THE THEORY OF WIMSHURST'S ALTERNATING STATIC MACHINE

By A. W. Simon[1]

Abstract

The quantitative theory of Wimshurst's alternating static machine is developed in detail according to the general method previously given by the author for other static machines. The theory predicts that the potential of any element of the machine will reverse at definite intervals and at the same time increase in value. While the exact period of reversal depends on the capacities involved in the machine, the approximate period can be calculated and is found to be the time of one revolution rather than the time of three quarters of a revolution as observed by Wimshurst. An actual machine was constructed and the reversal observed by means of the glow in discharge tubes. It was found that such a machine does reverse every revolution rather than every three quarters of a revolution. The theory given can be extended to other machines of the same type with more than four inductors. The period of reversal of a six-inductor machine is calculated to be approximately the same as that of a four-inductor machine.

SOME thirty years ago Wimshurst[2] described an experimental influence machine which had the peculiar property of reversing its polarity with the greatest regularity,—a reversal occurring, according to Wimshurst, every three quarters of a revolution. The present wo k is concerned with the explanation of the action of a machine of this type.

The machine studied in detail is represented diagrammatically in Fig. 1, from which its essential elements and mode of operation can be seen readily. If we number the elements as indicated in the figure, set up the equations for the charges for two successive quarter turns, and then eliminate the charges by substituting for each charge its value in terms of the electric coefficients and the corresponding potentials, according to the general method outlined in previous papers,[3] we have eight equations, the left members of which involve as unknowns linearly the eight potentials, $V_0(n+1)$, $V_1(n+1)$, $\cdots$ $V_7(n+1)$; and the right members of which involve as unknowns linearly the eight potentials $V_0(n)$, $V_1(n)$, $\cdots$ $V_7(n)$; and in which the matrix of the coefficients on either side is a square matrix of the eighth order, involving as elements certain sums of the electric coefficients.

[1] National Research Fellow.
[2] J. Wimshurst, Phil. Mag. **31**, 507 (1891).
[3] A. W. Simon, Phys. Rev. **24**, 690 (1924; also Phil. Mag. **49**, 257 (1925).

The two matrices in question can be represented very conveniently by means of the scheme already given in a previous paper;[4] in particular, the matrix of the coefficients of the *left* members is obtained by writing at the heads of the *columns* and also at the heads of the *rows* of an eighth order matrix the numbers: 0,8; 1,9;2,10; 3,11; 4; 5; 6; 7; while the matrix of the coefficients of the *right* members is given by writing at the heads of the *columns* of an eighth order matrix the numbers just given and at the heads of the rows the numbers 0,6; 1,7; 2,4; 3,5; 8,9; 10; 11. Any cell of these matrices then represents the coefficient occupying the corresponding place in the matrix of the coefficients of the fundamental

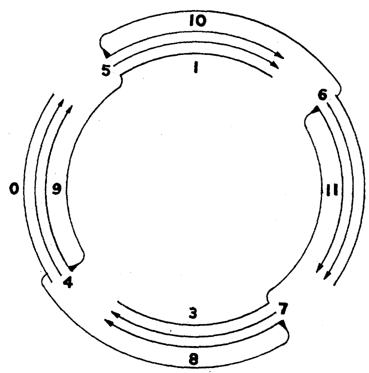

Fig. 1. Diagrammatic representation of Wimshurst's alternating static machine

potential equations, and this coefficient is obtained in terms of the coefficients of capacity and coefficients of induction of the elements of the machine by combining the numbers at the head of the column in which the coefficient stands with the numbers at the head of the row according to the notation[5]:

$$a_{p,\,q\,.\,r,\,s} \equiv a_{p,\,r} + a_{p,\,s} + a_{q,\,r} + a_{q,\,s}\ ,$$

where $a_{i,i}$ represents the coefficient of capacity of the conductor indexed i, and $a_{i,j}$ represents the coefficient of induction of the conductors indexed i and j.

Due to the fourfold geometrical symmetry of the machine the two matrices are, furthermore, respectively of the form

$$\begin{vmatrix} M_1 & M_2 \\ M_3 & M_4 \end{vmatrix}\ ,\qquad \begin{vmatrix} M_5 & M_6 \\ M_7 & M_8 \end{vmatrix}\ , \tag{1}$$

[4] A. W. Simon, Phys. Rev. **28**, 142 (1926).

[5] This modification of our previous notation is necessitated by the fact that in the present case some of the subscripts are numbers of two digits.

where each of the symbols M_i represents a cyclic matrix of the fourth order, i.e., M_i is of the form:

$$\begin{vmatrix} a_i & d_i & c_i & b_i \\ b_i & a_i & d_i & c_i \\ c_i & b_i & a_i & d_i \\ d_i & c_i & b_i & a_i \end{vmatrix} \tag{2}$$

The quantities a_i, b_i, c_i, d_i, in turn are given, for $i=1, 2, \cdots 8$, in terms of the coefficients of capacity and coefficients of induction of the elements of the machine in the configuration of Fig. 1, if we again make use of the representation defined in a previous work, by:

	0,8	1,9	2,10	3,11	4	5	6	7
0,8	a_1	d_1	c_1	b_1	a_2	d_2	c_2	b_2
4	a_3	d_3	c_3	b_3	a_4	d_4	c_4	b_4
0,6	a_5	d_5	c_5	b_5	a_6	d_6	c_6	b_6
8	a_7	d_7	c_7	b_7	a_8	d_8	c_8	b_8

(3)

I. Mathematical Solution

The method of solution of a system of equations of this type has already been pointed out in a previous paper.[6] If we employ the method there described, we derive a set of auxiliary equations of the form:

$$\begin{aligned} A_{1j}Z_{1j}(n+1)+A_{2j}Z_{2j}(n+1)&=A_{5j}Z_{1j}(n)+A_{6j}Z_{2j}(n)\ , \\ A_{3j}Z_{1j}(n+1)+A_{4j}Z_{2j}(n+1)&=A_{7j}Z_{1j}(n)+A_{8j}Z_{2j}(n)\ , \end{aligned} \tag{4}$$

where the Z's are defined by:

$$\begin{aligned} Z_{1j}(n)&\equiv[V_0(n)w_j^0+V_1(n)w_j^1+V_2(n)w_j^2+V_3(n)w_j^3]\ , \\ Z_{2j}(n)&\equiv[V_4(n)w_j^0+V_5(n)w_j^1+V_6(n)w_j^2+V_7(n)w_j^3]\ , \end{aligned} \tag{5}$$

and the quantities A_{ij} are defined by:

$$A_{ij}\equiv[a_iw_j^0+b_iw_j^1+c_iw_j^2+d_iw_j^3]\ . \tag{6}$$

As the solutions of (4) we have at once[7] a set of equations of the form:

$$\begin{aligned} Z_{1j}(n)&=C_{1j}r_{1j}^n+C_{2j}r_{2j}^n \\ Z_{2j}(n)&=C_{1j}'r_{1j}^n+C_{2j}'r_{2j}^n \end{aligned} \tag{7}$$

where r_{1j} and r_{2j} are the roots of the algebraic equation:

$$f_j(r)\equiv\begin{vmatrix} A_{1j}r-A_{5j} & A_{2j}r-A_{6j} \\ A_{3j}r-A_{7j} & A_{4j}r-A_{8j} \end{vmatrix}=0 \tag{8}$$

[6] A. W. Simon, Phys. Rev. **27** 747 (1926).

[7] G. Boole, "A Treatise on the Calculus of Finite Differences," Chap. XI

and the C's are certain arbitrary constants to be determined.

From (7) we can deduce at once the formulas for the quantities $V_s(n)$; we have merely to note that from the definition of the Z's namely (5), it follows:

$$4V_s(n) = \sum_{j=0}^{3} Z_{1j}(n)/w_{sj}\ , \qquad s=0,\ 1,\ 2,\ 3,$$
$$4V_s(n) = \sum_{j=0}^{3} Z_{2j}(n)/w_{sj}\ , \qquad s=4,\ 5,\ 6,\ 7\ ; \tag{9}$$

whence, taking account of (7), we have finally:

$$4V_s(n) = \sum_{j=0}^{3} [C_{1j}r_{1j}{}^n+C_{2j}r_{2j}{}^n]/w_{sj}\ , \qquad s=0,\ 1,\ 2,\ 3\ ;$$
$$4V_s(n) = \sum_{j=0}^{3} [C_{1j}'r_{1j}{}^n+C_{2j}'r_{2j}{}^n]/w_{sj}\ , \qquad s=4,\ 5,\ 6,\ 7\ . \tag{10}$$

The problem therefore reduces to the study of the right members of (7), in particular, to the study of the arbitrary constants.

II. Evaluation of the Arbitrary Constants

If now we put $n=0$ in (4), substitute for the Z's the values given by (7), and solve for C_{1j}' and C_{2j}' in terms of C_{1j} and C_{2j}, we have, since $f_j(r_{1j})=f_j(r_{2j})=0$:

$$C_{1j}'=\overline{\underline{K}}_{1j}C_{1j}\ , \qquad C_{2j}'=\overline{\underline{K}}_{2j}C_{2j}\ ; \tag{11}$$

where the K's are certain constants given by:

$$\overline{\underline{K}}_{1j}D = \begin{vmatrix} A_{1j}r_{1j}-A_{5j} & A_{2j}r_{2j}-A_{6j} \\ A_{3j}r_{1j}-A_{7j} & A_{4j}r_{2j}-A_{8j} \end{vmatrix}\ ,$$
$$\overline{\underline{K}}_{2j}D = \begin{vmatrix} A_{2j}r_{1j}-A_{6j} & A_{1j}r_{2j}-A_{5j} \\ A_{4j}r_{1j}-A_{8j} & A_{3j}r_{2j}-A_{7j} \end{vmatrix}\ , \tag{12}$$
$$D\equiv(r_{1j}-r_{2j}) \begin{vmatrix} A_{2j} & A_{6j} \\ A_{4j} & A_{8j} \end{vmatrix}\ .$$

If next we put $n=0$ in (7), substitute for C_{1j}' and C_{2j}' the values just found, and solve for C_{1j} and C_{2j}, we have:

$$C_{1j}=[\overline{\underline{K}}_{2j}Z_{1j}(0)-Z_{2j}(0)]/[\overline{\underline{K}}_{2j}-\overline{\underline{K}}_{1j}]\ ,$$
$$C_{2j}=[Z_{2j}(0)-\overline{\underline{K}}_{1j}Z_{1j}(0)]/[\overline{\underline{K}}_{2j}-\overline{\underline{K}}_{1j}]\ . \tag{13}$$

The relations (11) and (13) exhibit three important results: (1) If $C_{1j}=0$, then also $C_{1j}'=0$; if $C_{2j}=0$, then also $C_{2j}'=0$. (2) If $Z_{2j}(0)=\overline{\underline{K}}_{1j}Z_{1j}(0)$, then $C_{2j}=0$ and $C_{1j}=Z_{1j}(0)$; if $Z_{2j}(0)=\overline{\underline{K}}_{2j}Z_{1j}(0)$, then

$C_{1j}=0$ and $C_{2j}=Z_{1j}(0)$. (3) Finally, if $Z_{1j}(0)=Z_{2j}(0)=0$, then also $C_{1j}=C_{1j}'=C_{2j}=C_{2j}'=0$.

III. Particular Solutions of the Voltage Equations

The results just given show that for certain relations between the quantities $V_s(0)$, the general solution reduces to simple forms. In particular, if the original potentials satisfy the relations:

$$\begin{aligned} V_s(0) &= E(0)\cos(s-p)k2\pi/4, & s&=0,\ 1,\ 2,\ 3\ ; \\ V_s(0) &= \kappa_{ij}E(0)\cos[(s-p)k2\pi/4+\Psi_{ij}]\ , & s&=4,\ 5,\ 6,\ 7\ ; \end{aligned} \tag{14}$$

where s, p, and $-k$ take the values 0, 1, 2, 3, and κ_{ij} and ψ_{ij} are defined by:

$$\underline{\overline{K}}_{ij} \equiv \kappa_{ij}(\cos\Psi_{ij}+i\ \sin\Psi_{ij})\ , \tag{15}$$

the general solution reduces to:

$$\begin{aligned} V_s(n) &= \rho_{ij}{}^n E(0)\cos[n\theta_{ij}+(s-p)k2\pi/4]\ , & s&=0,\ 1,\ 2,\ 3\ ; \\ V_s(n) &= \rho_{ij}{}^n \kappa_{ij}E(0)\cos[n\theta_{ij}+(s-p)k2\pi/4+\Psi_{ij}]\ , & s&=4,\ 5,\ 6,\ 7\ ; \end{aligned} \tag{16}$$

where r_{ij} has been written in the form:

$$r_{ij} \equiv \rho_{ij}(\cos\theta_{ij}+i\ \sin\theta_{ij})\ . \tag{17}$$

IV. Computation of the Roots and Constants

For the interpretation of the results we require the numerical values of the quantities r_{ij} and $\underline{\overline{K}}_{ij}$. These are functions, by virtue of (8), (12), (6) and (3), only of the coefficients of capacity and coefficients of induction of the twelve elements of the machine in a certain position, namely that in which the carriers just break contact with the brushes (Fig. 1). These coefficients, of course, it is impossible to calculate, so that it would be necessary in order to complete the analysis of any given machine to actually measure them.

However, we can determine the approximate values of the quantities r_{ij} and $\underline{\overline{K}}_{i,j}$ if we put $a_{0,0}=a$, $a_{9,9}=ca$, $a_{4,4}=(c+1)a$, $a_{0,4}=(e-a)$, $a_{9,4}=c(e-a)$, and all the other coefficients $a_{i,j}$ equal to 0. These conditions, it may be noted, will be approximately satisfied if the inductors and carriers are identical in shape and dimensions, and if the distance between the members of each group of adjacent elements (0,4,9, for example) is small compared to their distance away from the elements of the other groups.

Substituting the values just given in (8) and (12), solving for r_{ij} and $\underline{\overline{K}}_{ij}$ in terms of c, a, e, and finally taking the limit as $e\rightarrow 0$ (the latter procedure is necessary to avoid indeterminate forms in the case of $f_0(r)$

and $f_2(r)$), we have the following set of values for the quantities r_{ij} and $\underline{\overline{K}}_{ij}$:

j	r_{1j}	r_{2j}	$\underline{\overline{K}}_{1j}$	$\underline{\overline{K}}_{2j}$
0	1	$-1/(1+c)$	1	-1
1	$1+i$	$-i$	$\dfrac{2c(c+2)-i(4c+5)}{2c(c+3)+5}$	1
2	1	0	1	-1
3	$1-i$	$+i$	$\dfrac{2c(c+2)+i(4c+5)}{2c(c+3)+5}$	1

In addition it is important to note that two of the r's are identically equal to unity irrespective of what the capacities involved in the machine are. In order to prove this it is convenient to introduce a new symbol s_{ij}, which is defined as the sum of all those coefficients which have as subscripts the combinations which result from the pairing off of i with the numbers of those elements of the machine which are similarly placed with respect to the element indexed j. As groups of similarly placed elements we have those indexed 0,1,2,3; 4,5,6,7; and 8,9,10,11. In terms of this notation $f_0(r)$ becomes:

$$f_0(r) = \begin{vmatrix} r(s_{00}+s_{09}+s_{90}+s_{99}) - (s_{00}+s_{04}+s_{90}+s_{94}), & r(s_{40}+s_{49}) - (s_{44}+s_{40}) \\ r(s_{04}+s_{94}) \qquad - (s_{09}+s_{99}) & \cdot\ , \ r(s_{44}) \qquad - (s_{49}) \end{vmatrix}$$

from which it is obvious that $f_0(r)$ vanishes for $r=1$.

The notation s_{ij} is very convenient when large numbers of elements must be dealt with; in fact the proof just given applies at once to any alternator of this type irrespective of the number of inductors.

A similar proof can be given for $f_2(r)$ provided we introduce the notation d_{ij} which we define to be similar to s_{ij} except that the signs of the successive terms of the sum are alternately $+$ and $-$. In particular for example: $d_{0,4} \equiv +a_{0,4} - a_{0,5} + a_{0,6} - a_{0,7}$.

V. Physical Interpretation of the Results

From (10) and the table of values of the r's, we note at once that for large values of n, (theoretically only after an infinite number of quarter turns but practically after a few revolutions) the potentials are given by:

$$\begin{aligned} V_s(n) &= C_{11}r_{11}^n/w_{s1} + C_{13}r_{13}^n/w_{s3}\,, \qquad & s=0,\ 1,\ 2,\ 3\ ; \\ V_s(n) &= \underline{\overline{K}}_{11}C_{11}r_{11}^n/w_{s1} + \underline{\overline{K}}_{13}C_{13}r_{13}/w_{s3}\,, \qquad & s=4,\ 5,\ 6,\ 7\ . \end{aligned} \tag{18}$$

If next we write each quantity appearing in (18) in trigonometric form and note that the two terms on the right-hand side are conjugate complex quantities, these equations become:

$$V_s(n)=R_{11}\rho_{11}{}^n[\cos n\,\theta_{11}+\Delta_{11}-s\pi/2]\ ,\qquad s=0,\ 1,\ 2,\ 3\ ;$$
$$V_s(n)=\kappa_{11}R_{11}\rho_{11}{}^n[\cos n\,\theta_{11}+\Delta_{11}-s\pi/2+\Psi_{11}]\ ,\ s=4,\ 5,\ 6,\ 7\ ; \qquad (19)$$

where, if we write C_{11} in the form $C_{11}\equiv A_{11}+iB_{11}$, R_{11} and Δ_{11} are defined by: $R_{11}\equiv\sqrt{A_{11}{}^2+B_{11}{}^2}$ and $\Delta_{11}\equiv\text{arc tan } B_{11}/A_{11}$.

From these equations it is seen that *the potential of any element will alternate and at the same time increase in value*—a result which is in accordance with the observation of Wimshurst that such a "machine is self-exciting notwithstanding that when at work its electrical charges alternate."

Moreover, we can add that eventually the potentials of the inductors ($s=0,1,2,3$) taken in the direction of rotation will be $\pi/2$ out of phase; that the same will be true of the potentials of the insulated carriers ($s=4,5,6,7$); while the potential of any insulated carrier will be proportional to the potential of the adjacent inductor but will differ in phase from it by an angle ψ.

Let us next inquire what the theoretical frequency of alternation will be. This will depend, for any particular machine, on the numerical values of the electric coefficients, and for the reasons given above can be calculated only approximately. However, if we use the approximate values for r_{11} given in the table above, we have $\theta_{11}=\pi/4$, whence it follows that *the potential of any element should reverse approximately every revolution*, rather than every three quarters of a revolution as was observed by Wimshurst.

It is also to be noted that ρ_{11} comes out approximately equal to $\sqrt{2}$ from which it follows that the potential of any element should be approximately quadrupled every revolution—a result which is interesting because it is the same as that obtained for the case of a four-inductor alternator of the type described by the writer.[8]

To the results just given we can add a number of others which we deduce from the particular solutions. Of these three are of interest:

(1) $V_s(0)=E_0(0)$, for $s=0,1,2,3$; $V_s(0)=E_4(0)$, for $s=4,5,6,7$; and, further, $E_4(0)=\overline{\underline{K}}_{10}E_0(0)$. The corresponding solution is, since $r_{10}=1$, for every n from 0 to ∞: $V_s(n)=E_0(0)$, for $s=0,1,2,3$; $V_s(n)=E_4(0)$, for $s=4,5,6,7$; that is to say, no change takes place.

(2) $V_s(0)=E_0(0)\cos(s-p)(-\pi)$, for $s=0,1,2,3$; $V_s(0)=E_4(0)$ co-$(s-p)(-\pi)$, for $s=4,5,6,7$; and further $E_4(0)=\overline{\underline{K}}_{12}E_0(0)$. The cors responding solution is, since $r_{12}\equiv1$, for every n from 0 to ∞: $V_s(n)=E_0(0)\cos(p-s)\pi$, for $s=0,1,2,3$: $V_s(n)=E_4(0)\cos(p-s)\pi$, for $s=4,5,6,7$; that is to say, for this case also, no change takes place.

[8] A. W. Simon, Phys. Rev. **25,** 368 (1925).

(3) $V_s(0) = E(0)\cos[(s-p)(-\pi/2)]$, for $s=0,1,2,3$; $V_s(0) = \kappa_{11}E(0)\cos[(s-p)(-\pi/2)+\psi_{11}]$, for $s=4,5,6,7$; for this case we have, for every n from 0 to ∞:

$$\begin{aligned} V_s(n) &= \rho_{11}{}^n E(0)\cos[n\,\theta_{11}+(p-s)\pi/2]\,, & s&=0,\ 1,\ 2,\ 3\ ; \\ V_s(n) &= \rho_{11}{}^n \kappa_{11} E(0)\cos[n\,\theta_{11}+(p-s)\pi/2+\Psi_{11}]\,, & s&=4,\ 5,\ 6,\ 7\ . \end{aligned} \tag{20}$$

Comparing (19) and (20) we note that the potentials of the various elements of the machine tend toward the state in which the potentials satisfy (19); that this state is attained when the transient terms (those corresponding to $|r|\leq 1$) have become negligible; and, finally, that if the system is originally in this state, then the transient terms are absent altogether.

VI. Experimental Test of the Theory

The discrepancy between our theoretical result, that the potential of any element should reverse approximately every revolution, and the experimental result of Wimshurst that it reverses every three quarters of a revolution, obviously requires further investigation. Toward this end an actual machine satisfying the conditions we have laid down in the previous discussion was constructed. In particular, it consisted of two fixed glass discs 30″ in diameter and a rotating glass disc 27″ in diameter, each of the fixed discs carrying two inductors placed with their centers 180° apart, and the rotating disc carrying eight sectors, four on each side, placed exactly opposite one another, with their centers 90° apart. The inductors and carriers were all of the same size and shape, being sections of a ring of tinfoil with an inner diameter of $6\frac{3}{4}$″ and an outer diameter of $12\frac{1}{2}$″, each section subtending an angle of 45° at the center.

The rotating disc was placed between the two fixed discs, and the latter were mounted so that the center lines of the successive inductors were 90° apart, but so that alternate inductors were on opposite sides of the rotating disc. The inductors were also on the outside faces of the fixed discs, so that a thickness of glass intervened between each inductor and the nearest carrier. The inductors and carriers were so mounted that as each pair of carriers came directly in front of an inductor, the three elements of the group exactly covered one another.

Brushes were so placed as to connect the outside carrier of each pair of rotating carriers to the inductor next following in the direction of rotation, and were adjusted to break contact at the instant when the carriers left the position directly in front of the inductors (Fig. 1). The glass discs were shellac varnished, and the brush holders insulated with hard rubber.

The reversal was observed first by means of small paper electroscopes according to the method of Wimshurst, but these were not found to work very consistently.

An improved method was to insert into one or more of the brush circuits, i.e., between brush and inductor, a straight gaseous discharge tube and observe the glow in this tube. In this way the period of reversal could be very accurately observed, and it was found that *the reversal actually did occur every revolution* rather than every three quarters of a revolution, as stated by Wimshurst.

VII. Extension of the Theory

As already pointed out in a previous paper[6] machines of the type just discussed can be constructed with more than four inductors, in particular with 6 inductors and 12 carriers, or, in general, with $2m$ inductors and $4m$ carriers, and the theory just given can be extended immediately to these machines. We have merely to let j take the values $0, 1, \ldots . 2m$, in the equations given above.

It is of interest to inquire what the frequency of alternation of a six-inductor machine would be. If the theory of such a machine (6 inductors, 12 carriers) is developed it is found that for this case also the theoretical frequency is approximately one alternation per revolution, and this is very likely true for every machine of this type, that is to say: increasing the number of inductors does not greatly change the frequency of alternation.

The theory given can also be extended to the case of a machine with 4 inductors and 16 carriers; or, in general, to the case of 2m inductors and 8m carriers. The previous case reduced, as we saw from Eq. (8), to the solution of quadratic equations, the present case would reduce to the solution of biquadratic equations.

For machines with a larger number of carriers than $8m$, the solution of algebraic equations of degree greater than the fourth would be required, so that algebraic methods would probably fail. Whether it is possible to solve these more complicated types has not been investigated; it would be desirable, however, to investigate these cases with a view to solving the case of a very large number (theoretically an infinite number) of carriers, since this would approximate a *sectorless*, i.e., a plain glass disc, machine. In fact, the theory developed by the author applies only to *sectored* machines; the theory of *sectorless* machines still remains to be developed.

Norman Bridge Laboratory,
California Institute of Technology
Pasadena, California.
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