11
FLOWS WITH GAS DYNAMICS

11.1 INTRODUCTION
This chapter addresses the class of compressible flows in which a gaseous continuous phase is seeded with droplets or particles and in which it is necessary to evaluate the relative motion between the disperse and continuous phases for a variety of possible reasons. In many such flows, the motivation is the erosion of the flow boundaries by particles or drops and this is directly related to the relative motion. In other cases, the purpose is to evaluate the change in the performance of the system or device. Still another motivation is the desire to evaluate changes in the instability boundaries caused by the presence of the disperse phase.

Examples include the potential for serious damage to steam turbine blades by impacting water droplets (e.g. Gardner 1963, Smith et al. 1967). In the context of aircraft engines, desert sand storms or clouds of volcanic dust can not only cause serious erosion to the gas turbine compressor (Tabakoff and Hussein 1971, Smialek et al. 1994, Dunn et al. 1996, Tabakoff and Hamed 1986) but can also deleteriously effect the stall margin and cause engine shutdown (Batcho et al. 1987). Other examples include the consequences of seeding the fuel of a solid-propelled rocket with metal particles in order to enhance its performance. This is a particularly complicated example because the particles may also melt and oxidize in the flow (Shorr and Zaehringer 1967).

In recent years considerable advancements have been made in the numerical models and methods available for the solution of dilute particle-laden flows. In this text, we present a survey of the analytical methods and the physical understanding that they generate; for a valuable survey of the numerical methods the reader is referred to Crowe (1982).
11.2 EQUATIONS FOR A DUSTY GAS

11.2.1 Basic equations

First we review the fundamental equations governing the flow of the individual phases or components in a dusty gas flow. The continuity equations (equations 1.21) may be written as

\[ \frac{\partial}{\partial t} (\rho_N \alpha_N) + \frac{\partial (\rho_N \alpha_N u_{Ni})}{\partial x_i} = I_N \]  \hspace{1cm} (11.1)

where \( N = C \) and \( N = D \) refer to the continuous and disperse phases respectively. We shall see that it is convenient to define a loading parameter, \( \xi \), as

\[ \xi = \frac{\rho_D \alpha_D}{\rho_C \alpha_C} \]  \hspace{1cm} (11.2)

and that the continuity equations have an important bearing on the variations in the value of \( \xi \) within the flow. Note that the mixture density, \( \rho \), is then

\[ \rho = \rho_C \alpha_C + \rho_D \alpha_D = (1 + \xi) \rho_C \alpha_C \]  \hspace{1cm} (11.3)

The momentum and energy equations for the individual phases (equations 1.45 and 1.69) are respectively

\[ \rho_N \alpha_N \left[ \frac{\partial u_{Nk}}{\partial t} + u_{Ni} \frac{\partial u_{Nk}}{\partial x_i} \right] = \alpha_N \rho_N g_k + F_N - I_N u_{Nk} - \delta_N \left[ \frac{\partial p}{\partial x_k} - \frac{\partial \sigma_D^{Cij}}{\partial x_i} \right] \]  \hspace{1cm} (11.4)

\[ \rho_N \alpha_N c_v_N \left[ \frac{\partial T_N}{\partial t} + u_{Ni} \frac{\partial T_N}{\partial x_i} \right] = \delta_N \sigma_{Ci,j} \frac{\partial u_{Cj}}{\partial x_j} + Q_N + W_N + Q_T - I_N (u_{Di} - u_{Ni}) - (e_N^* - u_{Ni} u_{Ni}) I_N \]  \hspace{1cm} (11.5)

and, when summed over all the phases, these lead to the following combined continuity, momentum and energy equations (equations 1.24, 1.46 and 1.70):

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \left( \sum_N \rho_N \alpha_N u_{Ni} \right) = 0 \]  \hspace{1cm} (11.6)
\[
\frac{\partial}{\partial t} \left( \sum_N \rho_N \alpha_N u_{NK} \right) + \frac{\partial}{\partial x_i} \left( \sum_N \rho_N \alpha_N u_{Ni} u_{NK} \right) = \rho g_k - \frac{\partial p}{\partial x_k} + \frac{\partial \sigma_{Ck_i}}{\partial x_i}
\]

\[\text{(11.7)}\]

\[
\sum_N \left[ \rho_N \alpha_N c_v N \left\{ \frac{\partial T}{\partial t} + u_{Ni} \frac{\partial T}{\partial x_i} \right\} \right] = \sigma_{Cij} \frac{\partial u_{Ci}}{\partial x_j} - F_{Di} (u_{Di} - u_{Ci}) - I_D (e^*_D - e^*_C) + \sum_N u_{Ni} u_{Ni} I_N
\]

\[\text{(11.8)}\]

To these equations of motion, we must add equations of state for both phases. Throughout this chapter it will be assumed that the continuous phase is an ideal gas and that the disperse phase is an incompressible solid. Moreover, temperature and velocity gradients in the vicinity of the interface will be neglected.

### 11.2.2 Homogeneous flow with gas dynamics

Though the focus in this chapter is on the effect of relative motion, we must begin by examining the simplest case in which both the relative motion between the phases or components and the temperature differences between the phases or components are sufficiently small that they can be neglected. This will establish the base state that, through perturbation methods, can be used to examine flows in which the relative motion and temperature differences are small. As we established in chapter 9, a flow with no relative motion or temperature differences is referred to as homogeneous. The effect of mass exchange will also be neglected in the present discussion and, in such a homogeneous flow, the governing equations, 11.6, 11.7 and 11.8 clearly reduce to

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0
\]

\[\text{(11.9)}\]

\[
\rho \left[ \frac{\partial u_k}{\partial t} + u_i \frac{\partial u_k}{\partial x_i} \right] = \rho g_k - \frac{\partial p}{\partial x_k} + \frac{\partial \sigma_{Ck_i}}{\partial x_i}
\]

\[\text{(11.10)}\]

\[
\left[ \sum_N \rho_N \alpha_N c_v N \right] \left\{ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} \right\} = \sigma_{Cij} \frac{\partial u_i}{\partial x_j}
\]

\[\text{(11.11)}\]
where \( u_i \) and \( T \) are the velocity and temperature common to all phases.

An important result that follows from the individual continuity equations 11.1 in the absence of exchange of mass (\( I_N = 0 \)) is that

\[
\frac{D}{Dt} \left\{ \frac{\rho D \alpha D}{\rho_C \alpha_C} \right\} = \frac{D \xi}{Dt} = 0 \tag{11.12}
\]

Consequently, if the flow develops from a uniform stream in which the loading \( \xi \) is constant and uniform, then \( \xi \) is uniform and constant everywhere and becomes a simple constant for the flow. We shall confine the remarks in this section to such flows.

At this point, one particular approximation is very advantageous. Since in many applications the volume occupied by the particles is very small, it is reasonable to set \( \alpha C \approx 1 \) in equation 11.2 and elsewhere. This approximation has the important consequence that equations 11.9, 11.10 and 11.11 are now those of a single phase flow of an effective gas whose thermodynamic and transport properties are as follows. The approximation allows the equation of state of the effective gas to be written as

\[
p = \rho \mathcal{R} T \tag{11.13}
\]

where \( \mathcal{R} \) is the gas constant of the effective gas. Setting \( \alpha C \approx 1 \), the thermodynamic properties of the effective gas are given by

\[
\rho = \rho_C (1 + \xi) \quad ; \quad \mathcal{R} = \mathcal{R}_C / (1 + \xi)
\]

\[
c_v = \frac{c_v C + \xi c_{sD}}{1 + \xi} \quad ; \quad c_p = \frac{c_p C + \xi c_{sD}}{1 + \xi} \quad ; \quad \gamma = \frac{c_p C + \xi c_{sD}}{c_v C + \xi c_{sD}} \tag{11.14}
\]

and the effective kinematic viscosity is

\[
\nu = \mu_C / \rho_C (1 + \xi) = \nu_C / (1 + \xi) \tag{11.15}
\]

Moreover, it follows from equations 11.14, that the relation between the isentropic speed of sound, \( c \), in the effective gas and that in the continuous phase, \( c_C \), is

\[
c = c_C \left[ 1 + \frac{\xi c_{sD} / c_p C}{(1 + \xi c_{sD} / c_v C)(1 + \xi)} \right]^{\frac{1}{2}} \tag{11.16}
\]

It also follows that the Reynolds, Mach and Prandtl numbers for the effective gas flow, \( Re, M \) and \( Pr \) (based on a typical dimension, \( \ell \), typical velocity, \( U \), and typical temperature, \( T_0 \), of the flow) are related to the Reynolds, Mach and Prandtl numbers for the flow of the continuous phase, \( Re_C, M_C \)
and $Pr_C$, by

$$Re = \frac{U\ell}{\nu} = Re_C(1 + \xi) \quad (11.17)$$

$$M = \frac{U}{c} = M_C \left[ \frac{(1 + \xi c_sD/c_{pC})(1 + \xi)}{(1 + \xi c_sD/c_{pC})} \right]^{\frac{1}{2}} \quad (11.18)$$

$$Pr = \frac{c_p \mu}{k} = Pr_C \left[ \frac{(1 + \xi c_sD/c_{pC})}{(1 + \xi)} \right] \quad (11.19)$$

Thus the first step in most investigations of this type of flow is to solve for the effective gas flow using the appropriate tools from single phase gas dynamics. Here, it is assumed that the reader is familiar with these basic methods. Thus we focus on the phenomena that constitute departures from single phase flow mechanics and, in particular, on the process and consequences of relative motion or slip.

### 11.2.3 Velocity and temperature relaxation

While the homogeneous model with effective gas properties may constitute a sufficiently accurate representation in some contexts, there are other technological problems in which the velocity and temperature differences between the phases are important either intrinsically or because of their consequences. The rest of the chapter is devoted to these effects. But, in order to proceed toward this end, it is necessary to stipulate particular forms for the mass, momentum and energy exchange processes represented by $I_N$, $F_{Nk}$ and $Q_{IN}$ in equations 11.1, 11.4 and 11.5. For simplicity, the remarks in this chapter are confined to flows in which there is no external heat added or work done so that $Q_N = 0$ and $W_N = 0$. Moreover, we shall assume that there is negligible mass exchange so that $I_N = 0$. It remains, therefore, to stipulate the force interaction, $F_{Nk}$ and the heat transfer between the components, $Q_{IN}$. In the present context it is assumed that the relative motion is at low Reynolds numbers so that the simple model of relative motion defined by a relaxation time (see section 2.4.1) may be used. Then:

$$F_{CK} = -F_{DK} = \frac{\rho D a D}{t_u} (u_{DK} - u_{CK}) \quad (11.20)$$

where $t_u$ is the velocity relaxation time given by equation 2.73 (neglecting the added mass of the gas):

$$t_u = m_p/12\pi R\mu C \quad (11.21)$$
It follows that the equation of motion for the disperse phase, equation 11.4, becomes

\[
\frac{Du_{Dk}}{Dt} = \frac{u_{Ck} - u_{Dk}}{t_u}
\]  

(11.22)

It is further assumed that the temperature relaxation may be modeled as described in section 1.2.9 so that

\[
Q_{IC} = -Q_{ID} = \frac{\rho_D \alpha_D c_{sD} N u}{t_T} (T_D - T_C)
\]  

(11.23)

where \(t_T\) is the temperature relaxation time given by equation 1.76:

\[
t_T = \rho_D c_{sD} R^2 / 3 k_C
\]  

(11.24)

It follows that the energy equation for the disperse phase is equation 1.75 or

\[
\frac{D T_D}{Dt} = \frac{N u (T_C - T_D)}{2 t_T}
\]  

(11.25)

In the context of droplet or particle laden gas flows these are commonly assumed forms for the velocity and temperature relaxation processes (Marble 1970). In his review Rudinger (1969) includes some evaluation of the sensitivity of the calculated results to the specifics of these assumptions.

### 11.3 NORMAL SHOCK WAVE

Normal shock waves not only constitute a flow of considerable practical interest but also provide an illustrative example of the important role that relative motion may play in particle or droplet laden gas flows. In a frame of reference fixed in the shock, the fundamental equations for this steady flow in one Cartesian direction (\(x\) with velocity \(u\) in that direction) are obtained from equations 11.1 to 11.8 as follows. Neglecting any mass interaction (\(I_N = 0\)) and assuming that there is one continuous and one disperse phase, the individual continuity equations 11.1 become

\[
\rho_N \alpha_N u_N = \dot{m}_N = \text{constant}
\]  

(11.26)

where \(\dot{m}_C\) and \(\dot{m}_D\) are the mass flow rates per unit area. Since the gravitational term and the deviatoric stresses are negligible, the combined phase momentum equation 11.7 may be integrated to obtain

\[
\dot{m}_C u_C + \dot{m}_D u_D + p = \text{constant}
\]  

(11.27)
Also, eliminating the external heat added ($Q = 0$) and the external work done ($W = 0$) the combined phase energy equation 11.8 may be integrated to obtain
\[ \dot{m}_C(c_vC + \frac{1}{2}u_C^2) + \dot{m}_D(c_sD + \frac{1}{2}u_D^2) + pu_C = \text{constant} \] (11.28)
and can be recast in the form
\[ \dot{m}_C(c_pC + \frac{1}{2}u_C^2) + pu_C(1 - \alpha_C) + \dot{m}_D(c_sD + \frac{1}{2}u_D^2) = \text{constant} \] (11.29)

In lieu of the individual phase momentum and energy equations, we use the velocity and temperature relaxation relations 11.22 and 11.25:
\[ \frac{Du_D}{Dt} = u_D \frac{du_D}{dx} = \frac{u_C - u_D}{t_u} \] (11.30)
\[ \frac{DT_D}{Dt} = u_D \frac{dT_D}{dx} = \frac{T_C - T_D}{t_T} \] (11.31)
where, for simplicity, we confine the present analysis to the pure conduction case, $Nu = 2$.

Carrier (1958) was the first to use these equations to explore the structure of a normal shock wave for a gas containing solid particles, a dusty gas in which the volume fraction of particles is negligible. Under such circumstances, the initial shock wave in the gas is unaffected by the particles and can have a thickness that is small compared to the particle size. We denote the conditions upstream of this structure by the subscript 1 so that
\[ u_{C1} = u_{D1} = u_1 ; \quad T_{C1} = T_{D1} = T_1 \] (11.32)
The conditions immediately downstream of the initial shock wave in the gas are denoted by the subscript 2. The normal single phase gas dynamic relations allow ready evaluation of $u_{C2}$, $T_{C2}$ and $p_2$ from $u_{C1}$, $T_{C1}$ and $p_1$.

Unlike the gas, the particles pass through this initial shock without significant change in velocity or temperature so that
\[ u_{D2} = u_{D1} ; \quad T_{D2} = T_{D1} \] (11.33)
Consequently, at the location 2 there are now substantial velocity and temperature differences, $u_{C2} - u_{D2}$ and $T_{C2} - T_{D2}$, equal to the velocity and temperature differences across the initial shock wave in the gas. These differences take time to decay and do so according to equations 11.30 and 11.31. Thus the structure downstream of the gas dynamic shock consists of a relaxation zone in which the particle velocity decreases and the particle
temperature increases, each asymptoting to a final downstream state that is denoted by the subscript 3. In this final state

\[ u_{C3} = u_{D3} = u_3 \quad ; \quad T_{C3} = T_{D3} = T_3 \quad (11.34) \]

As in any similar shock wave analysis the relations between the initial (1) and final (3) conditions, are independent of the structure and can be obtained directly from the basic conservation equations listed above. Making the small disperse phase volume approximation discussed in section 11.2.2 and using the definitions 11.14, the relations that determine both the structure of the relaxation zone and the asymptotic downstream conditions are

\[ \dot{m}_{C} = \rho_{C} u_{C} = \dot{m}_{C1} = \dot{m}_{C2} = \dot{m}_{C3} \quad ; \quad \dot{m}_{D} = \rho_{D} u_{D} = \xi \dot{m}_{C} \quad (11.35) \]

\[ \dot{m}_{C}(u_{C} + \xi u_{D}) + p = (1 + \xi)\dot{m}_{C}u_{C1} + p_{1} = (1 + \xi)\dot{m}_{C}u_{C3} + p_{3} \quad (11.36) \]

\[ (c_{pC}T_{C} + \frac{1}{2}u_{C}^{2}) + \xi(c_{sD}T_{D} + \frac{1}{2}u_{D}^{2}) = (1 + \xi)(c_{p}T_{1} + \frac{1}{2}u_{1}^{2}) = (1 + \xi)(c_{p}T_{3} + \frac{1}{2}u_{3}^{2}) \quad (11.37) \]

and it is a straightforward matter to integrate equations 11.30, 11.31, 11.35, 11.36 and 11.37 to obtain \( u_{C}(x), u_{D}(x), T_{C}(x), T_{D}(x) \) and \( p(x) \) in the relaxation zone.

First, we comment on the typical structure of the shock and the relaxation zone as revealed by this numerical integration. A typical example from the
review by Marble (1970) is included as figure 11.1. This shows the asymptotic behavior of the velocities and temperatures in the case $t_u/t_T = 1.0$. The nature of the relaxation processes is evident in this figure. Just downstream of the shock the particle temperature and velocity are the same as upstream of the shock; but the temperature and velocity of the gas has now changed and, over the subsequent distance, $x/c_1 t_u$, downstream of the shock, the particle temperature rises toward that of the gas and the particle velocity decreases toward that of the gas. The relative motion also causes a pressure rise in the gas, that, in turn, causes a temperature rise and a velocity decrease in the gas.

Clearly, there will be significant differences when the velocity and temperature relaxation times are not of the same order. When $t_u \ll t_T$ the velocity equilibration zone will be much thinner than the thermal relaxation zone and when $t_u \gg t_T$ the opposite will be true. Marble (1970) uses a perturbation analysis about the final downstream state to show that the two processes of velocity and temperature relaxation are not closely coupled, at least up to the second order in an expansion in $\xi$. Consequently, as a first approximation, one can regard the velocity and temperature relaxation zones as uncoupled. Marble also explores the effects of different particle sizes and the collisions that may ensue as a result of relative motion between the different sizes.

This normal shock wave analysis illustrates that the notions of velocity and temperature relaxation can be applied as modifications to the basic gas dynamic structure in order to synthesize, at least qualitatively, the structure of the multiphase flow.

**11.4 ACOUSTIC DAMPING**

Another important consequence of relative motion is the effect it has on the propagation of plane acoustic waves in a dusty gas. Here we will examine both the propagation velocity and damping of such waves. To do so we postulate a uniform dusty gas and denote the mean state of this mixture by an overbar so that $\bar{p}$, $\bar{T}$, $\bar{\rho}_C$, $\bar{\xi}$ are respectively the pressure, temperature, gas density and mass loading of the uniform dusty gas. Moreover we chose a frame of reference relative to the mean dusty gas so that $\bar{\mu}_C = \bar{\mu}_D = 0$. Then we investigate small, linearized perturbations to this mean state denoted by $\tilde{p}$, $\tilde{T}_C$, $\tilde{T}_D$, $\tilde{\rho}_C$, $\tilde{\alpha}_D$, $\tilde{\mu}_C$, and $\tilde{\mu}_D$. Substituting into the basic continuity, momentum and energy equations 11.1, 11.4 and 11.5, utilizing the expressions and assumptions of section 11.2.3 and retaining only terms
linear in the perturbations, the equations governing the propagation of plane acoustic waves become

\[
\frac{\partial \tilde{u}_C}{\partial x} + \frac{\partial \bar{p}}{\partial t} - \frac{1}{T} \frac{\partial \tilde{T}_C}{\partial t} = 0 \quad (11.38)
\]

\[
\rho_D \frac{\partial \tilde{\alpha}_D}{\partial t} + \frac{\partial \tilde{u}_D}{\partial x} = 0 \quad (11.39)
\]

\[
\frac{\partial \tilde{u}_C}{\partial t} + \frac{\tilde{u}_C}{t_u} - \frac{\tilde{u}_D}{t_u} + \frac{1}{\gamma} \frac{\partial \bar{p}}{\partial x} = 0 \quad (11.40)
\]

\[
\frac{\partial \tilde{u}_D}{\partial t} + \frac{\tilde{u}_D}{t_u} - \frac{\tilde{u}_C}{t_u} = 0 \quad (11.41)
\]

\[
\frac{\partial \tilde{T}_C}{\partial t} + \frac{\xi \tilde{T}_C}{t_T} - \frac{\xi \tilde{T}_D}{t_T} + \frac{(\gamma - 1) \bar{p}}{\gamma T} \frac{\partial \bar{p}}{\partial t} = 0 \quad (11.42)
\]

\[
\frac{\partial \tilde{T}_D}{\partial t} + \frac{c_p \tilde{T}_D}{c_s T_T} - \frac{c_p \tilde{T}_C}{c_s T_T} = 0 \quad (11.43)
\]

where \(\gamma = c_p/c_{eC}\). Note that the particle volume fraction perturbation only occurs in one of these, equation 11.39; consequently this equation may be set aside and used after the solution has been obtained in order to calculate \(\tilde{\alpha}_D\) and therefore the perturbations in the particle loading \(\tilde{\xi}\). The basic form of a plane acoustic wave is

\[
Q(x, t) = \bar{Q} + \tilde{Q}(x, t) = \bar{Q} + Re \left\{ Q(\omega) e^{i\kappa x + i\omega t} \right\} \quad (11.44)
\]

where \(Q(x, t)\) is a generic flow variable, \(\omega\) is the acoustic frequency and \(\kappa\) is a complex function of \(\omega\); clearly the phase velocity of the wave, \(c_\kappa\), is given by \(c_\kappa = Re\{-\omega/\kappa\}\) and the non-dimensional attenuation is given by \(Im\{-\kappa\}\). Then substitution of the expressions 11.44 into the five equations 11.38, 11.40, 11.41, 11.42, and 11.43 yields the following dispersion relation for \(\kappa\):

\[
\left( \frac{\omega}{\kappa c_C} \right)^2 = \frac{(1 + i\omega t_u) \left( \frac{c_pC}{c_{eD}} + \xi + i\omega t_T \right)}{(1 + \xi + i\omega t_u) \left( \frac{c_pC}{c_{eD}} \right) \xi + i\omega t_T} \quad (11.45)
\]

where \(c_C = (\gamma R_C T)^{1/2}\) is the speed of sound in the gas alone. Consequently, the phase velocity is readily obtained by taking the real part of the square root of the right hand side of equation 11.45. It is a function of frequency, \(\omega\), as well as the relaxation times, \(t_u\) and \(t_T\), the loading, \(\xi\), and the specific
heat ratios, $\gamma$ and $c_p c/c_s d$. Typical results are shown in figures 11.2 and 11.3.

The mechanics of the variation in the phase velocity (acoustic speed) are evident by inspection of equation 11.45 and figures 11.2 and 11.3. At very low frequencies such that $\omega t_u \ll 1$ and $\omega t_T \ll 1$, the velocity and temperature relaxations are essentially instantaneous. Then the phase velocity is simply obtained from the effective properties and is given by equation 11.16. These are the phase velocity asymptotes on the left-hand side of figures 11.2 and 11.3. On the other hand, at very high frequencies such that $\omega t_u \gg 1$ and
\[ \omega t_T \gg 1, \] there is negligible time for the particles to adjust and they simply do not participate in the propagation of the wave; consequently, the phase velocity is simply the acoustic velocity in the gas alone, \( c_C \). Thus all phase velocity lines asymptote to unity on the right in the figures. Other ranges of frequency may also exist (for example \( \omega t_u \gg 1 \) and \( \omega t_T \ll 1 \) or the reverse) in which other asymptotic expressions for the acoustic speed can be readily extracted from equation 11.45. One such intermediate asymptote can be detected in figure 11.3. It is also clear that the acoustic speed decreases with increased loading, \( \xi \), though only weakly in some frequency ranges. For small \( \xi \) the expression 11.45 may be expanded to obtain the linear change in the acoustic speed with loading, \( \xi \), as follows:

\[
\frac{c_K}{c_C} = 1 - \frac{\xi}{2} \left[ \frac{(\gamma - 1)c_p c}{csD} \left\{ \frac{(c_p c / c_s D)^2 + (\omega t_T)^2}{1 + (\omega t_T)^2} \right\} + 1 \right] + \ldots \quad (11.46)
\]

This expression shows why, in figures 11.2 and 11.3, the effect of the loading, \( \xi \), on the phase velocity is small at higher frequencies.

Now we examine the attenuation manifest in the dispersion relation 11.45. The same expansion for small \( \xi \) that led to equation 11.46 also leads to the following expression for the attenuation:

\[
Im\{-\kappa c_c / \omega\} = \frac{\xi \omega}{2c_c} \left[ \frac{(\gamma - 1)\omega t_T}{(c_p c / c_s D)^2 + (\omega t_T)^2} + \frac{\omega t_u}{1 + (\omega t_T)^2} \right] + \ldots \quad (11.47)
\]

In figures 11.2 and 11.3, a dimensionless attenuation, \( Im\{-\kappa c_c / \omega\} \), is plotted against the reduced frequency. This particular non-dimensionalization is somewhat misleading since, plotted without the \( \omega \) in the denominator, the attenuation increases monotonically with frequency. However, this presentation is commonly used to demonstrate the enhanced attenuations that occur in the neighborhoods of \( \omega = t_u^{-1} \) and \( \omega = t_T^{-1} \) and which are manifest in figures 11.2 and 11.3.

When the gas contains liquid droplets rather than solid particles, the same basic approach is appropriate except for the change that might be caused by the evaporation and condensation of the liquid during the passage of the wave. Marble and Wooten (1970) present a variation of the above analysis that includes the effect of phase change and show that an additional maximum in the attenuation can result as illustrated in figure 11.4. This additional peak results from another relaxation process embodied in the phase change process. As Marble (1970) points out it is only really separate
from the other relaxation times when the loading is small. At higher loadings
the effect merges with the velocity and temperature relaxation processes.

### 11.5 OTHER LINEAR PERTURBATION ANALYSES

In the preceding section we examined the behavior of small perturbations about a constant and uniform state of the mixture. The perturbation was a plane acoustic wave but the reader will recognize that an essentially similar methodology can be used (and has been) to study other types of flow involving small linear perturbations. An example is steady flow in which the deviation from a uniform stream is small. The equations governing the small deviations in a steady planar flow in, say, the \((x, y)\) plane are then quite analogous to the equations in \((x, t)\) derived in the preceding section.

#### 11.5.1 Stability of laminar flow

An important example of this type of solution is the effect that dust might have on the stability of a laminar flow (for instance a boundary layer flow) and, therefore, on the transition to turbulence. Saffman (1962) explored the effect of a small volume fraction of dust on the stability of a parallel flow. As expected and as described in section 1.3.2, when the response times
of the particles are short compared with the typical times associated with the fluid motion, the particles simply alter the effective properties of the fluid, its effective density, viscosity and compressibility. It follows that under these circumstances the stability is governed by the effective Reynolds number and effective Mach number. Saffman considered dusty gases at low volume concentrations, $\alpha$, and low Mach numbers; under those conditions the net effect of the dust is to change the density by $(1 + \alpha \rho_S/\rho_G)$ and the viscosity by $(1 + 2.5 \alpha)$. The effective Reynolds number therefore varies like $(1 + \alpha \rho_S/\rho_G)/(1 + 2.5 \alpha)$. Since $\rho_S \gg \rho_G$ the effective Reynolds number is increased and therefore, in the small relaxation time range, the dust is destabilizing. Conversely for large relaxation times, the dust stabilizes the flow.

11.5.2 Flow over a wavy wall

A second example of this type of solution that was investigated by Zung (1967) is steady particle-laden flow over a wavy wall of small amplitude (figure 11.5) so that only the terms that are linear in the amplitude need be retained. The solution takes the form

$$\exp(i\kappa_1 x - i\kappa_2 y)$$

where $2\pi/\kappa_1$ is the wavelength of the wall whose mean direction corresponds with the $x$ axis and $\kappa_2$ is a complex number whose real part determines the inclination of the characteristics or Mach waves and whose imaginary part determines the attenuation with distance from the wall. The value of $\kappa_2$ is obtained in the solution from a dispersion relation that has many similarities to equation 11.45. Typical computations of $\kappa_2$ are presented in figure 11.6. The asymptotic values for large $t_u$ that occur on the right in this figure correspond to cases in which the particle motion is constant and

![Figure 11.5. Schematic for flow over a wavy wall.](image-url)
Figure 11.6. Typical results from the wavy wall solution of Zung (1969). Real and imaginary parts of $\kappa_2/\kappa_1$ are plotted against $t_u U/\kappa_1$ for various mean Mach numbers, $M = U/c_C$, for the case of $t_T/t_u = 1$, $c_p c_D/c_s D = 1$, $\gamma = 1.4$ and a particle loading, $\xi = 1$.

unaffected by the waves. Consequently, in subsonic flows ($M = U/c_C < 1$) in which there are no characteristics, the value of $Re\{\kappa_2/\kappa_1\}$ asymptotes to zero and the waves decay with distance from the wall such that $Im\{\kappa_2/\kappa_1\}$ tends to $(1 - M^2)^{1/2}$. On the other hand in supersonic flows ($M = U/c_C > 1$) $Re\{\kappa_2/\kappa_1\}$ asymptotes to the tangent of the Mach wave angle in the gas alone, namely $(M^2 - 1)^{1/2}$, and the decay along these characteristics is zero.

At the other extreme, the asymptotic values as $t_u$ approaches zero correspond to the case of the effective gas whose properties are given in section 11.2.2. Then the appropriate Mach number, $M_0$, is that based on the speed of sound in the effective gas (equation 11.16). In the case of figure 11.6, $M_0^2 = 2.4 M^2$. Consequently, in subsonic flows ($M_0 < 1$), the real and imaginary parts of $\kappa_2/\kappa_1$ tend to zero and $(1 - M_0^2)^{1/2}$ respectively as $t_u$ tends to zero. In supersonic flows ($M_0 > 1$), they tend to $(M_0^2 - 1)^{1/2}$ and zero respectively.
11.6 SMALL SLIP PERTURBATION

The analyses described in the preceding two sections, 11.4 and 11.5, used a linearization about a uniform and constant mean state and assumed that the perturbations in the variables were small compared with their mean values. Another, different linearization known as the small slip approximation can be advantageous in other contexts in which the mean state is more complicated. It proceeds as follows. First recall that the solutions always asymptote to those for a single effective gas when \( t_u \) and \( t_T \) tend to zero. Therefore, when these quantities are small and the slip between the particles and the gas is correspondingly small, we can consider constructing solutions in which the flow variables are represented by power series expansions in one of these small quantities, say \( t_u \), and it is assumed that the other (\( t_T \)) is of similar order. Then, generically,

\[
Q(x_i, t) = Q^{(0)}(x_i, t) + t_u Q^{(1)}(x_i, t) + t_u^2 Q^{(2)}(x_i, t) + \ldots \quad (11.49)
\]

where \( Q \) represents any of the flow quantities, \( u_{Ci}, u_{Di}, T_C, T_D, p, \rho_C, \alpha_C, \alpha_D, \) etc. In addition, it is assumed for the reasons given above that the slip velocity and slip temperature, \( (u_{Ci} - u_{Di}) \) and \( (T_C - T_D) \), are of order \( t_u \) so that

\[
u_{Ci}^{(0)} = u_{Di}^{(0)} = u_i^{(0)} ; \quad T_C^{(0)} = T_D^{(0)} = T^{(0)} \quad (11.50)
\]

Substituting these expansions into the basic equations 11.6, 11.7 and 11.8 and gathering together the terms of like order in \( t_u \) we obtain the following zeroth order continuity, momentum and energy relations (omitting gravity):

\[
\frac{\partial}{\partial x_i} \left( (1 + \xi) \rho_C^{(0)} u_i^{(0)} \right) = 0 \quad (11.51)
\]

\[
(1 + \xi) \rho_C^{(0)} u_k^{(0)} \frac{\partial u_i^{(0)}}{\partial x_k} = -\frac{\partial p^{(0)}}{\partial x_i} + \frac{\partial \sigma_D^{(0)}}{\partial x_k} \quad (11.52)
\]

\[
\rho_C^{(0)} u_k^{(0)} (c_{pC} + \xi c_{aD}) \frac{\partial T^{(0)}}{\partial x_k} = u_k^{(0)} \frac{\partial p^{(0)}}{\partial x_k} + \sigma_D^{(0)} \frac{\partial u_i^{(0)}}{\partial x_k} \quad (11.53)
\]

Note that Marble (1970) also includes thermal conduction in the energy equation. Clearly the above are just the equations for single phase flow of the effective gas defined in section 11.2.2. Conventional single phase gas dynamic methods can therefore be deployed to obtain their solution.

Next, the relaxation equations 11.22 and 11.25 that are first order in \( t_u \)
yield:

\[ u_k^{(0)} \frac{\partial u_i^{(0)}}{\partial x_k} = u_{Ci}^{(1)} - u_{Di}^{(1)} \] (11.54)

\[ u_k^{(0)} \frac{\partial T^{(0)}}{\partial x_k} = \left( \frac{t_u}{t_T} \right) \left( \frac{Nu}{2} \right) (T_C^{(1)} - T_D^{(1)}) \] (11.55)

From these the slip velocity and slip temperature can be calculated once the zeroth order solution is known.

The third step is to evaluate the modification to the effective gas solution caused by the slip velocity and temperature; in other words, to evaluate the first order terms, \( u_{Ci}^{(1)} \), \( T_C^{(1)} \), etc. The relations for these are derived by extracting the \( O(t_u) \) terms from the continuity, momentum and energy equations. For example, the continuity equation yields

\[ \frac{\partial}{\partial x_i} \left[ \xi \rho_C^{(0)} (u_{Ci}^{(1)} - u_{Di}^{(1)}) + u_i^{(0)} (\rho_D \alpha_D^{(1)} - \xi \rho_C^{(1)}) \right] = 0 \] (11.56)

This and the corresponding first order momentum and energy equations can then be solved to find the \( O(t_u) \) slip perturbations to the gas and particle flow variables. For further details the reader is referred to Marble (1970).

A particular useful application of the slip perturbation method is to the one-dimensional steady flow in a convergent/divergent nozzle. The zeroth order, effective gas solution leads to pressure, velocity, temperature and density profiles that are straightforward functions of the Mach number which is, in turn, derived from the cross-sectional area. This area is used as a

Figure 11.7. The dimensionless choked mass flow rate as a function of loading, \( \xi \), for \( \gamma_C = 1.4 \) and various specific heat ratios, \( c_{pC}/c_{sD} \) as shown.
surrogate axial coordinate. Here we focus on just one part of this solution namely the choked mass flow rate, $\dot{m}$, that, according to the single phase, effective gas analysis will be given by

$$\frac{\dot{m}}{A_*(p_0\rho_{C0})^{\frac{1}{2}}} = (1 + \xi)^{\frac{1}{2}}\gamma^{\frac{1}{2}} \left( \frac{2}{1 + \gamma} \right)^{(\gamma+1)/2(\gamma-1)}$$

where $p_0$ and $\rho_{C0}$ refer to the pressure and gas density in the upstream reservoir, $A_*$ is the throat cross-sectional area and $\gamma$ is the effective specific heat ratio as given in equation 11.14. The dimensionless choked mass flow rate on the left of equation 11.57 is a function only of $\xi$, $\gamma_C$ and the specific heat ratio, $c_{pC}/c_{sD}$. As shown in figure 11.7, this is primarily a function of the loading $\xi$ and is only weakly dependent on the specific heat ratio.