Supplementary Material

Quantum Complexity Classes

We start with the relevant definitions. Let $x$ be a binary string. Then, we denote by the verifier $V \equiv V_x$ a quantum circuit of length $T = \text{poly}(|x|)$, $U = U_T \cdots U_1$ (with local gates $U_i$ acting on $m = \text{poly}(|x|)$ qubits), which is generated uniformly from $x$. The verifier takes an $n = \text{poly}(|x|)$ qubit quantum state $|\psi\rangle$ as an input (we will express everything in terms of $n$ instead of $|x|$ in the following), together with $m - n$ initialized ancillas, $|0\rangle_A \equiv |0 \cdots 0\rangle_A$, applies $U$, and finally measures the first qubit in the $\{|0\rangle, |1\rangle\}$ basis to return 1 (“proof accepted”) or 0 (“proof rejected”). The acceptance probability for a proof $|\psi\rangle$ is then given by $p_{\text{acc}}(V(\psi)) := \langle \psi|\Omega|\psi\rangle$, with

$$
\Omega = (I \otimes |0\rangle_A)U^\dagger(|1\rangle_1 \otimes I)U(I \otimes |0\rangle_A).
$$

**Definition 1 (QMA).** Let $0 \leq a, b \leq 1$ s.th. $a - b > \frac{1}{\text{poly}(n)}$ for some polynomial $p(n)$. A language $L$ is in QMA($a, b$) if there exists a verifier $s$-th.

$$
x \in L \Rightarrow \lambda_{\max}(\Omega) > a
$$

$$
x \notin L \Rightarrow \lambda_{\max}(\Omega) < b.
$$

**Definition 2 ($\#\text{BQP}$).** Let $0 \leq a, b \leq 1$ s.th. $a - b > \frac{1}{\text{poly}(n)}$, and let $\Omega$ be a verifier map with no eigenvalues between $a$ and $b$. Then, the class $\#\text{BQP}(a, b)$ consists of all problems of the form “compute the dimension of the space spanned by all eigenvectors of $\Omega$ with eigenvalues $\geq a$.”

An alternative definition for $\#\text{BQP}$ is the following:

**Definition 3 ($\#\text{BQP}$, alternate definition).** Consider a verifier $\Omega$ with the property that there exist subspaces $A \oplus R = C^N (N = 2^n)$ such that $\langle \psi|\Omega|\psi\rangle \geq a$ for all $|\psi\rangle \in A$, and $\langle \psi|\Omega|\psi\rangle \leq b$ for all $|\psi\rangle \in R$, where again $a - b > 1/\text{poly}(n)$. Then $\#\text{BQP}(a, b)$ consists of all problems of the form “compute $\dim A$”.

Note that $\dim A$ is well-defined: Consider two decompositions $C^N = A \oplus R$ and $C^N = A' \oplus R'$. Without loss of generality, if we assume $\dim A > \dim A'$, it follows $\dim A + \dim R' > N$, and thus there exists a non-trivial $|\mu\rangle \in A \cap R'$, which contradicts the definition.

**Theorem 4.** Definition 2 and Definition 3 are equivalent.

**Proof.** To show that Definition 2 implies Definition 3, let $A$ be spanned by the eigenvectors with eigenvalues $\geq a$. To show the converse, we use the minimax principle for eigenvalues [7], which states that the $k$th largest eigenvalue $\lambda_k$ of a Hermitian operator $\Omega$ in an $N$-dimensional Hilbert space can be obtained from either of the equivalent optimizations

$$
\lambda_k(\Omega) = \max_{\mathcal{M}_k} \min_{|x\rangle \in \mathcal{M}_k} \langle x|\Omega|x\rangle
$$

$$
= \min_{\mathcal{M}_{N-k+1}} \max_{|x\rangle \in \mathcal{M}_{N-k+1}} \langle x|\Omega|x\rangle,
$$

where $\mathcal{M}_k$ is a subspace of dimension $k$, and $|x\rangle$ is a unit vector. Now notice that Def. 3 implies that

$$
\min_{|x\rangle \in A} \langle x|\Omega|x\rangle \geq a \quad \text{and} \quad \max_{|x\rangle \in R} \langle x|\Omega|x\rangle \leq b.
$$

Next, consider the minimax theorem for $k = \dim A$. From Eq. (2) we have

$$
\lambda_{\dim A} = \max_{|x\rangle \in \mathcal{M}_{\dim A}} \min_{|x\rangle \in \mathcal{M}_{\dim A}} \langle x|\Omega|x\rangle
$$

$$
\geq \min_{|x\rangle \in A} \langle x|\Omega|x\rangle
$$

$$
\geq a.
$$

Now consider the case that $k = \dim A + 1$. From Eq. (3), using the fact that $N - (\dim A + 1) + 1 = \dim R$, we have

$$
\lambda_{\dim A + 1} = \min_{|x\rangle \in \mathcal{M}_{\dim R}} \max_{|x\rangle \in \mathcal{M}_{\dim R}} \langle x|\Omega|x\rangle
$$

$$
\leq \max_{|x\rangle \in R} \langle x|\Omega|x\rangle
$$

$$
\leq b.
$$

Thus we have

$$
\lambda_{\dim A} \geq a > b \geq \lambda_{\dim A + 1},
$$

since $a - b \geq 1/\text{poly}(n)$. It follows that $\lambda_{\dim A}$ is the smallest eigenvalue of $\Omega$ which is still larger than $a$, and therefore the span of the first $\dim A$ eigenvectors of $\Omega$ is equal to the span of all eigenvectors with eigenvalue $\geq a$. The equivalence follows.

The class $\#\text{BQP}(a, b)$ inherits the useful property of strong error reduction from QMA: the thresholds $(a, b)$ can be amplified to $(1 - 2^{-r}, 2^{-r})$, $r = \text{poly}(n)$ while keeping the size of the proof.

**Theorem 5.** Let $\#\text{BQP}_n(a, b)$ denote $\#\text{BQP}$ with an $n$ qubit witness. Then $\#\text{BQP}_n(a, b) \subseteq \#\text{BQP}_n(1 - 2^{-r}, 2^{-r})$ for every $r = \text{poly}(n)$.

This follows directly from the strong amplification procedure presented in [7], which describes a procedure to amplify any verifier map $\Omega$ such that any eigenvalue above $a$ (below $b$) is shifted above $1 - 2^{-r}$ (below $2^{-r}$) at an overhead polynomial in $r$.

Using Thm. 5, we will write $\#\text{BQP}$ instead of $\#\text{BQP}(a, b)$ from now on, where $a - b > \text{poly}(n)$, and $a, b$ can be exponentially close to 1 and 0, respectively.
The Complexity of the Density of States

We now use the class \#BQP to characterize the complexity of the density of states problem and the problem of counting the number of ground states of a local Hamiltonian. We start by defining these problems, as well as the notion of local Hamiltonian, and then show that both problems are \#BQP-complete.

Definition 6 (k-local Hamiltonian). Given a set of poly(n) quantum spins each with dimension bounded by a constant, a Hamiltonian \( H \) for the system is said to be k-local if \( H = \sum_i H_i \) is a sum of at most poly(n) Hermitian operators \( H_i \), \( \|H_i\| \leq 1 \), each of which acts nontrivially on at most k spins.

Note that k-local does not imply any geometric locality, only that each spin interacts with at most \( k-1 \) other spins for any given interaction term. However, we restrict ourselves to \( k = O(\log(n)) \) so that each \( H_i \) can be specified by an efficient classical description.

Definition 7 (dos). Let \( E_2 - E_1 > 1/\text{poly}(n) \), \( \Delta = (E_2 - E_1)/\text{poly}(n) \), and let \( H = \sum_i H_i \) be a k-local Hamiltonian such that \( H \) has no eigenvalues in the intervals \([E_1 - \frac{\Delta}{2}, E_1 + \frac{\Delta}{2}]\) and \([E_2 - \frac{\Delta}{2}, E_2 + \frac{\Delta}{2}]\). Then, the problem dos (density of states) is to compute the number of orthogonal eigenstates with eigenvalues in the interval \([E_1, E_2]\).

Definition 8 (\#LH). Let \( E_0 \leq E_1 < E_2 \), with \( E_2 - E_1 > 1/\text{poly}(n) \), and let \( H = \sum_i H_i \) be a k-local Hamiltonian s.th. \( H \geq E_0 \), and \( H \) has no eigenvalues between \( E_1 \) and \( E_2 \). Then, the problem \#LH \#LH(\( E_1 - E_0 \)) (sharp local Hamiltonian) is to compute the dimension of the eigenspace with eigenvalues \( \leq E_1 \).

Note that \#LH depends on the “energy splitting” \( E_1 - E_0 \) of the low-energy subspace. In particular, for \( E_1 - E_0 = 0 \), \#LH(0) corresponds to computing the degeneracy of the ground state subspace. As we will see in what follows, the class \#LH(\( \sigma \)) is the same for any splitting \( \exp(-\text{poly}(n)) \leq \sigma \leq 1/\text{poly}(n) \).

We now show that \#LH and DOS are both \#BQP-complete. We do so by giving reductions from \#LH(1/\text{poly}(n)) to DOS, from DOS to \#BQP, and from \#BQP to \#LH(\exp(-\text{poly}(n)))); this will at the same time prove the claimed independence of \#LH(\( \sigma \)) of the splitting \( \sigma \).

Theorem 9. \#LH(1/\text{poly}(n)) reduces to dos.

Proof. If we denote the parameters of the \#LH problem by \( E_0, E_1, E_2 \), then we can simply relate them to the parameters \( E_1, E_2, \Delta \) of a DOS problem by \( \Delta = E_2 - E_1 \), \( E_1 = E_0 - \frac{\Delta}{2} \) and \( E_2 = E_1 + \frac{\Delta}{2} \), and the result follows directly. \( \square \)

Theorem 10. DOS is contained in \#BQP.

Proof. We start with a k-local Hamiltonian \( H' := \nu(H^2 - (E_1 + E_2)H + E_1 E_2) \).

\( H' \) is a 2k-local Hamiltonian; here, \( \nu = 1/\text{poly}(n) \) is chosen such that each term in \( H' \) is subnormalized. Any eigenvalue of \( H \) in the interval \([E_1 + \frac{\Delta}{2}, E_2 - \frac{\Delta}{2}]\) translates into an eigenvalue of \( H' \) which is below

\[ A := -\nu \Delta (E_2 - E_1 - \frac{\Delta}{2}) \leq -1/\text{poly}(n) \],

whereas any eigenvalue outside \([E_1 - \frac{\Delta}{2}, E_2 + \frac{\Delta}{2}]\) translates into an eigenvalues of \( H' \) above

\[ B := \nu \Delta (E_2 - E_1 + \frac{\Delta}{2}) \geq 1/\text{poly}(n) \]

The original DOS problem now translates into counting the number of eigenstates of \( H' \) with negative energy, given a spectral gap in \([1/\text{poly}(n)], \text{A,B} \) around zero. We now use the circuit which was used in [?] to prove that log-local Hamiltonian is in QMA; it accepts any input state \(|\psi\rangle\) with probability

\[ p(|\psi\rangle) = \frac{1}{2} - \frac{|\langle\psi|H'|\psi\rangle|^2}{2m}, \]

where \( m = \text{poly}(n) \) is the number of terms in \( H' \). [The idea is to randomly pick one term \( H' \) in the Hamiltonian and toss a coin with probability \((1 - (\langle\psi|H'|\psi\rangle))/2\].

Computing the answer to the original DOS problem then translates to counting the number of states with acceptance probability \( \geq a = \frac{1}{2} + \frac{\Delta}{2m} \), given that there are no eigenstates between \( a,b = (A - B)/2m \geq 1/\text{poly}(n) \), this shows that this number can be computed in \#BQP. \( \square \)

Theorem 11. \#LH(\exp(-\text{poly}(n))) is \#BQP-hard.

Proof. To show \#BQP-hardness of \#LH, we need to start with an arbitrary QMA verifier circuit \( U = U_T \ldots U_1 \) and construct a Hamiltonian with as many ground states as the circuit has accepting inputs. By amplification, we can assume that the acceptance and rejection thresholds for the verifier are \( a = 1 - \epsilon \) and \( b = \epsilon \), where we can choose \( \epsilon = \text{O}(\text{exp}(-p(n))) \) for any polynomial \( p(n) \). As before, let \( \mathcal{A} \) and \( \mathcal{R} \) be the eigenspaces of \( \Omega \) with eigenvalues \( \geq a \) and \( \leq b \), respectively. Define

\[ U[\mathcal{R}] := \{U|\psi\rangle | 0\}_{\mathcal{A}} : |\psi\rangle \in \mathcal{R}\]  \( (6) \)

and denote the projector onto this space by \( \Pi_U[\mathcal{R}] \). Notice that for any state \(|\chi\rangle \in U[\mathcal{R}] \), due to our rejection threshold \( b = \epsilon \), we have

\[ \langle\chi|(|1\rangle\langle1| \otimes 1)|\chi\rangle \leq \epsilon. \]  \( (7) \)
We now follow Kitaev’s original construction to encode a QMA verifier circuit into a Hamiltonian which has a “proof history” as its ground state for any proof $|\psi\rangle \in A | ? ? ]$. That is, the ground states of the Hamiltonian are given by

$$|\Phi\rangle = \sum_{t=0}^{T} U_t \ldots U_1 |\psi\rangle |0\rangle_A |t\rangle_T$$

where the third register is used as a “clock”. The Hamiltonian has the form

$$H = H_{\text{init}} + \sum_{t=1}^{T} H_{\text{eval}}(t) + H_{\text{final}}$$

where

- $H_{\text{init}} = 1_I \otimes (1 - |0\rangle\langle 0|_A) \otimes |0\rangle\langle 0|_T$ checks that the ancilla is properly initialized, penalizing states without properly initialized ancillas;
- $H_{\text{eval}}(t) = -\frac{1}{2} U_t \otimes |t\rangle\langle t-1|_T - \frac{1}{2} U_t^\dagger \otimes |t-1\rangle\langle t|_T + \frac{1}{2} |1\rangle\langle 1|_T$ checks that the propagation from time $t-1$ to $t$ is correct, penalizing states with erroneous propagation;
- $H_{\text{final}} = \Pi_U [R] \otimes |T\rangle\langle T|_T$ causes each state $|\phi\rangle$ built from an input $|\psi\rangle_I \in R$ (but which nonetheless has a correctly initialized ancilla) to receive an energy penalty.

As we show in Lemma 12, the total Hamiltonian $H$ has a spectral gap $1/\text{poly}(n)$ above the $\text{dim} A$-dimensional ground state subspace. However, $H_{\text{final}}$ is not a local Hamiltonian, but as we argue in the following, it can be replaced by the usual term $H_{\text{final}} = [0\rangle\langle 0|_{\text{prim}} \otimes |T\rangle\langle T|_T$ while keeping the ground space dimension (up to small splitting in energies) and the $1/\text{poly}(n)$ spectral gap. As we prove in Lemma 13,

$$H_{\text{final}}^\text{std} \geq H_{\text{final}} - \sqrt{\epsilon} 1.$$  

Thus, replacing $H_{\text{final}}$ by $H_{\text{final}}^\text{std}$ will decrease the energy of any excited state by at most $\sqrt{\epsilon} = O(\text{exp}(-p(n)/2))$. (The energy of the ground states is already minimal and cannot decrease.) On the other hand, the energy of any proper proof history for $H$ cannot increase by more than

$$\langle \chi | H_{\text{final}}^\text{std} | \chi \rangle = \langle \chi | 0\rangle\langle 0|_A \otimes 1_1 | \chi \rangle \leq \epsilon = O(\text{exp}(-p(n))),$$

due to our choice of acceptance threshold, i.e., the low energy subspace has dimension $\text{dim} A$. We thus obtain a Hamiltonian with a $\text{dim} A$ dimensional ground state subspace with energy splitting $\leq \epsilon = \text{exp}(-p(n))$ and a $1/\text{poly}(n)$ spectral gap above. \[\Box\]

The following two lemmas are used in the preceding proof of Theorem 11.

**Lemma 12.** $H$ has a spectral gap of size $1/\text{poly}(n)$.

**Proof.** Our proof follows closely the discussion in Ref. [?], cf. also [?]. We can block diagonalize $H$ by the (conjugate) action of the following unitary operator,

$$W = \sum_{j=0}^{T} U_j \ldots U_1 \otimes |j\rangle\langle j|_T,$$

which maps $H \rightarrow H' = W^\dagger H W$. As this has no effect on the spectrum, we can work with the simpler $H'$ from now on.

Let us explicitly write the effect of conjugation by $W$ on the terms of $H$. The first term is unaffected, $H'_{\text{init}} = H_{\text{init}}$. The final term becomes $H'_{\text{final}} = \Pi_R \otimes |0\rangle\langle 0|_A \otimes |T\rangle\langle T|_T$, where $\Pi_R$ is simply the projector onto the space $R$, and the ancillas are in the correct initial state.

We can now conjugate each of the terms in $H_{\text{eval}}(t)$ separately. For example, the first term gives

$$W^\dagger (U_t \otimes |t\rangle \langle t-1|_T) W = 1 \otimes |t\rangle \langle t-1|.$$  

The other terms are exactly analogous, and we find that

$$H'_{\text{eval}}(t) = 1 \otimes \frac{1}{2} [|t\rangle\langle t-1| + |t\rangle\langle t| - |t\rangle\langle t-1| - |t-1\rangle\langle t|].$$

The total evolution Hamiltonian is then block-diagonal a matrix which looks like a hopping Hamiltonian in the clock register,

$$\sum_t H'_{\text{eval}}(t) = 1 \otimes E,$$

where the $(T + 1)$-by-$(T + 1)$-dimensional tri-diagonal matrix $E$ is given by

$$E = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ & -\frac{1}{2} & \ddots \\ & & & 1 & -\frac{1}{2} \\ & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$  

To discuss the spectrum of $H'$, let

$$S_1 = A \otimes |0\rangle\langle 0|_A \otimes C^{T+1},$$

$$S_2 = R \otimes |0\rangle\langle 0|_A \otimes C^{T+1},$$

and $S_3$ the orthogonal complement of $S_1 \oplus S_2$; i.e., $S_1$ corresponds to evolutions starting from good proofs, $S_2$ to those starting from wrong proofs, and $S_3$ to evolutions with wrongly initialized ancillas. Note that $H' = H|_{S_1} \oplus H|_{S_2} \oplus H|_{S_3}$, thus, we can analyze the spectrum for $H|_{S_p}$ separately. Note that the restriction to $S_p$ does not affect $H'_{\text{eval}}(t)$. Since we expect the ground state subspace to
occur on $S_1$; we need to compute ground state energy and gap of $H'|S_1$, as well as lower bound the ground state energies of $H'|S_p$ and $H'|S_0$.

First, $H'_{\text{init}}|S = H'_{\text{final}}|S = 0$, i.e., the spectrum of $H'|S$ equals the spectrum of $E'$, which can be straightforwardly determined to be $1 - \cos \omega_n$, with $\omega_n = n\pi/(T + 1)$, and eigenfunctions $(\cos \frac{3}{2}\omega_n, \cos \frac{5}{2}\omega_n, \ldots)$; the ground state degeneracy is indeed $\dim A$ as desired.

On the other hand, $H'_{\text{final}}|S_2 = 1$, and $H'_{\text{init}}|S_3 \geq 1$, i.e., in both cases the ground state energy of $H'|S_p$ is lower bounded by the ground state energy of

$$E' = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & \cdots & \cdots & -\frac{1}{2} \\
-\frac{1}{2} & 1 & -\frac{1}{2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{1}{2} & \cdots & \cdots & \cdots & \cdots \\
-\frac{1}{2} & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},$$

which has eigenvalues $1 - \cos \vartheta_n$, with $\vartheta_n = (n + \frac{1}{2})\pi/(T + 1)$, and eigenfunctions $(\cos \frac{3}{2}\vartheta_n, \cos \frac{5}{2}\vartheta_n, \ldots)$.

It follows that $H'$ (and thus $H$) has a ground state energy of $1 - \cos \omega_0 = 0$, and a gap $1 - \cos \frac{\pi}{2T + 3} = O(1/T^2) = O(1/poly(n))$ above.

It remains to prove Eq. (10), which follows from the following Lemma by choosing $P = \langle 0 | 0 \rangle \otimes 1$, $Q = H_U|\pi \rangle$, and using Eq. (7).

**Lemma 13.** Let $P$ and $Q$ be projectors such that $\|Q(1 - P)Q\|_\infty \leq \epsilon$. Then

$$P - Q \geq -\sqrt{\epsilon} I. \quad \text{(15)}$$

**Proof.** We begin by recalling the result due to Jordan [? ] (see Ref. [? ] for a more modern treatment) for the simultaneous canonical form of two projectors. In the subspace where $P$ and $Q$ commute, both operators are diagonal in a common basis and the spectrum is either $(0, 0), (0, 1), (1, 0)$, or $(1, 1)$, and direct sums of those terms. In the subspace where they don’t commute, the problem decomposes into a direct sum of two-by-two blocks given by

$$P_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_j = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}, \quad \text{(16)}$$

where $s = \sin(\theta_j)$ for some angle $\theta_j$, $c^2 + s^2 = 1$, and the subscript $j$ just labels a generic block.

In fact, this two-by-two block form is completely general if we allow embedding our projectors into a larger space while preserving their rank. The rank-preserving condition guarantees that our bound is unchanged, since we are only appending blocks of zeros, and so we will consider this two-by-two form without loss of generality.

The constraint that $\|Q(1 - P)Q\|_\infty \leq \epsilon$ implies constraints on the values that $\sin(\theta_j)$ can take. In particular, we can directly compute this operator norm in each block separately, and we find that for all $j$

$$\|Q_j(1 - P_j)Q_j\|_\infty = \sin^2(\theta_j) \leq \epsilon. \quad \text{(17)}$$

We can also directly compute in each block that

$$P_j - Q_j = \begin{pmatrix} 1 - c^2 & 1 - cs \\ -cs & -s^2 \end{pmatrix}. \quad \text{(18)}$$

The spectrum of this operator is easily computed to be $\pm |\sin(\theta_j)|$. Thus, the least eigenvalue of $P - Q$ is bounded from below by $-\sqrt{\epsilon}$. \Box

**Quantum vs. Classical Counting Complexity**

We finally want to relate #BQP to the classical counting class #P. It is clear that any #P problem can be solved in #BQP by using a classical circuit. We will now show that conversely, #BQP can be reduced to #P under weakly parsimonious reductions: That is, for any function $f \in \#BQP$ there exist polynomial-time computable functions $\alpha$ and $\beta$, and a function $g \in \#P$, such that $f = \alpha \circ g \circ \beta$. This differs from Karp reductions where no postprocessing is allowed, $\alpha = \text{Id}$, but still only requires a single call to a #P oracle, in contrast to Turing reductions.

**Theorem 14.** There exists a weakly parsimonious reduction from #BQP to #P.

**Proof.** We start from a verifier operator $\Omega$ for a #BQP problem. First, we use strong error reduction to let $a = 1 - 2^{-(n - 2)}$ and $b = 2^{-(n + 2)}$. It follows that

$$|\dim A - \text{tr} \Omega| \leq 2^n 2^{-(n + 2)} = \frac{1}{4} \quad \text{(19)}$$

and thus we need to compute $\text{tr} \Omega$ to accuracy $1/4$. This can be done using the “path integral” method previously used to show containments of quantum classes inside PP and #P [? ]. We rewrite $\text{tr} \Omega = \sum \zeta f(\zeta)$, where the sum is over products of transition probabilities along a path, which we label

$$\zeta \equiv (i_0, \ldots, i_N, j_1, \ldots, j_N), \quad \text{(20)}$$

so that

$$f(\zeta) = \langle i_0 | f(1) | U^+_1 | j_1 \rangle \langle j_1 | U^+_2 | j_2 \rangle \cdots \langle j_{N - 1} | f(1) | i_N \rangle = \langle i_T | [0 | 0 \rangle \otimes 1 | i_T \rangle | i_T \rangle | U_{T} \cdots U_{1} | i_0 \rangle | 0 \rangle A. \quad \text{(21)}$$

(cf. Fig. 1 in the main manuscript for an illustration).

Since any quantum circuit can be recast in terms of real gates at the cost of doubling the number of qubits [? ]
we can simplify the proof by assuming \( f(\zeta) \) to be real. To achieve the desired accuracy it is sufficient to approximate \( f \) up to \(|\zeta| + 2 \) digits, where \(|\zeta| = \text{poly}(n)\) is the number of bits in \( \zeta \). Now define
\[
g(\zeta) := \text{round} \left[ 2^{\lvert \zeta \rvert + 2} (f(\zeta) + 1) \right]
\]
and note that \( g(\zeta) \) is a positive and integer-valued function satisfying
\[
\left| 2^{-|\zeta|-2} \sum_{\zeta} g(\zeta) - 1 - \sum_{\zeta} f(\zeta) \right| \leq \frac{1}{4}.
\]
Finally, by defining a boolean indicator function,
\[
h(\zeta, \xi) = \begin{cases} 
1 & \text{if } 0 \leq \xi < g(\zeta) \\
0 & \text{otherwise} 
\end{cases}
\]
we can write \( g(\zeta) = \sum_{\xi \geq 0} h(\zeta, \xi) \), and thus,
\[
\sum_{\zeta} g(\zeta) = \sum_{\xi \zeta} h(\zeta, \xi).
\]
This shows that \( \text{tr } \Omega \) can be approximated to accuracy \( \frac{1}{4} \), and thus \( \dim A \) can be determined by counting the number of satisfying assignments of a single boolean function \( h(\zeta, \xi) \) that can be efficiently constructed from \( \Omega \), i.e., by a single query to a black box solving \#P problems, together with the efficient postprocessing described by Eq. (23). \( \square \)