# ON THE THEORY OF ELECTROSTATIC ALTERNATORS

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Abstract

A more elegant method than that previously given for the solution of the equations involved in the theory of electrostatic alternators is given, and illustrated by application to an electrostatic alternator of eight inductors and eight carriers, in which each carrier as it comes in front of an inductor is connected to the next following inductor by means of a brush and insulated arm. The method can be extended to the solution of electrostatic alternators of $m$ carriers and $m$ inductors, where each carrier is connected to the inductor next following, also where each carrier is connected by means of brushes properly placed to the $p$th inductor following. It may also be applied to electrostatic alternators with $m$ inductors and $2m$, $3m$, or $4m$ carriers, and to electrostatic alternators of the Wimshurst type. With regard to the eight inductor alternator it is shown that three waves travel around the machine with periods corresponding to the time of 2 revolutions, 1 revolution, and 2/3 revolutions. The amplitude factor for each wave per eighth revolution is calculated approximately. Finally the effect of the initial conditions (potentials) on the action of the machine is discussed. It is shown that the potentials of the inductors of the machine eventually lag $\pi/4$ behind each other.

THE quantitative theory of a four inductor alternator of the type to be discussed in the present work has already been given,[2] and a general method of solution of alternators of this type outlined.[3] Since the publication of this work, however, it has been possible to develop a much more elegant method of solving the equations involved in the theory of these machines, and our present work is concerned both with an exposition of this method as well as a discussion of the physical interpretation of the results obtained by its application.

As the subject of our analysis we shall take an eight inductor electrostatic alternator, and present the theory of this machine in a form which admits of ready extension to alternators of this type with any number of inductors.

The machine studied is represented diagrammatically in Fig. 1. It consists of eight fixed inductors, and eight revolving carriers, and operates in a manner easily seen from the figure. The inductors are assumed to be identical in shape and symmetrically mounted, and the same is assumed to apply to the carriers.

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[2] A. W. Simon, Phys. Rev. **25**, 368 (1925).
[3] A. W. Simon, Phys. Rev. **26**, 111 (1925).

By exactly the same method as was employed in the case of the four inductor alternator already cited,[4] it can be shown that the potentials of the inductors after $(n+1)$ and $n$ cycles satisfy a set of equations of which the left members take the form:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $aV_0(n+1)$ | $+hV_1(n+1)$ | $+gV_2(n+1)$ | $+fV_3(n+1)$ | $+eV_4(n+1)$ | $+dV_5(n+1)$ | $+cV_6(n+1)$ | $+bV_7(n+1)$ |
| $b$ " | $+a$ " | $+h$ " | $+g$ " | $+f$ " | $+e$ " | $+d$ " | $+c$ " |
| $c$ " | $+b$ " | $+a$ " | $+h$ " | $+g$ " | $+f$ " | $+e$ " | $+d$ " |
| $d$ " | $+c$ " | $+b$ " | $+a$ " | $+h$ " | $+g$ " | $+f$ " | $+e$ " |
| $e$ " | $+d$ " | $+c$ " | $+b$ " | $+a$ " | $+h$ " | $+g$ " | $+f$ " |
| $f$ " | $+e$ " | $+d$ " | $+c$ " | $+b$ " | $+a$ " | $+h$ " | $+g$ " |
| $g$ " | $+f$ " | $+e$ " | $+d$ " | $+c$ " | $+b$ " | $+a$ " | $+h$ " |
| $h$ " | $+g$ " | $+f$ " | $+e$ " | $+d$ " | $+c$ " | $+b$ " | $+a$ " |

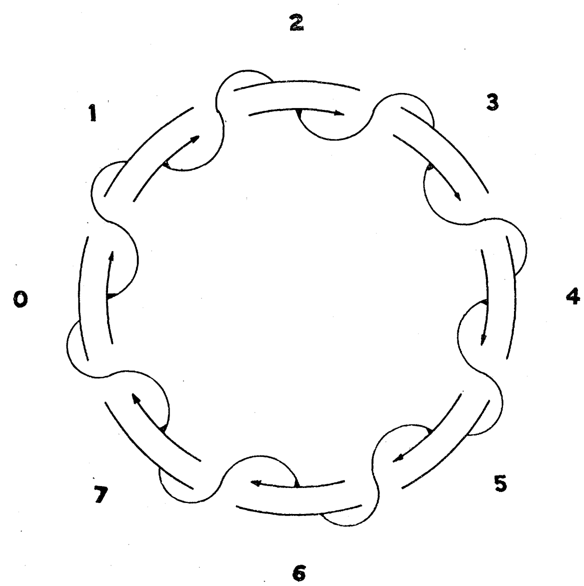


Fig. 1. An eight inductor electrostatic alternator.

and the right members take exactly the same form except that the coefficients $a$, $b$, $c$, etc. are replaced by $a'$, $b'$, $c'$, etc. respectively, and the argument $(n+1)$ of the functions $V_s(n+1)$ is replaced by $n$.

## I. General Solution of the Voltage Equations

Of fundamental importance for the mathematical solution of these equations is the following matrix, in which $\omega_j$ stands for the eighth root of unity corresponding to the amplitude of $j2\pi/8$.

$$\begin{array}{cccccccc} \omega_0^0 & \omega_0^1 & \omega_0^2 & \omega_0^3 & \omega_0^4 & \omega_0^5 & \omega_0^6 & \omega_0^7 \\ \omega_1^0 & \omega_1^1 & \omega_1^2 & \omega_1^3 & \omega_1^4 & \omega_1^5 & \omega_1^6 & \omega_1^7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \omega_7^0 & \omega_7^1 & \omega_7^2 & \omega_7^3 & \omega_7^4 & \omega_7^5 & \omega_7^6 & \omega_7^7 \end{array} \tag{2}$$

[4] A. W. Simon, Phys. Rev. **24,** 690 (1924).

For the purpose of the following work we shall number the rows and columns from 0 to 7, instead of from 1 to 8, as is the usual custom.

If we multiply the equations in order by the elements in the $j$th row of this matrix, and add, we obtain an equation of the form:

$$[a\omega_j^0+b\omega_j^1+c\omega_j^2+\cdots+h\omega_j^7][\omega_j^0V_0(n+1)+\omega_j^1V_1(n+1)+\cdots]=$$
$$[a'\omega_j^0+b'\omega_j^1+c'\omega_j^2+\cdots+h'\omega_j^7][\omega_j^0V_0(n)+\omega_j^1V_1(n)+\cdots]$$

the solution of which is obviously:

$$[\omega_j^0V_0(n)+\omega_j^1V_1(n)+\cdots]=\frac{[a'\omega_j^0+b'\omega_j^1+\cdots]^n}{[a\omega_j^0+b\omega_j^1+\cdots]^n}[\omega_j^0V_0(0)+\omega_j^1V_1(0)+\cdots] \quad (3)$$

and, if we do this for every row, (i. e., if we let $j$ run through the sequence of values 0, 1, $\cdots$ 7) we obtain eight simultaneous linear equations, the left members of which involve the functions $V_s(n)$ and the $\omega$'s, and in which the matrix of the coefficients of the left members is exactly the matrix (2). By elimination from this set we can then obtain each of the functions $V_s(n)$ in terms of the quantities $V_s(0)$ and the quantities $[a'\omega_j^0+b'\omega_j^1+\cdots]^n/[a\omega_j^0+b\omega_j^1+\cdots]^n$.

Now the matrix (2) has two important properties which allow us to carry out the process of elimination very easily. (1) If we examine the sum of the quantities in any column, we note that it is the sum of like powers of the roots of the algebraic equation $x^8-1=0$, and from the formula for the sum of like powers of the roots of an algebraic equation, it follows directly that the sum of the quantities in every column of the matrix (or row, if we replace every element of the form of $\omega_p{}^q$ by $\omega_q{}^p$), except the first, is zero. (2) If we divide each row through by the factor which will make the elements in the $s$th column all unity, the resulting matrix can be obtained from the matrix (2) by $s$ cyclic permutations of the columns of the matrix (2), and has the property that the sum of the coefficients in every column except the $s$th is zero.

To solve for any $V_s(n)$ then we have merely to divide the equations (3) through by such factors as will make the elements in the $s$th column of the matrix (2) all equal to unity, and then add the resulting equations.

Accordingly, we have as the general formula for $V_s(n)$:

$$8V_s(n)=\sum_{j=0}^{7}[\omega_j^0V_0(0)+\omega_j^1V_1(0)+\cdots][a'\omega_j^0+b'\omega_j^1+\cdots]^n/\omega_{sj}[a\omega_j^0+b\omega_j^1+\cdots]^n \quad (4)$$

where $\omega_{sj}$ is the element in the $s$th column and $j$th row of the matrix (2).

Now while each term of this sum is complex, the sum itself is real, for we can group the complex terms into pairs of conjugate terms, such that the imaginary parts will cancel out in the sum.

This is readily seen from the matrix (2) if in addition we note that each of the expressions in brackets in formula (4) is obtained by combining a sequence of eight quantities, (namely the sequences $V_0(0)$, $V_1(0), \cdots, V_7(0)$; $a', b', \cdots, h'$; and $a, b, \cdots, h$) with the elements in any row of the matrix. As pairs of rows which give rise to conjugate terms we have 1,7; 2,6; and 3,5. The rows 0,4, it may be pointed out, give rise to real terms. For the purpose of our further calculations it is convenient to rewrite Eq. (4) in another form. If namely we put:

$$[\omega_j^0 V_0(0)+\omega_j^1 V_1(0)+\cdots]=[A_j(0)+iB_j(0)]=R_j(0)[\cos\Delta_j+i\sin\Delta_j]$$

$$[a'\omega_j^0+b'\omega_j^1+\cdots]=[\alpha_j'+i\beta_j']=\sigma_j'[\cos\varphi_j'+i\sin\varphi_j']$$

$$[a\omega_j^0+b\omega_j^1+\cdots]=[\alpha_j+i\beta_j]=\sigma_j[\cos\varphi_j+i\sin\varphi_j]$$

$$\omega_{sj}=\cos\partial_{sj}+i\sin\partial_{sj}$$

and, further, let $\rho_j=\sigma_j'/\sigma_j$ and $\theta_j=\varphi_j'-\varphi_j$, we can rewrite Eq. (4) as:

$$8V_s(n)=\sum_{j=0}^{7}[A_j(0)+iB_j(0)]\rho_j^n[\cos(n\theta_j-\partial_{sj})+i\sin(n\theta_j-\partial_{sj})]$$

## II. Particular Solutions of the Voltage Equations

For various assumed initial conditions, i. e., sets of values of the quantities $V_s(0)$, the general solution reduces to simple forms. In particular consider the sets of values given by:

$$V_s(0)=E(0)\cos(s-p)k2\pi/8 \tag{5}$$

where $s$, $p$, and $-k$, take the values 0, 1, 2, $\cdots$, 7. We then have for $A_j(0)$ and $B_j(0)$, respectively:

$$A_j(0)=E(0)\sum_{s=0}^{7}[\cos(s-p)k2\pi/8][\cos sj2\pi/8]\ ,$$

$$B_j(0)=E(0)\sum_{s=0}^{s=7}[\cos(s-p)k2\pi/8][\sin sj2\pi/8]\ .$$

Now from the fact that $\sum_{s=0}^{7}\omega_p\omega_s^r=0$, or $8\omega_p$, respectively, according as $\pm r\neq 0$, 8, 16, etc., or as $\pm r=0$, 8, 16, etc., and from trigonometry, it is possible to prove readily the following theorem:

Theorem I. For any given $k$ such that $-k\neq 0$ or 4, we have $A_j(0)=0$ and $B_j(0)=0$, for every value of $j$ except $j=-k$ and $j=8+K$; for $j=-k$ and $j=8+K$, $A_j(0)=4E(0)\cos(-pk)2\pi/8$ and $B_j(0)=4E(0)\sin$

$(-pk)2\pi/8$; while for $k=0$ or $-4$, $A_j(0)=0$ and $B_j(0)=0$, for every value of $j$ except 0 or $+4$; finally for $kj=0$ or $-4$, $A_j(0)=8E(0)\cos(-pk)2\pi/8$, $B_j(0)=0$.

## III. Physical Interpretation of the Results

For the physical interpretation of the results we need to know the relative magnitudes of the quantities $\rho_j$ and $\theta_j$. These are obtained by substituting in the formulas for $\rho_j$ and $\theta_j$, the values of the quantities $a$, $a'$, $b$, $b'$, etc. in terms of the coefficients of capacity and coefficients of induction of the elements of the machine in that configuration at which the carriers just break contact with the brushes. In order to calculate exactly the various values of $\rho_j$ and $\theta_j$ we would require the numerical values of these coefficients. It is, of course, impossible to calculate these. To complete the analysis of any machine it would be necessary to actually measure them. However, we can gain some idea of the relative magnitudes of the $\rho$'s and $\theta$'s involved, if we make the same assumptions as were made in the case of the four inductor alternator, namely, that the inductors nearly completely surround the carriers and that the inductors are relatively far apart. Under these conditions we can calculate the following set of values for the $\rho$'s and $\theta$'s involved.

| $j$ | $\tan\theta$ | $\theta$ | $\rho$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 1, 7 | $\sqrt{2}-1$ | $\pi/8$ | $\sqrt{2}/\sqrt{2-\sqrt{2}}$ |
| 2, 6 | 1 | $\pi/4$ | $\sqrt{2}$ |
| 3, 5 | $\sqrt{2}+1$ | $3\pi/8$ | $\sqrt{2}/\sqrt{2+\sqrt{2}}$ |
| 4 | 0 | 0 | 0 |

The values obtained on the basis of these assumptions are really ideal values which in practice we can only approximate. Nevertheless, they correctly indicate the relative magnitudes of the quantities in question, so that for the purpose of our further discussion we shall assume:

$$\rho_0=1,\ \rho_1>\rho_2>1,\ \rho_3<1,\ \rho_4<1;$$
$$\theta_0=0,\ \theta_1\doteq\pi/8,\ \theta_2\doteq\pi/4,\ \theta_3\doteq3\pi/8,\ \theta_4=0.$$

The phase angles $\partial_{sj}$, on the other hand, are independent of the capacities involved in the machine, and are readily determined from the matrix (2). In particular, we have

$$\partial_{s0}=0,\ \partial_{s1}=s\pi/4,\ \partial_{s2}=s\pi/2,\ \partial_{s3}=s3\pi/4,\ \partial_{s4}=s\pi.$$

With all these relations before us we can inquire as to the action of the alternator. We see that in general the action of the machine may be described by asserting that three potential waves of different periods travel around the machine, two of which continually increase in amplitude ($\rho > 1$) and one of which continually decreases in amplitude ($\rho < 1$). Furthermore, eventually, one only of these waves will be perceptible, the others becoming either vanishingly small or negligible in comparison. The wave in question is that corresponding to $\rho_1$ and $\theta_1$, for we have, for $n$ large,

$$V_s(n) = R_1(0)\rho_1{}^n \cos\,(n\theta_1 + \Delta_1 - s\pi/4)\ .$$

However, the action of the alternator will depend, theoretically at least, very markedly on the initial conditions. For example, if we choose the original potentials so as to satisfy the set of values given by equation (5); that is, if we adjust the original potentials so that there will be a difference of phase of $k2\pi/8$ between each pair of successive inductors and if, furthermore, we put the maximum potential on the $p$th inductor, then only that wave corresponding to $j = -k$ will be present.

If we let $-k$ vary from 0 to 4 and apply Theorem I, we obtain a number of interesting special cases, which it should be possible to verify experimentally.

(1) $k = 0$, $V_s(0) = E(0) = \text{const.}$ The original potentials are all the same. For this case, if in addition we note that it can be shown by the method of a previous paper,[4] $\rho_0 \equiv 1$, no matter what the capacities involved in the machine are, as long as it is symmetrical, we have for every $n$ from 0 to $\infty$: $V_s(n) = E(0) = \text{const.}$, that is to say, no change takes place, as we might expect also from qualitative considerations.

(2) $-k = 1$, $V_s(0) = E(0)\cos(s-p)(-\pi/4)$. The corresponding solution is, for every $n$ from 0 to $\infty$:

$$V_s(n) = E(0)\rho_1{}^n \cos\,[n\theta_1 - (s-p)\pi/4]\ .$$

Under these conditions the potential will reverse approximately every revolution, and the machine will build up very rapidly.

(3) $-k = 2$, $V_s(0) = E(0)\cos(s-p)(-\pi/2)$. The solution in this case is given by

$$V_s(n) = E(0)\rho_2{}^n \cos\,[n\theta_2 - (s-p)\pi/2]\ .$$

The potential now reverses approximately every half revolution, but the machine does not build up as rapidly as in case (2), since $\rho_2 < \rho_1$.

(4) $-k = 3$, $V_s(n) = E(0)\rho_3{}^n \cos[n\theta_3 - (s-p)3\pi/4]$. The potential reverses approximately every third of a revolution, but the potentials continually decrease, ($\rho_3 < 1$), and eventually become vanishingly small. In practice they would very likely decrease until the ever-present contact e.m.f.'s took hold and re-excited the machine.

(5) $-k=4$, $V_s(n)=E(o)\rho_4^n \cos[-(s-p)\pi]$. Under these conditions also the potentials would theoretically decrease to zero, or in practice until the contact e.m.f.'s took hold.

## IV. Application of the Method to Other Machines

(1) *Multiple Inductor Machines of the type just discussed.* It has already been stated in our opening paragraph that the method of analysis used in the present work for an eight inductor alternator can be applied to a machine of this type with any number of inductors. For the general case of $m$ inductors and $m$ carriers the matrix (2) would be simply a square matrix of the $m$th order, involving the $m$th roots of unity. A theorem corresponding to Theorem I could also be set up very easily for the general case. We have merely to note that if $\omega_j$ is the $m$th root of unity which has the amplitude $j2\pi/m$, we have $\sum_{s=0}^{m} \omega_p \omega_s^r = 0$ or $m\omega_p$, respectively, according as $\pm r$ is, or as it is not, zero or a multiple of $m$. This follows directly from the fact that $\sum \omega_s^r$ is the sum of like powers of the roots of the equation $x^m - 1 = 0$.

(2) *Solution for Other Schemes of Connections.* Furthermore, if in the case of the eight inductor system just discussed each carrier is connected not to the next following inductor but to the second, third, or $p$th one following, we obtain equations of exactly the same form as Eq. (1) except that the constants $a$, $a'$, $b$, $b'$, etc. have different values. This means, that the solution is of exactly the same form as before but that the $\rho$'s and $\theta$'s have different values, and this again holds in general; namely, that once the solution for one alternator of $m$ inductors is determined for one scheme of connections, the solutions for all other schemes of connections are obtained by assigning to the $\rho$'s and $\theta$'s their proper values. Of course, the scheme of connections must be symmetrical and not haphazard.

(3) *Machines with Double the Number of Carriers.* If the eight inductor machine just discussed were built up with 16 carriers instead of 8, the matrix of the coefficients of the voltage equations (sixteen in number) would take the form

$$\begin{matrix} M_1 & M_2 \\ M_3 & M_4 \end{matrix}$$

where each of the matrices $M_i$ is a cyclic matrix of the 8th order. To solve this case, we divide the set of sixteen equations into two sets of eight (the first set involving $M_1$, $M_2$, the second $M_3$, $M_4$) and apply the method by which Eq. (3) is obtained to each set separately, but instead of deriving an equation of the form (3), we would now have from the first set one of the form:

$$(\alpha_1+i\beta_1)Z_1(n+1)+(\alpha_2+i\beta_2)Z_2(n+1) = (\alpha_1'+i\beta_1')Z_1(n)+(\alpha_2'+i\beta_2')Z_2(n),$$

and from the second set another of the same type except that the $\alpha$'s and $\beta$'s have different values. Each of these pairs (and there would be eight such pairs) can be solved by the method of finite differences, as already outlined in previous papers. This method again can be generalized to apply to the alternator with any number of inductors.

(4) and (5) *Machines in which the number of carriers is three or four times the number of inductors.* The method just given for case (3) can be further generalized to apply to the case of 24 and 32 carriers, or in general to the case in which the number of carriers is three or four times the number of inductors. Case (3) above reduces, if the method of finite differences is applied, to the solution of quadratic equations, while the two cases now under consideration reduce to the solution of cubic and biquadratic equations respectively.

(6) *Electrostatic Alternators of the Wimshurst Type.* What has just been said applied also to alternators of the Wimshurst type, since they give rise to equations of exactly the same type as (3), (4), (5).

In general, of course, as the number of inductors and carriers is increased the detailed solution becomes more and more laborious although theoretically it can be carried out. However, it is to be expected that the general features of these machines are not greatly changed by an increase in the number of elements; for example, it is to be expected that doubling the number of carriers will make the voltage wave more continuous without greatly affecting the period of alternation or the transformation ratio.

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