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A Brief Note on Linearized, Unsteady, Supercavitating Flows

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Three different models for the unsteady fluctuations of a slender cavity in the limit of small reduced frequency are compared with the results of quasi-steady calculations. Tulin's kinematically closed model in unsteady flow is seen to tend smoothly to a limiting quasi-steady motion having the same value for the compliance of the cavitating flow, unlike other models that have been used in the past.

Introduction

THE STEADY THEORY for linearized supercavitating flows has now enjoyed a wide popularity with application to a multitude of problems in naval hydrodynamics since its introduction by Tulin 25 years ago (Tulin, 1953).[3] Shortly thereafter well-known extensions to unsteady flows were made by Parkin (1957), Wu (1957), and Guerst (1961), with the result that unsteady flows with unbounded cavities were readily treated either by use of the acceleration potential or by use of the perturbation complex velocity itself. Nevertheless difficulties were encountered when the cavities were of finite extent. In this event it would appear that 'sources' would be needed to permit the cavity to change volume dynamically, yet these same sources would lead—in strict two-dimensional unbounded flow—to a logarithmic pressure at infinity which was considered nonphysical at the time. This particular question led to a wide range of interesting works which are thoroughly reviewed and explained by Wu (1970), who points out that two-dimensional supercavitating flows are not so in the "large" and that they must be considered as "inner" flows in some three-dimensional contexts. Indeed, if this were not the case, it would be hard to explain how cavities of different size could be made to appear in "two-dimensional" water tunnel test sections at all. With this point of view then, it is perfectly reasonable that unsteady cavitating flows can have cavities whose boundaries can fluctuate in shape as well as in volume. The strength of any net source effect must be determined from physical conditions in the three-dimensional flow at large. In the same way, the pressure at points remote from the cavity is determined by the flow there. Thus, the logarithmic behavior of the two-dimensional source flow becomes modified, and it is neither necessary nor correct to assume, as Guerst did, that there be *no* net fluctuating volume source in an unsteady cavity flow merely to avoid this singular behavior.

There is another difficulty in the formulation of these unsteady cavity flows, and this is the establishment of a proper physical model for the analysis. It is now generally agreed that some sort of a "wake" flow is desirable for *steady* cavity flow analysis to simulate the momentum defect associated with the forebody drag. However, the body of experimental data that led to this conclusion in steady flow does not exist for unsteady flows. But it does seem plausible, certainly for slowly varying flows, that the cavity boundary should be well defined kinematically and have a definite end. Parkin further assumed that this boundary should be closed just as in Tulin's original steady-flow cavity model. As we shall see shortly, this leads to the evaluation of a rather tricky retarded integral. Parkin did not actually evaluate the closure integral, as his principal results were for cavities of infinite extent.

Leehey (1962) in a discussion of boundary conditions for unsteady flows treated a rather similar problem and also adopted the closed-cavity model. In order to secure numerical results, however, he adopted the strategem of fixing the end of the cavity (that is, the length of the cavity was not allowed to change with time), but the volume of the cavity was permitted to change through the fluctuation of the cavity ordinates. In fact, this approach appeared so attractive that Kim et al (1975) used it in an analysis of unsteady surging flow through a cavitating inducer cascade. It is difficult to see, however, how the Leehey assumption can be correct as the frequency becomes very low, since then a quasi-steady oscillation of cavity length as observed in a water tunnel would not be possible. Since Kim's work more recent quasi-steady calculations of cavitating cascade behavior have been carried out by Brennen (1976). A feature of special interest in both calculations is the "compliance" of the cavity flow. The compliance is the rate of change in cavity volume in respect to inlet pressure; it is a physical quantity of utmost importance for dynamic analysis of systems including cavitating pumps, propellers, etc. As perhaps might be expected, the low-frequency limit of Kim's calculation does not agree with the quasi-steady one; in fact Kim's calculation may be shown to exhibit a singular behavior of the compliance as frequency vanishes, contrary to experience and the quasi-steady limit.

The purpose of the present note is to show by means of a simple example that it is possible to carry through an unsteady linearized free-streamline calculation in which the kinematics of the cavity surface are consistently accounted for. The model used is that of Tulin; that is, it is assumed that during the unsteady motions the cavity forms a closed material surface.

Sample problem

For simplicity, we consider the unsteady linearized free-streamline flow past a blunt-nosed body. The forebody is stationary, the pressure within the cavity, p_c, is steady, but the pres-

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sure far away varies with time. We take the forebody to be the parabolic wedge with the wetted surface

$$Y = \sqrt{2R_L x}$$

and the cavity length $l(t) \gg R_L$, the leading-edge radius. We have the usual linearizations

$$(u,v) = (U_\infty + u', v')$$

with subscript ∞ denoting far upstream quantities from which the kinematic boundary conditions on the steady forebody are

$$\frac{dY}{dx} = \frac{v'}{U_\infty}, \qquad v' = U_\infty \sqrt{\frac{R_L}{2x}} \tag{1}$$

The constant-pressure free streamline obeys the linearized Euler equation of motion

$$\frac{\partial u'}{\partial t} + U_\infty \frac{\partial u'}{\partial x} = 0 \tag{2}$$

We imagine now that on the free surface the velocity may be separated into spatial part and a harmonic time part, $u' = \hat{u}(x)e^{j\omega t}$. By substitution into (2) we find on the free streamline that with g being a constant

$$u'(x,t) = g e^{j\omega(t - x/U_\infty)} + \text{const.} \tag{3}$$

Equations (1) and (3) form a mixed boundary-value problem whose solution is (after Guerst)

$$W = u - iv = U_\infty + A\sqrt{\frac{z-l}{z}} + B\sqrt{\frac{z}{z-l}} + C - \frac{g}{\pi} e^{j\omega t}\sqrt{z(z-l)} \int_0^l \frac{d\xi e^{-j\omega\xi/U_\infty}}{\sqrt{\xi(l-\xi)}(\xi - z)} \tag{4}$$

where A, B, and C are constants to be determined. Note that the cavity length $l = l(t)$. The unknown constants are found from the kinematic closure condition, the steady-state pressure far from the body, and the flow tangency condition, equation (1).

Closure

The ordinates $Y(x,t)$ of the cavity boundary are governed by the relation

$$v'(x,t) = \frac{\partial Y}{\partial t} + U_\infty \frac{\partial Y}{\partial x},$$

whose solution is

$$Y(x,t) = \frac{1}{U_\infty} \int_0^x v'\left(\xi, t - \frac{x-\xi}{U_\infty}\right) d\xi$$

(Parkin 1957). By closure we mean that $Y(l,t) = 0$; hence we need

$$0 = \int_0^{l(t)} v'\left(\xi, t - \frac{l(t)-\xi}{U_\infty}\right) d\xi \tag{5}$$

This highly implicit, retarded integral is quite difficult to evaluate given v' from equation (4). In the present work we are concerned with the low-frequency limit of the unsteady motion and it therefore is appropriate to expand equation (5) in the series (dropping the prime)

$$0 = \int_0^{l(t)} d\xi \left\{ v(\xi,t) - \frac{l-\xi}{U_\infty}\frac{\partial v}{\partial t}(\xi,t) + \frac{1}{2}\left(\frac{l-\xi}{U_\infty}\right)^2 \frac{\partial^2 v}{\partial t^2}(\xi,t) + \ldots \right\} \tag{6}$$

and for our present purposes we will terminate the series after the second term. It will be seen shortly that this is an expansion in reduced frequency based on mean cavity length l_0, namely, $k = \omega l_0/U_\infty$, so that the velocity function, equation (4), need only be known through the first term in k. To effect this, it is convenient to introduce a scaled variable

$$\eta \equiv z/l(t) \tag{7a}$$

whence equation (4) appears

$$W = U_\infty + A\sqrt{\frac{\eta-1}{\eta}} + B\sqrt{\frac{\eta}{\eta-1}} + C - \frac{g e^{j\omega t}}{\pi}\sqrt{\eta(\eta-1)} \int_0^1 \frac{e^{-j\omega\xi/U_\infty l} d\xi}{\sqrt{\xi(1-\xi)}(\xi-\eta)} \tag{7b}$$

We now carry out Guerst's second linearization by putting

$$l = l_0 + l_1 e^{j\omega t}, \quad |l_1| \ll l_0$$

We are going to require that $k = \omega l_0/U_\infty \ll 1$, but first we have that

$$W = U_\infty + A\sqrt{\frac{\eta-1}{\eta}} + B\sqrt{\frac{\eta}{\eta-1}} + C - \frac{g}{\pi} e^{j\omega t} R(\eta,k) \tag{8a}$$

$$v(\eta,t) = -A\sqrt{\frac{1-\eta}{\eta}} + B\sqrt{\frac{\eta}{1-\eta}} + \frac{g}{\pi} e^{j\omega t} R(\eta,k) \tag{8b}$$

with

$$R(\eta,k) = \sqrt{\eta(1-\eta)} \int_0^1 \frac{d\xi e^{-jk\xi}}{\sqrt{\xi(1-\xi)}(\xi-\eta)}$$

and as k vanishes (8c)

$$R(\eta,k) = -jk\pi\sqrt{\eta(1-\eta)} + 0(k^2)$$

It may now be seen that the velocity function W is only a function of the scaled variables η and t. From equation (6) it is also seen that the closure integral will likewise be only a function of η and t. In what follows we now assume that

$$A = A_0 + A_1 e^{j\omega t}, \qquad |A_1| \ll A_0$$
$$B = B_0 + B_1 e^{j\omega t}, \qquad |B_1| \ll B_0$$

and substitute equation (8) into equation (6). In this, only linear terms in A_0, A_1, etc. are retained through k and only terms proportional to $e^{j\omega t}$ are retained in the unsteady part (that is, no higher harmonics are considered). We get

$$0 = \int_0^1 d\eta \left\{ -A_0\sqrt{\frac{1-\eta}{\eta}} + B_0\sqrt{\frac{\eta}{1-\eta}} \right\} \tag{9a}$$

and

$$0 = \int_0^1 d\eta \left\{ -A_1\sqrt{\frac{1-\eta}{\eta}} + B_1\sqrt{\frac{\eta}{1-\eta}} + \frac{g}{\pi}R(\eta,k) - jk(1-\eta)\left[-A_1\sqrt{\frac{1-\eta}{\eta}} + B_1\sqrt{\frac{\eta}{1-\eta}} + \frac{g}{\pi}R(\eta,k) - \eta\frac{l_1}{l_0} \times \left(-A_0\frac{d}{d\eta}\sqrt{\frac{1-\eta}{\eta}} + B_0\frac{d}{d\eta}\sqrt{\frac{\eta}{1-\eta}} \right) \right] \right\} \tag{9b}$$

The first of these is the usual "steady" closure term; the second is the unsteady contribution through $0(k)$. (In the latter the dependence of η on t is taken into account.)

The velocity perturbations are required to vanish at infinity; we find that as $z \to \infty$

$$W \to U_\infty + A + B + C + g e^{j\omega t}\left(1 - \frac{jk}{2}\right)$$

or

$$\left.\begin{aligned} A_0 + B_0 + C &= 0 \\ A_1 + B_1 + g\left(1 - \frac{jk}{2}\right) &= 0 \end{aligned}\right\} \tag{10}$$

The kinematic condition, equation (1), gives further that as $x \to 0$

$$v(\eta,t) \to -\frac{A}{\sqrt{\eta}} = U_\infty \sqrt{\frac{R_L}{2x}}$$

With $\eta = x/(l_0 + l_1 e^{j\omega t})$, we have on separating steady and unsteady parts and again retaining only the fundamental frequency

$$\left.\begin{aligned} -A_0\sqrt{l_0} &= U_\infty \sqrt{\frac{R_L}{2}} \\ \frac{l_1}{l_0} &= -2\frac{A_1}{A_0} \end{aligned}\right\} \tag{11}$$

Finally, from equation (4), we see that just behind the forebody on the free streamline ($x = 0^+$) the horizontal velocity is

$$u'(0^+,0) \equiv u'_{\text{cav}} = C + ge^{j\omega t} \tag{12}$$

Thus, given R_L and u'_{cav}, all other parameters are determined from equations (11), (10), and (9). In particular

$$A_1 \doteq -\frac{g}{2}(1 - jk)$$

Source strength

The velocity function far from the body has the expansion

$$W \to U_\infty + \frac{l(t)}{z}\left\{\frac{B_0 - A_0}{2} + e^{j\omega t}\left(\frac{B_1 - A_1}{2} - \frac{jkg}{8}\right)\right\} + 0\left(\frac{1}{z^2}\right) \tag{13}$$

and the second term of this expression may be recognized as the contribution of a source. In the present model there is no net steady source term [$B_0 = A_0$ from equation (9a)], but there is a fluctuating source strength which we designate as

$$q(t) = q_1 e^{j\omega t}$$

The amplitude of this source strength is, after substituting for B_1 and A_1 in the foregoing,

$$q_1 \doteq -j\frac{3}{4}\pi g l_0 k \tag{14}$$

a value proportional to the unsteady velocity perturbation on the cavity.

Discussion

The present results include in a natural way the zero frequency or steady limit. In this special case the reduced frequency is zero and the added dynamic term [the last term of equation (8a)] is seen to vanish with the result of equation (8c). Then A, B, and C are evaluated as for the purely steady case from the steady closure integral of equation (9a). In what may be termed the "quasi-steady" approximation, these constants are evaluated for a sequence of different steady states with time (or frequency) then appearing only as an implicit parameter. The function $R(\eta,k)$ representing the unsteady motion is then neglected in the quasi-steady calculation.

It should be noted that it was not necessary to separate the velocity function into a steady part and an unsteady part. Once Guerst's second linearization is made, the velocity function is seen to be simply a function of the similarity variable η and time. As a further result the closure integral is readily evaluated for small reduced frequency and the solution is seen to tend smoothly to the quasi-steady one as the reduced frequency vanishes.

We now return to the compliance of the flow. Define

$$C_B \equiv \frac{\partial V_{\text{cav}}}{\partial p_\infty} = \frac{\partial V_{\text{cav}}}{\partial t} \bigg/ \frac{\partial p_\infty}{\partial t} \tag{15}$$

where V_{cav} is the volume of cavity. The first factor is just the volume source $q_1 e^{j\omega t}$. The pressure at a remote point is got from integration of the Euler equation, and its time derivative is

$$\frac{\partial p_\infty}{\partial t} = \rho U_\infty \frac{\partial u'_{\text{cav}}}{\partial t} - \rho \int_{0+}^{\infty} \frac{\partial^2 u'}{\partial t^2} dx$$

The first term is the "quasi-steady" value $gj\omega e^{j\omega t}$. The remaining term, the one due to the acceleration of the fluid, is in quadrature to the linear term. This term may be bounded if the flow is coupled to a three-dimensional flow or to a two-dimensional one with a neighboring constant-pressure surface. It may also be logarithmically singular as for an unbounded two-dimensional flow, or, if coupled to one-dimensional channels as in Kim's case, become linear in x. The pressure fluctuations due to these latter uniform slug flow mass oscillations may be removed or separately accounted for, leaving a finite residual inertial effect proportional to k^2 for small reduced frequency. Thus, this acceleration effect may be neglected for small frequencies, for flows largely three-dimensional, or in the channel oscillations of Kim. Then we have

$$C_B \doteq \frac{q_1}{\rho U_\infty j\omega g} \tag{16}$$

With this result we evaluate the compliance for four cases:

(i) The kinematically closed model (Tulin).

(ii) The quasi-static approximation (based on a sequence of steady states).

(iii) Guerst's model.

(iv) The Leehey fixed-cavity terminus model.

For Case (i) we find from equations (14) and (16) that

$$C_{Bi} = -\frac{3}{4}\frac{\pi l_0^2}{\rho U_\infty^2}$$

It may be shown that for Case (ii)

$$C_{Bii} = -\frac{3}{4}\pi\frac{l_0^2}{\rho U_\infty^2}$$

which is precisely the same as in Case (i) through $0(k^2)$! For Case (iii), by definition,

$$C_{Biii} \equiv 0$$

The case for (iv) follows from requiring $l_1 = 0$ in equation (11). Then from equation (13) it is readily found that

$$C_{Biv} = \frac{\pi l_0^2}{\rho U_\infty^2}\left(\frac{1}{4} + \frac{j}{k}\right)$$

showing that the compliance is singular as k approaches zero, and the real part has the incorrect sign! A further simple comparison of Cases (i) and (ii) may be made of the fluctuating cavity length l_1; we find

$$\frac{l_{1(\text{i})}}{l_{1(\text{ii})}} = 1 - jk$$

so that the amplitude of the fluctuation is the same; only the phase is different to $0(k)$.

The present discussion has emphasized the compliance of the flow as a principal factor in determining which flow model may be the more physically realistic. It is quite likely that other features of the flow may not be so sensitively affected by the model. Indeed, this is no doubt the case as the calculations of unsteady lift forces in cavity flow (Leehey 1962) show. This is perhaps only a manifestation of the prevailing observation that lift forces are insensitive to the cavity model for long cavities. The same is not quite so true, however, for other dynamic features of internal flows.

Concluding remarks

The results of this sample calculation show that the kinematically closed unsteady cavity model has a very plausible low-frequency behavior, namely, one that agrees with the quasi-steady limit. The evaluation of the closure integral is effected by an expansion in reduced frequency; this precludes drawing conclusions concerning high-frequency limits for the closed-cavity model. No doubt other physically plausible models may be adopted; however, there is not at the present time an adequate experimental basis to decide between alternatives. Finally, one of the authors (O.F.) has carried through a low-frequency analysis of unsteady flow in an inducer cascade employing the model described herein; there too, a plausible low-frequency limit is found for the complex matrix connecting input and output fluctuations across the cascade.

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