with the results of Figure 2, in which we have previously shown the composite pressure disturbance. Note that both P^+ and P^- components have their maxima at $\Omega = 0$, zero reduced frequency. These values are indeed the magnitudes of P^+ and P^- given for the compact solution by equations (34) and (35). From these values, they drop monotonically with increasing Ω. Shown also in Figures 3 and 4 is the phase angle of each wave, taken with respect to the solution at the nozzle throat. This phase is zero for the compact nozzle ($\Omega = 0$) and increases quite rapidly with reduced frequency. Note that the two waves are 180° out of phase at the nozzle throat and also for the compact nozzle.

The question now is whether this phase difference is modified as we follow the wave pattern down the nozzle. This change in phase relationship is shown in Figure 5, for three values of the Mach number (which may be interpreted as discharge values for discrete nozzles or as Mach numbers at discrete values of x in a specific nozzle) as a function of reduced frequency. Clearly, there is a tendency for the two disturbances to combine more nearly in phase as the reduced frequency rises, confirming the intuitive indication mentioned earlier. The effect is shown even more clearly in the vector phase diagrams of Figure 6. Here, it may be observed how the tendency toward alignment increases the magnitude of the resultant vector which, in turn, leads to the maximum disturbances shown in Figures 2 and 3. It is the reduction with reduced frequency of the individual wave amplitudes, P_2^+ and P_2^- shown in Figures 4 and 5, that account for the drop in composite disturbance $p'/\gamma p$ for large reduced frequencies. Thus, it appears that the rise of pressure disturbance with reduced frequency may be well estimated from the two wave strengths calculated from compact analysis, using the phase distortion accumulated as the waves progress toward the nozzle discharge.

The calculations utilized in Figures 2 and 3 were carried out by S. Candel[3] and those shown in Figures 4 - 6 were performed by M. Bohn, graduate student at the California Institute of Technology.

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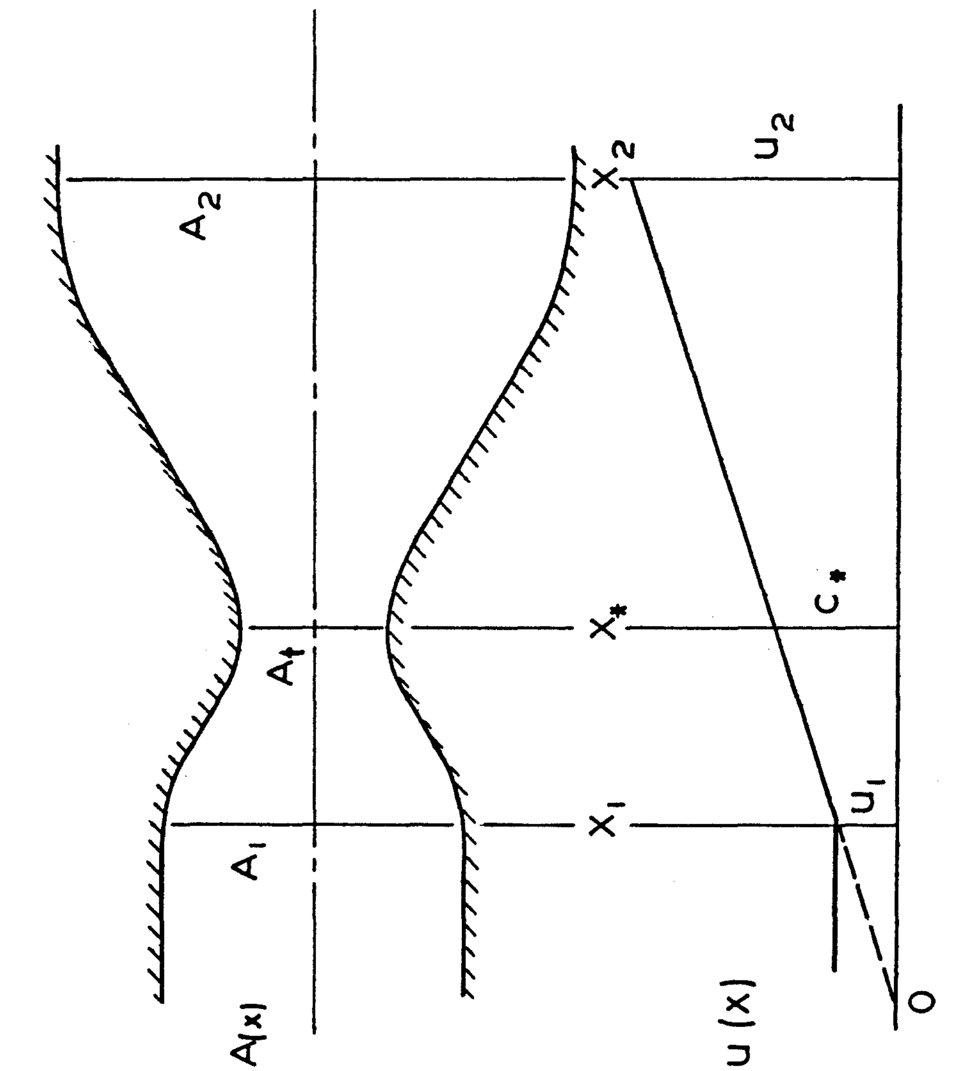

Fig. 1. Notation and Geometry for Nozzle with Finite Length.

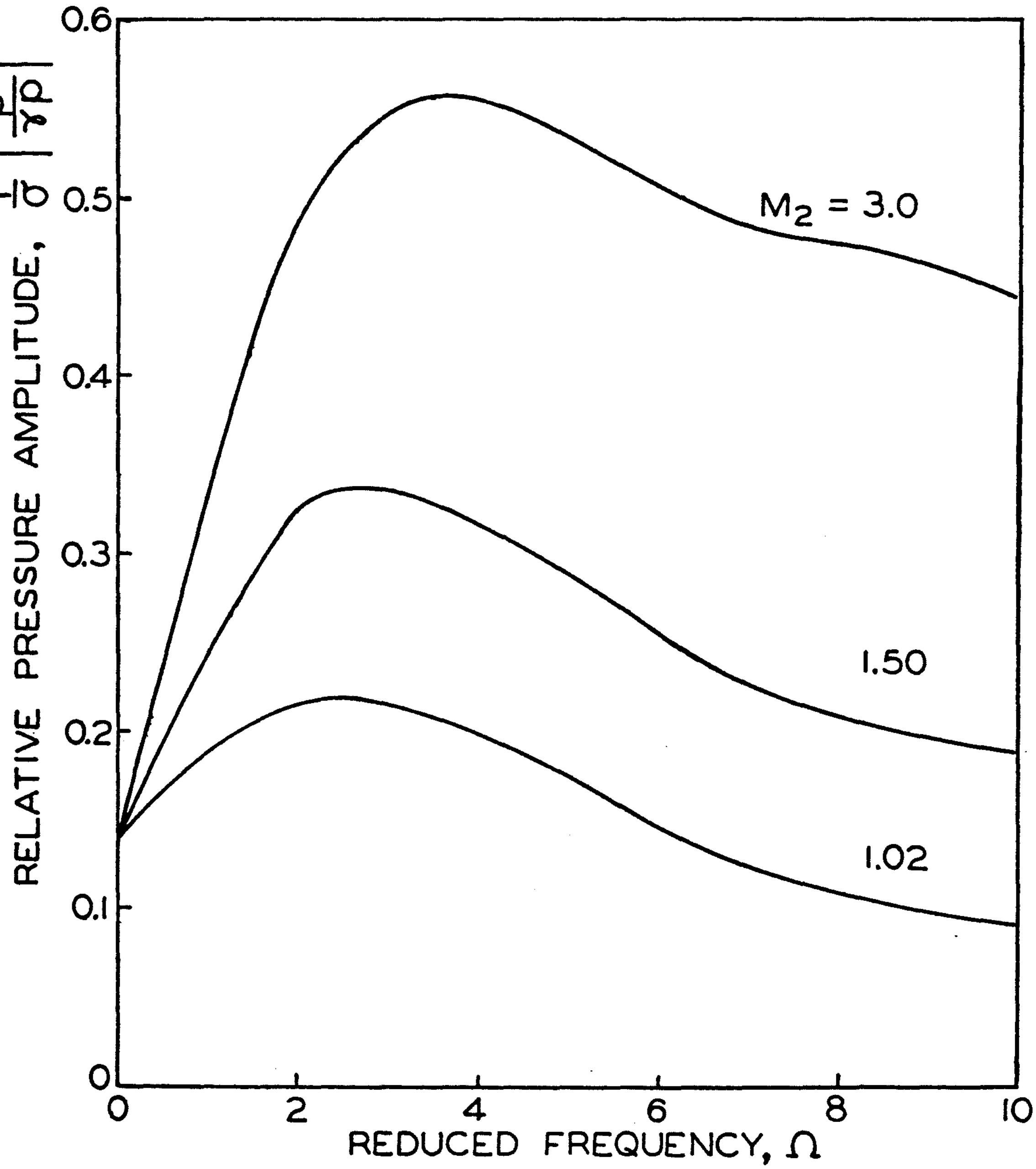

Fig. 2. Dependence of Relative Pressure Amplitude at Nozzle Discharge upon Reduced Frequency for Several Values of Nozzle Discharge Mach Number, M_2. Inlet Mach Number, $M_1 = 0.29$.

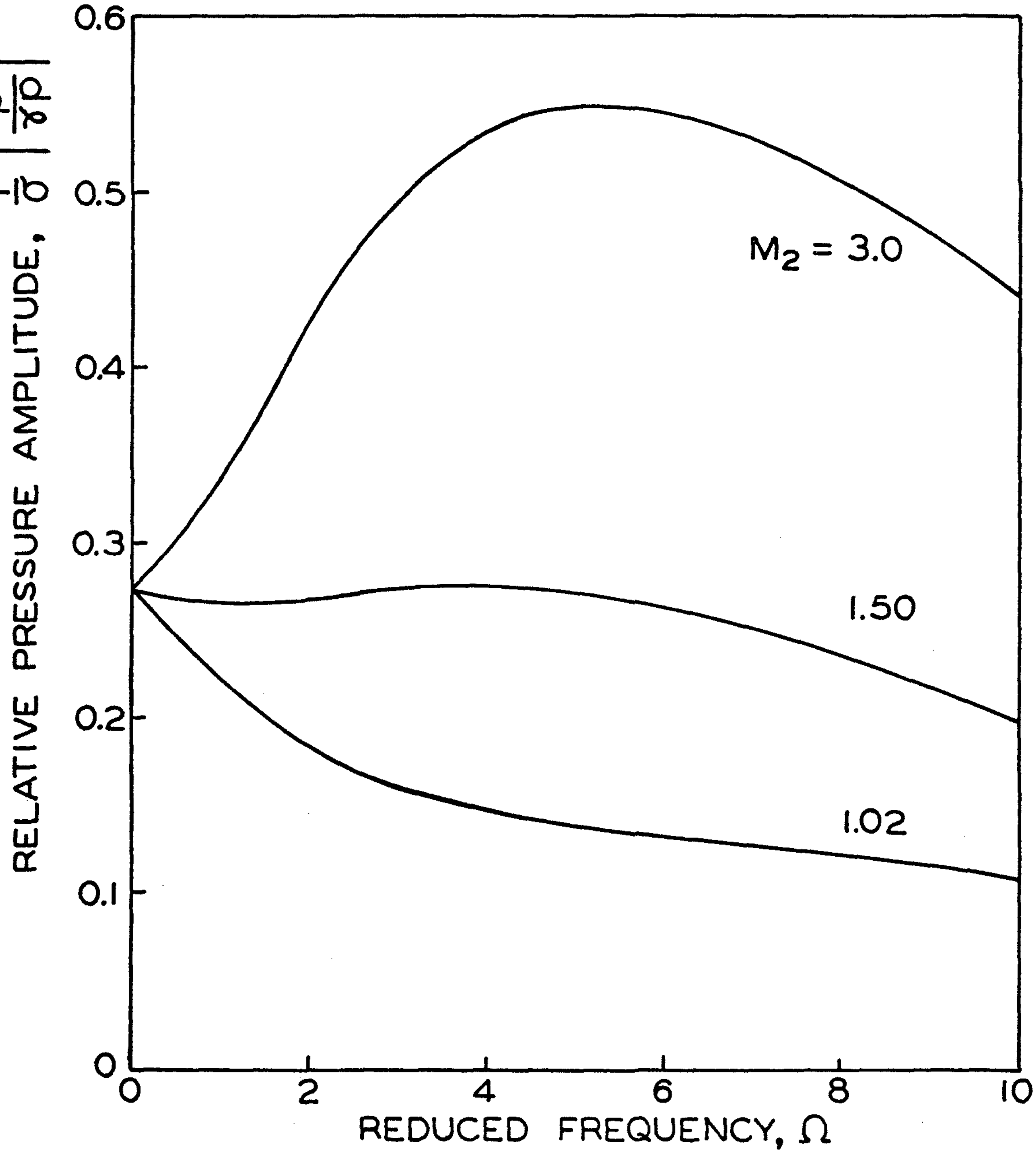

Fig. 3. Dependence of Relative Pressure Amplitude at Nozzle Discharge upon Reduced Frequency for Several Values of Nozzle Discharge Mach Number, M_2. Inlet Mach Number, $M_1 = 0.61$.

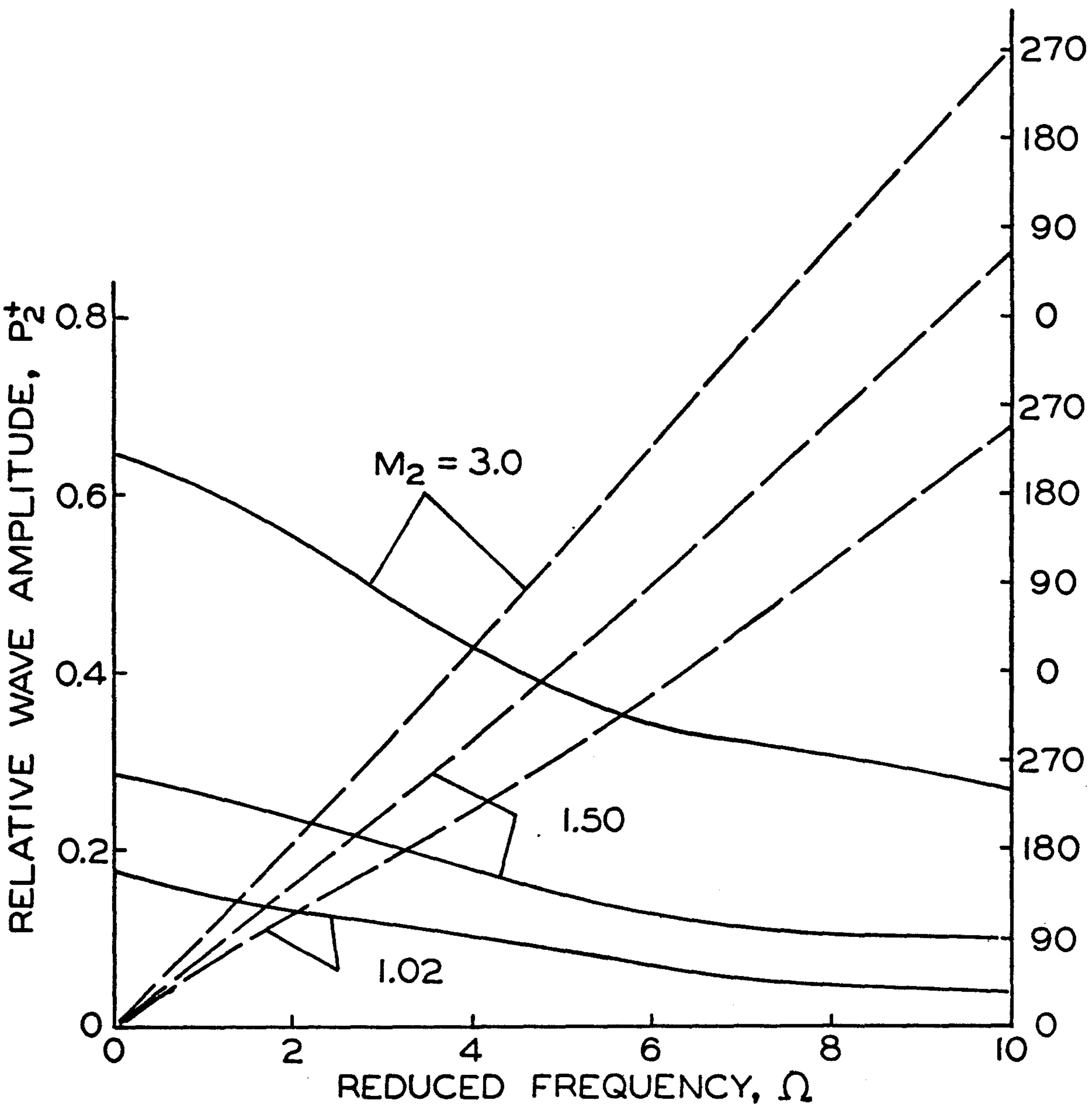

Fig. 4. Relative Pressure Amplitude and Phase of Downstream-Facing Wave, P_2^+. Inlet Mach Number, $M_1 = 0.29$.

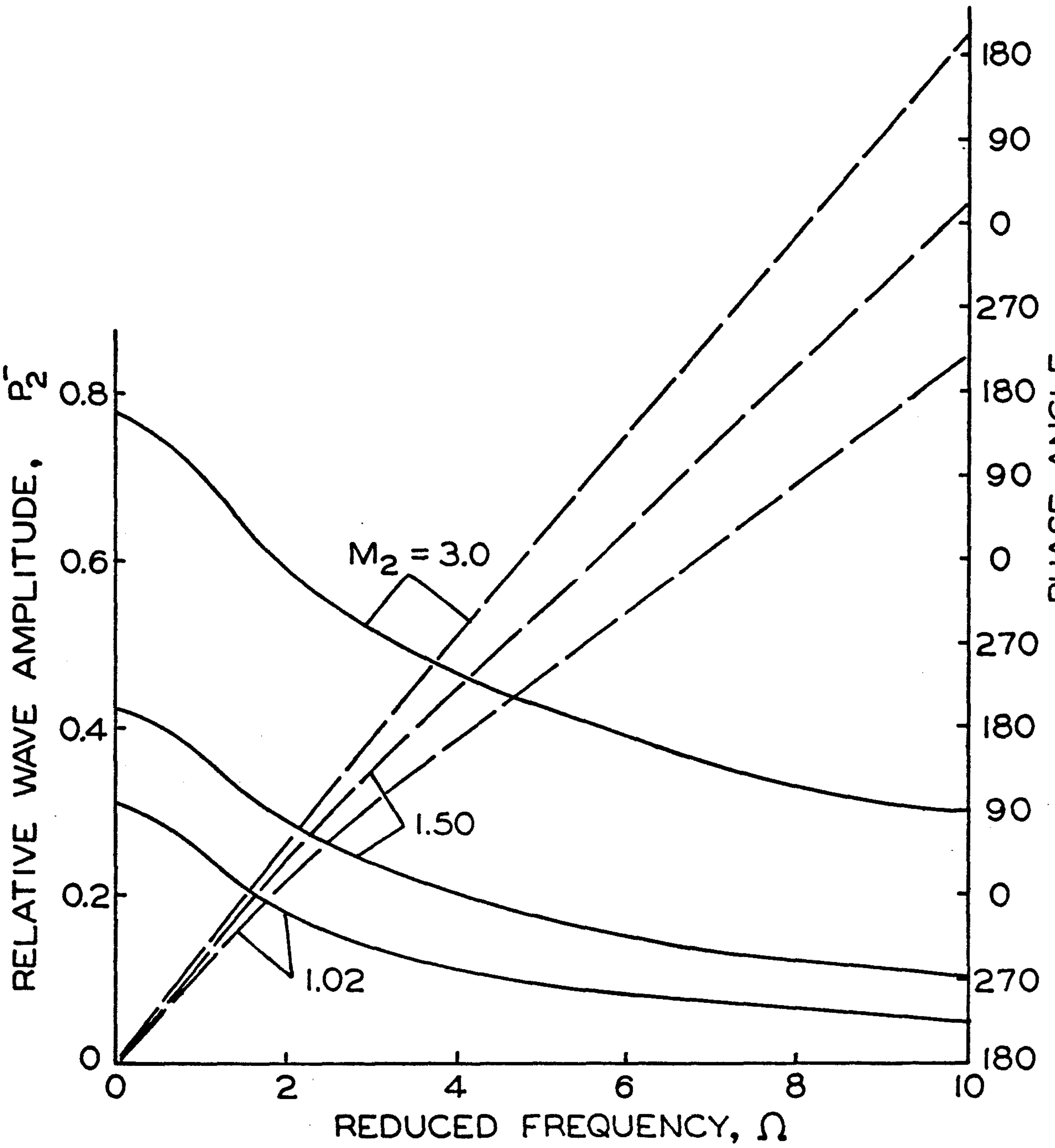

Fig. 5. Relative Pressure Amplitude and Phase of Upstream-Facing Wave, P_2^-. Inlet Mach Number, $M_1 = 0.29$.

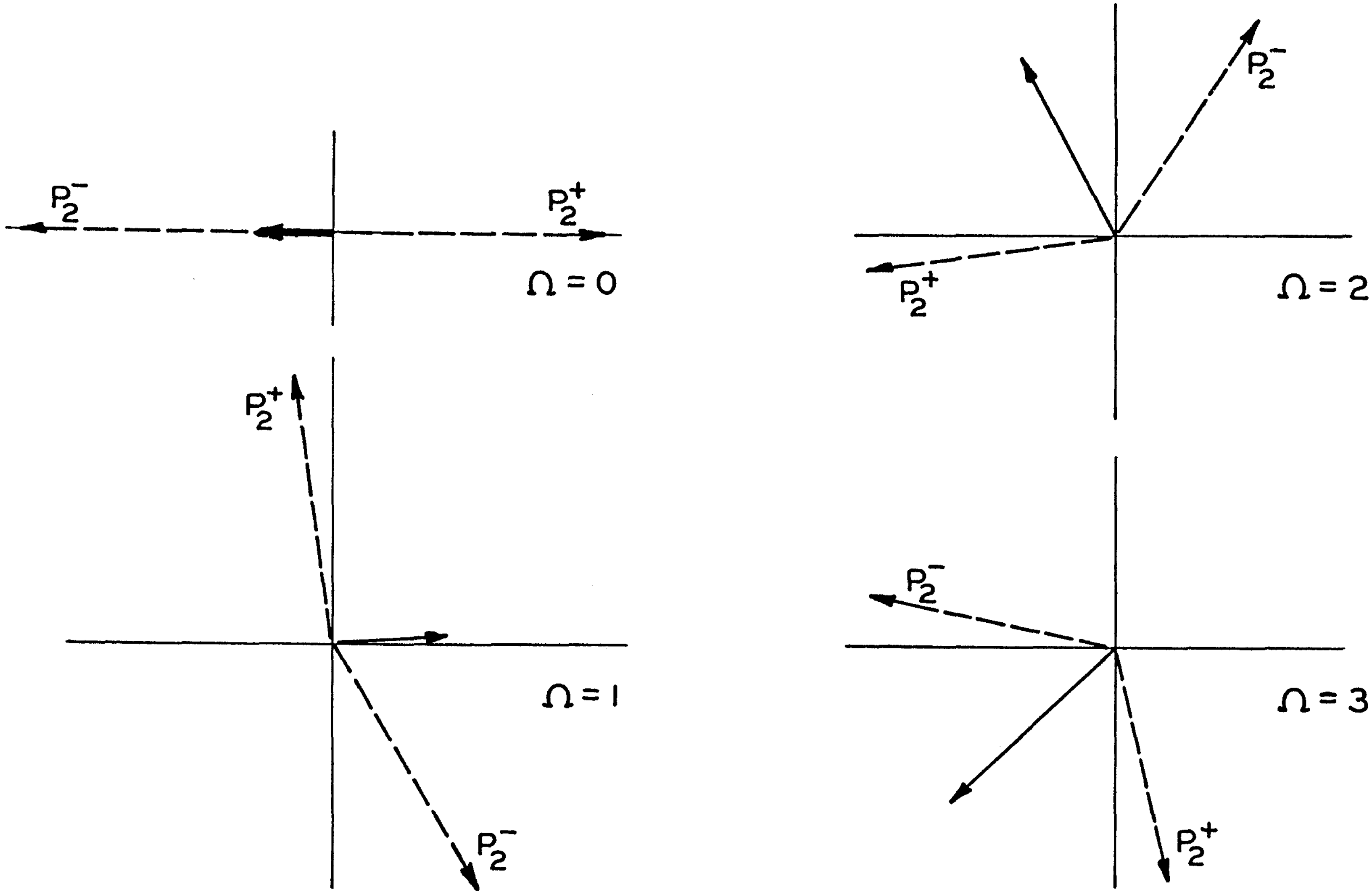

Fig. 6. Phase Diagrams Showing Composition of P_2^+ and P_2^- Waves to Form Pressure Fluctuation at Nozzle Discharge. $M_1 = 0.29$, $M_2 = 3.0$.