The proportion of various graphs in graph-designs

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Abstract. Let \( \mathcal{G} \) be a family of simple graphs. A \( \mathcal{G} \)-design on \( n \) points is a decomposition of the edges of \( K_n \) into copies of graphs in \( \mathcal{G} \). In case that \( \mathcal{G} \) consists of complete graphs \( K_k \) with \( k \) in some set of positive integers, such a \( \mathcal{G} \)-design is called a pairwise balanced design (PBD) on \( n \) points with block sizes from \( K \). Here we are concerned with the possible proportions of the numbers of copies of graphs \( G \in \mathcal{G} \) that appear in decompositions for large \( n \). We extend a result of Colbourn and Rödl on PBDs to \( \mathcal{G} \)-designs, and give a further result on the possible numbers of copies of \( G \) in a \( \mathcal{G} \)-design containing each vertex of the complete graph \( K_n \).

1. Introduction

For a positive integer \( n \) and a set \( K \) of positive integers, a 2\(-(n, K, 1)\) design consists of a set \( X \) of \( n \) points and a family \( \mathcal{A} \) of subsets of \( X \), called blocks, so that \( |A| \in K \) for every \( A \in \mathcal{A} \), and every subset \( \{x, y\} \) of two points in \( X \) is contained in a unique member of \( \mathcal{A} \). These may also be called pairwise balanced designs (PBDs) with block sizes in \( K \).

We use \( \alpha(K) \) for the gcd (greatest common divisor) of \( \{k - 1 : k \in K\} \) and \( \beta(K) \) for the gcd of \( \{k(k - 1) : k \in K\} \). It is known, see [4], that 2\-(n, K, 1) designs exist for all integers \( n \) that are sufficiently large with respect to \( K \) and such that
\[
\begin{align*}
n - 1 &\equiv 0 \pmod{\alpha(K)}, \\
n(n - 1) &\equiv 0 \pmod{\beta(K)}.
\end{align*}
\]
These congruences are necessary conditions for the existence of a 2\-(n, K, 1) design for any \( n \).

The following theorem was proved by Colbourn and Rödl in [2].

Theorem 1.1. Let \( K = \{k_1, k_2, \ldots, k_\ell\} \) be given, where the integers \( k_i \) are distinct and at least 2. Let \( p_1, p_2, \ldots, p_\ell \) be nonnegative real numbers that sum to 1, and let \( \epsilon > 0 \). For every sufficiently large integer \( n \) satisfying (1.1), there exists a 2\-(n, K, 1) design in which the proportion of blocks of size \( k_i \) is within \( \epsilon \) of \( p_i \), simultaneously for all \( i = 1, 2, \ldots, \ell \).

We give another proof of this theorem. It is no extra work to prove an extension of their result to graph-designs.

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Let $\mathcal{G}$ be a family of simple graphs. A $\mathcal{G}$-decomposition of a graph $H$ is a set $\mathcal{D}$ of edge-disjoint subgraphs of $H$, each subgraph in $\mathcal{D}$ isomorphic to a member of $\mathcal{G}$, so that every edge belongs to exactly one member of $\mathcal{D}$. A $\mathcal{G}$-design on $n$ points is a $\mathcal{G}$-decomposition of the complete graph $K_n$.

We use $\alpha(\mathcal{G})$ for the gcd of the degrees of the vertices of graphs in $\mathcal{G}$, and $\beta(\mathcal{G})$ for the gcd of $\{2|E(G)| : G \in \mathcal{G}\}$. It is known [3] that $\mathcal{G}$-designs on $n$ points exist for all sufficiently large (with respect to $\mathcal{G}$) integers $n$ satisfying

$$n - 1 \equiv 0 \pmod{\alpha(\mathcal{G})},$$

$$n(n - 1) \equiv 0 \pmod{\beta(\mathcal{G})}.$$  

(1.2)

The congruences (1.2) are necessary conditions for the existence of $\mathcal{G}$-designs on $n$ points for any $n$. In general, if there exists a $\mathcal{G}$-decomposition of $H$, then

$$\alpha(H) \equiv 0 \pmod{\alpha(\mathcal{G})},$$

$$\beta(H) \equiv 0 \pmod{\beta(\mathcal{G})}.$$  

(1.3)

**Theorem 1.2.** Let $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$ be given, where the graphs $G_i$ are pairwise nonisomorphic and where each has at least one edge. Let $p_1, p_2, \ldots, p_\ell$ be nonnegative real numbers that sum to 1, and let $c > 0$. For every sufficiently large integer $n$ satisfying the congruences (1.2), there exists a $\mathcal{G}$-design in which the proportion of copies of $G_i$ used in the decomposition is within $c$ of $p_i$ for all $i = 1, 2, \ldots, \ell$.

It should be clear that Theorem 1.1 is the consequence of Theorem 1.2, when we take $G_i$ to be a complete graph on $\ell_i$ points. Theorem 1.2 and a corollary will be proved in Section 2. In Section 3, we prove the following theorem. It is stronger than Theorem 1.2.

**Theorem 1.3.** Let $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$ be given, where the graphs $G_i$ are pairwise nonisomorphic and where each has at least one edge. Let $p_1, p_2, \ldots, p_\ell$ be nonnegative real numbers that sum to 1, and let $c > 0$. For every sufficiently large integer $n$ satisfying the congruences (1.2), there exists a $\mathcal{G}$-design in which for every point $x$, the proportion of copies of $G_i$ that appear in the decomposition and that contain $x$ is within $c$ of $p_i$ for all $i = 1, 2, \ldots, \ell$.

2. Proof of Theorem 1.2

Assume the hypothesis and notation of Theorem 1.2. It is sufficient to prove the theorem in the case that the $p_i$'s are rational numbers. Suppose that $p_i = s_i/t$ (with a common denominator $t$) and that $t$ is large enough so that $1/t < \epsilon$.

Let $H_0$ be the vertex-disjoint union of $s_1$ copies of $G_1$, $i = 1, 2, \ldots, \ell$. Let $H_i$ be the vertex-disjoint union of $H_0$ and one additional copy of $G_i$, $i = 1, 2, \ldots, \ell$, and let $\mathcal{H} = \{H_0, H_1, \ldots, H_\ell\}$. In any graph $H_i$, the proportion of copies of $G_i$ that appear is one of $s_i/(t + 1)$, $s_i/t$, or $(s_i + 1)/(t + 1)$. Every $\mathcal{H}$-design immediately gives us a $\mathcal{G}$-design, in which the proportion of copies of $G_i$ that appear is between $s_i/(t + 1)$ and $(s_i + 1)/(t + 1)$, and this is within $c$ of $p_i$.

From [3], $\mathcal{H}$-designs on $n$ points exist for all large integers $n$ satisfying

$$n - 1 \equiv 0 \pmod{\alpha(\mathcal{H})},$$

$$n(n - 1) \equiv 0 \pmod{\beta(\mathcal{H})}.$$  

(2.1)
We claim that
\[(2.2) \quad \alpha(H) = \alpha(G) \quad \text{and} \quad \beta(H) = \beta(G),\]
i.e. that the congruences (1.2) and (2.1) are identical. This will complete the proof of Theorem 1.2.

The set of degrees of vertices in graphs in \(H\) is identical with the set of degrees of vertices in graphs in \(G\), so the left hand equation in (2.2) is trivial. It is also trivial that \(\beta(G)\) divides \(\beta(H)\). Finally, since \(\beta(H)\) divides \(2|E(H_i)|\) for all \(j\), it divides the difference
\[2|E(H_i)| - 2|E(H_0)| = 2|E(G_i)|\]
for each \(i\). Hence \(\beta(H)\) divides the gcd of \(2|E(G_i)|, i = 1, 2, \ldots, \ell\), which is \(\beta(G)\). This establishes the right hand equation in (2.2).

By a \(G\)-packing \(P\) in \(K_n\), we mean a set of edge-disjoint isomorphic copies of \(G\) in \(K_n\).

**Corollary 1.** Let \(G\) be a simple graph with at least one edge, and let \(\varepsilon > 0\) be given. For every sufficiently large integer \(n\), there exists a \(G\)-packing \(P\) of \(K_n\) so that the ratio of the number of edges that occur in copies of \(G\) in \(P\) to \(n(n-1)/2\) is more than \(1 - \varepsilon\).

**Proof.** Apply Theorem 1.2 with \(G_1 = G\), with \(G_2\) a graph with a single edge, \(p_1 = 1\), and \(p_2 = 0\). For \(G = \{G_1, G_2\}\), we have \(\alpha(G) = 1\) and \(\beta(G) = 2\), so all integers \(n\) satisfy the congruences (1.2) in this case. \(\Box\)

A much stronger result about packings of complete graphs into \(K_n\) will appear in [1].

**3. Proof of Theorem 1.3**

As in Section 2, we use the fact that if we have an \(A\)-decomposition \(D_B\) of each graph \(B \in E\), and a \(B\)-decomposition of a graph \(H\), then we naturally obtain an \(A\)-decomposition \(D\) of \(H\), namely
\[D = \bigcup_{B \in E} D_B.\]

Let \(A \in A\) and suppose that for every vertex \(x\) of a graph \(B \in E\), the ratio of the number of copies of \(A\) in \(D_B\) that contain \(x\) to the total number of graph in \(D_B\) that contain \(x\) is within \(\varepsilon\) of a number \(p\). Then for every vertex \(y\) of \(H\), the ratio of the number of copies of \(A\) in \(D\) that contain \(y\) to the total number of graph in \(D\) that contain \(y\) is within \(\varepsilon\) of \(p\).

It is sufficient to prove Theorem 1.3 in the case that the \(p_i\)'s are rational numbers, and we may also assume that they are positive. Suppose that \(p_i = s_i/t\) (with a common denominator \(t\), and where \(s_i > 0\) for all \(i\)) and that \(t\) is large enough so that \(1/t < \varepsilon\).

Let \(u_i\) be the number of vertices of \(G_i\). Let \(J\) be the edge disjoint union of \(Cs_i/u_i\) copies of \(G_i, i = 1, 2, \ldots, \ell\), where \(C \geq 2\) is an integer chosen so that \(Cs_i/u_i\) is an integer for each \(i\).

Label the vertices of \(J\) with positive integers in the range from \(1\) to \(N\) for some integer \(N\) so that the absolute values of the differences of the labels on adjacent vertices are distinct. For example, an (inefficient) way to do this is to use labels \(2^0, 2^1, 2^2, \ldots, 2^{u-1}\) in any order, where \(u\) is the number of vertices of \(J\); here \(N = \ldots\)
$2^{v-1}$. Identify the vertices of $J$ with their labels in the (additive) group $Z_{2N+1}$ of integers modulo $2N + 1$, so that $J$ is now a subgraph of the complete graph on vertex set $Z_{2N+1}$.

Let $L_0$ be the union of all translations $J + a$, $a \in Z_{2N+1}$. The condition on absolute values of the differences of labels ensures that the graphs $J + a$ are pairwise edge-disjoint. Because if $x, y \in Z_{2N+1}$ are adjacent in both $J + a$ and $J + b$, then $x - a, y - a$ are adjacent in $J$ and $x - b, y - b$ are adjacent in $J$; if these are not the ends of the same edge of $J$, then $d = \pm(x - y)$ modulo $2N + 1$ appears as the difference of the labels of the ends of two edges of $J$; but then $|d|$ is the difference of the labels of the ends of two edges of $J$.

So $L_0$ admits a decomposition into $2N + 1$ copies of $J$, and then we obtain a $G$-decomposition $D_0$ of $L_0$. The number of translates of a single copy of $G$ that contain any given point $x \in Z_{2N+1}$ is $u_i$, so the number of copies of $G_i$ in $D_0$ that contain any point $x$ is $C_{si}$. The total number of graphs in $D_0$ that contain $x$ is $Ct$.

Let $L_j$ be obtained from $L_0$ by deleting the edges of one copy of $G_j$ from $L_0$, $i = 1, 2, \ldots, \ell$. Of course, $L_j$ has a $G$-decomposition $D_j$ obtained by deleting that one copy of $G_j$ from $D$. Each point $x \in Z_{2N+1}$ is contained in either $C_{si}$ or $C_{si} - 1$ copies of $G_i$ in $D_j$; the total number of graphs in $D_j$ that contain $x$ is $Ct$ or $Ct - 1$.

In any case, the proportion of copies of $G_i$ among the graphs in $D_j$ that contain $x$ is between $(C_{si} - 1)/(Ct)$ and $(C_{si})/(Ct - 1)$, and is within $1/t < \epsilon$ of $p_i$.

Let $\mathcal{L} = \{L_0, L_1, \ldots, L_\ell\}$. From [2], $\mathcal{L}$-designs on $n$ points exist for all large integers $n$ satisfying

$$n - 1 \equiv 0 \pmod{\alpha(\mathcal{L})},$$

$$n(n - 1) \equiv 0 \pmod{\beta(\mathcal{L})}.$$  

From a $\mathcal{L}$-design, we obtain a $G$-design using the $G$-decompositions of $L_i$ described above. For any point $y$, the proportion of copies of $G_i$ among the graphs in the $G$-decomposition that contain $y$ will be within $\epsilon$ of $p_i$.

We claim that

$$\alpha(\mathcal{L}) = \alpha(G) \quad \text{and} \quad \beta(\mathcal{L}) = \beta(G),$$  

i.e. that the congruences (1.3) and (3.1) are identical. This will complete the proof of Theorem 1.3.

First, since each $L_i$ has a $G$-decomposition, $\alpha(G)$ divides $\alpha(\{L_i\})$ and $\beta(G)$ divides $\beta(\{L_i\})$ for each $i = 1, 2, \ldots, \ell$. Hence $\alpha(G)$ divides $\alpha(\mathcal{L})$ and $\beta(G)$ divides $\beta(\mathcal{L})$.

If there is a vertex of degree $d$ in some $G_i$, then, since one copy of $G_i$ was deleted from $L_0$ to obtain $L_i$ and $L_0$ is regular of degree $Ct$, then some point in $Z_{2N+1}$ has degree $Ct$ in $L_0$ and degree $Ct - d$ in $L_i$. Then $\alpha(\mathcal{L})$ divides these degrees and so divides the difference $d$. Since this is true for the degree $d$ of every vertex of any graph in $\mathcal{G}$, $\alpha(\mathcal{L})$ divides $\alpha(G)$. Also, $\beta(\mathcal{L})$ divides $2|E(L_0)|$ and $2|E(L_i)|$, so it divides $2|E(L_0)| - 2|E(L_i)| = 2|E(G_i)|$ for each $i$, and hence $\beta(\mathcal{L})$ divides $\beta(G)$. This confirms (3.2). \hfill \Box

References


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