Local non–Calderbank-Shor-Steane quantum error-correcting code on a three-dimensional lattice

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We present a family of non-Calderbank-Shor-Steane quantum error-correcting code consisting of geometrically local stabilizer generators on a 3D lattice. We study the Hamiltonian constructed from ferromagnetic interaction of overcomplete set of local stabilizer generators. The degenerate ground state of the system is characterized by a quantum error-correcting code whose number of encoded qubits are equal to the second Betti number of the manifold. These models (i) have solely local interactions; (ii) admit a strong–weak duality relation with an Ising model on a dual lattice; (iii) have topological order in the ground state, some of which survive at finite temperature; and (iv) behave as classical memory at finite temperature.

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I. INTRODUCTION

One of the motivations for studying quantum error-correcting code on lattice is to protect quantum information without active correction. Many models on 2D lattices have been proposed and analyzed [1–7] but the no-go theorem rules out all finite-range finite-strength Hamiltonian systems in 2D as a self-correcting quantum memory [8,9]. This does not apply to higher dimensions. For instance, it was shown that 4D toric code is a self-correcting quantum memory [10,11]. Bombin et al. showed that there is also a 6D model that exhibits similar behavior [12]. Whether such a thermally protected model exists in 3D remains an open problem. Three-dimensional toric code can store classical information at finite temperatures but it fails to do so for quantum information [13]. The topological color code in 3D, albeit lacking a rigorous proof, is believed to show a similar behavior: There exists a stringlike logical operator that is thermally unstable [14]. The 3D model proposed by Nussinov and Ortiz shows similar behavior [15,16]. Another model was proposed by Chamon and analyzed recently by Bravyi et al. This model may be able to protect quantum information but not in a thermodynamic sense [17,18].

It is worth noting that all the listed 3D models except Chamon’s model share a similar property: The quantum error correcting code defining the ground state of the system is a Calderbank-Shor-Steane (CSS) code [19,20], meaning that it can be decomposed into two classical codes. CSS code is a special kind of quantum error-correcting code that can be described by stabilizer formalism [21]. These quantum error-correcting codes can be “stabilized” by a set of stabilizer group generators, meaning they are simultaneous $+1$ eigenstate of the group elements. If there exists a set of generators that can be written as either a product of $X$s or product of $Z$s, these are called as CSS code. From this definition, one can see that majority of the proposed models for quantum memories fall into this category [1,2,12–16]. When studying the stability of these models, one can show that one of the codes can protect classical information from thermal fluctuation while the other one cannot. This means that there is a manifest difference between how the models treat the bit-flip error and the phase-flip error. Chamon’s model treats $X$, $Y$, and $Z$ error in an identical manner but it lacks stability in the thermal sense [17,18]. Since we expect a singular behavior at the phase boundary between an “ordered state” and “disordered state” for thermally stable quantum memory, the absence of a finite-temperature phase transition seems troublesome unless there is an argument that can evade this logic. Motivated by these ideas, we present a spin-$\frac{1}{2}$ model with a finite-temperature phase transition whose ground state is a non-CSS quantum error-correcting code. Our model exhibits a topological order, but only the classical part survives in finite temperatures.

The outline of the paper is as follows. We set the stage by introducing the Hamiltonian in Sec. II. In Sec. III, we study the quantum code that defines the ground state of the Hamiltonian. We calculate the number of qubits and find the logical operators. In Sec. IV, we study the low-energy excitation of the Hamiltonian that consists of particles and closed strings. We construct a duality relation with classical Ising model in Sec. V to show the finite-temperature phase transition.

II. MODEL

We place qubits on a vertices of a four-valent 3D lattice. Using the notation $X_i \equiv \sigma_x^i$, $Y_i \equiv \sigma_y^i$, $Z_i \equiv \sigma_z^i$ stabilizer generators are

$$B^p_x = \prod_{i \in p} X_i,$$

$$B^p_y = \prod_{i \in p} Y_i,$$

$$B^p_z = \prod_{i \in p} Z_i,$$

where $p$ is the plaquette and $\{i \in p\}$ denotes a set of vertices on plaquette $p$. We shall partition a set of plaquettes into $P_x$, $P_y$, $P_z$, which corresponds to a set of nontrivial supports for $B^p_x$, $B^p_y$, $B^p_z$. We shall call elements of these sets $X$, $Y$, $Z$ plaquettes.

Our model is inspired by the construction of topological color code in 3D [14]. For this quantum code, qubits reside on the vertices of the lattice, and the lattice is locally four-valent. The stabilizer generators are either a product of $X$s or a product of $Z$s, and they correspond to the unit cells of different dimensions; in one example, generators are either in cubic form or plaquette form. Our approach differs in the sense that we allow only plaquette operators as stabilizer generators.

A local description of our model can be seen in Fig. 1(a). At each vertex, there are six plaquette operators that have nontrivial support on it. Each plaquette operator meets with...
FIG. 1. (Color online) The vertex figure and unit cell of our model. Qubits reside on the vertices. One can see that $B_x$ meets with another $B_x$ at one vertex, whereas it meets with $B_y$ and $B_z$ at two vertices.

a same kind of plaquette operator on each vertices and meet with four other plaquette operators on two vertices. Thus the assignment in Fig. 1(a) guarantees commutativity between the stabilizer operators. We must point out that not every lattice structure allows vertex figure like Fig. 1(a). There are only four translationally invariant convex tessellations that have tetrahedral vertex figure: bitruncated cubic honeycomb, cantitruncated cubic honeycomb, omnitruncated cubic honeycomb, and cantitruncated alternated cubic honeycomb [22]. Only the first three admits an arrangement of plaquette operators similar to Fig. 1(a) at every vertex. In this paper, we mainly study the bitruncated cubic honeycomb model for its simplicity but analogous results shall be discussed in full generality if possible. Unit cell is shown in Fig. 1(b) and tessellation is shown in Fig. 2. Btruncated qubic honeycomb is a space-filling tessellation made up of truncated octahedra. It has 14 faces, 36 edges, and 24 vertices. There are 6 square faces and 8 hexagonal faces. Without loss of generality, one can set the 6 square faces to be $Y$-plaquette operators, 4 of the hexagonal faces to be $X$-plaquette operators, and the 4 remaining hexagonal faces to be $Z$-plaquette operators. The Hamiltonian is a sum over the plaquette operators:

$$H = -J \left( \sum_{p_x \in P_x} B_x^{p_x} + \sum_{p_y \in P_y} B_y^{p_y} + \sum_{p_z \in P_z} B_z^{p_z} \right).$$

III. QUANTUM CODE

The purpose of this section is to study the quantum code generated by a set of group generators $\{B_x^{p_x}, B_y^{p_y}, B_z^{p_z}\}$. We start by introducing the notation and definition that shall be used throughout the analysis. The rest of the section is mainly divided into two parts. In Sec. III B, we count the number of encoded qubits. In Sec. III C, we completely specify a set of logical operators for each qubit.

A. Preliminary results

Given a CW complex, Euler characteristic $\chi$ can be defined as an alternating sum of $k_n$s, where $k_n$ denotes a number of cells of dimension $n$:

$$\chi = \sum_{i=0}^{d} k_i (-1)^i.$$

For instance, if we consider a two-dimensional manifold, $k_0$ is a number of vertices, $k_1$ a number of edges, and $k_2$ a number of faces. One of the main ideas that we use in this paper is that $\chi$ can be also written as an alternating sum of Betti number $b_i$s:

$$\chi = \sum_{i=0}^{d} b_i (-1)^i,$$

where $b_i$ is the rank of the $n$-th singular homotopy group, but for odd-dimensional closed orientable manifold it is not necessary to calculate each individual $b_i$s. This is due to Poincaré duality: Although it has various different forms, for the purpose of our paper, we can use the one originally introduced by Poincaré himself.

Theorem 1 (Poincaré, 1895). $b_k = b_{d-k}$ for a closed orientable $d$-dimensional manifold.

From this theorem, one can easily deduce that $\chi = 0$ for the odd dimensional closed orientable manifold.

B. Number of encoded qubits

The number of encoded qubits can be computed from the size of the stabilizer group and the number of physical qubits. Since the plaquette operators are not independent of each other, we must count the number of independent relations. In such a pursuits, geometrical interpretation of our model becomes useful. We would first like to point out that multiplying all the plaquette operators on a unit cell reduces to identity. One can see this from Fig. 1(b). Since any contractible closed surface on the lattice can be represented as a union of unit cells, one can see that multiplication of plaquette operators on any contractible closed surface reduces to identity. Therefore
we have $C-1$ independent relations which generate smooth deformation, where $C$ is the number of unit cells. We must subtract 1 because multiplying all but one cell results in a relation for that very cell.

Let us consider a periodic boundary condition on all three directions. There exists a noncontractible surface that reduces to identity as one can see in Figs. 3(a) and 3(b). Since there are three topologically distinguishable noncontractible surfaces, we have three independent relations, resulting in $C+2$ independent relations. Finally, multiplying all X-like operators adds one independent relation. One can check that multiplication of $Y$s and multiplication of $Z$s are implied by the previously mentioned relations.

Accounting for these relations, the number of encoded qubits is $V = F + C + 3 = 3$, where $V$ is the number of vertices, $F$ is the number of faces, and $C$ is the number of unit cells. The first two correspond to the number of qubits and number of plaquette operators. The remaining terms represent a number of independent relations between plaquette operators. We shall show that in fact the number of encoded qubits depends only on the second Betti number, $b_2$.

Lemma 1. For stabilizer group $\{B^x_{p_i}, B^y_{p_i}, B^z_{p_i}\}$, number of encoded qubits is $b_2$.

Proof. Let us consider the dual lattice. This can be constructed by replacing $k$-dimensional object into a $(d-k)$-dimensional object. For instance, vertices of the dual lattice resides on the center of the unit cells of the original lattice. Faces on the dual lattice can be constructed by connecting the edges so the resulting surface is perpendicular to the edges in the original lattice. Euler characteristic $\chi$ is trivially 0 due to Poincaré duality. The unit cells of the resulting dual lattice is an irregular tetrahedron. Let us denote $k_i$s to be number of $i$-dimensional cells on the dual lattice. One can see that $V$, the total number of vertices in the original lattice becomes $k_1$, a number of unit cells in the dual lattice. Similarly, $F$ is identical to $k_3$ and $C$ is identical to $k_0$. Note that $k_2 = k_3$, for each cell contains four faces and each faces meet with two tetrahedral cells. Therefore, we have

$$V = F - C = k_3 - k_1 + k_0 \quad \text{(3.3)}$$
$$= -k_3 + k_2 - k_1 + k_0 = 0. \quad \text{(3.4)}$$

Hence

$$k = V - (F - (C - 1 + 1 + b_2)) \quad \text{(3.5)}$$
$$= b_2, \quad \text{(3.6)}$$

where $b_2$ is the second Betti number of the manifold. One can also use this hypothesis to prove that the group generated by the plaquette operators does not contain $-I$.

Lemma 2. $\langle B^x_{p_i}, B^y_{p_i}, B^z_{p_i} \rangle$ does not contain $-I$.

Proof. Consider a product of plaquette operators that is proportional to the identity operator. Any such configuration can be generated by the product of all $X$-plaquette operators, the product of all $Y$-plaquette operators, the product of all $Z$-plaquette operators, and the product of plaquettes along a closed surface. The first three are trivially $+I$. For unit cells, we have 24 vertices at which $X$, $Y$, and $Z$ meet. Since all the generators commute with each other, we can arrange the product to be in the following canonical form:

$$\prod_{p_i} B^x_{p_i}, \prod_{p_i} B^y_{p_i}, \prod_{p_i} B^z_{p_i}.$$

Since $XYZ = i$, the product of plaquette operators on a unit cell is 1. Similarly, for the product of plaquette operators on a noncontractible surface described in Figs. 3(a) and 3(b), we have $4n$ vertices where $X$, $Y$, and $Z$ meet. Hence we arrive at the same conclusion. Since any product of plaquette operators that results in a trivial operator can be constructed by these constraints, the group does not contain $-I$.

C. Logical operators

There are two logical operators that are reminiscent of the surface and string operator of 3D toric code. These are drawn in Fig. 4. One can see the surface operator on the top of the

FIG. 4. (Color online) There is one surface operator and one string operator for each qubits. The surface operator corresponds to the product of $ZZZZ$ on $Y$ plaquettes. The string operator is the line perpendicular to this surface, showing a sequence $YZYXYZYX \cdots$. 
lattice system which is a product of $B_{p_i}^\pm$s on one layer of $Y$ plaquettes. The complementary logical operator to this is the string operator that has a sequence of $YZXYXYZYXYZYX\cdots$ along the line perpendicular to the surface operator. This string winds around the torus and completes a noncontractible loop. These two operators anticommute with each other and both of them commute with the stabilizer generators.

We can similarly define two sets of complementary operators in other directions. One can easily check the expected commutation and anticommutation relations.

### IV. Low-Energy Excitation

Quasiparticles excitations in 2D typically arise as anyons. For instance, in Kitaev’s toric code, two quasiparticles are created in pair, and when fused together, they vanish [1]. There are two kind of particles analogous to electric and magnetic charge, and when one particle winds around another one, the system attains a nontrivial global phase. In 3D, trajectory of winding around another particle can be deformed into a trivial contour. Hence one needs higher-dimensional object to attain a similar topological action. In 3D there are closed stringlike excitations and particlelike excitations [2,13]. When the particle winds around the string so the trajectory and the string together forms a knot, the system attains a nontrivial global phase.

Our model presents a similar picture. Particle-like excitations are created in a pair. If we truncate a stringlike logical operator, excitations form at the end points. When the particle-antiparticle pair is created, they can diffuse without any extra energy cost. Closed stringlike excitations can be similarly thought as a truncated surfacelike logical operator. Near the boundary of the surface, there are excitations and hence the energy cost grows linearly with the size of the surface. When a particle penetrates the closed string, we find that

$$|\psi_{\text{initial}}\rangle = SP|\Phi\rangle$$  \hspace{1cm} (4.1)

$$|\psi_{\text{final}}\rangle = USP|\Phi\rangle = -|\psi_{\text{initial}}\rangle.$$  \hspace{1cm} (4.2)

where $S$ is a closed-string excitation, $P$ is a particle excitation, and $U$ is a trajectory of the particle. Thus system gains $e^{i\pi}$ phase factor. This is illustrated in Fig. 5. One can see that as a particle penetrates through the surface operator and returns to the original position, it coincides with the surface operator at one vertex, thus giving the anticommutation relation.

Low-energy excitation in terms of elementary objects provides us an intuitive picture for the thermal stability. Particles can be created out of vacuum in pair and propagate freely. They can diffuse and wind around the torus to induce logical error. Closed strings, on the other hand, need energy that is proportional to its perimeter. Given a closed stringlike excitation as in Fig. 5, the stabilizer generators anticommuting with the surface operator reside only near the boundary of the surface. $Z$ plaquettes trivially commute with the surface operator. $X$ plaquettes commute with the surface operator since they meet at two vertices. However, there are $Y$ plaquettes meeting at exactly one vertex at the boundary. Hence we expect our system to be a stable classical memory.

### V. Duality

The typical strong-weak duality relation relates a strong coupling limit of one model to a weak coupling limit of another model: We use a slightly different strategy here. We first show that our model can be mapped into an Ising gauge theory, from which we can use the Wegner-type duality relation with the Ising model. Mapping from our model to Ising gauge theory is not exact for a finite-sized lattice, but this difference vanishes in the thermodynamic limit. Starting from the partition function of our model,

$$Z = \text{tr}[\exp(-\beta H)]$$  \hspace{1cm} (5.1)

$$= \text{tr}[\prod_{s \in S}(\cosh \beta J + \sinh \beta J)],$$  \hspace{1cm} (5.2)

where $S_i \in \{B_{p_i}^+, B_{p_i}^-, B_{p_i}^{\pm}\}$,

$$Z = (\cosh \beta J)^n \text{tr}[\Pi_i(1 + \alpha S_i)]$$  \hspace{1cm} (5.3)

$$= (\cosh \beta J)^n \text{tr}\left(\sum_{i=0}^1 \Pi_i \alpha^i S_i^i\right).$$  \hspace{1cm} (5.4)

Since the Pauli operators are traceless, the nonvanishing terms correspond to the nontrivial constraints presented in Sec. III B. Note that there were two kind of constraints: constraints coming from the closed two-manifold and constraints coming from space-filling products of $X$s, $Y$s, or $Z$s. Using this, we can write down the partition function in the following form.

$$Z = (2\cosh \beta J)^n \left[\sum_c \alpha^{A_c} + (1 + \alpha^x)(1 + \alpha^y)\right]$$

$$\times (1 + \alpha^z) - 1 + \text{C.T.}.$$  \hspace{1cm} (5.5)

$\Sigma_c$ is a sum over a configuration of closed two-manifolds. $A_c$ is the number of plaquettes for each configurations. C.T. denotes the cross terms between closed two-manifolds and the space-filling product of $X$s, $Y$s, or $Z$s. $n_{x,y,z}$ denotes the number of $X$-, $Y$-, and $Z$-plaquette operators. The main idea

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**FIG. 5.** (Color online) Representation of particle penetrating through a stringlike excitation. Truncated surface operator is a product of $Z$ plaquettes in white. Trajectory of the particle is a nontrivial support of the colored plaquette operators, which coincides with the $Z$ surface.
is that the partition function is dominated by the first term in the thermodynamic limit. We show this in Appendix A.

Lemma 3. \( Z - \text{C.T.} - (a^{x_i} + a^{y_i} + a^{z_i}) = Z_{\text{IG}}(\beta J) \), where \( Z_{\text{IG}} \) is a partition function of Ising gauge theory on the same lattice with temperature \( \beta \) and coupling constant \( J \).

Proof. Consider a mapping \( B_{p_i}^x \to ZZZZZZ, B_{p_i}^y \to ZZZZZ, \) and \( B_{p_i}^z \to ZZZZZZ, \) where \( Z \cdots Z \) are products of \( Z \) on the edges of each plaquettes. The resulting model is an Ising gauge theory on a bitrucated cubic honeycomb. The partition function is

\[
Z_{\text{IG}}(\beta J) = Z - \text{C.T.} - (a^{x_i} + a^{y_i} + a^{z_i}).
\]

Using the duality relation between Ising gauge theory and the Ising model, we can map our model into an Ising model. We show the duality relation in Appendix B.

Theorem 2. Our model with coupling constant \( \beta J \) is dual to the classical Ising model on a dual lattice with a dual coupling constant \( \beta J = -\frac{1}{2} \ln \beta J \).

Since the Ising model undergoes a finite-temperature phase transition, so does our model. This is analogous to the behavior of 3D toric code under a temperature change. As in our model, one can show that 3D toric code has critical temperature by using the duality relation with the Ising model. Below the critical temperature, there is symmetry breaking with respect to a surfacelike logical operator. Symmetry associated to the stringlike logical operator is broken only at the ground state.

One glaring difference, though, is that 3D toric code can be decomposed into two classical Hamiltonians without spoiling the phase transition: the Hamiltonian responsible for the bit-flip error is identical to Ising gauge theory, which has a finite-temperature phase transition. On the other hand, the Hamiltonian responsible for correcting the phase-flip error does not have a phase transition. Hence one can intuitively understand that 3D toric code can only correct bit-flip errors but not phase-flip errors under thermal equilibrium. Our model does not allow such decomposition. Once we get rid of any of \( B_{p_i}^x, B_{p_i}^y, \) or \( B_{p_i}^z \), the partition function does not exhibit a phase transition any more. This shows that non-CSS code with a finite-temperature phase transition in 3D does not necessarily provide a self-correcting quantum memory.

VI. CONCLUSION

In this paper, we studied an exactly solvable 3D spin model and studied its topological order. The ground state of the system defines a non-CSS quantum error correcting code. At finite temperature, this system is expected to behave as a stable classical memory, but not as a stable quantum memory. This is mainly due to the fact that there exists a stringlike logical operator. In light of studying the possibility of self-correcting quantum memory, this reconfirms the general properties that have been found in 3D stabilizer codes so far: For each encoded qubit, there exists one surfacelike logical operator and one stringlike logical operator. It seems that we cannot avoid such outcomes unless the shape of the logical operator changes as the system size changes, as in Chamon’s model [17,18]. This in fact was recently argued to be the general feature of stabilizer codes whose number of encoded qubits remain invariant under system size change [23].

It is worth noting that the thermal stability analysis of our model is not rigorous at this stage, even though the energy barrier increasing at the perimeter of the surface is compelling evidence that this must be true. It would be desirable to make a rigorous estimate of thermal relaxation rate using the method introduced by Chesi et al. [24]. We expect the stringlike logical operator to be thermally fragile and the surfacelike logical operator to be stable. As in 3D toric code [13], we also expect the topological entropy of our model to show a singular behavior near the critical point. These singular behaviors arise due to the existence of finite-temperature phase transitions, which we can show rigorously by the strong-weak duality relation between our quantum model to a classical Ising model on the dual lattice.

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APPENDIX A: BOUND FOR THE CROSS TERMS

Cross terms can be written as

\[
\text{C.T.} = \sum_c \alpha^c \sum_{i \in \{x,y,z\}} a^{n_i-2n_i^c},
\]

where \( n_x, n_y, n_z \) are the total number of \( X,Y,Z \) plaquettes and \( n_x^c, n_y^c, n_z^c \) are the number of \( X,Y,Z \) plaquettes for configuration \( c \).

Lemma 4. There exists \( 0 < \epsilon_{1,2} < 1 \) such that

\[
A_c + n_i - 2n_i^c \geq \epsilon_1 A_c + \epsilon_2 n_i,
\]

for \( \forall c, i \).

Proof. Consider \( i = x \). The left-hand side of the inequality is

\[
n_y^c + n_z^c - n_x^c + n_x \geq n_y^c + n_z^c - (1 - \epsilon)n_x^c + (1 - \epsilon)n_x,
\]

\[
\geq \left( \frac{\epsilon}{2} \right) A_c + (1 - \epsilon)n_x.
\]

On the second line, we used the fact that the minimum is achieved in the case where \( n_x^c = 0 \), implying \( n_x^c = n_x^c = \frac{1}{2} A_c \). The same logic can be applied to \( i = y, z \).

\[
n_x^c + n_z^c - n_x^c + n_y \geq n_x^c + n_z^c - (1 - \epsilon)n_y^c + (1 - \epsilon)n_y,
\]

\[
\geq \left[ \frac{2}{5} - \frac{3}{5} (1 - \epsilon) \right] A_c + (1 - \epsilon)n_y.
\]
Similarly, we used the fact that the minimum is achieved in the case where one of \( n_i \) or \( n_f \) is 0. Then we have a 2:3 ratio between the \( X(Z) \) plaquettes and \( Y \) plaquettes. Therefore, for \( \epsilon > \frac{1}{2} \), we have such \((\epsilon_1,\epsilon_2)\).

**Lemma 5.**

\[
\lim_{\text{vol} \to \infty} \frac{Z(\beta J)}{Z_{\text{IG}}(\beta J)} = 1, \tag{A7}
\]

where \( Z_{\text{IG}}(\beta J) \) is a partition function for Ising gauge theory with temperature \( \beta \) and coupling constant \( J \) and where \text{vol} is the volume of the lattice.

**Proof.**

We use

\[
\sum_{\alpha} \alpha^{\epsilon_1 A_i} = (\frac{2 \cos \beta J}{2 \cos \beta J})^n \sum_{\alpha} \alpha^{\epsilon_1 A_i}, \tag{A8}
\]

\[
= \left(\frac{1}{2 \cosh \beta J}\right)^n Z_{\text{IG}}(\beta J'), \tag{A9}
\]

where

\[
\tanh \beta J' = (\tanh \beta J)^{t_i} . \tag{A10}
\]

Thus the cross terms can be bound by

\[
Z_{\text{IG}}(\beta J') \left(\frac{\cosh \beta J'}{\cos \beta J'}\right)^n \alpha^{\delta_{i,n}}, \tag{A11}
\]

where \( \delta_i = \frac{n_i}{n} \) and where \( n \) is the total number of plaquettes. This becomes

\[
Z_{\text{IG}}(\beta J') \left[ \left( \frac{1 - t^2 \cosh \beta J'}{1 - t \cosh \beta J'}\right)^{\frac{1}{2}} \right]^{\frac{1}{t_i}}. \tag{A12}
\]

where \( t = \tanh \beta J' \). One can show that \( \frac{1 - t^2 \cosh \beta J'}{1 - t \cosh \beta J'} < 1 \) for \( \beta J > 0 \). Since the renormalized coupling constant \( J' \) is larger than \( J \), we can see that these correction terms become negligible in thermodynamic limit. Therefore,

\[
\lim_{\text{vol} \to \infty} \left| \frac{Z(\beta J) - Z_{\text{IG}}(\beta J)}{Z_{\text{IG}}(\beta J)} \right| \leq \left| \frac{Z_{\text{IG}}(\beta J')}{Z_{\text{IG}}(\beta J)} \right|^n + O(\alpha^n), \tag{A13}
\]

where \( J' > J \) and \( 0 < \lambda < 1 \). In \( n \to \infty \) limit, we get the desired result.

**APPENDIX B: DUALITY BETWEEN ISING GAUGE THEORY AND THE ISING MODEL**

**Lemma 6.** Ising gauge theory on a bitruncated cubic honeycomb is dual to the Ising model on the dual lattice.

**Proof.**

\[
Z = (\cosh \beta J)^n \text{tr}[\Pi_i (1 + \tanh \beta J S_i)] \tag{B1}
\]

\[
= (\cosh \beta J)^n \text{tr}\left( \sum_{\{k_j\}=0} \Pi_i \alpha^{k_j} S_i^k \right) \tag{B2}
\]

\[
= (2 \cosh \beta J)^n \sum_{\{k_j\}=0} \Pi_i \alpha^{k_j} \Pi_q \delta_2 \left( \sum_{j} k_j, \Pi_q \right), \tag{B3}
\]

where \( \Pi_q \) is a product over all the edges and \( \sum_j k_j, \Pi_q \) is a sum over \( k_j, \Pi_q \) that have nontrivial support on edge \( e \). There are three such \( k_j, \Pi_q \). One can use \( k_j, \Pi_q = \frac{1}{2} (1 - ZZ) \), where \( ZZ \) is a product of \( Zs \) on qubits that reside on the vertices of the dual lattice. For eight spin configurations \((Z_1, Z_2, Z_3) = (-1, -1, -1), (1, 1, 1), (-1, -1, -1), (-1, -1, 1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, 1)\), one can see that all of these configurations satisfy the delta function. Furthermore, we have two combinations for \((k_1, k_2, k_3) = (0, 0, 0)\) and two combinations for \((0, 1, 1), (1, 0, 1), (1, 1, 0)\). Plugging this in, we get

\[
Z = (\cosh \beta J)^n \sum_{\{Z_i, \Pi_i\}} \Pi_i \alpha^{\frac{1}{2} Z_i + \Pi_i Z_i - \Pi_i}, \tag{B4}
\]

where \( Z_{\Pi} \) is the \( Z \) operator on the dual sites of plaquette \( i \) and \( \Pi_i \) is the unit normal vector to the plaquette. Therefore, up to a constant, the partition function is identical to the partition of Ising model with \( \beta J = -\frac{1}{2} \tanh \beta J' \).