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THE ANALYSIS OF STRESS AND DEFORMATION

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PREFACE

This book was prepared for a course in the mechanics of deformable bodies at the authors' institution, and is at a level suitable for advanced undergraduate or first-year graduate students. It differs from the traditional treatment by going more deeply into the fundamentals and giving less emphasis to the design aspects of the subject. In the first two chapters the principles of stress and strain are presented and a sufficient introduction is given to the theory of elasticity so that the student can see how exact solutions of problems can be derived, and can appreciate the nature of the approximations embodied in some commonly used simplified solutions. The third chapter is devoted to the bending of beams, and the fourth chapter treats the instability of elastic systems. Applications to axially symmetric problems, curved beams, and stress concentrations are discussed in Chapter 5; applications to torsion problems are discussed in Chapter 6; applications to problems of plates and shells are discussed in Chapter 7. Applications to problems involving viscous and plastic behavior are treated in Chapter 8, and problems of wave propagation are treated in Chapter 9. An introduction to numerical methods of solving problems is given in Chapter 10. An introduction to tensor notation by means of the equations of elasticity is given in Appendix I. Experimental methods of determining stresses by means of strain gages, brittle coatings, and photoelasticity are described in Appendices II and III. A brief introduction to variational methods is presented in Appendix IV. The material in the book is laid out so that a short course can be based on Chapters 1, 2, 3, 4, and 8 and Appendices II and III.

Some of the special aspects of the subject and some of the details of the derivations are left to the problems; the assignment of homework should be made with this in mind.

To indicate to the student the nature of the more advanced parts of the subject, some topics are included that would not necessarily be covered in the formal course work.

The book is aimed primarily at those students who will pursue graduate work, and it is intended to give a good preparation for advanced studies in the field. It should also give a good foundation to students primarily interested in design who would cover the more applied aspects of the subject in courses on design.

The authors wish to express their appreciation to those members of the California Institute of Technology community whose suggestions and efforts have helped to bring our lecture notes into book form.

G. W. H.
T. V., Jr.

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CHAPTER 1

BASIC PRINCIPLES OF STRESS AND STRAIN

1-1 INTRODUCTION

The mechanics of deformable bodies deals with the stresses and strains produced in bodies by external actions. In its practical aspects, the subject deals with such questions as how large a force can a body withstand without collapsing; how far out of shape will the body be deformed by the action of prescribed forces; what is the most efficient shape of the body for withstanding the forces? The answers to these and related questions are required in all phases of a technically advanced society. We come into contact with numerous examples of the application of stress analysis every day. Bridges and buildings are examples, as are machines, airplanes, missiles, etc. In short, any solid body whose weight, strength, or deformation is an item for consideration must be studied from the point of view of stress analysis. In its theoretical aspects, *the subject is concerned with investigating the differential equations*, and their solutions, that describe the states of stress and strain in bodies of various shapes and materials under the actions of various external agencies.

The study of stress and strain began with Galileo (1564–1642) who published the results of his studies in his book *Two New Sciences* (1638).* One of the new sciences was *dynamics* and the other was *stress analysis.*

* The notebooks of Leonardo da Vinci (1452–1519) show that he studied the breaking strength of iron wires and the strength of beams and columns, but he did not publish any of his work.

Galileo tried to determine the stresses in a cantilever beam, but he was unaware that the stress distribution could not be determined without considering the deformation of the beam. In effect, his analysis assumed that the material was infinitely rigid. Robert Hooke (1635–1703), in his paper *De Potentia Restitutiva* (1678), was the first to point out that a body is deformed if a force acts upon it. Actually he restricted his consideration to bodies for which the deformation was proportional to the force and, hence, a linear relation between stress and strain is called *Hooke's Law*. This forms the basis for the development of the *Theory of Elasticity*, which is the subject of stress analysis of linearly elastic materials.

In the years following Galileo many engineers, physicists, and mathematicians worked on problems of stress analysis.* The development of the subject proceeded along two lines. The theory of elasticity aimed at analyzing the exact stress distribution in a loaded body and it formed the more mathematical part of the subject. The mathematical difficulties, however, were often too great in the case of important practical problems, and consequently alongside of the theory of elasticity there developed a branch of stress analysis that concerned itself largely with more or less approximate solutions of practical problems. This branch of the subject is often called *Strength of Materials*, to distinguish it from the Theory of Elasticity, although the name is not really appropriate. A better name would be *strength of bodies* or *applied stress analysis*.

There are many books dealing with various aspects of the theory of elasticity and strength of materials.† New work in the field of stress analysis appears in such publications as the *Journal of Applied Mechanics*, and the *Journal of Mechanics and Physics of Solids*. This book will give the reader a sufficient introduction to the theory of elasticity so that he will understand what the subject involves and will be prepared to undertake more advanced studies if he has a special interest in this field. The major portion of the book is devoted to more applied aspects of the subject, such as problems of beams, columns, torque shafts, and pressure vessels.

The first portion of the book presents definitions of important concepts, quantities and nomenclature, and it is essential that the reader understand

* S. Timoshenko, *A History of Strength of Materials*, New York: McGraw-Hill Book Company (1953).

I. Todhunter and K. Pearson, *History of the Theory of Elasticity and of the Strength of Materials*, New York: Dover (1960).

C. Truesdell, *The Rational Mechanics of Flexible or Elastic Bodies, 1638–1788*, O. Fussli: Switzerland (1960).

† Especially well-known books are: A. E. H. Love, *The Mathematical Theory of Elasticity*; S. P. Timoshenko, *Strength of Materials*; and Timoshenko and Goodier, *Theory of Elasticity*.

these thoroughly. He will find that many of the concepts, such as stress, strain, and energy are not simple, but require careful thought for their comprehension.

1-2 DEFINITION OF STRESS

Stress on a Surface. The stress on a surface is defined as follows: Let a curve that encloses a small area A be drawn on the surface and let the vector **F** be the resultant force acting on the surface area A as shown in Fig. 1.1a, then the average stress on A is

$$\mathbf{s}_{\text{ave}} = \frac{\mathbf{F}}{A} \tag{1.1}$$

If, now, the area A is reduced in size so that in the limit $A \to 0$, then the stress at this point on the surface is defined to be

$$\mathbf{s} = \lim_{A \to 0} \frac{\mathbf{F}}{A} \tag{1.2}$$

This value of **s** gives the magnitude and direction of the stress at a point on the surface as indicated in Fig. 1.1b. The vector stress **s** can be resolved into a normal stress component σ and a tangential stress component τ, as shown in Fig. 1.1b.

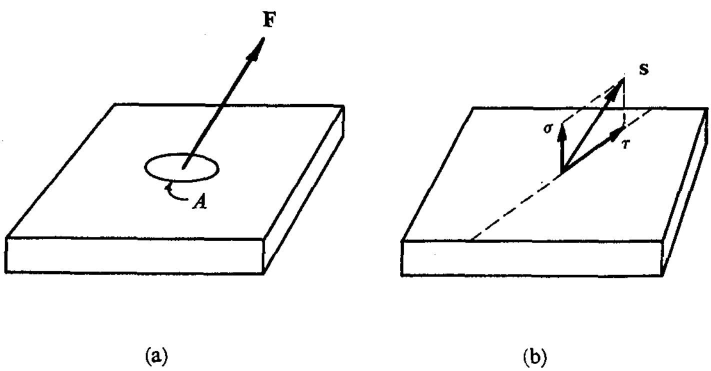


FIGURE 1.1

The limit of $A \to 0$ is of course an idealization since the surface itself is not continuous on an atomic scale. We will limit our consideration to the average stress on areas whose dimensions, while small compared to the dimensions of the body, are large compared to the distance between atoms in the body.

Stresses acting on the surface of a body give rise to stresses in the interior. To illustrate these internal stresses, we may make an imaginary cut through the body. Stresses acting on the surfaces exposed by the cut must be such as to hold the cut portions of the body in equilibrium. A simple example of an internal stress is found in a long, straight, circular bar in tension under the action of two co-axial forces of magnitude F, as shown in Fig. 1.2. A freebody diagram of the right half of the bar is shown in Fig. 1.2b. The cut is perpendicular to the axis of the bar and the stress intensity s_1 is, in this case, uniform over the cross-sectional area A of the bar. The loading conditions which produce a uniform stress intensity in the bar are given in Section 3-2. Equilibrium requires that

$$s_1 A = F$$

If the cut is made at 45° as shown in Fig. 1.2c, the stress on the cut surface is again uniformly distributed and equilibrium requires

$$s_2 A_2 = F$$

and, since $A_2 = A\sqrt{2}$, the stress has the magnitude

$$s_2 = \frac{F}{A\sqrt{2}}$$

and is directed along the axis of the bar. It is seen that, depending upon how the cut is made, different values of stress intensity are obtained. The stress

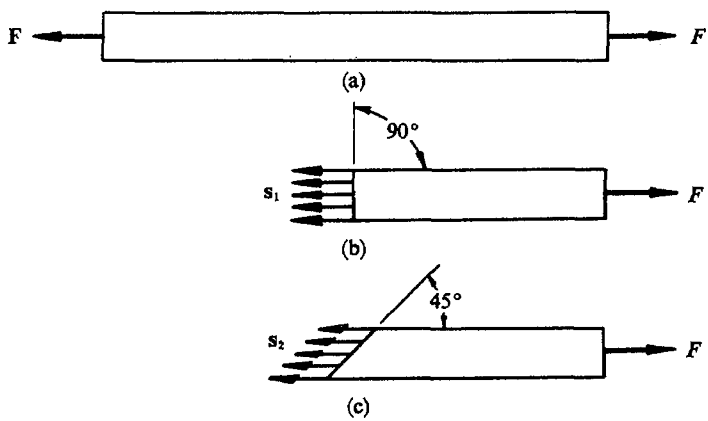


FIGURE 1.2

vector is directed along the axis of the bar and therefore makes different angles with differently oriented surfaces.

A more complicated example is that of an eccentrically loaded bar as shown in Fig. 1.3. The stress on the cut surface is directed along the axis of the bar,

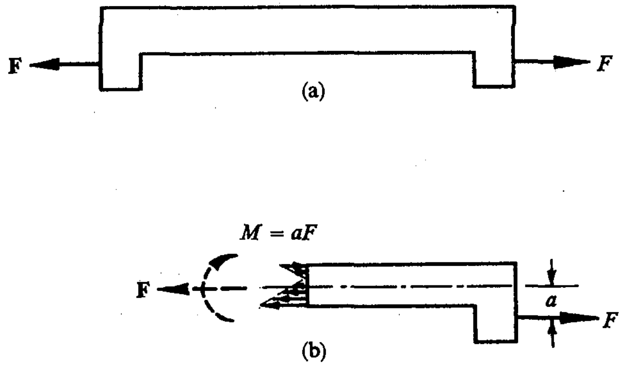


FIGURE 1.3

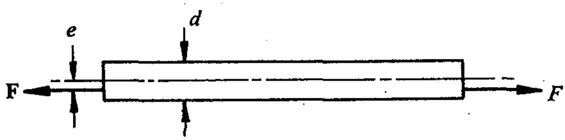
but it is not uniformly distributed. Its intensity must vary over the surface for otherwise the resultant of the stress could not hold the cut section, shown in Fig. 1.3b, in equilibrium. Usually a freebody cut through a stressed body will disclose a nonuniform distribution of stress. It is the objective of stress analysis to determine the internal distribution of stress.

Problems

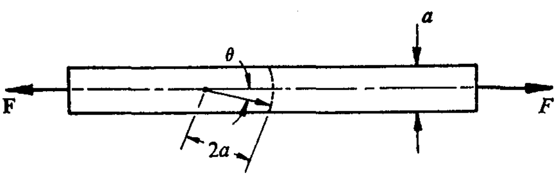
1.1 A straight bar with a circular cross section of 1 in. radius is pulled in tension by two forces of 50,000 lb each that produce a uniform stress in the bar. On what plane is the stress a maximum? On what plane is it a minimum? What is the value of the stress on a plane midway between the maximum and minimum planes?

1.2 A straight bar of constant cross-sectional area A is pulled in tension by two forces F that produce a uniform tension. What stress acts on the plane whose normal makes an angle θ with the axis of the bar?

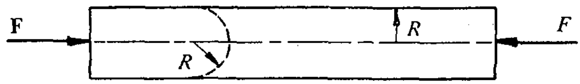
1.3 A bar of circular cross section is pulled by two collinear forces F whose line of action is offset a distance e from the axis of the bar. Determine the stress distribution over a plane normal to the axis, assuming that the stress varies linearly over the plane.



1.4 A bar with square cross section is pulled in tension by two axial forces F that produce a uniform stress on any plane through the bar. Consider the imaginary cylindrical cut shown in the diagram. What is the stress distribution over this cut?



1.5 A straight bar with circular cross section is put in compression by two equal axial forces that produce a uniform stress on any plane through the bar. An imaginary spherical cut is made through the bar as shown. What is the stress distribution over this cut?



Stress at a Point. If we wish to examine the stress at some point P in the interior of a body we may make a freebody cut, passing the plane a_1–a_1 through the point P as shown in Fig. 1.4. The stress $\mathbf{s}_1$, shown in the diagram

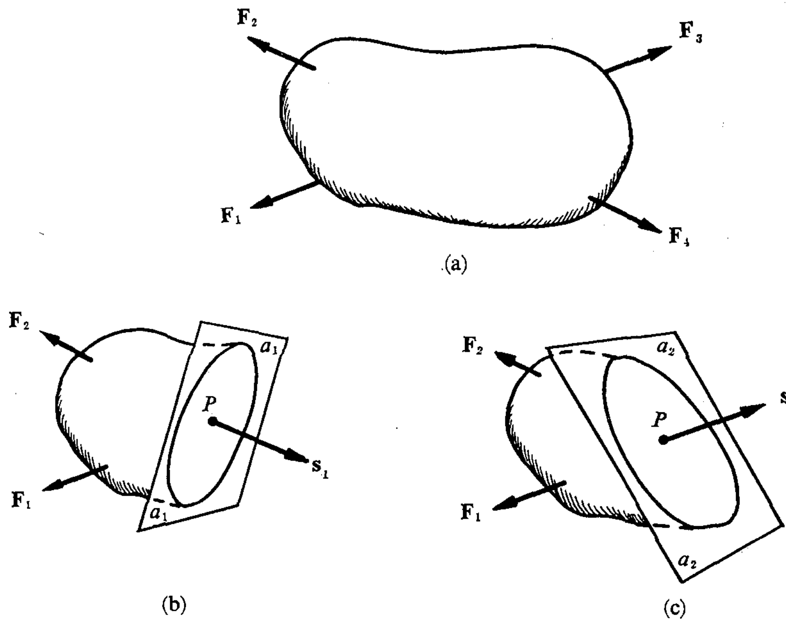


FIGURE 1.4

is the stress acting on the left-hand side of the cut body at the point *P*. *If a* different cut, say a_2–a_2, is made through P, there will be a different stress $\mathbf{s}_2$ acting on the cut surface at P. There is an unlimited number of differently oriented planes that can be passed through the point P, each of which will have a different stress at that point. Therefore, knowledge of the stress on only one of these planes is but partial information about the stress at the point P. A complete description of the *state of stress at the point* requires that the stress on all possible planes through P be specified. The simplest description of the state of stress at a point is given by a specification of the stress on each of three different planes through the point. In Fig. 1.5a is shown the point P whose cartesian coordinates are x, y, z. Infinitesimal segments of three planes passing through P are also shown, the three planes being parallel to the three coordinate planes, $\mathbf{s}_x$ is the stress at point P acting on a plane that is perpendicular to the x-axis, and $\mathbf{s}_y$ and $\mathbf{s}_z$ are the stresses acting on the planes perpendicular to the y and z axes.* The stresses are not necessarily normal to the faces on which they act. That these three stresses do indeed specify the state of stress at the point is easily proven. In Fig. 1.5b is shown

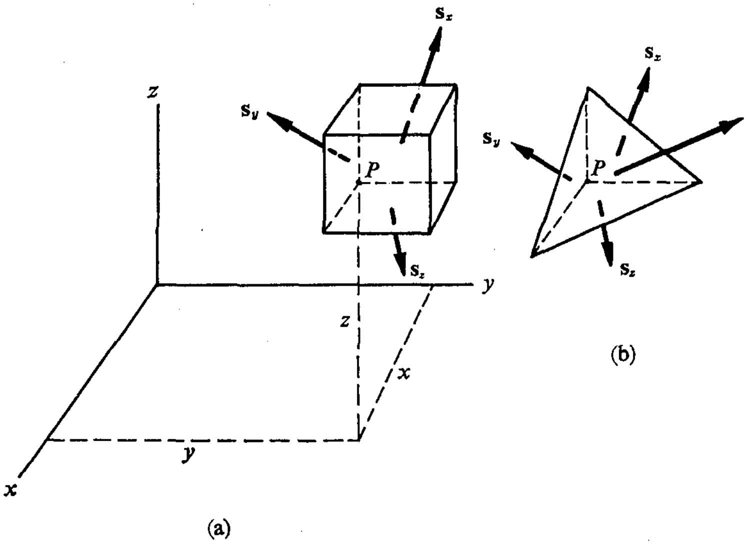


FIGURE 1.5

the tetrahedron formed by passing an inclined plane an infinitesimal distance this side of point P. The figure is a freebody diagram of a small element of the body and, since equilibrium must be satisfied, the sum of the forces must be zero

$$\mathbf{s}_n\, dA_n + \mathbf{s}_x\, dA_x + \mathbf{s}_y\, dA_y + \mathbf{s}_z\, dA_z = 0 \tag{1.3}$$

where dA_n is the area of the inclined face on which the stress vector $\mathbf{s}_n$ acts, dA_x is the area of the x-face, etc. It can be shown that the areas of the faces are related by

$$\begin{aligned} dA_x &= l_{xn}\, dA_n \\ dA_y &= l_{yn}\, dA_n \\ dA_z &= l_{zn}\, dA_n \end{aligned} \tag{1.4}$$

* If the area of the segment is infinitesimal and the rate of change of stress is finite, the stress can be taken to be constant over the segment. In special cases, such as near a sharp re-entrant corner or near the tip of a crack, there may be a stress singularity and, in this case, the approximation of constant stress over the segment may not be valid.

where l_{xn} is the direction cosine (cosine of the angle between the $+x$-axis and the perpendicular to the n-face). Substituting these in Eq. (1.3) there is obtained

$$\mathbf{s}_n = -(l_{xn}\mathbf{s}_x + l_{yn}\mathbf{s}_y + l_{zn}\mathbf{s}_z) \tag{1.5}$$

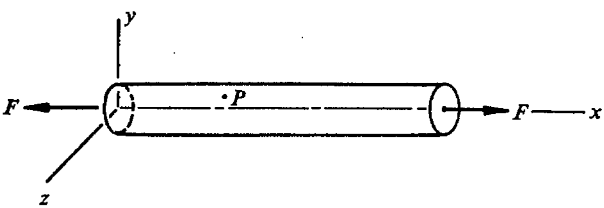
In the limit as the areas dA_n, dA_x, etc. approach zero, the inclined plane passes through P and Eq. (1.5) gives the stress on the inclined plane at point P. We thus see that the stress on any plane (n = any direction) through P can be expressed in terms of $\mathbf{s}_x$, $\mathbf{s}_y$, and $\mathbf{s}_z$ and, in general, the state of stress at a point is described by three stress vectors. When we speak of the *stress at a point* we mean such a set of three stress vectors. Since the stresses on any set of three planes through the point are equivalent to the stresses on any other set of three planes through the point, we may say that the stress at a point is described by any triplet of stress vectors corresponding to three planes through the point. The three planes must form a corner of the element but need not be orthogonal.

The stress vectors $\mathbf{s}_x$, $\mathbf{s}_y$, $\mathbf{s}_z$ will, in general, have different magnitudes and directions at every point in the body.

Problems

1.6 Prove that Eqs. (1.4) are correct.

1.7 A circular bar is pulled in tension by two forces F as shown in the diagram. What is the state of stress at the point P?



1.8 In Prob. 1.7, what is the state of stress at point P expressed in terms of a rotated orthogonal coordinate system with axes x', y', z', and stresses $\mathbf{s}_{x'}$, $\mathbf{s}_{y'}$, $\mathbf{s}_{z'}$, where $l_{xx'} = \frac{1}{2}$; $l_{yx'} = 0$; $l_{zx'} = -\sqrt{3}/2$; $l_{xz'} = \sqrt{3}/2$; $l_{yz'} = 0$; $l_{zz'} = \frac{1}{2}$; $l_{xy'} = 0$; $l_{yy'} = 1$; $l_{zy'} = 0$.

The Stress Element. It is customary when describing the stress at a point to give the stress vectors on three sides of an infinitesimal element whose faces are perpendicular to the coordinate directions. For rectangular coordinates the element is a cube whose volume is $(dx\, dy\, dz)$ as shown in Fig. 1.6a. The stress at the point whose coordinates are x, y, z is described by $\mathbf{s}_x$, $\mathbf{s}_y$, $\mathbf{s}_z$. As a matter of convenience, the stress vectors are shown as if they were acting at the centers of the sides rather than at the point P. This is

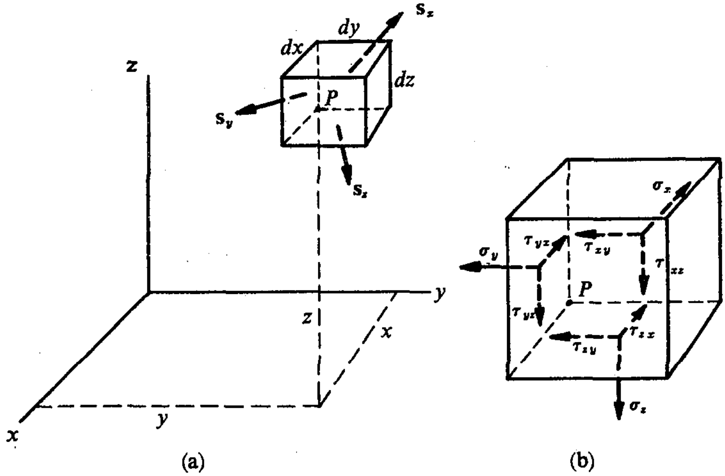


FIGURE 1.6

permissible as the sides of the element are of infinitesimal dimensions so that the stress may be taken to be uniform over the side. In the absence of a stress singularity, the stress can vary at most by an infinitesimal quantity from one side of the element to the other and, unless we are particularly concerned with the infinitesimal quantities, we shall neglect them.

It is customary to specify the three stress vectors in terms of components parallel to the coordinate directions; for example, in cartesian components

$$
\begin{aligned}
|\mathbf{s}_x| &= |\sigma_x \mathbf{i} + \tau_{xy}\mathbf{j} + \tau_{xz}\mathbf{k}| \\
|\mathbf{s}_y| &= |\tau_{yx}\mathbf{i} + \sigma_y \mathbf{j} + \tau_{yz}\mathbf{k}| \\
|\mathbf{s}_z| &= |\tau_{zx}\mathbf{i} + \tau_{zy}\mathbf{j} + \sigma_z \mathbf{k}|
\end{aligned}
\tag{1.6}
$$

The absolute value signs must be used in Eq. (1.6) because we have not yet defined a sign convention for the stress components. These components are shown in Fig. 1.6b. The letters σ_x, σ_y, σ_z, are customary notation for *normal stresses*; that is, σ_x is a stress normal to the x-face on which it is acting, etc. The stresses that are tangent to the faces on which they act are called *shear stresses* and are denoted τ_{xy}, etc. The subscripts on τ_{xy} specify the face on which the stress acts and the direction of the stress. For example, τ_{xy} is the stress acting on the x-face in a direction parallel to the y-axis. The *resultant* shear on a face is given by the resultant of the two shear stress components.

From Eqs. (1.6) it is seen that a set of nine scalar stress components specify the state of stress at a point

$$
\begin{vmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{vmatrix} = \text{stress} \tag{1.7}
$$

The stress components σ_x, σ_y, etc., will, in general, have different values at every point in the body. If mathematical expressions can be found (functions of x, y, and z) which give the correct values of the stress components at every point in the body, they are said to describe the *state of stress in the body.*

It should be noted that although the elements in Fig. 1.6 are shown with stresses on three faces only, there are also stresses on the other three faces. These are not shown because they are not the stresses at the point x, y, z, but are the stresses associated with the point $x + dx$, $y + dy$, $z + dz$.

Sign Conventions for Stresses. The vector sign convention for stress components is not as useful as the convention which we shall adopt, because it does not relate to the deformational effect of the stress. For example, an element of a bar in tension shown in Fig. 1.7a has a stress $\mathbf{s} = \sigma_y\mathbf{j}$ on its right face, and a stress $-\mathbf{s} = -\sigma_y\mathbf{j}$ on its left face according to the vector sign convention. Both $\mathbf{s}$ and $-\mathbf{s}$ are acting to elongate the bar, and therefore are equal insofar as their deformational effect on the bar is concerned.

The sign convention we shall use for stresses is based on defining positive and negative faces of the stress element. This is done by noting that each coordinate axis has a positive and a negative direction. A side of the element facing in a positive direction is called a positive face and a side facing in a negative direction is called a negative face. The sign convention for stresses is then: *A positive stress component acts on a positive face in a positive direction, or acts on a negative face in a negative direction.* Correspondingly, a negative stress component acts on a positive face in a negative direction, or on a negative face in a positive direction. The reason for adopting this sign convention is explained by the following remarks.

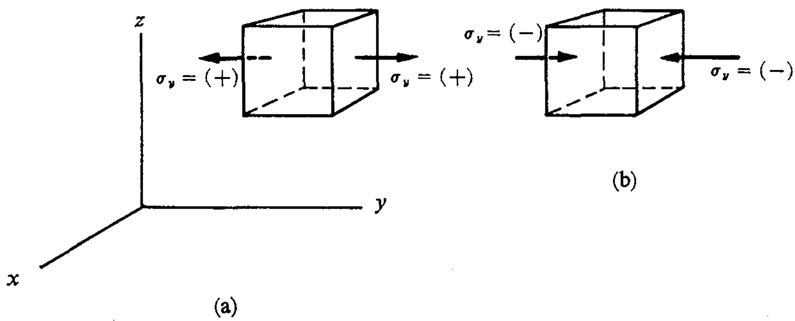


FIGURE 1.7

Let us consider a state of stress in which all stress components are zero except σ_y and let it be given that σ_y is constant throughout the body; for example, the body could be a straight bar subjected to tension. A freebody diagram of a stress element would then be as shown in Fig. 1.7a. By our sign convention each tension stress in Fig. 1.7a is a positive stress. If the bar were in compression the stresses would appear as shown in (b) and they would be negative. The sign convention is chosen so that a tension stress is positive and a compression stress is negative.

Let us now suppose that only tangential stress components are acting, say τ_{xy} and τ_{yx}, and that these are uniform throughout the body. A freebody diagram of the element would then appear as shown in Fig. 1.8. According to our sign convention, the four stresses shown are positive since they act either on a positive face in a positive direction, or on a negative face in a negative direction.

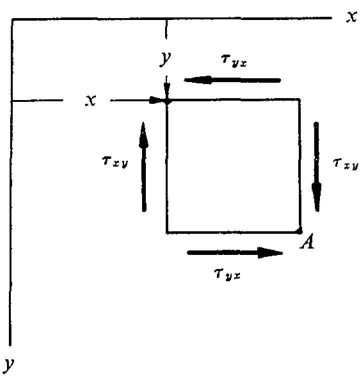


FIGURE 1.8

1-3 PROPERTIES OF STRESSES

Equality of Shear Stresses. The element at point x,y, shown in Fig. 1.8, is in equilibrium so far as the sum of the forces in the x-direction and in the y-direction are concerned, since the two $\tau_{xy}\, dy\, dz$ forces cancel each other, and the two $\tau_{yx}\, dx\, dz$ forces also cancel. The element must also be in equilibrium for moments. Taking moments about the corner A, the stress τ_{xy} multiplied by the area on which it acts gives the force $\tau_{xy}\, dy\, dz$, and this multiplied by dx gives its moment about A (clockwise). The stress τ_{yx} multiplied by the area upon which it acts gives a resultant force $\tau_{yx}\, dx\, dz$, and this multiplied by dy gives a counterclockwise moment $\tau_{yx}\, dx\, dz\, dy$. Equilibrium requires that the two moments balance each other

$$\tau_{xy}\, dy\, dz\, dx = \tau_{yx}\, dx\, dz\, dy$$

Dividing both sides of this equation by $dx\, dy\, dz$ gives

$$\tau_{xy} = \tau_{yx}$$

This equation states that the two in-plane components of shear stress at a point are always equal, that is,

$$\tau_{xy} = \tau_{yx}; \quad \tau_{yz} = \tau_{zy}; \quad \tau_{zx} = \tau_{xz} \tag{1.8}$$

Equation (1.8) is valid for a general state of stress. This may be verified by setting the sum of the moments equal to zero, canceling terms and dropping higher order infinitesimals. These equations are important for they effectively reduce by one-third the number of stress components that must be specified in order to prescribe the stress at a point. Instead of the nine different stresses of Eq. (1.7), the stress at a point is described by six different stresses

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} = \text{stress} \tag{1.9}$$

Hereafter, we shall not distinguish between τ_{xy} and τ_{yx} but shall write τ_{xy} for both.

The stress array of Eq. (1.9) describes the stress at a point with reference to the cartesian coordinates x, y, z. The stresses σ_x, τ_{xy}, and τ_{xz} act on a plane perpendicular to the x-axis which passes through the point x, y, z, and similarly the stresses σ_y, τ_{xy}, τ_{yz}, and σ_z, τ_{xz}, τ_{yz} act on planes perpendicular to the y and z axes respectively. Similar arrays may be written for the stress at a point which give the stresses with reference to any three coordinate directions. For example, the stress array in cylindrical coordinates, r, θ, and z with the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ shown in Fig. 1.9a is written

$$\begin{vmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{vmatrix} = \text{stress} \tag{1.9a}$$

The stress array in spherical coordinates r, φ, and θ with the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\varphi$, and $\mathbf{e}_\theta$ shown in Fig. 1.9b is

$$\begin{vmatrix} \sigma_r & \tau_{r\varphi} & \tau_{r\theta} \\ \tau_{r\varphi} & \sigma_\varphi & \tau_{\varphi\theta} \\ \tau_{r\theta} & \tau_{\varphi\theta} & \sigma_\theta \end{vmatrix} = \text{stress} \tag{1.9b}$$

A more concise notation for the stress at a point is given in Appendix I where it is shown that stress may be represented by a cartesian tensor of the second rank.

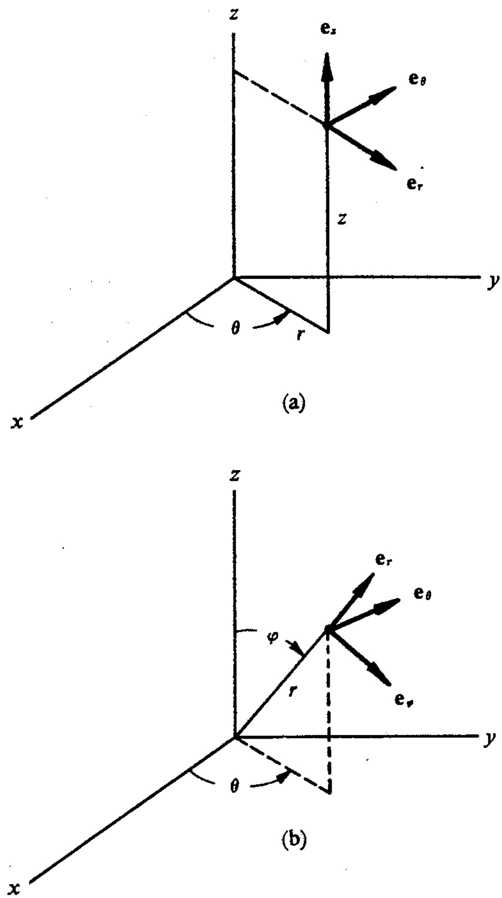


FIGURE 1.9

Principal Stresses. The stress at a point is described by a set of three stress vectors, and it might be asked whether, out of all possible sets, there are any that are particularly simple. There is one set, the so-called set of *principal stresses*, that is particularly simple and easily visualized. Let us consider a general state of stress at a point P as shown in Fig. 1.10a, where the stress vectors $\mathbf{s}_x$, $\mathbf{s}_y$, $\mathbf{s}_z$ make some arbitrary angles with the surfaces on which they act. As was shown in Section 1-2, the stress vector $\mathbf{s}_n$, on an inclined plane through the point P can be expressed in terms of the stresses $\mathbf{s}_x$, $\mathbf{s}_y$, $\mathbf{s}_z$ as

$$\mathbf{s}_n = -(\mathbf{s}_x l_{nx} + \mathbf{s}_y l_{ny} + \mathbf{s}_z l_{nz}) \tag{1.10}$$

By changing the orientation of the plane, that is, by changing l_{nx}, l_{ny}, and l_{nz}, the magnitude and direction of $\mathbf{s}_n$ can be made to change. By choosing the appropriate values of the direction cosines we can find a plane for which $\mathbf{s}_n$ is normal to the surface on which it acts. Referring to Fig. 1.10b we see that the condition for $\mathbf{s}_n$ to be normal to the n-plane is that it can be written in component form as

$$\mathbf{s}_n = s_n l_{nx}\mathbf{i} + s_n l_{ny}\mathbf{j} + s_n l_{nz}\mathbf{k} \tag{1.11}$$

Therefore, the right-hand side of Eq. (1.11) must be equal to the right-hand side of Eq. (1.10). If these two expressions are equated, the resulting vector equation can be solved for the principal stresses and the direction cosines* in terms of the components of $\mathbf{s}_x$, $\mathbf{s}_y$, $\mathbf{s}_z$. The vector equation is equivalent to three simultaneous scalar equations whose solution gives three different values of principal stress, $s_n = \sigma_1$, σ_2, σ_3, and the planes on which they act are *mutually perpendicular*. What is found, then, is that at any point in a body there is a particular orientation of stress element for which the stress vectors are normal to the faces on which they act. Every state of stress at a point is therefore equivalent to a set of three normal stresses acting on a cube of material as shown in Fig. 1.10c. Equation (1.9) can, therefore, always be put in the form

$$\begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{vmatrix} = \text{stress}$$

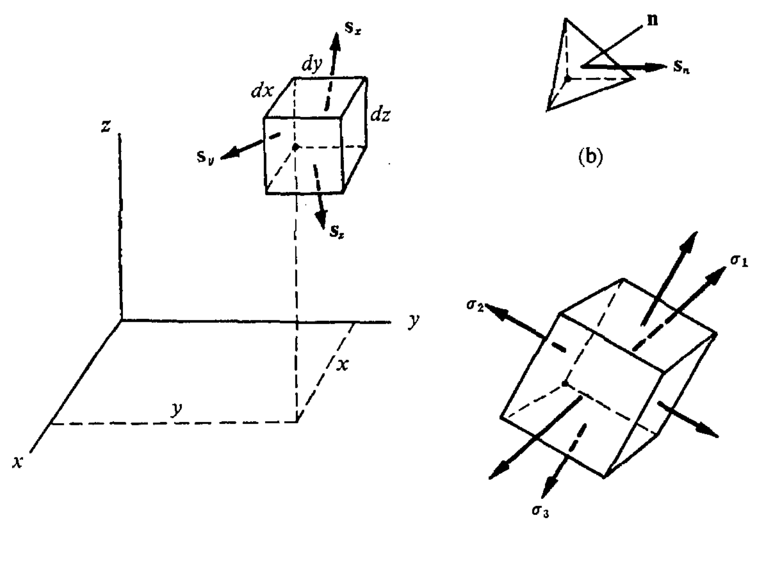


FIGURE 1.10

* The algebra of this procedure is given in Appendix I. It is analogous to finding the principal moments of inertia of a body.

It may be verified that the principal stresses have the following properties:

(a) One of the principal stresses is the largest normal stress on any plane through the point. By largest normal stress is meant largest in algebraic sense; that is, largest tension stress or smallest compression stress.

(b) One of the principal stresses is the smallest normal stress on any plane through the point; that is, one of the principal stresses is the smallest (positive) tension stress (or it may be the largest compression stress).

The use of principal stresses is very convenient when considering the effect of stresses on a material. By considering the principal stresses, every state of stress is seen to be similar to every other state of stress, the only difference being in the values of the three normal stresses σ_1, σ_2, σ_3 (and in the orientation of the element).

1-4 PROPERTIES OF PLANAR STRESS SYSTEMS

Many problems in stress analysis deal with bodies that may be considered to have a planar system of stresses. A *planar stress* system is defined to be one for which the normal stress σ_z is a principal stress throughout the body. In other words, the z-face of any element is acted upon only by σ_z and, therefore, τ_{xz} and τ_{yz} are zero throughout the body.

$$\begin{vmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{vmatrix} = \text{planar stress}$$

The simplest example of planar stress is when σ_z is zero. This is called *plane stress*.

$$\begin{vmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{vmatrix} = \text{plane stress}$$

In this case an element of the body has stresses σ_x, σ_y, τ_{xy} as shown in Fig. 1.11 and these stresses are constant over the thickness of the body; that is, the values of σ_x, σ_y, τ_{xy} are independent* of z. The special case of planar stress in which σ_z = constant may be considered to be a superposition of *plane stress* and the stress state in which the only non-zero stress is σ_z = constant.

* This follows from the fact that τ_{xz} and τ_{yz} are zero. It can be proved from the equilibrium equations and stress-strain relations. A necessary condition for plane stress in the plate is that the z-dimension be sufficiently small compared to the other dimensions.

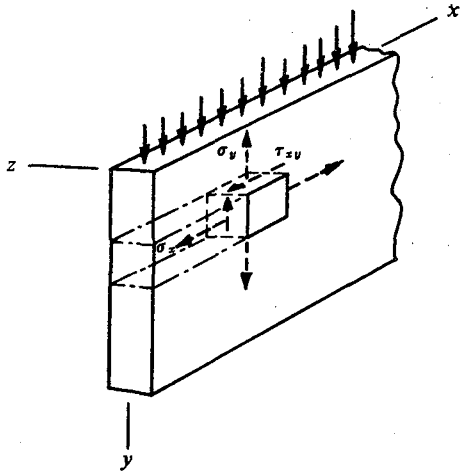
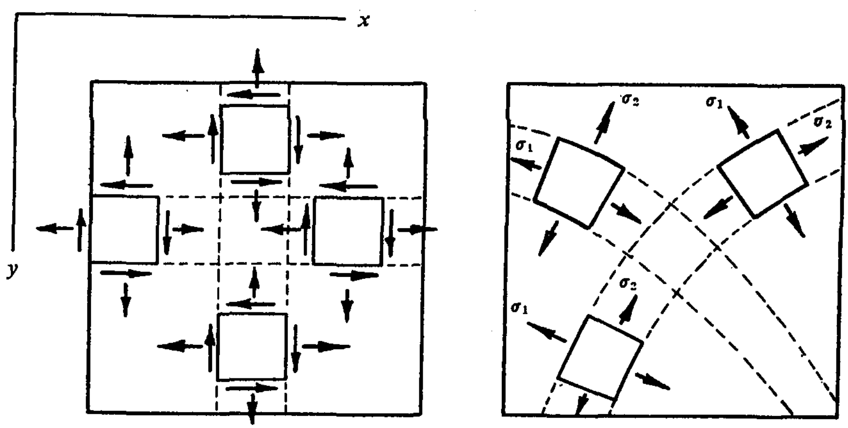


FIGURE 1.11

Another frequently encountered planar stress problem is the so-called plane strain problem which is described in Section 2-2.

Stress Field. When applied forces act upon a body stresses are generated throughout the body; that is, there is a state of stress at every point in the body. For example, σ_x will have a value at every point in the body and so also will σ_y, τ_{xy}, etc. The totality of stresses within a body is called a stress field by analogy with an electric field or a magnetic field. The existence of the stress field is indicated in Fig. 1.12 which shows a diagram of a thin plate under a condition of plane stress. In (a) the σ_x, σ_y, τ_{xy} stresses are indicated at various



(a) (b)

FIGURE 1.12

points. In (b), each element is oriented to show principal stresses, σ_1 and σ_2. The dotted lines in (b) are at every point tangent to the direction of the principal stresses at that point. These dotted lines are called stress trajectories by analogy with projectile motion, for if σ_1 were the velocity of a particle, then the dotted line would be the path described by the particle. The principal stress trajectories are, of course, orthogonal where they cross.

Transformation of Stresses. If a plate is in a condition of plane stress a point in the body will have only stresses σ_x, σ_y, τ_{xy}. Suppose the numerical values of these stresses are

$$\sigma_x = +10{,}000 \text{ psi}; \quad \sigma_y = -5000 \text{ psi}; \quad \tau_{xy} = +5000 \text{ psi*}$$

A freebody diagram of the stress element is then as shown in Fig. 1.13. Since

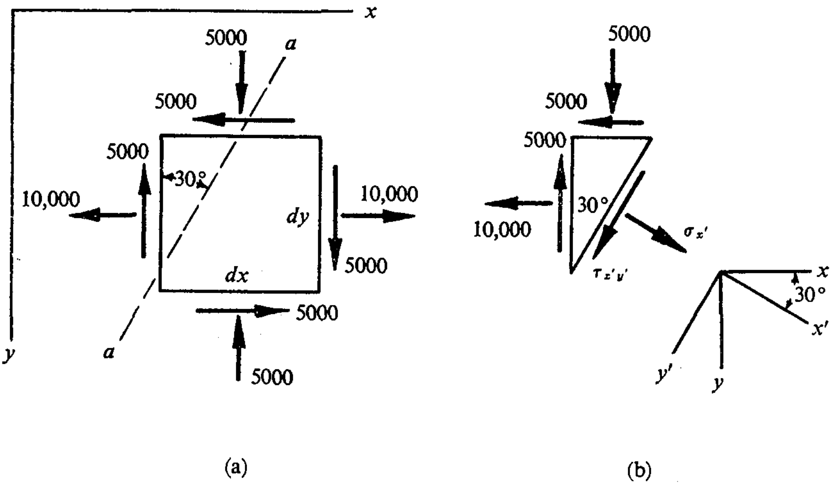


FIGURE 1.13

we are treating the stress at a point, we are not concerned with the infinitesimal variation of stress across the infinitesimal element. To determine the stress on the 30° plane indicated by the dotted line, we begin by drawing the freebody diagram of a corner of the element as shown in (b), where $\sigma_{x'}$ and $\tau_{x'y'}$ are the unknown stresses on the 30° plane. The element must be in equilibrium and this determines the values of $\sigma_{x'}(= +10{,}580)$ and $\tau_{x'y'}(= -4000)$. If we take a freebody of the lower left corner of the element in (a), as shown in Fig. 1.14a, we can determine from equilibrium that $\sigma_{y'} = -4580$ psi. In Fig. 1.14 are shown freebodies of the two stress elements $dx\,dy$ and $dx'\,dy'$. They both represent the same state of stress at the point P.

* A commonly used unit of stress, pounds per square inch is designated psi.

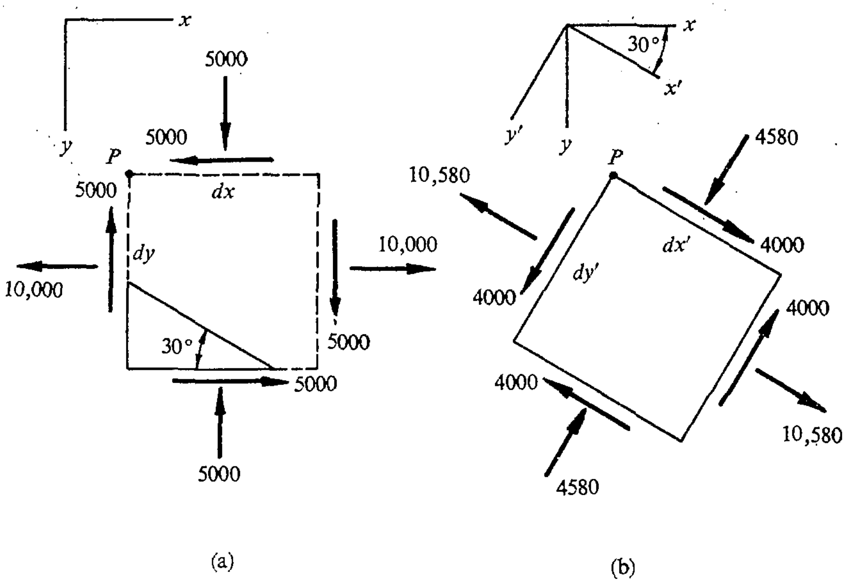


FIGURE 1.14

The general equations for the transformation of stresses in the x-y plane can be derived by writing the equations of equilibrium for the general state of stress shown in Fig. 1.15. Note that all of the stresses are shown as being positive, and θ is positive when measured from the x-axis toward the y-axis.

From equilibrium of the freebody (b) the following relations are obtained

$$\sigma_{x'} = \sigma_x \cos^2 \theta + 2\tau_{xy} \sin \theta \cos \theta + \sigma_y \sin^2 \theta \tag{1.12}$$

$$\tau_{x'y'} = \tau_{xy} \cos^2 \theta + (\sigma_y - \sigma_x) \sin \theta \cos \theta - \tau_{xy} \sin^2 \theta \tag{1.13}$$

The stress corresponding to $\sigma_{y'}$, is given by Eq. (1.12) when $(\theta + \pi/2)$ is put in place of θ. The foregoing equations are sometimes written in alternate forms by using trigonometric relations*

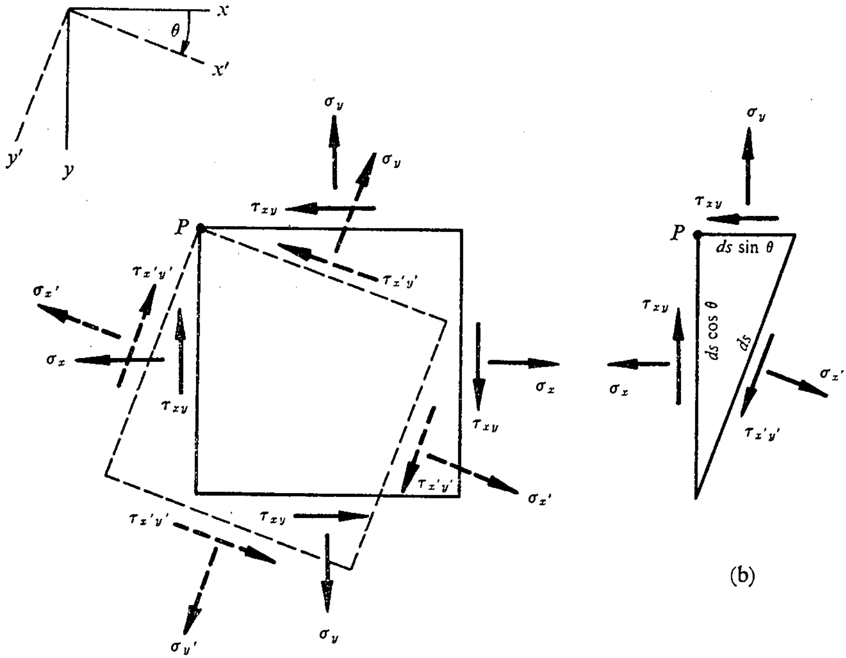
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \tag{1.14}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \tag{1.15}$$

where θ is measured from the x-axis to the x'-axis.

Particularly simple forms of the equations result when they are written in terms of the principal stresses. Consider the element shown in Fig. 1.16a,

* $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$; $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$; $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$.



(a)

(b)

FIGURE 1.15

where σ_1 and σ_2 are the principal stresses. The transformation equations then have the form

$$\sigma_{x'} = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\varphi \tag{1.16}$$

$$\tau_{x'y'} = -\frac{\sigma_1 - \sigma_2}{2} \sin 2\varphi \tag{1.17}$$

where φ is measured from the 1-axis to the x'-axis.

A graphical representation of Eqs. (1.16) and (1.17) is shown in Fig. 1.16b, where the stresses are plotted on σ-τ axes. This representation is called a Mohr's circle diagram for plane stress.* The stresses on the principal planes plot as points on the horizontal axis ($\tau = 0$) at the stress values σ_1 and σ_2. The stresses $\sigma_{x'}$, $\tau_{x'y'}$, on a plane at an angle φ from the plane of the principal stress σ_1, plot as a point on a circle whose center is at $(\sigma_1 + \sigma_2)/2$ as indicated in Fig. 1.16b. Note that the point $\sigma_{x'}$, $\tau_{x'y'}$, is on the circle at an angle of 2φ from the point representing the stress on the principal plane $\tau = 0$, $\sigma = \sigma_1$.

* O. Mohr (1835–1918), Professor of Engineering at Dresden Polytechnic.

This construction represents Eqs. (1.16) and (1.17) as may be verified. The stresses $\sigma_{x''}$ and $\tau_{x''y''}$ on a plane at $\varphi + 90°$ from the axis of maximum principal stress plot as a point on Mohr's circle at 180° from $\sigma_{y'}$ and $\tau_{x'y'}$. It follows that $\sigma_{x''} = \sigma_{y'}$, and $\tau_{x''y''} = -\tau_{x'y'}$ from our sign convention for shear stress. The relationship between the positive coordinate axes and the positive stress axes of Mohr's circle should be noted. They are chosen such that a clockwise rotation of the coordinate axes through φ, is represented on Mohr's circle as a clockwise rotation through 2φ. A simple method of finding the angle between the principal *axis* and the x'-*axis* from Mohr's circle is shown in Fig. 1.16c.

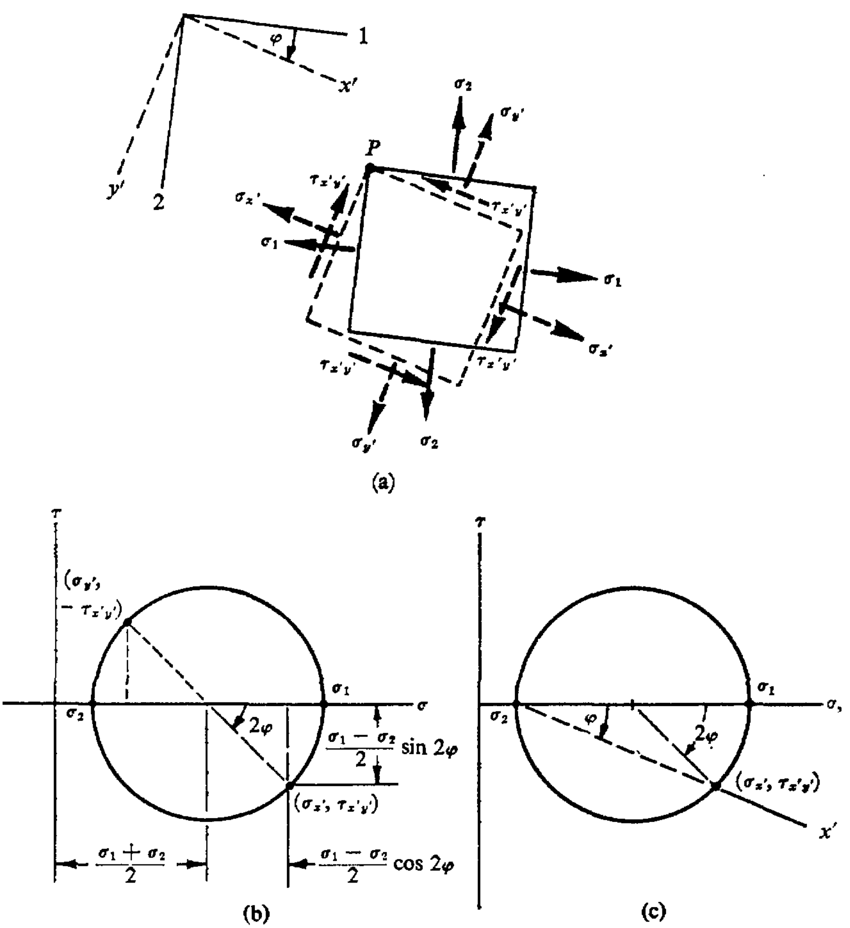
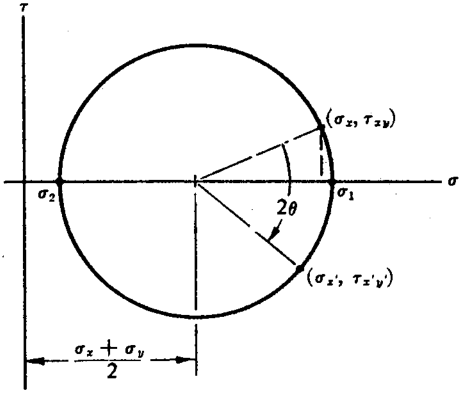
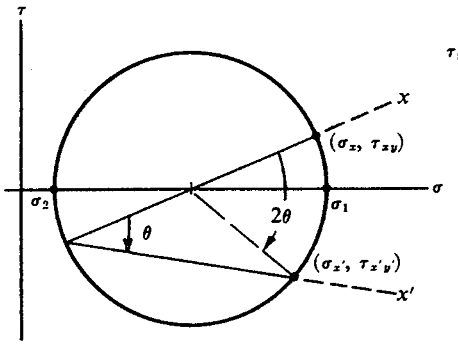


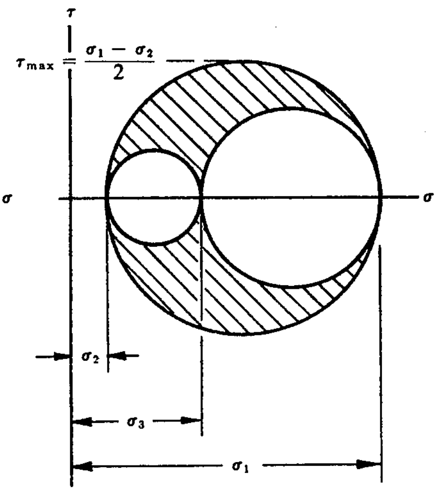
FIGURE 1.16

Equations (1.14) and (1.15) are also represented by Mohr's circle as shown in Fig. 1.17a. The construction of Mohr's circle for the general state of plane stress σ_x, σ_y, τ_{xy} can be deduced from this figure. The center of the circle is located at $\sigma = (\sigma_x + \sigma_y)/2$, $\tau = 0$. The state of stress σ_x, τ_{xy} on the x-face is plotted and the circle is then drawn. The state of stress, $\sigma_{x'}$, $\tau_{x'y'}$, on a plane normal to the x' axis lies on the circle at an angle 2θ from the point σ_x, τ_{xy}. The relative orientation between the x and x' axes is indicated in Fig. 1.17b.



(a)





(b) (c)

FIGURE 1.17

Mohr's circle is a convenient representation of the equations of transformation of stresses and makes their meaning apparent.

From the foregoing equations, or from Mohr's circle, we can deduce certain properties of plane stresses:

(a) The sum of the two normal stresses is a constant for all values of θ. This is sometimes stated by saying $(\sigma_{x'} + \sigma_{y'})$ is invariant.

(b) The maximum and minimum normal stresses occur when the element is so oriented that the shear stress is zero.

(c) The maximum planar shear stress, sometimes called the principal shear stress, occurs at 45° from the principal normal stresses.

(d) The plane of principal normal stresses is specified by

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{1.18}$$

and the plane of maximum planar shear stress is specified by

$$\tan 2\theta = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} \tag{1.19}$$

(e) If the shear stress on one face of the stress element is zero, the shear stresses on the other three faces are also zero.

(f) If the principal stresses are such that $\sigma_1 = \sigma_2$, then $\tau_{x'y'} = 0$ for all θ. This state of stress is called *pure biaxial compression* or *tension*.

(g) If the principal stresses are such that $\sigma_1 = -\sigma_2$, then at 45° from the principal plane the maximum planar shear stress has the value σ_1, and the stresses $\sigma_{x'}$ and $\sigma_{y'}$ are zero. This state of stress is called *pure shear*.

Mohr's circle may be used for the transformation of the general state of stress at a point

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$

when the stress element is rotated about the x, y, or z axis. A Mohr's circle for rotation of the stress element about each of the three principal axes is drawn in Fig. 1.17c. It may be shown by considering the general transformation of stress that the stress components for any rotation of the stress element lie within the shaded portion of the diagram. It is apparent from this that the maximum shear stress at a point is $\tau_{\max} = (\sigma_1 - \sigma_2)/2$ where σ_1 and σ_2 are the maximum and minimum principal stresses at that point.

The transformation of stresses for a general rotation of the stress element can also be derived by considering equilibrium of the stress element. Since more stress components and more faces of the stress element are involved, the resulting expressions are more complex. The general transformation of stresses is given in Appendix I.

Problems

1.9 Derive Eqs. (1.12) and (1.13) from the freebody diagram shown in Fig. 1.15b.

1.10 Show that Eqs. (1.14) and (1.15) are derivable from Eqs. (1.12) and (1.13).

1.11 Show that Fig. 1.17a is a geometrical representation of Eqs. (1.14) and (1.15).

1.12 Show:

$$\sigma_{x''} = \sigma_{y'}$$

$$\tau_{x''y''} = -\tau_{x'y'}$$

when the x''-axis is 90° from the x'-axis.

1.13 Draw Mohr's circle for:

(a) Pure uniaxial tension
(b) Pure uniaxial compression
(c) Pure biaxial tension
(d) Pure shear

1.14 Prove that $\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$; that is, prove that in a planar stress distribution the sum of the two normal stresses is an invariant.

1.15 Prove that in a plane stress distribution the principal stresses σ_1 and σ_2 are the maximum and minimum normal planar stresses.

1.16 Prove that the maximum planar shear stress occurs on planes that make 45° with the principal normal stresses, σ_1, σ_2.

1.17 Show that the plane of principal normal stress σ_1 makes an angle φ with the plane that is perpendicular to the x-axis, where $\tan 2\varphi = +2\tau_{xy}/(\sigma_x - \sigma_y)$. Show that the plane of maximum planar shear stress is specified by $\tan 2\varphi = -(\sigma_x - \sigma_y)/2\tau_{xy}$.

1.18 A plane stress distribution has principal stresses σ_1 and σ_2 with $\sigma_3 = 0$, where $\sigma_1 = -\sigma_2$. Deduce the maximum planar shear stress and the plane on which it acts.

1.19 The stress at a point is $\sigma_x = 8000$ psi, $\sigma_y = -9000$ psi, $\tau_{xy} = -4000$ psi. What are the $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$ stresses if the x'-axis makes an angle of $+60°$ with the $+x$-axis, and the z'-axis coincides with the z-axis?

1.20 The stress at a point is $\sigma_x = 0$, $\sigma_y = 0$, $\tau_{xy} = 10{,}000$ psi. What are $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$, if the x'-axis makes an angle of $+30°$ with the x-axis, and the z'-axis coincides with the z-axis?

1.21 The stress at a point is $\sigma_x = 5000$ psi, $\sigma_y = -5000$ psi, $\tau_{xy} = 5000$ psi. Determine $\sigma_{x'}$ and $\sigma_{y'}$ if the x'-axis makes an angle of $+20°$ with the x-axis and the z'-axis coincides with the z-axis. Does $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$?

1.22 The stress at a point is $\sigma_x = -6000$ psi, $\sigma_y = 4000$ psi, $\tau_{xy} = -8000$ psi. What is the maximum tension stress at the point? What is the maximum planar shear stress?

1.23 The stress at a point is $\sigma_x = 5000$ psi, $\sigma_y = 5000$ psi, $\tau_{xy} = \sigma_z = \tau_{xz} = \tau_{yz} = 0$. Show that the maximum shear stress in the material at the point is 2500 psi.

Decomposition of Stress at a Point. A plane state of stress at a point will, in general, have two normal stresses and a shearing stress. This can be considered to be composed of a superposition of the two states of stress shown in Fig. 1.18. The stress on element (a) is a pure biaxial tension and on element (b) there is a pure shear as may be verified from Mohr's circle.* A general state of stress is therefore equivalent to a pure planar compression or tension (a) plus pure shear (b). A similar statement can be shown to hold for a general nonplanar state of stress (see Eq. (1.37)).

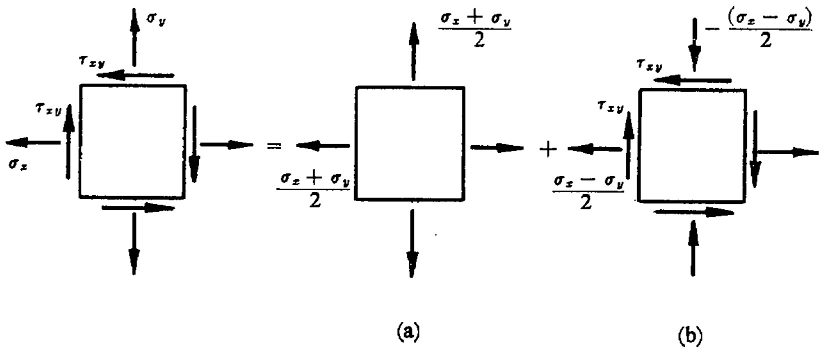


FIGURE 1.18

Problems

1.24 Prove that the state of stress formed by (b) of Fig. 1.18 is equivalent to the state of stress $\sigma_{x'} = \sigma_{y'} = 0$; $\tau_{x'y'} \neq 0$.

1.25 The plane stresses at a point are $\sigma_x = +8000$, $\sigma_y = +2000$, $\tau_{xy} = -4000$. Decompose this into a state of pure planar tension and pure planar shear.

1-5 DISPLACEMENTS AND STRAINS IN A CONTINUUM

Definition of a Continuum. A body is said to be a continuum if all physical properties are specified at every point and are continuous throughout the

* For a state of pure shear, the center of Mohr's circle lies on the origin of the σ-τ axes.

body. We may then define such quantities as density and stress as follows

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}, \quad \sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$

where ρ = density, V = volume, m = mass, and A = area. This is an idealization since no real materials have properties which are continuous over a distance approaching that of the spacing between atoms. When we treat a material as a continuum we are in effect saying that the average properties of a small volume of the material may be used to characterize that volume, and that adjacent small volumes do not have large differences in their average properties. The microscopic volumes used to define the properties have dimensions which are large compared to atomic spacings.

Displacements in a Continuum. Displacements of points in a body may result from deformation as well as from rigid body translation and rotation. The deformation may change the size of the body (dilatation) or change the shape of the body (distortion). For a complete description of the translation, rotation, dilatation, and distortion we must specify the displacement at each point in the body.

The location of an infinitesimal element in the unstressed body is designated as P in Fig. 1.19a, and as a consequence of the body being stressed and deformed, this element will move to a new position P'. The cartesian components of the displacement of the element (point x, y, z) are u, v, and w, and these components may vary from point to point in the body. We may describe the displacement components associated with a particular *element* of the body, by means of a *Lagrangian* or material coordinate system in which the x, y, z coordinates refer to a particular material element of the *unstressed* body. Displacement functions $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ then describe the displacement of this element which was at the point x, y, z *in the unstressed body*. The displaced point P' has the coordinates $x + u$, $y + v$, and $z + w$, and the displacement vector is written as

$$\mathbf{d} = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$

One of the common problems of stress analysis is to find expressions for u, v, and w that are such functions of x, y, and z that they will represent the correct displacement of every element in the stressed body. To relate the rotation, dilatation, and distortion to the displacement we shall consider how the components u, v, and w vary throughout the body. We shall restrict consideration to displacements that are continuous so that the derivatives of of the displacement exist at every point.

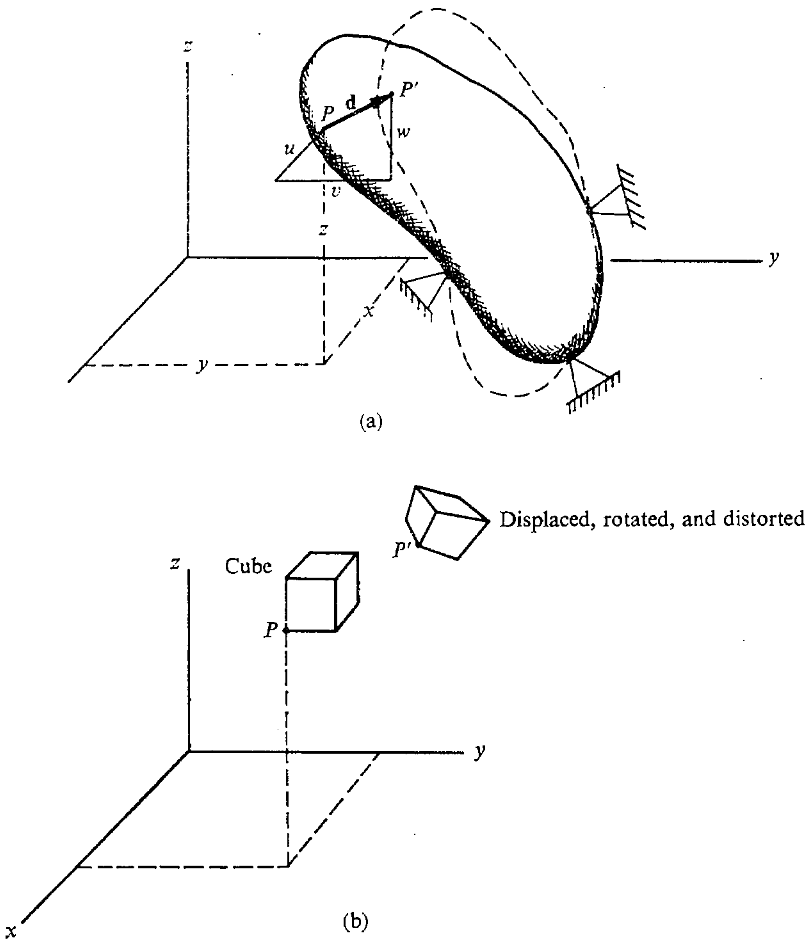


FIGURE 1.19

A difficulty is encountered in the use of a Lagrangian coordinate system because an element dx, dy, dz, such as that shown in Fig. 1.19b, changes its orientation and its size and shape when stressed so that the areas of its sides are changed. This should be taken into account when writing the equations of equilibrium. When earlier we wrote the equations of equilibrium of the stress at a point we did so for a *stressed* element dx, dy, dz, that is, we used a coordinate system in which the x, y, z coordinates located the displaced point P' and we examined the material that happened to be in the volume element. Such a coordinate system is a so-called *Eulerian* coordinate system in which x, y, z locates a point fixed in space, and dx, dy, dz is the element of volume that is at that fixed point in space. The treatment of stresses in the Eulerian coordinate system was uncomplicated by distortion of the element, which would not be true if we used Lagrangian coordinates. However, Lagrangian coordinates are much easier to employ when we try to describe the condition at the surface elements of a body.* The use of Eulerian coordinates is difficult when we write these conditions, since the coordinate boundaries of the deformed body are not known until the problem is solved. We see that there are serious difficulties associated with both Eulerian and Lagrangian coordinates. However, if the deformations of the body are sufficiently small, the Eulerian coordinates may be used without difficulty by taking the displaced boundaries of the body to be essentially the same shape as the boundaries of the undeformed body. For small deformations, the Lagrangian coordinates may also be used without difficulty since the shape of the deformed stress element will differ only negligibly from the shape of the undeformed element. Therefore, if we restrict consideration to small deformations, there is no substantial inconsistency in our approach if we claim to be using Lagrangian coordinates throughout.

Rotation. In Fig. 1.20a there is shown a small rigid-body rotation ω_z of an element about the point A. This rotation produces a displacement of the corner B relative to A as shown. More generally, for the element shown in

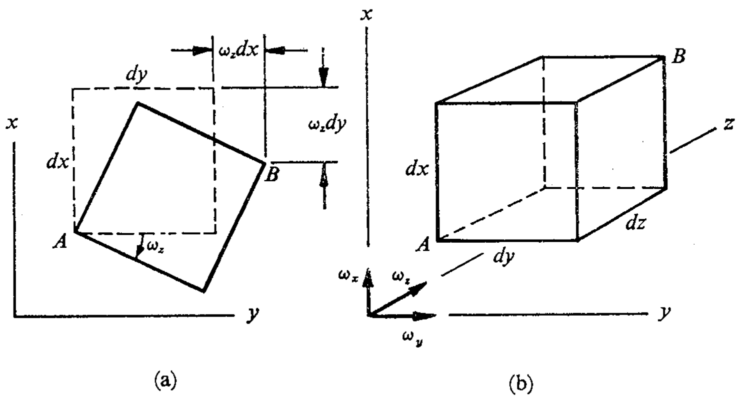


FIGURE 1.20

Fig. 1.20b, the displacement components of B relative to A due to rotations ω_x, ω_y, and ω_z are

$$\begin{aligned}(du)_{\text{rot}} &= \omega_y\,dz - \omega_z\,dy \\ (dv)_{\text{rot}} &= \omega_z\,dx - \omega_x\,dz \\ (dw)_{\text{rot}} &= \omega_x\,dy - \omega_y\,dx\end{aligned} \tag{1.20}$$

* The conditions at the surface elements, called the boundary conditions, are discussed in Section 2-2.

Small rotations are assumed in Eq. (1.20) so that $\sin \omega_z \cong \omega_z$, etc. In vector notation the rotations are related to the displacement by

$$\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} = \tfrac{1}{2} \operatorname{curl} \mathbf{d}$$

The components of rotation, as may be verified, are given in terms of the displacements by (see Prob. 1.27)

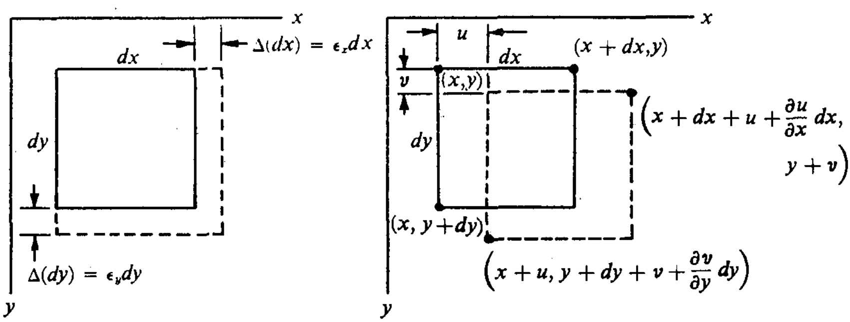
$$\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned} \tag{1.21}$$

Normal Strain and Dilatation. Each of the normal strains ϵ_x, ϵ_y, ϵ_z is defined as the incremental change in length divided by the original length. An extension is defined as a positive strain and a contraction as a negative strain. The positive strains shown in the element $dx\,dy$ in Fig. 1.21a, where each side has been extended a small fraction of its original length, are

$$\epsilon_x = \frac{\Delta(dx)}{dx}, \quad \epsilon_y = \frac{\Delta(dy)}{dy}, \quad \epsilon_z = \frac{\Delta(dz)}{dz} \tag{1.22}$$

The x- and y-coordinates of the corners of an element which has been strained in the x- and y-directions are shown in Fig. 1.21b. The element has been stretched in the x-direction by an amount

$$(du)_{\text{normal strain}} = \Delta(dx) = \left(x + dx + u + \frac{\partial u}{\partial x}\,dx\right) - (x + dx + u)$$



(a) (b)

FIGURE 1.21

Canceling terms, the following expression is found for du, and similar expressions for dv and dw

$$(du)_{\text{normal strain}} = \frac{\partial u}{\partial x}\,dx$$

$$(dv)_{\text{normal strain}} = \frac{\partial v}{\partial y}\,dy \tag{1.23}$$

$$(dw)_{\text{normal strain}} = \frac{\partial w}{\partial z}\,dz$$

The normal strains, ϵ_x, ϵ_y, ϵ_z at point (x, y) are therefore related to the displacements by:

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z} \tag{1.24}$$

The foregoing discussion assumes that the strains are small, $\epsilon_x \ll 1$, etc.

The dilatation e at a point is defined as the change in volume per unit volume

$$e = \frac{(dx + \epsilon_x\,dx)(dy + \epsilon_y\,dy)(dz + \epsilon_z\,dz) - dx\,dy\,dz}{dx\,dy\,dz}$$

For small strains the dilatation is given by the sum of the three normal strains

$$e = \operatorname{div}\mathbf{d} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \epsilon_x + \epsilon_y + \epsilon_z \tag{1.25}$$

Shearing Strains and Distortion. In addition to the extension and contraction of normal strains, the element can deform without changing the lengths of its sides. This deformation, called distortion, or pure shear, is illustrated in Fig. 1.22a where a top view of the element shows it deformed as a four-bar linkage would change shape. Note that the displacement and rotation are not involved in this definition of the distortion of the element. We see that the sides of the element have not changed in length, but that one diagonal has increased in length and one has decreased an equal amount (assuming that the deformations are very small). The deformation can thus be described as an extension $\epsilon_{x'}$ plus an equal contraction $\epsilon_{y'}$. It can also be described as an angular deformation, ϵ_{xy} of the side dx, plus an equal angular deformation, ϵ_{yx} of the side dy produced by the shearing strain of the element. The quantities ϵ_{xy} and ϵ_{yx} are called the xy-components of shearing strain of the element, and since no rotation is involved in the definition of shearing strain $\epsilon_{xy} = \epsilon_{yx}$.

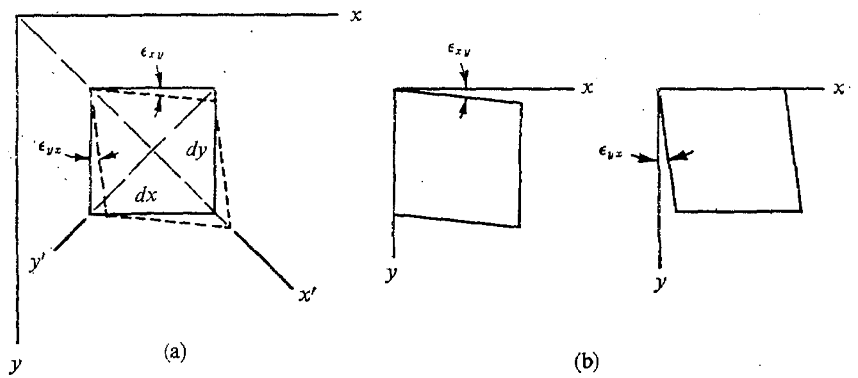


FIGURE 1.22

The shearing strain of the element in Fig. 1.22a can also be considered as being composed of the two *simple* shearing deformations as shown in Fig. 1.22b. The strains shown in Fig. 1.22 are positive. Similar to the sign convention for shearing stresses, the sign convention for positive shearing strain is that the positive side of the element moves in a positive direction; and thus a positive shearing strain is produced by a positive shearing stress. We say that the total shearing strain of an element is

$$\gamma_{xy} = \epsilon_{xy} + \epsilon_{yx}$$

where

$$\epsilon_{xy} = \tfrac{1}{2}\gamma_{xy}, \quad \epsilon_{yx} = \tfrac{1}{2}\gamma_{xy} \tag{1.26}$$

The quantity γ_{xy} represents the total angular change from the original right angle of the corner of the element. It is customary to use γ_{xy} to describe the shearing strain, so that when shearing strain is mentioned it is to be understood that it means γ_{xy} unless it is specifically stated to mean ϵ_{xy}.

In Fig. 1.23 is shown an element with only shearing strains. The displacements of the corners of the element are noted on the figure and from these

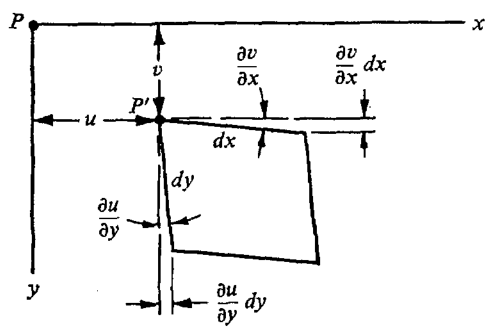


FIGURE 1.23

we see that the total change from a right angle at point P for small strains is $\partial u/\partial y + \partial v/\partial x$. The shearing strains are therefore expressed in terms of the displacements by

$$\begin{aligned}\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\end{aligned} \tag{1.27}$$

These expressions are appropriate, of course, only for small strains, $\gamma_{xy} \ll 1$, etc. The state of strain at a point is specified by the following set of strain components*

$$\begin{vmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_z \end{vmatrix} = \text{strain} \tag{1.28}$$

where $\epsilon_{yx} = \epsilon_{xy} = \frac{1}{2}\gamma_{xy}$, etc. As might be expected by analogy with stresses, the strains at a point involve three normal components and three shearing components. The strain array of Eq. (1.28) should be compared with the stress array of Eq. (1.9).

The relative displacement of points A and B in Fig. 1.20b due to the shearing strains alone, are

$$\begin{aligned}(du)_{\text{shear}} &= \epsilon_{xy}\,dy + \epsilon_{xz}\,dz = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) dz \\ (dv)_{\text{shear}} &= \epsilon_{yx}\,dx + \epsilon_{yz}\,dz = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) dz \\ (dw)_{\text{shear}} &= \epsilon_{zx}\,dx + \epsilon_{zy}\,dy = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) dy\end{aligned} \tag{1.29}$$

where it is assumed that the strains are small.

The total displacement of B relative to A in Fig. 1.20b, due to rotation, normal strain, and shearing strain is given by Eqs. (1.20), (1.23), and (1.29)

* When writing a strain array it is customary to use the ϵ_{xy}, ϵ_{xz}, etc., notation rather than γ_{xy}, γ_{xz}, etc. The reason for this is explained in Appendix I where the stresses and strains are shown to be tensors. The ϵ_{xy}, ϵ_{xz}, ϵ_{yz} are used in Mohr's circle for strains in the following section.

$$du = (du)_{\text{rot}} + (du)_{\text{normal}} + (du)_{\text{shear}}$$

$$du = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) dz - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dy + \frac{\partial u}{\partial x} dx + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) dz$$

Terms can be combined to obtain a simpler expression for du and corresponding expressions can be derived for dv and dw

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

These expressions are just the total differentials of u, v, and w. It is therefore evident that the rotations ω_x, ω_y, ω_z, normal strains ϵ_x, ϵ_y, ϵ_z, and shearing strains ϵ_{xy}, ϵ_{xz}, ϵ_{yz}, at each point, completely specify the variation in displacement throughout a continuum.

Transformations of Strains. A general state of plane strain (defined to have ϵ_z = constant and $\gamma_{xz} = \gamma_{yz} = 0$) can be considered to be composed of the three x, y strains shown in Fig. 1.24. The strain shown in (a) is a planar

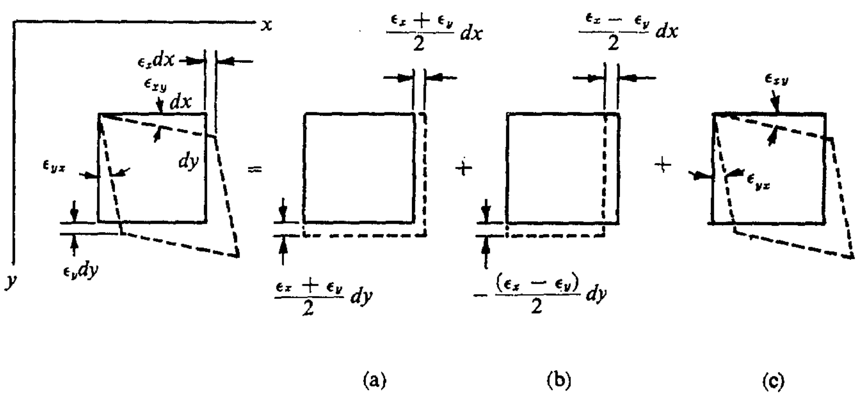


FIGURE 1.24

dilatation which is a uniform expansion in the xy-plane, that is, a circle of radius R in the xy-plane is expanded to a circle of radius $R + \frac{1}{2}(\epsilon_x + \epsilon_y)R$. The planar dilatation in (a) is independent of the orientation of the element.

Each normal strain in (a) is given by $\epsilon_a = (\epsilon_x + \epsilon_y)/2 = (\epsilon_{x'} + \epsilon_{y'})/2$. The strains shown in (b) and (c) are pure distortional strains. That (b) is a pure distortion can be shown by transforming (c) into normal strains equivalent to (b) as follows. In Fig. 1.25a is shown the pure shearing strain (c) which shortens the semi-diagonal AB by an amount

$$\Delta AB = \frac{\epsilon_{xy}\,dx}{\sqrt{2}}$$

The normal strain in the y'-direction is thus a contraction

$$\epsilon_{y'} = -\frac{\Delta AB}{AB} = -\frac{\epsilon_{xy}\,dx}{\sqrt{2}}\frac{\sqrt{2}}{dx} = -\epsilon_{xy}$$

Similarly the strain in the x'-direction is an extension

$$\epsilon_{x'} = +\frac{\Delta AC}{AC} = +\epsilon_{xy}$$

A pure shearing strain (Fig. 1.25a) can thus be resolved into a pair of equal and opposite normal strains at 45°, as shown in Fig. 1.25b.

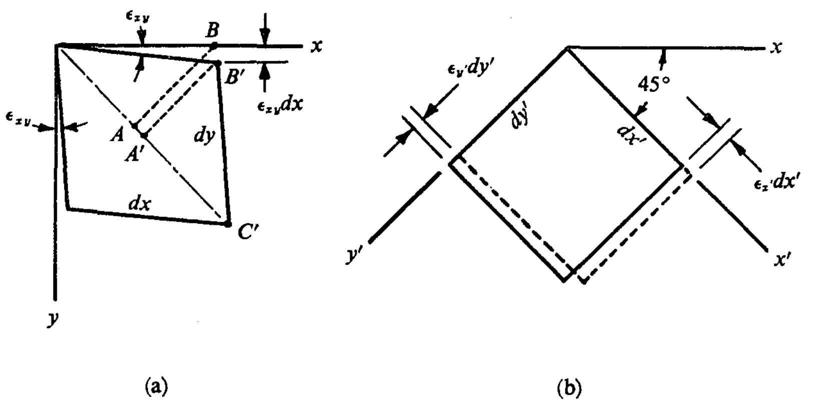


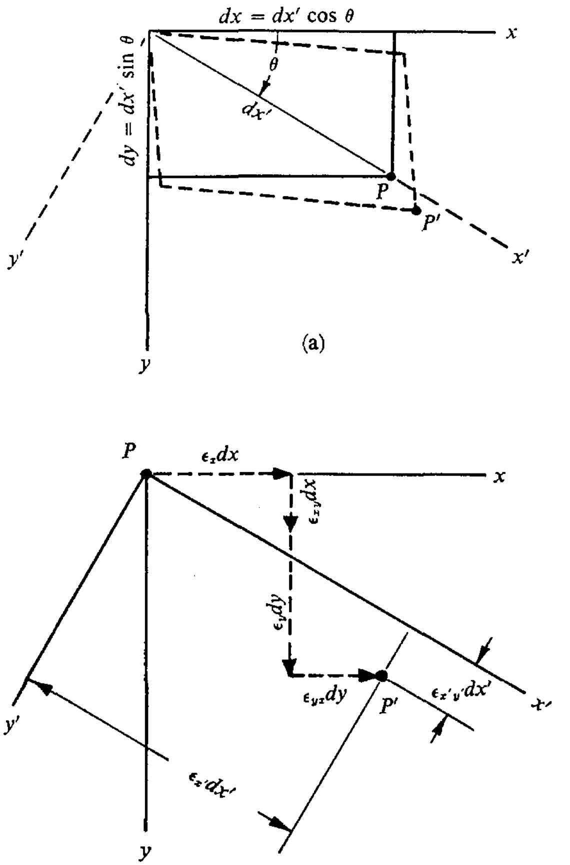
FIGURE 1.25

The *principal normal strains* are obtained when the element is so oriented that the shearing strain is zero. There is an exact correspondence between strains and stresses, and strains transform according to the same rules as stresses. This may be shown as follows. Consider the element shown in Fig. 1.26a, with a diagonal of length dx', and sides of length $dx = \cos\theta\,dx'$ and $dy = \sin\theta\,dx'$. The strains ϵ_x, ϵ_y, ϵ_{xy}, and ϵ_{yx} move the point P to point P' and when referred to the x'- and y'-axes the strains must give the same displacement. Consider the component of displacement along x' which is $\epsilon_{x'}\, dx'$. This displacement is made up of the components in the x- and y-directions shown in Fig. 1.26b where it is seen that

$$\epsilon_{x'}\, dx' = \epsilon_x\, dx \cos\theta + \epsilon_{xy}\, dx \sin\theta + \epsilon_y\, dy \sin\theta + \epsilon_{yx}\, dy \cos\theta$$

which, after simplification gives

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\epsilon_{xy} + \epsilon_{yx}}{2}\sin 2\theta$$



(b)

FIGURE 1.26

Treating the component of displacement $\epsilon_{x'y'}\, dx'$ in a similar manner gives

$$\epsilon_{x'y'} = \frac{\epsilon_{xy} - \epsilon_{yx}}{2} + \frac{\epsilon_{xy} + \epsilon_{yx}}{2}\cos 2\theta - \frac{\epsilon_x - \epsilon_y}{2}\sin 2\theta$$

We note that the shearing strains are defined so that $\epsilon_{xy} = \epsilon_{yx}$, and therefore, the transformation equations become

$$\begin{aligned} \epsilon_{x'} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \epsilon_{xy} \sin 2\theta \\ \epsilon_{x'y'} &= -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \epsilon_{xy} \cos 2\theta \end{aligned} \tag{1.30}$$

The transformation equations in terms of the principal normal strains ϵ_1 and ϵ_2 are

$$\begin{aligned} \epsilon_{x'} &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\varphi \\ \epsilon_{x'y'} &= -\frac{\epsilon_1 - \epsilon_2}{2} \sin 2\varphi \end{aligned} \tag{1.31}$$

where φ is measured from the 1-axis to the x'-axis. These equations should be compared with the equations of stress transformation, (1.14) through (1.17), where it will be seen that they have the same form. Everything that was concluded about the transformation of stresses applies to the transformation of strains. The transformation equations have the same form for strains as they do for stresses and Mohr's circle is the same. It should be noted that it is the strains ϵ_{xy} that are transformed, not γ_{xy}. The general three-dimensional equations of transformation of strains are given in Appendix I.

Problems

1.26 Show that Eqs. (1.20) represent the displacements due to rotations ω_x, ω_y, ω_z. Are these equations valid for large values of ω_x, ω_y, and ω_z?

1.27 Show geometrically that Eqs. (1.21) give the rotations ω_x, ω_y, and ω_z.

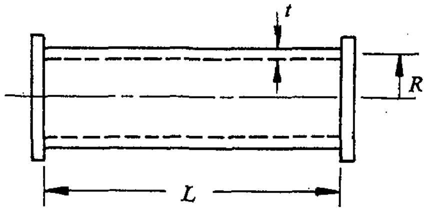
1.28 Show geometrically that γ_{yz} in Eq. (1.27) represents the change in angle between dy and dz, Fig. 1.20b.

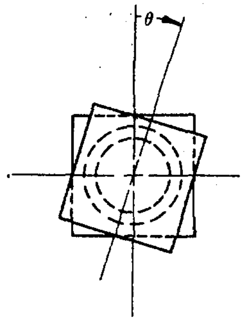
1.29 Derive an expression for the change in volume of an element of unit volume subjected to a small pure shear strain γ_{xy}.

1.30 Show geometrically how $(dv)_{\text{shear}}$ is related to ϵ_{yx} and ϵ_{yz}.

1.31 Carry out the steps used to obtain Eqs. (1.30).

1.32 A steel tube of circular cross section with a thin wall has flat plates welded to its ends. The top plate is then rotated through an angle θ with respect to the bottom plate as shown. Relate the strain in the tube to the angle and the tube dimensions.





1.33 A thin-walled tube is subjected to internal pressure, and the radius is increased by ΔR. What is the circumferential strain?

1.34 Derive the expression including higher-order terms for the change in volume of an element whose unstressed volume is unity.

1.35 The planar strains at a point are $\epsilon_x = +0.001$, $\epsilon_y = -0.0005$, $\epsilon_{xy} = +0.0004$. What are the principal normal strains?

1.36 The planar strains at a point are $\epsilon_x = 0.005$, $\epsilon_y = 0.001$, $\gamma_{xy} = 0$. What is the maximum shearing strain $\gamma_{x'y'}$ and what is the orientation of the x', y'-axes?

1.37 Prove that for planar strains $\epsilon_x + \epsilon_y = \epsilon_{x'} + \epsilon_{y'}$, for all x', y'.

1-6 RELATIONS BETWEEN STRESSES AND STRAINS

When a material is stressed it undergoes deformation and thus strains are produced. The strains will depend upon the state of stress in the body, its stress history, and the physical properties of the material of which the body is formed. There exist many materials with widely differing properties; for example, steel, copper, lead, wood, concrete, plastics, putty, gelatin, etc. We shall concern ourselves only with a few of the properties common to engineering materials. The properties of materials such as putty, gelatin, etc., may be found in publications on rheology.

Isotropy. A material is said to be *isotropic** if its properties are independent of direction. This can best be explained by considering some nonisotropic materials. Wood is nonisotropic as it is much softer normal to the grain as shown in Fig. 1.27a, than parallel to the grain as shown in (b).

Single crystals of most metals are anisotropic, having different properties in the different crystallographic directions. However, polycrystalline aggregates of these anisotropic metals exhibit isotropic properties provided that the individual crystals are randomly oriented, and that the crystal size is small

* From Greek *isos*: equal, and *tropos*: direction.

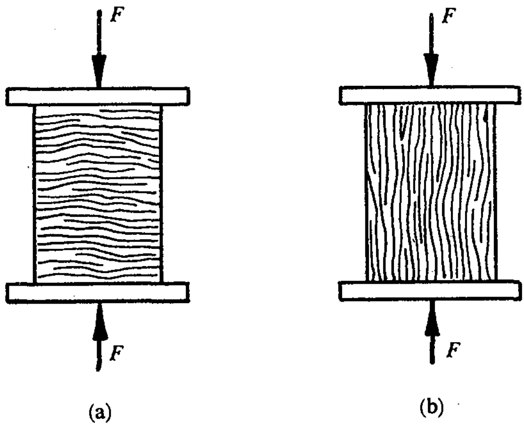


FIGURE 1.27

compared to the sample tested. Mechanical working, such as cold rolling of sheet stock, produces preferential grain orientations, and somewhat different properties are found parallel and perpendicular to the direction of rolling. However, for most applications metal bars, sheets, etc., are assumed to be isotropic.

Homogeneity. A body is said to be *homogeneous* if its properties are the same at every point in the body. If a piece of wood is glued to a piece of steel, the resulting body is nonhomogeneous. A reinforced concrete beam is also a nonhomogeneous body.

Elasticity. A body is said to be elastic if it returns to its original unloaded shape when the applied forces are removed. If the body is part of a conservative system, that is, if no energy is dissipated when the body is slowly loaded and unloaded, then there is an explicit functional relation between the stresses and the resulting strains. The simplest is a linear relation between the stresses and the strains. Such a material is said to be *linearly elastic*. A steel coilspring is an example of a body having a linear relation between the stresses and strains, as well as having a linear relation between the applied force and the elongation of the spring. Some materials are nonlinearly elastic; rubber is an example of this as its stiffness increases with increasing strain.

Visco-Elasticity. A material is said to be visco-elastic if the stresses are dependent upon the rates of stress and strain as well as upon the strains themselves (energy is dissipated during straining).* A visco-elastic solid returns to its initial undeformed condition when unloaded. A visco-elastic fluid is a viscous fluid which has elastic deformations under the stress. If the viscosity is very high the material may appear more like a solid than a fluid.

Plasticity. A material is said to be plastic if, being unbroken, it does not return to its original shape when unloaded. A highly plastic material is putty. For very small pressures putty behaves essentially as if it were linearly elastic, but for larger pressures it is squeezed out of shape and remains that way when the pressure is removed. Lead is also highly plastic. A low-carbon steel is also quite plastic under high stresses; on the other hand, at very low temperatures it will fail in a brittle manner without exhibiting plastic deformations. At sufficiently high temperatures all solids become plastic. Chalk, rock, concrete, and glass are nonplastic, brittle materials at room temperature. Analysis of the behavior of plastic materials is discussed in Chapter 8.

Hooke's Law. A linear relation between stress and strain is called Hooke's Law. In its most general form, a linear stress-strain relation would mean that each stress, for example, σ_x, is a linear function of each of the strains, ϵ_x, ϵ_y, etc., and vice versa

$$\epsilon_x = C_1\sigma_x + C_2\sigma_y + C_3\sigma_z + C_4\tau_{yz} + C_5\tau_{xz} + C_6\tau_{xy}$$

The values of the *elastic coefficients* C_1, C_2, etc. depend upon the properties of the material and must be determined experimentally. In the particular case of a linearly elastic, homogeneous, and isotropic body, certain of the coefficients are zero and the stress-strain relations have the form (derived in

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\epsilon_y &= \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} & \gamma_{yz} &= \frac{\tau_{yz}}{G} \qquad (1.32) \\
\epsilon_z &= \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} & \gamma_{zx} &= \frac{\tau_{zx}}{G}
\end{aligned}$$

Appendix I) where the constant E is called *Young's modulus*† or the *modulus of elasticity;* the constant ν is called *Poisson's ratio*;‡ and G is called the *shearing modulus of rigidity*.

* The springs and shock absorbers on an automobile form a visco-elastic system. Analysis of the behavior of visco-elastic material is discussed in Chapter 8.

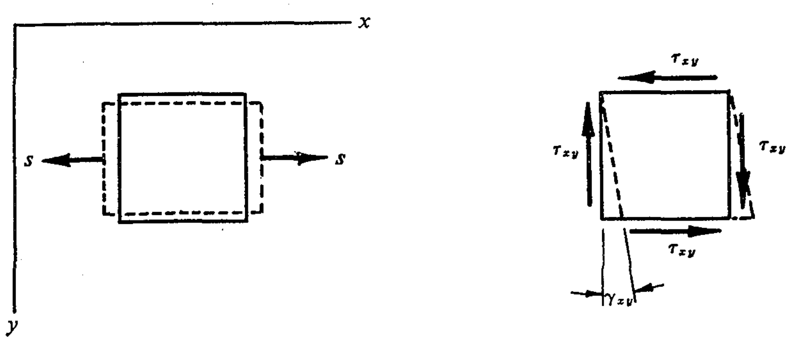
† Thomas Young (1773–1829) in his book, *A Course of Lectures on Natural Philosophy and the Mechanical Arts* (1807), defined the modulus of elasticity.

‡ S. Poisson (1781–1840) published many papers in the theory of elasticity, chiefly dealing with problems of vibrating beams and plates.

The physical meaning of E and ν are made clear by considering the cube shown in side view in Fig. 1.28a. The only stress acting is a uniform $\sigma_x = s$. The effect of this tension is to stretch the body by an amount

$$\Delta L_x = \epsilon_x L = \frac{\sigma_x}{E} L = \frac{sL}{E}$$

If the stress s is doubled, the elongation is doubled, etc. It is an experimentally observed fact that when a linearly elastic body is stretched thus in the x-direction it contracts in the y- and z-directions. If the stress s is doubled, the



(a) (b)

FIGURE 1.28

contraction is doubled. The ratio of the contractive strains ϵ_y, ϵ_z to the tensile strain ϵ_x is called Poisson's ratio and is denoted by ν. Poisson's ratio for steel is approximately 0.3. For an incompressible material, $\nu = 0.5$. For most engineering materials Poisson's ratio is in the range $0.15 < \nu < 0.35$.

If the body is subjected to a uniform shearing stress τ_{xy} as shown in (b) it will deform through the angle γ_{xy}. This might be, for example, a deformed element in a twisted steel tube. The linear relation between shearing stress and shearing strain in Eqs. (1.32) gives

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

It is seen that the shearing modulus G must be related to E and ν for the pure shear, $\tau_{xy} = s$, shown in Fig. 1.29b is the same as the state of stress, $\sigma_{x'} = +s$, $\sigma_{y'} = -s$, shown in Fig. 1.29a The state of strain is also the same for the two elements, and from Mohr's circle for strains it is seen that $\epsilon_{x'} = +\gamma_{xy}/2$, $\epsilon_{y'} = -\gamma_{xy}/2$ and, hence,

$$\epsilon_{x'} = +\frac{\gamma_{xy}}{2} = +\frac{\tau_{xy}}{2G} = +\frac{s}{2G}$$

$$\epsilon_{y'} = -\frac{\gamma_{xy}}{2} = -\frac{\tau_{xy}}{2G} = -\frac{s}{2G}$$

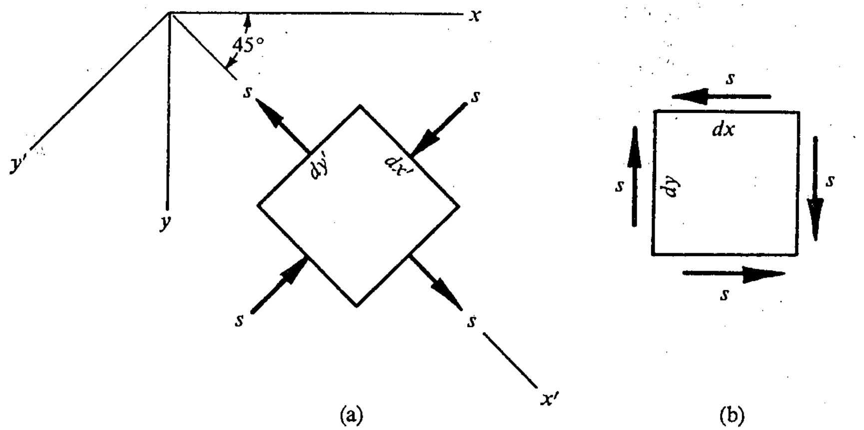


FIGURE 1.29

On the other hand, from Hooke's law

$$\epsilon_{x'} = \frac{\sigma_{x'}}{E} - \nu\frac{\sigma_{y'}}{E} = \frac{s(1+\nu)}{E}$$

$$\epsilon_{y'} = \frac{\sigma_{y'}}{E} - \nu\frac{\sigma_{x'}}{E} = -\frac{s(1+\nu)}{E}$$

The two ways of computing the normal strains must give the same result, so

$$\frac{s}{2G} = \frac{s(1+\nu)}{E}$$

or

$$G = \frac{E}{2(1+\nu)} \tag{1.33}$$

The elastic properties of an isotropic material are thus completely described by two material constants, for example E and ν. Additional material constants are required to describe nonisotropic materials, as discussed in Appendix I.

Hooke's law can be stated in an alternative form by inverting Eqs. (1.32) to give the stresses in terms of the strains

$$\begin{aligned}
\sigma_x &= (\lambda + 2G)\epsilon_x + \lambda\epsilon_y + \lambda\epsilon_z \qquad & \tau_{xy} &= G\gamma_{xy} \\
\sigma_y &= (\lambda + 2G)\epsilon_y + \lambda\epsilon_z + \lambda\epsilon_x \qquad & \tau_{yz} &= G\gamma_{yz} \\
\sigma_z &= (\lambda + 2G)\epsilon_z + \lambda\epsilon_x + \lambda\epsilon_y \qquad & \tau_{zx} &= G\gamma_{zx}
\end{aligned} \tag{1.34}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

The coefficient λ is called Lamé's constant.*

Another way of looking at the stress-strain relations is as follows. An element of material whose sides have lengths dx, dy, dz has a volume $(dx\,dy\,dz)$. After the stressing the sides have lengths $dx(1 + \epsilon_x)$, $dy(1 + \epsilon_y)$, and $dz(1 + \epsilon_z)$, and the net change in volume is

$$dx\,dy\,dz(1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z) - dx\,dy\,dz = (\epsilon_x + \epsilon_y + \epsilon_z)dx\,dy\,dz$$

where small quantities of higher order have been neglected. The change in volume per unit volume is thus given by the sum of the three normal strains, and adding together the first three of Eqs. (1.32), we obtain

$$e = (\epsilon_x + \epsilon_y + \epsilon_z) = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \qquad (1.35)$$

The volume change (dilatation) of the element is thus determined by the sum of the three normal stresses. Letting $\bar{\epsilon}$ represent the average normal strain and $\bar{\sigma}$ represent the average normal stress

$$\bar{\epsilon} = \tfrac{1}{3}(\epsilon_x + \epsilon_y + \epsilon_z)$$

$$\bar{\sigma} = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

we may write Eq. (1.35) in the form

$$\bar{\epsilon} = \frac{1 - 2\nu}{E}\bar{\sigma} = \frac{1}{C}\bar{\sigma} \qquad (1.36)$$

The coefficient C is called the modulus of compression. Since the change in volume of a given element is independent of the orientation of the coordinate axes, the average strain $\bar{\epsilon}$, and the average stress $\bar{\sigma}$, at a point in a body are invariant; that is, their values are the same for all orientations of the coordinate axes with respect to which the stresses and strains are given.

The stress at a point, shown with principal stresses in Fig. 1.30 may be thought of as the superposition of the dilatational stress (a) and the distortional stress (b). As the sum of the three normal stresses in (b) is equal to zero these stresses produce no volume change and, hence, they produce only distortion.† The stress and strain at a point can, therefore, be thought of as

* G. Lamé (1795–1870). His book *Leçons sur la Théorie Mathématique de l'Elasticité des Corps Solides*, was the first book on the theory of elasticity.

† Since these stresses represent the deviation from a pure dilatational state of stress, they are sometimes called deviator stresses.

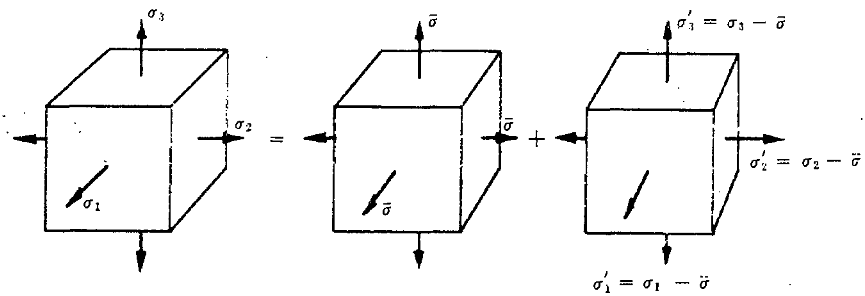


FIGURE 1.30

being a pure volume change with no distortion of shape, plus a pure distortion with no change of volume

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} = \begin{vmatrix} \bar{\sigma} & 0 & 0 \\ 0 & \bar{\sigma} & 0 \\ 0 & 0 & \bar{\sigma} \end{vmatrix} + \begin{vmatrix} \sigma_x' & \tau_{xy}' & \tau_{xz}' \\ \tau_{xy}' & \sigma_y' & \tau_{yz}' \\ \tau_{xz}' & \tau_{yz}' & \sigma_z' \end{vmatrix}$$

$$\begin{vmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_z \end{vmatrix} = \begin{vmatrix} \bar{\epsilon} & 0 & 0 \\ 0 & \bar{\epsilon} & 0 \\ 0 & 0 & \bar{\epsilon} \end{vmatrix} + \begin{vmatrix} \epsilon_x' & \epsilon_{xy}' & \epsilon_{xz}' \\ \epsilon_{xy}' & \epsilon_y' & \epsilon_{yz}' \\ \epsilon_{xz}' & \epsilon_{yz}' & \epsilon_z' \end{vmatrix} \tag{1.37}$$

where $\sigma_x' = \sigma_x - \bar{\sigma}$, $\tau_{xy}' = \tau_{xy}$, etc. With this way of decomposing the state of stress and strain, the stress-strain relations of Eqs. (1.34) are sometimes written

$$\bar{\sigma} = C\bar{\epsilon} \qquad \text{dilatation}$$

$$\left.\begin{aligned} \sigma_x' &= 2G\epsilon_x'; & \sigma_y' &= 2G\epsilon_y'; & \sigma_z' &= 2G\epsilon_z' \\ \tau_{xy}' &= 2G\epsilon_{xy}'; & \tau_{yz}' &= 2G\epsilon_{yz}'; & \tau_{zx}' &= 2G\epsilon_{zx}' \end{aligned}\right\} \text{distortion} \tag{1.38}$$

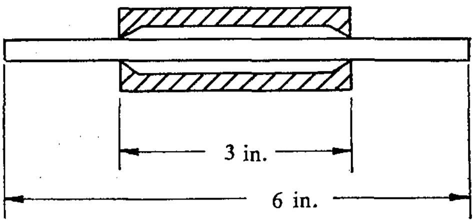
where $\epsilon_x' = \epsilon_x - \bar{\epsilon}$, etc., and $\epsilon_{xy}' = \epsilon_{xy}$, etc. These equations exhibit the effect of the stress at a point in producing dilatation and distortion.

Problems

1.38 Prove, in detail, that $\lambda = \nu E/(1 + \nu)(1 - 2\nu)$.

1.39 Is the orientation of the principal stresses the same as that of the principal strains in (a) an isotropic material, (b) an anisotropic material?

1.40 A pressure chamber is constructed with two coaxial holes as shown. A steel rod passes through frictionless seals in the holes and the chamber is pressurized at 50,000 psi. What change in length of the rod occurs?



1.41 The state of stress at a point is $\sigma_x = +10{,}000$ psi, $\sigma_y = -4000$ psi, $\sigma_z = +6000$ psi, $\tau_{xy} = +5000$ psi. $\tau_{yz} = -3000$ psi, $\tau_{zx} = 0$. Decompose this into a set of pure dilatational stresses and a set of pure distortional stresses.

1.42 What is the change in volume per unit volume for the stresses of Prob. 1.41 if $E = 30 \times 10^6$ and $G = 11.5 \times 10^6$ psi?

1.43 Show that the distortion stress-strain relations are $\sigma'_x = 2G\epsilon'_x$, etc.

1.44 A straight tube of circular cross section has a length L, an inner radius R, and wall thickness t. It is subjected to a uniform axial tension σ_x. What is the change in internal volume of the tube?

1.45 Cork has a Poisson's ratio that is essentially zero so that under axial tension or compression there is no lateral contraction or extension. What is the largest possible value of Poisson's ratio?

1.46 A surface of a stressed, isotropic body lies in the xy-plane. At a point on the surface where there are no applied loads acting, the strains ϵ_x, ϵ_y, and γ_{xy} are measured. Write the arrays for the state of stress and strain at the point in terms of the measured strains.

Strain-Measurements. If a material is isotropic, its elastic properties are described by the two constants E and G. These physical constants can be measured easily in the laboratory. If a bar is pulled in tension in a testing machine, its elongation per inch of length, when correlated with the applied force, will determine the value of E. The value of G can be determined by putting a thin, circular tube in torsion and measuring the twist per unit length produced by an applied twisting moment. If a material is neither isotropic nor linearly elastic, the experimental determination of its properties is much more difficult.

The strains on the surface of a body are commonly measured by means of bonded strain gages whose electrical resistance is sensitive to strain. These are described in Appendix II.

The strains in plates with plane stress systems can also be measured optically using models made of a transparent plastic, and passing polarized light through them. This so-called photoelastic method of measuring strains is described in Appendix III.

Average values of elastic moduli for some engineering materials are:

	E in psi	G in psi	ν
Steel	30,000,000	11,500,000	0.29
Aluminum	10,500,000	3,800,000	0.33
Copper	17,500,000	6,300,000	0.34
Brass	15,000,000	5,500,000	0.34
Titanium	16,000,000	6,000,000	0.34

The actual values of E, G, and ν may vary by 5% or so from the values listed depending upon the actual constitution of the material and the method of manufacture of the test specimen.

Stress-Strain Diagrams. The uniaxial stress-strain properties of a material are usually described by a diagram that gives the results of a tension test on a bar of the material. If the x-coordinate is taken to lie along the axis of the test specimen, the stress-strain diagram is a plot of the measured σ_x vs. ϵ_x. Figure 1.31 shows idealized stress-strain diagrams that describe the following types of materials:

(a) linearly elastic
(b) nonlinearly elastic
(c) elastic-plastic
(d) partly plastic

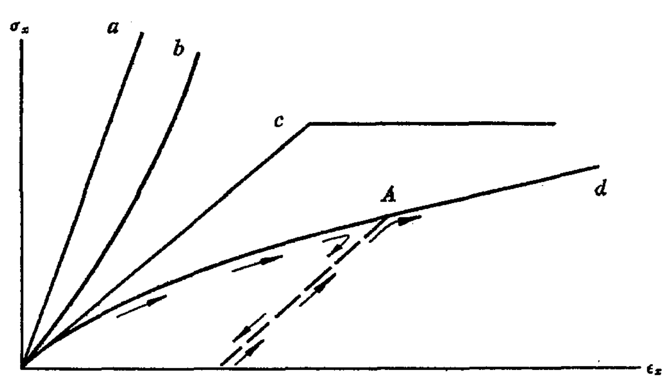


FIGURE 1.31

The two elastic curves represent both the stressing and the unstressing of the material but the elastic-plastic and partly plastic curves represent only the stressing of the material. The unstressing does not follow these curves. For example, if the partly plastic material is stressed to point A on the curve and the stress is then decreased, the strain follows along the line AB and at zero stress there is a permanent strain OB. Upon restressing, the strain goes from B to A and then proceeds along the curve. The fact that an increasing stress is required to produce a strain beyond point A is described by saying that the material is strain-hardening. The stress-strain diagram for a so-called perfectly plastic (or rigid-plastic) material would be like that for the elastic-plastic material with the modulus of elasticity infinitely large. The stress-strain diagram would then be a horizontal line extending to the σ_x-axis.

The stress-strain diagrams for some real materials are indicated in Fig. 1.32. The precise shapes of these curves depend upon the composition of the material, its prior thermal and strain treatment and the characteristics of the loading system. It is seen that for small strains steel can be considered to be linearly elastic. For strains somewhat beyond the yield point steel is a reasonable approximation to an elastic-plastic material. The point of initial yielding is called the upper yield point to distinguish it from the lower yield point which corresponds to the horizontal portion of the curve in Fig. 1.32. If the stressing of a steel bar is done quickly, the stress at which yielding begins is raised.

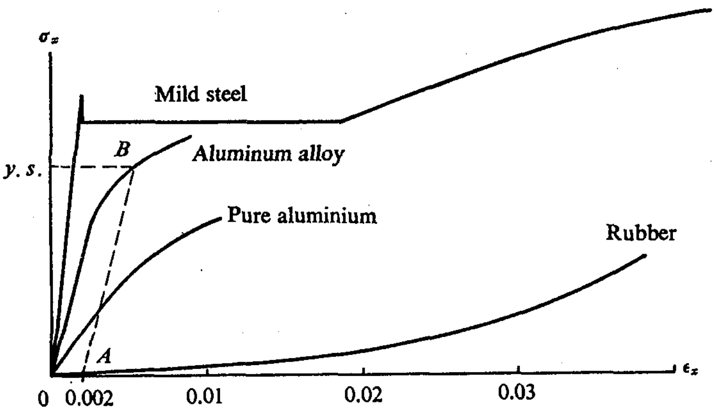


FIGURE 1.32

For materials which do not exhibit a yield point the yield strength is used to characterize the stress-strain behavior. The yield strength is defined to be the stress required to produce a specified plastic strain, usually taken as 0.002 (0.2% strain). This stress level (point B on the stress-strain curve for the aluminum alloy in Fig. 1.32) is determined by constructing the line AB

with the slope of the elastic portion of the curve (E) offset by the specified plastic strain (0.002).

Usually the stress-strain diagram is plotted with σ_x taken to be equal to the force divided by the original area of the test specimen, and ϵ_x taken to be equal to ΔL divided by L_0, where L_0 is the initial length. This procedure is satisfactory for small strains, but does not give a diagram of true stress and strain if the strain is large, since, in this case, the area of the specimen is reduced from its original value and F/A_0 is not the true stress. Instead of the conventional strain $\Delta L/L_0$, the so-called natural strain is sometimes used. This strain has an increment dL/L, where L is the length at the time the increment dL is imposed

$$d\epsilon_{x_n} = \frac{dL}{L}$$

Integrating this equation, we obtain

$$\epsilon_{x_n} = \log_e \frac{L}{L_0}$$

where L_0 is the length at zero stress. For small strains, $\log_e L/L_0 \to \Delta L/L_0$ and the natural strain is the same as the conventional strain. If the strains are large, it is important to know if a stress-strain diagram is plotted in conventional stress and strain or in true stress and natural strain.

Problems

1.47 A bar is made of mild steel whose stress-strain diagram is as shown in Fig. 1.32. The bar is stretched to a strain $\epsilon_x = 0.01$ and is then unloaded. It is next given a compressive strain of $\Delta\epsilon_x = -0.01$ from the unloaded condition. The bar is then unloaded. What residual strain is left in the bar? The yield point stress for the steel is $\sigma_x = 35{,}000$ psi.

1.48 An isotropic elastic bar is pulled in tension with a strain $\epsilon_x = \Delta L/L_0$. What should the conventional stress be multiplied by to give true stress?

1.49 An element of material is deformed with conventional strains ϵ_x, ϵ_y, ϵ_z. The natural strains are ϵ_{x_n}, ϵ_{y_n}, ϵ_{z_n}. Write the complete expression for the volume change of the element in terms of the conventional strains and in terms of the natural strains.

Thermal Strains. If an isotropic material, such as steel, is heated to a uniform temperature, it undergoes a dilatation without distortion. If an element of the material having dimensions dx, dy, dz is raised T degrees in temperature, its dimensions will be

$$dx(1 + \alpha T), \quad dy(1 + \alpha T), \quad dz(1 + \alpha T)$$

where α is the coefficient of thermal expansion for the material. Writing ϵ_T for αT we see that the effect of the temperature rise is to give the element a uniform dilatation, $3\epsilon_T$, without producing normal stresses. Therefore, if a stressed body is also heated, the stresses will not be proportional to the actual total strains ϵ_x, ϵ_y, ϵ_z, but will be proportional to $\epsilon_x - \epsilon_T$, $\epsilon_y - \epsilon_T$, $\epsilon_z - \epsilon_T$, and, hence, that part of the strain that is produced by the stresses is given by

$$
\begin{aligned}
\epsilon_x - \epsilon_T &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\epsilon_y - \epsilon_T &= \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} & \gamma_{yz} &= \frac{\tau_{yz}}{G} \qquad (1.39) \\
\epsilon_z - \epsilon_T &= \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} & \gamma_{zx} &= \frac{\tau_{zx}}{G}
\end{aligned}
$$

The average values of the coefficients of thermal expansion for some common materials are

	For degrees Fahrenheit
Steel	6×10^{-6}
Brass	11×10^{-6}
Aluminum	12×10^{-6}
Copper	9×10^{-6}
Titanium	4.9×10^{-6}

When a steel test specimen is pulled in tension, the change of pressure and volume produce a slight decrease in the temperature of the specimen. When the stress is released and the bar contracts, its temperature is slightly raised. This is analogous to the behavior of a perfect gas, but the effect is much smaller in the elastic solid. Careful measurements during a test will disclose the presence of temperature strains in addition to the strains directly produced by the stresses. The modulus measured in such a test will be slightly larger under adiabatic conditions (no heat flow) than under isothermal conditions (constant temperature).

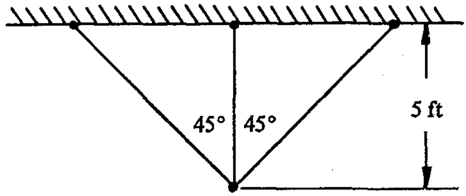
The elastic modulus is found to decrease with an increase in temperature. At very high temperatures metals lose their elasticity and behave as viscous fluids. The onset of the viscous behavior is gradual, and at lower temperatures, before the elasticity is lost, there occurs a small progressive straining under constant stress. This is called *creep*, and the creep rate is an important factor in the design of machine parts that must be stressed at elevated temperatures.

Problems

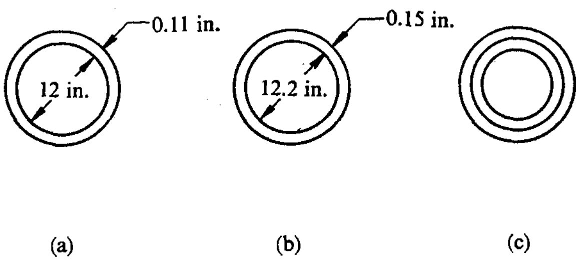
1.50 A thick-walled steel tube has an inner radius of 1 in. and an outer radius of 12 in. The tube is heated 50°F. What is the change in the area of the hole?

1.51 A straight, above-ground, steel pipeline for transporting oil across the Arabian desert is anchored against axial movement at intervals of two miles ($\epsilon_x = 0$). The temperature rises 100°F above the installation (no stress) temperature. What is the total axial force developed in the pipe if its diameter is 30 in. and its wall thickness is $\frac{3}{8}$ in.?

1.52 Three steel bars each having 2 sq. in. of cross-sectional area are pinned at their ends as shown. If the temperature is raised 50°F, what stresses are developed in the bars?



1.53 Two steel rings, both 1 in. wide have, at room temperature, the dimensions shown in the figure. Ring (b) is heated and slipped over ring (a). When the assembly has cooled to room temperature, what is the contact pressure between the rings? What are the stresses in the rings? (Assume σ_θ is constant over the thickness of the rings.)



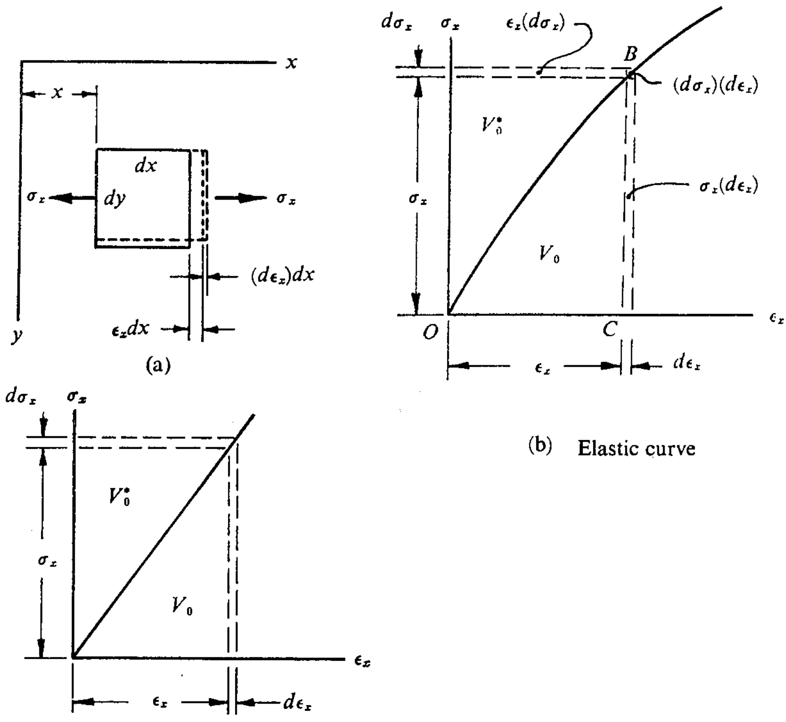
1-7 STRAIN ENERGY

When an element of elastic material is stressed as shown in Fig. 1.33, the element will be strained and, as the planes on which the σ_x stresses act are displaced, the stresses do work. The amount of work done is calculated as follows. The force acting on each x-face is $\sigma_x\, dy\, dz$. If the element were rigid, the work done during any displacement of the element would be equal to zero as the two stresses are oppositely directed and the work done by one would be canceled by that done by the other. Because of the straining of the element, however, the two stresses undergo different displacements and the work does not cancel. The work done by the force ($\sigma_x\, dy\, dz$) because of the straining is

$$\int_0^{\epsilon_x} (\sigma_x\, dy\, dz)(d\epsilon_x\, dx) = \int_0^{\epsilon_x} \sigma_x\, d\epsilon_x (dx\, dy\, dz)$$

If there is no dissipation of energy during the stressing, the work is stored in the element as *strain energy* and it can be recovered during the unloading process. The *strain energy density* (strain energy per unit volume) is defined as

$$V_0 = \int_0^{\epsilon_x} \sigma_x\, d\epsilon_x$$



(a)

(b) Elastic curve

(c) Linearly elastic

FIGURE 1.33

As seen in Fig. 1.33b, V_0 is the area OBC under the stress-strain diagram. In the case of plane stress there are three stresses to be considered and the strain energy density is

$$V_0 = \int_0^{\epsilon_x} \sigma_x\, d\epsilon_x + \int_0^{\epsilon_y} \sigma_y\, d\epsilon_y + \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy} \tag{1.40}$$

The area above the stress-strain diagram, see Fig. 1.33b, is called the complementary energy density, V_0^* which, for plane stress, is*

$$V_0^* = \int_0^{\sigma_x} \epsilon_x \, d\sigma_x + \int_0^{\sigma_y} \epsilon_y \, d\sigma_y + \int_0^{\tau_{xy}} \gamma_{xy} \, d\tau_{xy} \tag{1.41}$$

If the material is linearly elastic, as shown in Fig. 1.33c, the strain energy density for plane stress is

$$V_0 = \tfrac{1}{2}\sigma_x\epsilon_x + \tfrac{1}{2}\sigma_y\epsilon_y + \tfrac{1}{2}\tau_{xy}\gamma_{xy} \tag{1.42}$$

In this expression it is understood that the strains ϵ_x, ϵ_y, γ_{xy} are those produced by the combined actions of σ_x, σ_y, and τ_{xy}, which are the final values of the stresses. That the above equation for V_0 is correct can be seen from the following consideration. Let the state of stress and strain be

$$\begin{aligned} \sigma_x' &= k\sigma_x & \epsilon_x' &= k\epsilon_x \\ \sigma_y' &= k\sigma_y & \epsilon_y' &= k\epsilon_y \\ \tau_{xy}' &= k\tau_{xy} & \gamma_{xy}' &= k\gamma_{xy} \end{aligned}$$

and let k increase uniformly from zero to unity. The strain energy density is then given by

$$\begin{aligned} V_0 &= \int_0^{\epsilon_x} \sigma_x' \, d\epsilon_x' + \int_0^{\epsilon_y} \sigma_y' \, d\epsilon_y' + \int_0^{\gamma_{xy}} \tau_{xy}' \, d\gamma_{xy}' \\ &= \int_0^1 (k\sigma_x) \, d(k\epsilon_x) + \cdots \\ &= \sigma_x\epsilon_x \int_0^1 k \, dk + \cdots \\ &= \tfrac{1}{2}\sigma_x\epsilon_x + \tfrac{1}{2}\sigma_y\epsilon_y + \tfrac{1}{2}\tau_{xy}\gamma_{xy} \end{aligned}$$

For the general three-dimensional state of stress of a linearly elastic material

$$V_0 = \tfrac{1}{2}\sigma_x\epsilon_x + \tfrac{1}{2}\sigma_y\epsilon_y + \tfrac{1}{2}\sigma_z\epsilon_z + \tfrac{1}{2}\tau_{xy}\gamma_{xy} + \tfrac{1}{2}\tau_{yz}\gamma_{yz} + \tfrac{1}{2}\tau_{zx}\gamma_{zx} \tag{1.43}$$

By using Hooke's law, V_0 can be expressed either in terms of stresses only or in terms of strains only (see Problems 1.54–1.57).

* Strain energy was first used in analysis by L. Euler (1707–1783), and the concept of complementary energy was introduced by the German engineer F. Engesser (1848–1931). The strain energy density function was introduced by the British mathematician George Green (1793–1841).

It is seen from the definitions that

$$\frac{\partial V_0}{\partial \epsilon_x} = \sigma_x, \quad \frac{\partial V_0}{\partial \epsilon_y} = \sigma_y, \text{ etc.}$$
$$\frac{\partial V_0^*}{\partial \sigma_x} = \epsilon_x, \quad \frac{\partial V_0^*}{\partial \sigma_y} = \epsilon_y, \text{ etc.} \tag{1.44}$$

The total strain energy in a body is given by the integral of $dV = V_0 \, dx \, dy \, dz$ throughout the body

$$V = \int dV = \iiint V_0 \, dx \, dy \, dz$$

Similarly the total complementary energy is the integral of dV^* throughout the body.

If a straight bar of length L and a cross-sectional area A is pulled in tension by an axially applied force P so that a uniform axial stress $\sigma = P/A$ is developed, the total strain energy is

$$V = \iiint V_0 \, dx \, dy \, dz = \iiint \tfrac{1}{2}\sigma_x \epsilon_x \, dx \, dy \, dz$$
$$= \iiint \frac{1}{2} \frac{P}{A} \cdot \frac{P}{AE} \, dx \, dy \, dz = \frac{1}{2} \frac{P^2 L}{AE}$$

Note that the strain energy is a nonlinear function of stress even though the material is linearly elastic. Therefore, the principle of superposition does not hold for strain energy.

Problems

1.54 Show that for linearly elastic material in plane stress the strain energy per unit volume can be written

$$V_0 = \frac{1}{2}\left(\frac{\sigma_x^2}{E} + \frac{\tau_{xy}^2}{G} + \frac{\sigma_y^2}{E} - \frac{2\nu\sigma_x\sigma_y}{E}\right) \tag{1.45}$$

1.55 Show that for a general three-dimensional state of stress the strain energy density for linearly elastic material can be written

$$V_0 = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \tag{1.46}$$

1.56 Show that the strain energy density for a linearly elastic material can be written

$$V_0 = \tfrac{1}{2}\lambda(3\bar{\epsilon})^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \tfrac{1}{2}G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2) \tag{1.47}$$

1.57 Show that the strain energy density for a linearly elastic material can be written

$$V_0 = \frac{1}{2G}\frac{2}{3}\left[\left(\frac{\sigma_1' - \sigma_2'}{2}\right)^2 + \left(\frac{\sigma_2' - \sigma_3'}{2}\right)^2 + \left(\frac{\sigma_3' - \sigma_1'}{2}\right)^2\right] + \frac{1}{2E}3(1 - 2\nu)\bar{\sigma}^2 \tag{1.48}$$

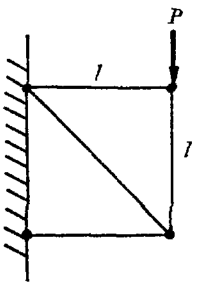
where σ_1, σ_2, σ_3 are the principal stresses, and $\sigma_1' = \sigma_1 - \bar{\sigma}$, $\sigma_2' = \sigma_2 - \bar{\sigma}$, $\sigma_3' = \sigma_3 - \bar{\sigma}$, and $\bar{\sigma} = (\sigma_1 + \sigma_2 + \sigma_3)/3$.

1.58 Show that for a linearly elastic material $\sigma_x\, d\epsilon_x = \epsilon_x\, d\sigma_x$.

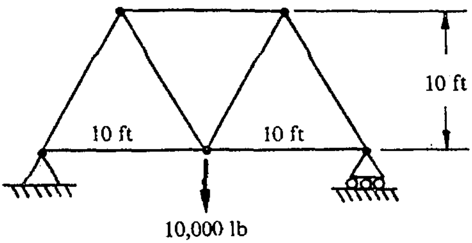
1.59 A steel bar is stressed in tension, $\sigma_x = 20{,}000$ psi. What is the strain-energy per cubic inch?

1.60 A circular, steel tube having 0.2 in. wall thickness, an outer diameter of 12.2 in., and a length of 6 ft is twisted to a stress of $\tau = 10{,}000$ psi. What is the strain energy stored in the tube?

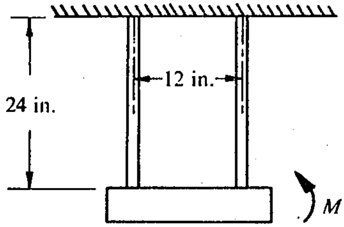
1.61 The square pin-jointed frame shown in the diagram is loaded by a force P. All members have the same cross-sectional area A, and modulus E. What is the total strain energy stored?



1.62 The pin-jointed truss shown in the diagram is formed of steel members each having a cross-sectional area of 1.5 sq. in. What is the strain energy stored?



1.63 Two steel bars with cross-sectional areas of 1 sq. in. each are connected by a rigid crosspiece to which a moment $M = 5000$ ft-lb is applied. What is the stored strain energy?



1.64 A 10-ft steel bar made of the mild steel whose stress-strain diagram is shown in Fig. 1.32 is pulled in tension to a strain $\epsilon_x = 0.02$. What is the total work done on the bar if the cross-sectional area is 1 sq. in.? How much of this is stored as strain energy and how much is dissipated in plastic deformation? Assume $\sigma_{yp} = 35{,}000$ psi.

Energy Methods. The strain energy of a body has certain properties that make it very informative as to the behavior of stressed bodies and it is a powerful tool for solving certain types of problems. We shall discuss these properties briefly here and later shall apply them to specific problems.

Strain Energy Equal to Work. We shall consider only conservative systems so that no energy is dissipated during the stressing and unstressing of a body, and we shall consider only statically loaded problems for which the forces are applied to the body so slowly that the kinetic energy of the body never exceeds a negligibly small amount. In this case the work W done by the external forces in deforming the body is equal to the strain energy V stored in the body

$$V = W \tag{1.49}$$

That this must be so can be seen in Fig. 1.34. The strain energy stored in an element is equal to the work done by the stresses on the faces of the element as the forces are displaced during the deformation. In the case of adjacent elements, such as 1 and 2, it is seen that the work done by the stress on the right-hand face of 1 is equal, and of opposite sign, to the work done by the stress on the left-hand face of element 2. Therefore, the work done by all of the stresses on the faces of the elements will cancel except for those faces that lie in the surface of the body. Actually, it is the same quantity that is expressed either in terms of the internal stresses

$$V = \iiint \left(\sum_{j=1}^{6} \int_0^{\epsilon_j} \sigma_j \, d\epsilon_j \right) dx \, dy \, dz \qquad j = 1, 2, \ldots, 6$$

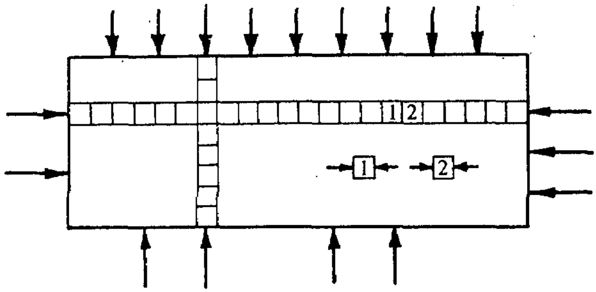


FIGURE 1.34

or expressed in terms of the external forces

$$W = \sum_{i=1}^{n} \int_0^{e_i} P_i \, de_i$$

The generalized notation σ_j, ϵ_j, $\sum_j$ in the expression for V means to sum the work done by all of the stresses, shearing as well as normal

$$\sigma_1 = \sigma_x, \quad \sigma_2 = \sigma_y, \quad \sigma_3 = \sigma_z, \quad \sigma_4 = \tau_{yz}, \quad \sigma_5 = \tau_{xz}, \quad \sigma_6 = \tau_{xy}$$

$$\epsilon_1 = \epsilon_x, \quad \epsilon_2 = \epsilon_y, \quad \epsilon_3 = \epsilon_z, \quad \epsilon_4 = \gamma_{yz}, \quad \epsilon_5 = \gamma_{xz}, \quad \epsilon_6 = \gamma_{xy}$$

The displacement e_i is the component, in the direction of P_i, of the displacement of point i at which P_i is applied, $e_i = \mathbf{d}_i \cdot \mathbf{P}_i / |\mathbf{P}_i|$. We may also regard P_i as an applied moment, if we take e_i to be the rotation of point i.

As an example of equating work to strain energy, consider the linearly elastic bar of length L and cross-sectional area A shown in Fig. 1.35, to which

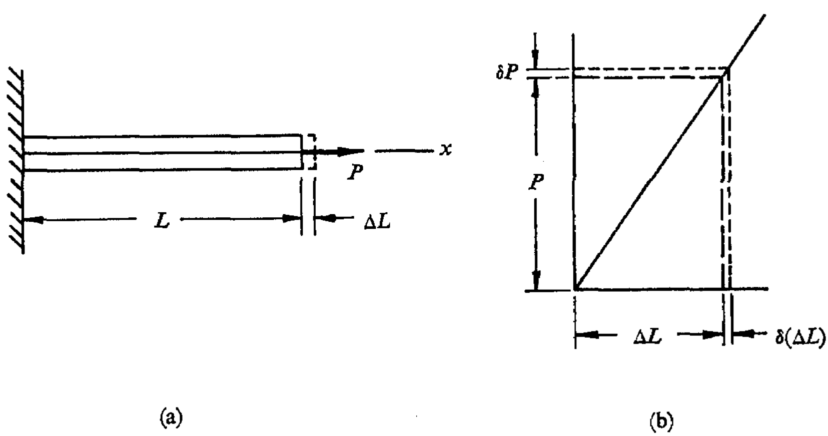


FIGURE 1.35

is applied an axial force that is gradually increased to a value P. The force P produces an elongation of the bar, ΔL, and produces a uniform tension stress $\sigma_x = P/A$. The strain energy and the work are:

$$V = \int_0^L \frac{1}{2}\sigma_x \epsilon_x A\,dx = \frac{1}{2}\sigma_x \epsilon_x AL = \frac{1}{2}\frac{\sigma_x^2}{E}AL = \frac{1}{2}\frac{P^2L}{AE}$$

$$W = \tfrac{1}{2}P(\Delta L)$$

The equation, $V = W$, is thus

$$\frac{1}{2}\frac{P^2L}{AE} = \frac{1}{2}P(\Delta L)$$

and solving for (ΔL) gives

$$\Delta L = \frac{PL}{AE}$$

This, of course, agrees with the elongation that would be computed directly from the strain in the bar. It should be noted that in this case the deformation is proportional to the load. It does not necessarily follow, however, that a body has linear load-deformation characteristics if it is made of linearly elastic material. For example, consider the two-bar system shown in Fig. 1.36 which will deform as indicated by the dotted lines under the action

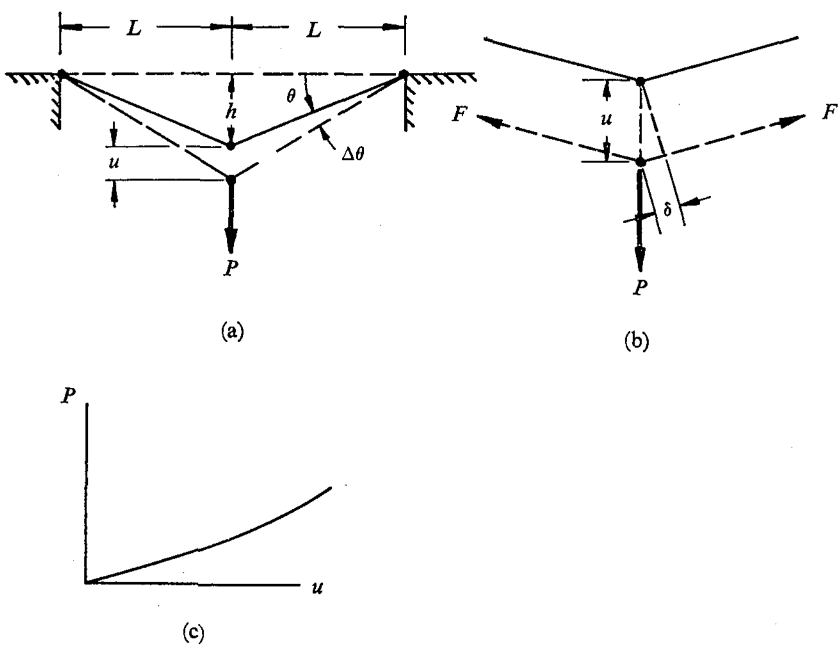


FIGURE 1.36

of the force P. For exact values of the forces and stresses in the members the calculations must be made for the equilibrium configuration, that is, the displacement must be taken into account. For example, equilibrium requires

$$F \sin(\theta + \Delta\theta) = \frac{P}{2}$$

so the force in the bar is

$$F = \frac{P}{2\sin(\theta + \Delta\theta)} = \frac{P}{2}\frac{[L^2 + (h+u)^2]^{1/2}}{(h+u)}$$

The displacement u is related to F by considering the deformations shown in Fig. 1.36b

$$\sin(\theta + \Delta\theta) = \frac{h+u}{[L^2 + (h+u)^2]^{1/2}} \cong \frac{\delta}{u} = \frac{FL(L^2 + h^2)^{1/2}}{uAEL} \qquad \text{for } \Delta\theta \ll 1$$

Eliminating F from the above two equations gives

$$P = \frac{2AE}{L} u \left\{ \frac{L}{(L^2 + h^2)^{1/2}} \frac{(h+u)^2}{[L^2 + (h+u)^2]} \right\}$$

It is seen that there is a nonlinear relation between the force (and stress) in the member and the applied load and, hence, this is a nonlinear, elastic problem. However, if the displacement u is sufficiently small compared to h, negligible error will be introduced if its effect is neglected and P is computed as

$$P = \frac{2AEh^2}{(L^2 + h^2)^{3/2}} u$$

However, the problem cannot be linearized if h is too small. For example, if h is zero, and $\Delta\theta \ll 1$,

$$P = 2AE \frac{(u/L)^3}{1 + (u/L)^2}$$

This expression cannot be linearized by keeping u small. In general, a linear relation between *load* and *deflection* will result when stresses can be calculated on the basis of the *initial*, unloaded geometry of the system, and the material follows Hooke's law.

Variation of Energy. A plane state of stress, σ_x, σ_y, τ_{xy}, will have energy density V_0 as given by Eq. (1.40). If now the stresses are changed by infinitesimal increments $\delta\sigma_x$, $\delta\sigma_y$, $\delta\tau_{xy}$, *the strains will change by increments* $\delta\epsilon_x$, $\delta\epsilon_y$, $\delta\gamma_{xy}$. As can be seen in Fig. 1.33b, the energy density then changes by

$$\delta V_0 = \sigma_x\,\delta\epsilon_x + \sigma_y\,\delta\epsilon_y + \tau_{xy}\,\delta\gamma_{xy}$$

where terms of the order $(\delta\sigma_x)(\delta\epsilon_x)$ have been neglected. If the quantity δV_0 is integrated throughout the body, there will be obtained the incremental change in energy, δV. The quantity δV is said to be the variation in strain energy produced by a variation in stresses, $\delta\sigma_x$, $\delta\sigma_y$, $\delta\tau_{xy}$. These changes in stresses are, of course, produced by corresponding changes in the external forces acting on the body.

Corresponding to a variation in strain energy there will be a variation in the work done by the applied forces. For example, if a bar of length L and area A is subjected to an axial force P, as shown in Fig. 1.35, the work done is

$$W = \tfrac{1}{2}P(\Delta L)$$

If the load is increased to $P + \delta P$ the change in length (ΔL) will be increased to $(\Delta L) + \delta(\Delta L)$ and the increment change in work will be

$$\begin{aligned}\delta W &= \tfrac{1}{2}(P + \delta P)[\Delta L + \delta(\Delta L)] - \tfrac{1}{2}P(\Delta L)\\ &= \tfrac{1}{2}P\,\delta(\Delta L) + \tfrac{1}{2}(\Delta L)\,\delta P + \tfrac{1}{2}\,\delta P\,\delta(\Delta L)\end{aligned}$$

It can be seen from Fig. 1.35b that for a linearly elastic system $P\,\delta(\Delta L) = (\Delta L)\,\delta P$. Hence, neglecting the second-order term, the increment of work is

$$\delta W = \tfrac{1}{2}P\,\delta(\Delta L) + \tfrac{1}{2}(\Delta L)\,\delta P = P\,\delta(\Delta L)$$

The term $P\,\delta(\Delta L)$ is, of course, just the work done by the applied load P as it moves through the displacement $\delta(\Delta L)$.

Principle of Virtual Work. If an increment of work δW is performed on a loaded system of one or more bodies, part of the increment will be stored as strain energy δV, and part may be dissipated δE

$$\delta W = \delta V + \delta E$$

If no energy is dissipated this relation becomes:

$$\delta V = \delta W \tag{1.50}$$

Equation (1.50) states that if the forces F acting on a body are changed by δF, the corresponding changes in the stresses and strains are such that the increment of work done by the applied forces is equal to the change in strain energy. Since the work done by the δF is a second-order quantity as compared to the work done by the F, we may avoid any mention of the δF and state the principle of virtual work* as: *during any infinitesimal virtual displacement*

* First stated in general form by John Bernoulli (1667–1748).

the increment of work δW done by the applied forces is equal to the increment δV of strain energy stored.

If the applied forces acting on the body have a potential energy U_a, then $W = -U_a$ since the work expended by the forces must be equal to the decrease in their potential energy. Equation (1.50) may then be written

$$\delta V - \delta W = \delta V + \delta U_a = \delta U = 0$$

where U is the *total* potential energy of the system. The equation

$$\delta U = 0 \tag{1.51}$$

expresses the condition of stationary potential energy of the system. This is an alternate form of Eq. (1.50). The principle of virtual work is a powerful method of analyzing problems as demonstrated in the following paragraphs (see also Appendix IV).

Castigliano's Theorem. Let an elastic body be acted upon by forces $P_1, P_2, \ldots, P_n$, as shown in Fig. 1.37 and let the resultant displacement of point 1 in the direction of P_1 be e_1, etc., as shown. Let infinitesimal increments $\delta P_1, \delta P_2, \ldots, \delta P_n$ be given to the forces. These will produce infinitesimal changes $\delta e_1, \delta e_2, \ldots, \delta e_n$ in the displacements. Either the δP_i or the δe_i can

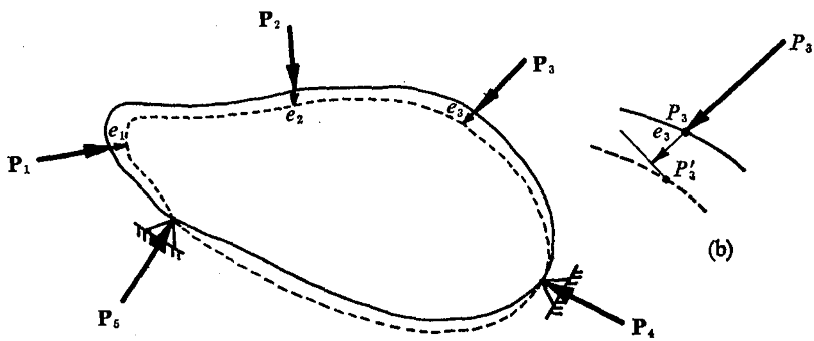


FIGURE 1.37

be arbitrarily specified provided they are consistent with the conditions of the problem. The increment of work is

$$\delta W = \sum_{i=1}^{n} P_i \, \delta e_i$$

The work done by δP_i being a higher order infinitesimal ($\frac{1}{2}\,\delta P_i\,\delta e_i$) is neglected. The strain energy may be thought of as being either a function of the applied forces, or a function of the displacements $e_1, e_2, \ldots, e_n$. In the case under consideration, the increment change in strain energy can be written

$$\delta V = \sum_{i=1}^{n} \frac{\partial V}{\partial e_i}\,\delta e_i$$

By the principle of virtual work $\delta V - \delta W = 0$, hence

$$\sum_{i=1}^{n} \left(\frac{\partial V}{\partial e_i} - P_i\right) \delta e_i = 0$$

Since the δe_i may be independently specified this sum can be equal to zero only if the expression in the parentheses is equal to zero, therefore

$$\frac{\partial V}{\partial e_i} = P_i \tag{1.52}$$

This is the first form of Castigliano's theorem* and it states that the derivative of the strain energy with respect to the displacement, e_i, is equal to the force, acting at the point, in the direction of e_i.

The second form of Castigliano's theorem is an analogous expression involving the variation of forces. The incremental change in energy is

$$\delta V = \sum_{i=1}^{n} \frac{\partial V}{\partial P_i}\,\delta P_i$$

The variation of work is

$$\delta W = \sum_{i=1}^{n} P_i\,\delta e_i$$

If the problem is linear (first [illegible]) .

$$\delta W = \sum_{i=1}^{n} e_i\,\delta P_i$$

Writing $\delta V = \delta W$ there is obtained

$$\frac{\partial V}{\partial P_i} = e_i \tag{1.53}$$

* A. Castigliano (1847–1884), *Theorie de Equilibré des Systèmes Elastiques*, Turin (1879).

This states that the derivative of the strain energy with respect to an applied force gives the displacement, of the point of application of the force, in the direction of the force. This relation, Eq. (1.53) does not hold for a nonlinear problem. The first form, Eq. (1.52) of Castigliano's theorem is completely general in that it is applicable to nonlinear elastic systems as well as to linear elastic systems.

Thus, for a linear system, we may write

$$\frac{\partial V}{\partial e_i} = P_i \qquad \frac{\partial V}{\partial P_i} = e_i \tag{1.54}$$

The equation on the right is commonly used to compute the displacement of a point in a loaded body. When the displacement component to be found is e_j, at the point of application of P_j, the procedure is to compute the *total* strain energy due to all of the loads, and determine $\partial V/\partial P_j$ which is the desired component e_j. To find a displacement component normal to $\mathbf{P}_j$ using Castigliano's theorem, we must apply a force Q_j at the point in the appropriate direction, compute the total strain energy, determine $\partial V/\partial Q_j$ and then evaluate $\partial V/\partial Q_j$ with Q_j set equal to zero. This procedure also applies to the problem of finding a displacement component at a point on the body where no loads are acting.

We shall apply Castigliano's theorem to find the horizontal displacement at the point of application of the vertical load in the pin-jointed frame shown in Fig. 1.38a. Since we wish to find the horizontal displacement, we must apply

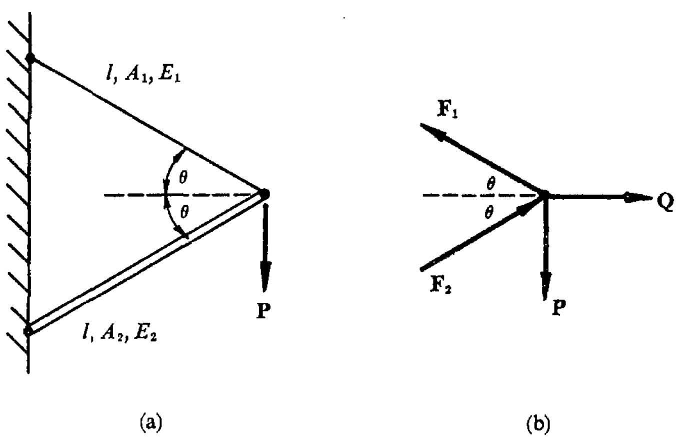


FIGURE 1.38

a horizontal load, say Q, at the point of application of P. We determine the forces in the two bars by considering equilibrium of the pin shown in Fig. 1.38b

$$(F_1 + F_2) \sin \theta = P$$

$$(F_1 - F_2) \cos \theta = Q$$

or

$$F_1 = \frac{1}{2}\left(\frac{P}{\sin \theta} + \frac{Q}{\cos \theta}\right)$$

$$F_2 = \frac{1}{2}\left(\frac{P}{\sin \theta} - \frac{Q}{\cos \theta}\right)$$

The total strain energy is given by

$$V = \sum_{i=1}^{2} \frac{F_i^2 L_i}{2A_i E_i}$$

and Castigliano's theorem gives for the horizontal displacement at the pin

$$e_Q = \frac{\partial V}{\partial Q} = \sum_{i=1}^{2} \frac{F_i(\partial F_i/\partial Q)L_i}{A_i E_i}$$

$$= \frac{l}{4 \cos \theta}\left[\left(\frac{P}{\sin \theta} + \frac{Q}{\cos \theta}\right)\frac{1}{A_1 E_1} - \left(\frac{P}{\sin \theta} - \frac{Q}{\cos \theta}\right)\frac{1}{A_2 E_2}\right]$$

The horizontal displacement of the point of application of P, with P acting alone, is then

$$e_Q = \left.\frac{\partial V}{\partial Q}\right|_{Q=0} = \frac{PL}{4 \cos \theta \sin \theta}\left(\frac{1}{A_1 E_1} - \frac{1}{A_2 E_2}\right)$$

Principle of Minimum Strain Energy. The principle of virtual work states that for any variation of the forces acting on a body the variation of internal strain energy is equal to the variation of external work, that is, $\delta V = \delta W$. If the variation is such that δW is zero, then it follows that

$$\delta V = 0 \tag{1.55}$$

This is the condition for the strain energy to be a minimum.* The principle of minimum strain energy is particularly useful in analyzing statically indeterminate problems.

An example of the application of the principle is illustrated in Fig. 1.39. The force P is applied to a rigid block which is restrained by three steel bars and, since two bars would be sufficient, this is a statically indeterminate system. The system with the center bar cut, as shown in Fig. 1.39b, is statically

* This is sometimes called the principle of least work.

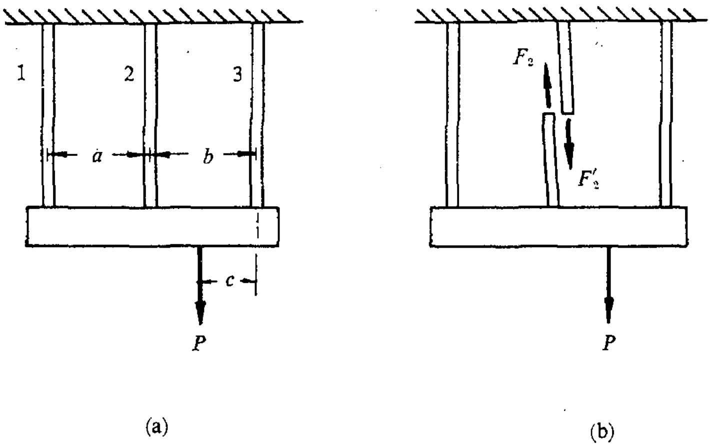


FIGURE 1.39

determinate if the values of F_2 and F_2' are known, and in this case the forces in the bars can be determined and the strain energy of the system computed

$$V = f(P, F_2, F_2', E, A_1, A_2, A_3, L_1, L_2, L_3)$$

According to the principle of virtual work

$$\frac{\partial V}{\partial F_2}\,\delta F_2 = \delta W_2; \quad \frac{\partial V}{\partial F_2'}\,\delta F_2' = \delta W_2'$$

However, it is a condition of the problem that $F_2 = F_2'$ and that the displacements of F_2 and F_2' are the same (the bar remains continuous) so $W_2 = -W_2'$ and $\delta W_2 = -\delta W_2'$. Hence, adding the two equations and writing $F_2' = F_2$, there is obtained

$$\frac{\partial V}{\partial F_2} = 0$$

This states that the force F_2 must have the value that minimizes the strain energy of the system. Castigliano's theorem applied to the reaction at the support gives the same result since $e_2 = 0$ at that point.

Another example is the statically indeterminate beam on three supports shown in Fig. 1.40. In this case $V = f(P_1, P_2, P_3, F)$, but from statics, we can eliminate two of the three unknown loads P_1, P_2, and P_3. So we can write

$$V = f(P_2, F)$$

By reasoning similar to that used in the preceding paragraph it may be concluded that

$$\frac{\partial V}{\partial P_2} = 0$$

Castigliano's theorem applied to the reaction P_2 gives the same result since the deflection $e_2 = 0$. This equation plus the two applicable equations of static equilibrium suffice to determine the three unknowns P_1, P_2, P_3. Methods of calculating the stresses, strains and strain energy in a beam are given in Chapter 3.

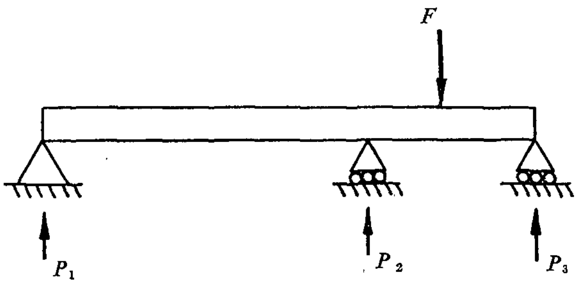


FIGURE 1.40

Reciprocal Theorem. Suppose that a linearly elastic body is acted upon by a set of forces P_i ($i = 1, 2, \ldots, n$), as shown in Fig. 1.37. The resulting displacements of the points of application of the forces have components e_i *in the direction of the forces.* These displacements are expressible as functions of the applied loads, and for a linear system

$$e_1 = C_{11}P_1 + C_{12}P_2 + C_{13}P_3 + \cdots + C_{1n}P_n$$

or, in general, for a linear system

$$e_i = \sum_{j=1}^{n} C_{ij}P_j$$

The constants C_{ij} are called influence coefficients as C_{ij} indicates the influence of P_j on the displacement e_i. It is seen that C_{ij} is equal to the displacement e_i produced at point i by a unit force ($P_j = 1$) at point j.

Suppose that two sets of forces are acting on the body, set (1) being the P_i' and set (2) being the P_j''. When these are applied to the body they will do a certain amount of work

$$W = \sum_{i=1}^{n} \tfrac{1}{2}P_i'e_i + \sum_{j=1}^{m} \tfrac{1}{2}P_j''e_j$$

where e_i and e_j are the displacements produced by both P_i' and P_j''. If the P_i' only are applied to the body, the work done is

$$W_1 = \sum_{i=1}^{n} \tfrac{1}{2}P_i'e_i'$$

where the e_i' are the displacements produced by the P_i'. Similarly, the work done by the P_j'' only is

$$W_2 = \sum_{j=1}^{m} \tfrac{1}{2}P_j''e_j''$$

Suppose now that the forces of set (1) are applied first and then the forces of set (2) are applied. The total work done will be

$$W = W_1 + W_2 + W_{1,2}$$

where $W_{1,2}$ is the work done by set (1) as it moves through the displacements produced by set (2). If the order of loading is reversed, the work done is

$$W = W_2 + W_1 + W_{2,1}$$

where $W_{2,1}$ is the work done by set (2) as it moves through the displacements produced by set (1). Since the total work done must be independent of the order of loading, it follows that

$$W_{1,2} = W_{2,1} \tag{1.56}$$

This is the reciprocal theorem,* which states: for a linear, elastic body, the work done by a set of forces (1) as a result of the application of a set of forces (2) is equal to the work that would be done by (2) as a consequence of applying (1).

If $W_{1,2}$ and $W_{2,1}$ are expressed in terms of the forces and displacements they are

$$W_{1,2} = \sum_{i=1}^{n} P_i'e_i'' = \sum_{i=1}^{n} P_i' \sum_{j=1}^{m} C_{ij}P_j''$$

$$W_{2,1} = \sum_{j=1}^{m} P_j''e_j' = \sum_{j=1}^{m} P_j'' \sum_{i=1}^{n} C_{ji}P_i'$$

Equation (1.56) thus states

$$\sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij}P_i'P_j'' = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ji}P_i'P_j''$$

or

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (C_{ij} - C_{ji})P_i'P_j'' = 0$$

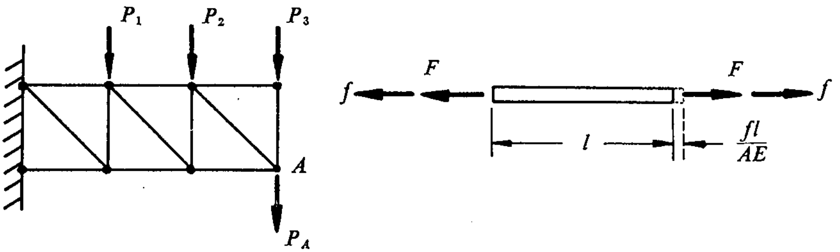
* First stated by E. Betti (1872).

Since this must hold for arbitrary P_i' and P_j'' it requires $C_{ij} - C_{ji} = 0$, or

$$C_{ij} = C_{ji} \tag{1.57}$$

The elastic coefficients are thus symmetrical in i and j. This is sometimes called Maxwell's reciprocal theorem.* It can be generalized to include the case when moments, as well as forces, are applied to the body. Equation (1.57) states that the component of displacement $e_i = (\mathbf{d}_i \cdot \mathbf{P}_i)/|\mathbf{P}_i|$ due to a unit load P_j is equal to the component of displacement $e_j = (\mathbf{d}_j \cdot \mathbf{P}_j)/|\mathbf{P}_j|$ at j due to a unit load P_i.

As an example, the reciprocal theorem will be used to calculate the deflection of a point on a truss. Suppose the truss shown in Fig. 1.41a is loaded



(a) (b)

FIGURE 1.41

by a set of forces P_1, P_2, P_3, and it is desired to calculate the vertical displacement, e_A, at point A. This may be done by applying a force P_A at the point A (this is the second set of forces). According to the reciprocal theorem

$$P_A e_A = P_1 e_1 + P_2 e_2 + P_3 e_3$$

where e_1, e_2, e_3 are the vertical displacements produced at points 1, 2, 3 by the load P_A. The right-hand side of the equation represents the work done by P_1, P_2, P_3 due to the application of P_A. It is, of course, no easier to calculate e_1, e_2, e_3 than to calculate e_A directly, however, the increment of work done by P_1, P_2, P_3 is equal to the corresponding increment of strain energy stored in the truss members, so that we may write

$$P_A e_A = \sum_{i=1}^{12} F_i \frac{f_i l_i}{A_i E_i}$$

* James Clerk Maxwell (1831–1879) deduced this specific relationship before Betti worked out the general result of Eq. (1.56).

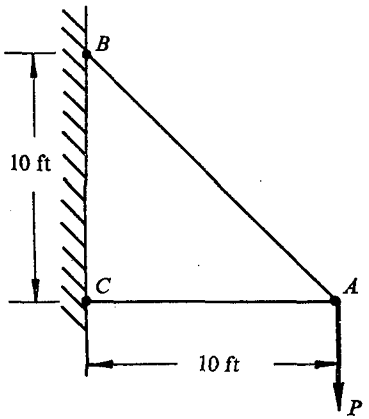
where F_i is the force produced in the ith truss member by P_1, P_2, P_3 and f_i is the force produced in the ith member by P_A. The extension of the member by P_A is given by fl/AE and this is the distance through which the force F moves, as shown in Fig. 1.41b. In particular, if P_A is a unit force, the displacement is given by

$$e_A = \sum_{i=1}^{12} F_i \frac{f_i l_i}{A_i E_i}$$

where f_i is the force produced in the ith truss member by the unit force applied at point A, and F_i is the force produced by loads P_1, P_2, P_3. The same result would be obtained by the application of Castigliano's theorem. See also Prob. 1.70 for another example of the application of the reciprocal theorem.

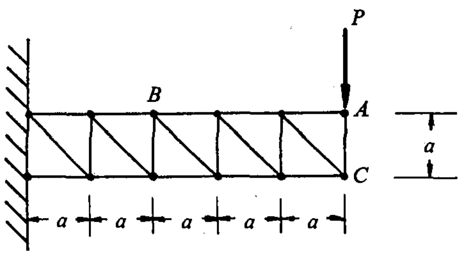
Problems

1.65 Compute the vertical displacement of point A under the action of the load $P = 10{,}000$ lb. Both members of the pin-jointed frame are made of steel and AB has 0.8 sq. in. cross-sectional area and AC has 1.5 sq. in. Solve this problem by using $V = W$.



1.66 Show that for a linear problem $\sum_{i=1}^{n} P_i\, \delta e_i = \sum_{i=1}^{n} e_i\, \delta P_i$.

1.67 Compute the vertical deflection of point A under the action of P. Each member of the pin-jointed truss is made of the same material and has the same cross-sectional area.

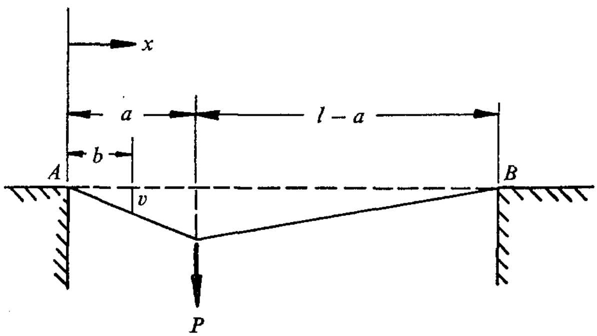


1.68 Show by applying Castigliano's theorem to the computation of the vertical displacement of point A on the truss in Fig. 1.41 that:

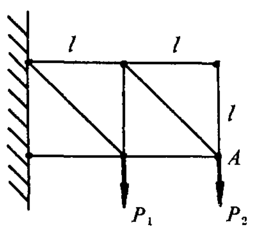
$$e_A = \sum_i F_i \frac{f_i l_i}{A_i E_i}$$

1.69 A thin-wall tube has an average radius R, a wall thickness t, and a length L. A flat plate is welded to each end and the bottom plate is anchored to the ground. A twisting moment M is applied to the top plate. What is the angle of rotation θ of the top plate as computed by the energy method?

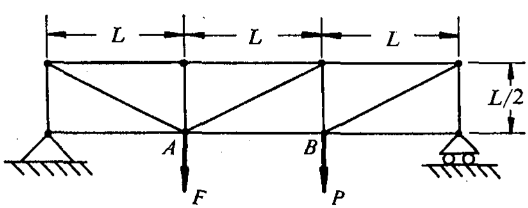
1.70 An elastic rubber band is stretched tightly between points A and B. A small vertical load P is applied at $x = a$ and this produces a displacement $v = f(x)$ at points along the span. Draw a diagram for the deflection at $x = b$ as a varies from zero to L. Use the reciprocal relation. The deflections are sufficiently small so that the system can be assumed to be linear.



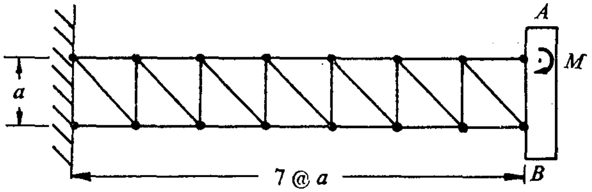
1.71 What is the vertical displacement of point A under the action of the forces P_1 and P_2? All members of the pin-jointed truss are made of the same material and have the same cross-sectional areas.



1.72 What are the vertical deflections at A and B? Each member of the pin-jointed truss is made of the same material and has the same cross-sectional area.

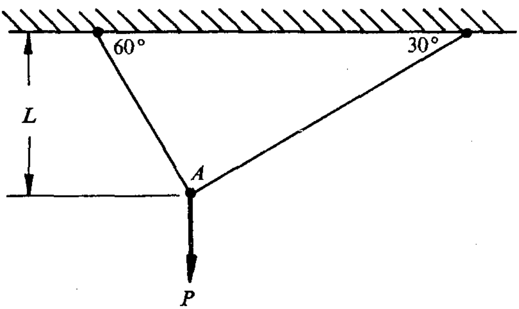


1.73 The cantilever, pin-jointed truss has a rigid member AB at its end. A moment M acts upon the member AB. What rotation of AB does it produce? All members have the same cross-sectional area and are made of the same material.



1.74 Determine the horizontal displacement of point A in Prob. 1.65.

1.75 Determine the horizontal displacement of point A under the action of force P. The pin-jointed members are made of the same material and have the same cross-sectional area.

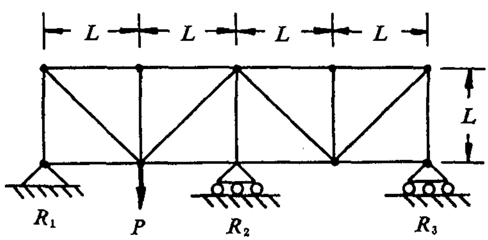


1.76 Determine the vertical displacement of point B in Prob. 1.67.

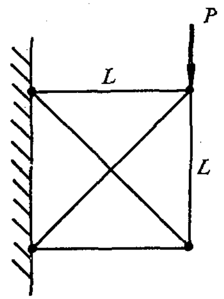
1.77 Determine the horizontal displacement of point C in Prob. 1.67.

1.78 Determine the vertical displacement of P in Fig. 1.36 using Castigliano's theorem.

1.79 Determine the support forces R_1, R_2, R_3. All members of the pin-connected truss are the same except for their lengths.



1.80 Determine the forces in the diagonal members of the pin-jointed frame. All the members are made of the same material and all have the same cross-sectional areas.



1-8 STRESS FAILURES

In following chapters we shall investigate how the stresses, strains, and displacements vary throughout bodies that are made of homogeneous, isotropic, and linearly elastic material. The resulting equations can be applied to actual bodies only insofar as the material properties agree with those assumed in the analysis. Many materials will have the specified properties if the stresses and strains to which they are subjected are sufficiently small, but if the magnitude of stress is increased, the material will eventually depart from linear elasticity and at sufficiently high stresses the material will fracture. It is important to be able to tell when the material properties will change and under what conditions failure may occur.

If a tension test specimen is pulled with an increasing force, it must eventually break into two pieces. This is a particularly simple type of failure that is readily determined in the laboratory. There are other types of failures that are much more subtle. For example, machine parts have been known to undergo several million operations of the machine and then fail. This is called a *fatigue failure* and its cause is explained as follows. If an element of material, say steel, is stressed to the point where some plastic strain is developed, all the work done by the stresses does not go into recoverable strain energy, for the work that went into producing the plastic (irrecoverable) strain is dissipated. Part of the work is expended in disrupting the internal structure of the material and part is lost in the associated production of heat. If a reversed stress of equal magnitude is then applied, there will be a similiar energy loss, etc. If the process is repeated a sufficient number of times, a small crack will form and then spread across the body and produce failure. This is just the way a piece of wire is broken when it is bent back and forth. The number of stress cycles that a material can withstand depends upon the magnitude of the plastic strain produced and the number of cycles can thus range from a few to an infinite number. This information is usually presented in so-called *s-n* diagrams which give the number n of stress cycles to failure under stress s. The *endurance limit* is the highest stress level that will not cause failure regardless of the number of times it is applied. Steel has an endurance limit somewhat below the yield point, and the *s-n* curve becomes horizontal at that point. Aluminum and its alloys generally have no endurance limit.

When a material is strained cyclically into the plastic range, the fact that energy is dissipated will be evident on the plotted stress-strain diagram. For example, suppose an element is stressed as indicated in Fig. 1.42. The

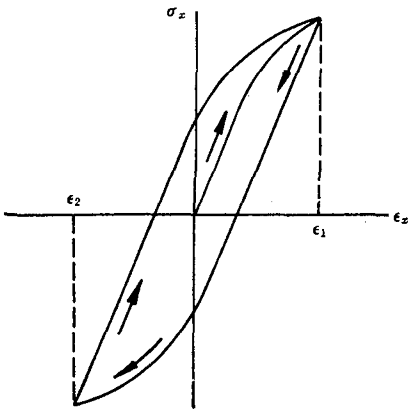


FIGURE 1.42

material is stressed in tension into the plastic range up to strain ϵ_1. The tension stress is then reduced and compressive stress is built up until a compressive strain ϵ_2 is reached. The stressing is then reversed until strain ϵ_1 is again reached, at which point one cycle has been completed. As may be verified, the area within the loop is the energy dissipated per unit volume during one cycle. A material that behaves as shown in Fig. 1.42 is said to exhibit hysteresis* and the loop is called a hysteresis loop.

When a material such as chalk, glass, concrete, cast iron, etc., is pulled in tension it experiences a *brittle failure*. These materials also have failing stresses for shear and for axial compression but these are, in general, larger than the failing stress in tension. By definition, a brittle material is one that fails in tension without exhibiting ductility. A failure criterion describes the combination of stresses that causes a stress failure. The criterion which is most commonly used for brittle materials subjected to a general state of stress is

$$\sigma_1 = \sigma_f \tag{1.58}$$

where σ_1 is the maximum principal stress, and σ_f is the stress required to fracture the material in a tension test. According to this criterion a crack will start at the most highly stressed point in a brittle material when the largest principal stress at that point reaches σ_f.

* From the Greek word meaning "to lag behind," since the stress seems to lag behind the strain as compared with the behavior of an elastic material.

A ductile material is one that can undergo large plastic deformations, for example, mild steel at room temperature. Under some conditions it may be considered satisfactory for a body to be stressed in the plastic range but in other cases it may be required that the maximum stress always be in the linearly elastic range. In this latter case, the design may be said to be a failure if yielding occurs and, hence, it is important to be able to estimate when yielding will occur. In a tension test of mild steel, yielding occurs when the axial stress reaches σ_0, the yield point stress. Suppose that a body made of this steel has at a highly stressed point the three principal stresses σ_1, σ_2, and σ_3. We wish to determine a failure criterion for this ductile material which relates σ_1, σ_2, and σ_3 for yielding to the yield point stress σ_0. Plastic deformation of a solid occurs by processes involving the plastic shearing or slip of atoms past each other. Large shear stresses cause such slip or plastic shearing, so one criterion which can be applied is the so-called maximum shear stress criterion suggested by Tresca.* For example, in the tensile test the maximum shear stress when yielding occurs is $\sigma_0/2$, and this is written

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_0}{2}$$

or

$$\sigma_1 - \sigma_2 = \sigma_0 \tag{1.59}$$

where σ_1 and σ_2 are the maximum and minimum principal stresses. This is called the Tresca yield criterion, or the maximum shear stress yield criterion. The Tresca criterion is quite easy to apply and agrees quite well with experiments. An even better agreement is obtained by taking the yield criterion to be a critical value of the shear strain energy, as was suggested by Hencky and by Mises.† The strain energy of distortion is given by Eq. (1.48) when $\bar{\sigma}$ is set equal to zero and, since $(\sigma_1 - \sigma_2) = (\sigma_1' - \sigma_2')$, etc., the expression for the shearing energy may be written

$$V_s = \frac{1}{2G}\frac{2}{3}\left[\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 + \left(\frac{\sigma_3 - \sigma_1}{2}\right)^2\right]$$

It is known from tension tests that yielding occurs when $\sigma_1 = \sigma_0$, $\sigma_2 = \sigma_3 = 0$. According to the criterion, therefore, yielding should begin when $V_s = (1/6G)\sigma_0^2$ and, hence, the yield condition is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2 \tag{1.60}$$

* H. Tresca, *Comptes Rendus Acad. Sci.*, Paris (1864).

† R. von Mises proposed the use of Eq. (1.60) in 1913 and H. Hencky, in 1924, gave the interpretation in terms of shear strain energy.

This is the so-called shear-distortion energy criterion for ductile materials.* While Eq. (1.60) agrees with experiment somewhat better than Eq. (1.59), it is not so easy to apply in the analysis of some problems and, hence, the Tresca criterion is often used as a matter of convenience. A yield criterion should not be applied to a material unless it is known that the criterion is applicable. For example, Eq. (1.60) gives good results when applied to mild steel at room temperatures, but it does not apply at very low temperatures for then mild steel becomes brittle.

When designing a machine or a structure, it is of practical importance to know how close to failure it is under the conditions for which it is designed. This is specified by the *factor of safety*, which is defined to be the factor by which the forces and weights used in design must be increased to reach the failing point. A factor of safety of 2 means that the applied load for which the body was designed must be doubled to reach failure.

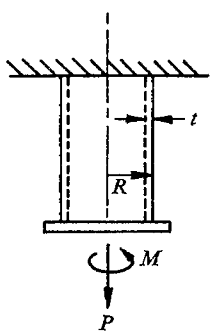
The failures of machine parts frequently begin at points of high stress concentration (see Section 5-6). Because of this, parts should be proportioned so as to minimize stress concentrations whenever possible.

Problems

1.81 Does it seem reasonable that the dilatational strain energy does not affect yielding?

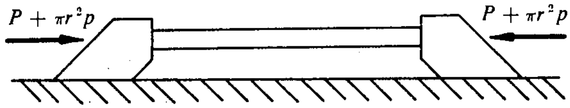
1.82 A certain steel is found to have a yield point in tension $\sigma_0 = 45{,}000$ psi. A thin-walled tube made of the same material is twisted in torsion. At what shear stress will the tube yield?

1.83 A thin-walled tube has the dimensions $R = 12$ in. and $t = 0.1$ in. The yield point of the material in tension is $\sigma_0 = 50{,}000$ psi. An axial weight, $P = 200{,}000$ lb, is hung as shown in the diagram, and then a twisting moment M is applied as shown. For what value of M will the tube yield, according to the yield criterion of Eq. (1.60)?



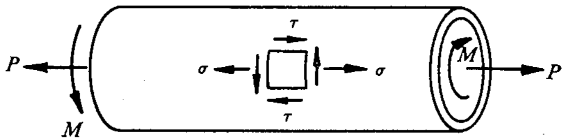
* It can be shown that this equation also expresses the condition for a material to yield isotropically.

1.84 A standard way of testing welded steel pipe is to mount a length of pipe in a jig that can exert a net axial force P in the pipe. The pipe can also be filled with water at an internal pressure p. A pipe whose inner diameter is 30 in. and whose wall thickness is $\frac{1}{4}$ in. is tested with $p = 800$ psi. What value of net axial force in the pipe will cause it to yield if $\sigma_0 = 65{,}000$ psi for the steel? What happens if the operator tries to make P larger than this value?



1.85 A crack initiated at a point on the free surface of an isotropic, brittle material when the strains at that point were $\epsilon_x = 0.0005$, $\epsilon_y = -0.0003$, and $\gamma_{xy} = -0.0006$. The crack formed perpendicular to the maximum principal strain at the point. Sketch the relative orientation of the crack and the x-y axes at that point. What stress would cause fracture of this material in a tensile test if $E = 10^7$ psi, $\nu = 0.2$?

1.86 A thin-walled tube of radius R is subjected to a twisting moment M and an axial force P, so that each element has stresses σ and τ as shown in the diagram. Show that the yield condition of Eq. (1.60) for this state of stress can be written $\sigma^2 + 3\tau^2 = \sigma_0^2$ (use Mohr's Circle). Sketch a diagram of σ/σ_0 vs. τ/τ_0, where τ_0 is the yield point stress in shear.



CHAPTER 2

EQUATIONS OF THE THEORY OF ELASTICITY

2-1 INTRODUCTION

This chapter deals with the equations which must be satisfied by the stresses and strains in an elastic body. The body may be, for example, a machine element or a structural element composed of a specified material with known stress-strain properties, and from an analysis of the entire machine or structure the forces which act on the element in question are determined. If the size and shape of the body are specified, it may be required to determine the distribution of stress and strain produced by the applied forces or to determine the displacements of various points in the body. This problem is one of analysis, and the equations that are required to determine these quantities will be derived in this chapter. It may be pointed out that it is a more difficult problem when the material and the shape of the body are to be determined to withstand certain applied forces, subject to limitations on the size or weight, or on the maximum stress in the body, or on displacements of the body. This latter problem is encountered in design and both analysis and synthesis must be applied to obtain a satisfactory solution.

A problem in analysis of stress and strain arises from a situation in which a body is subjected to certain actions. These may be applied forces, temperature changes, accelerations, or specified displacements of certain points of the body, and the resulting stresses, strains, and displacement are to be determined. The applied forces may act at various points on the surface of the body (concentrated forces), or they may be distributed over part of the surface of the body (distributed loads), or they may act on elements within the body (body forces) as in the case of gravity, or a magnetic field, or D'Alembert forces if the body is accelerating.

In this chapter we shall derive the equations of elasticity whose solution will give the stresses, strains, and displacements of a loaded body. A statement of the conditions imposed on a problem in elasticity is as follows:

(a) Every element of mass in the body is in a state of equilibrium or, more generally, Newton's second law must be satisfied.
(b) The material of which the body is composed has specified stress-strain relations.
(c) The strains are functions of the derivatives of the displacements.
(d) The stresses, strains, and displacements must be consistent with the prescribed loading and constraints of the body.

The first of these is a physical law that must be satisfied, the second is a statement of the properties of the material, the third is a geometrical condition, and the last is the loading condition or, as it is sometimes called, the boundary condition. Here we shall consider material which obeys Hooke's law. We shall also impose the condition that the strains be small, so that the expressions for the strains given in Section 1-5 can be used.

Mathematical Formulation. In order to make a stress analysis of a body, the four conditions listed above must be stated in mathematical terms. The first step is to choose a coordinate system. Considerable simplification can be achieved if an appropriate coordinate system is chosen in which the loading condition may be written in a relatively simple form. We shall use cartesian coordinates in this chapter, but other coordinate systems will be used in subsequent chapters. A freebody diagram of an element at point x, y, z in the body will appear as in Fig. 2.1.* Six equations of equilibrium can be written: the sums of the forces in the x-, y-, and z-directions must be equal to zero, and the sums of the moments about the x-, y-, z-axes must be equal to zero. The three moment equations of equilibrium were used in Chapter 1 in establishing that $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, $\tau_{zx} = \tau_{xz}$, so only the three force equations of equilibrium need be written. The element is acted upon by the stresses shown in Fig. 2.1, where

* Note that s_x, s_y, s_z appear on the negative faces of the element at point x, y, z, and hence the stress vectors on the positive faces of the $dx\,dy\,dz$ element are s_{x+dx}, s_{y+dy}, and s_{z+dz}.

$$\mathbf{s}_{x+dx} = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\,dx\right)\mathbf{i} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\,dx\right)\mathbf{j} + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}\,dx\right)\mathbf{k}$$

$$\mathbf{s}_{y+dy} = \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\,dy\right)\mathbf{i} + \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}\,dy\right)\mathbf{j} + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\,dy\right)\mathbf{k}$$

$$\mathbf{s}_{z+dz} = \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\,dz\right)\mathbf{i} + \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\,dz\right)\mathbf{j} + \left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\,dz\right)\mathbf{k}$$

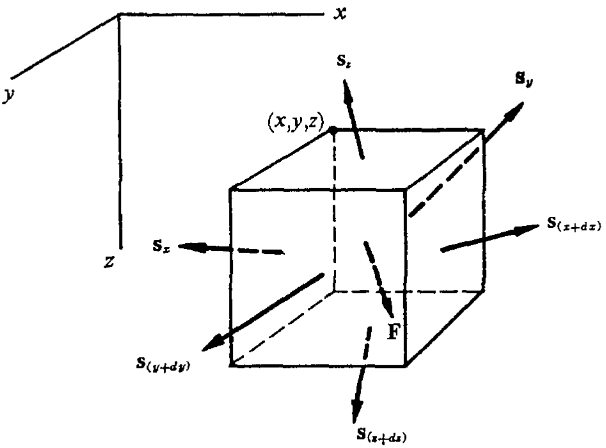


FIGURE 2.1

As may be deduced from Fig. 2.1 the vector equation of *equilibrium is*

$$(\mathbf{s}_{x+dx} - \mathbf{s}_x)\,dy\,dz + (\mathbf{s}_{y+dy} - \mathbf{s}_y)\,dx\,dz$$
$$+ (\mathbf{s}_{z+dz} - \mathbf{s}_z)\,dx\,dy + \mathbf{F}\,dx\,dy\,dz = 0$$

where $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ is the intensity of body force and the increments $\mathbf{s}_{x+dx} - \mathbf{s}_x$, etc., are given by

$$\mathbf{s}_{x+dx} - \mathbf{s}_x = \frac{\partial \sigma_x}{\partial x}\,dx\,\mathbf{i} + \frac{\partial \tau_{xy}}{\partial x}\,dx\,\mathbf{j} + \frac{\partial \tau_{xz}}{\partial x}\,dx\,\mathbf{k}$$

$$\mathbf{s}_{y+dy} - \mathbf{s}_y = \frac{\partial \tau_{xy}}{\partial y}\,dy\,\mathbf{i} + \frac{\partial \sigma_y}{\partial y}\,dy\,\mathbf{j} + \frac{\partial \tau_{yz}}{\partial y}\,dy\,\mathbf{k}$$

$$\mathbf{s}_{z+dz} - \mathbf{s}_z = \frac{\partial \tau_{zx}}{\partial z}\,dz\,\mathbf{i} + \frac{\partial \tau_{zy}}{\partial z}\,dz\,\mathbf{j} + \frac{\partial \sigma_z}{\partial z}\,dz\,\mathbf{k}$$

In scalar form, after some simplification, the equations of equilibrium can be written

$$
\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{yx}}{\partial x} + Y = 0 \\
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0
\end{aligned}
\tag{2.1}
$$

where X, Y, Z are the components of body force intensity in the x-, y-, z-directions. These three equations involve six unknown stresses and hence the three equilibrium equations are not sufficient, that is, the problem is statically indeterminant. Additional equations are provided by the fact that the material obeys Hooke's law, which for an isotropic material is

$$
\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \qquad & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\epsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) \qquad & \gamma_{yz} &= \frac{\tau_{yz}}{G} \\
\epsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) \qquad & \gamma_{xz} &= \frac{\tau_{xz}}{G}
\end{aligned}
\tag{2.2}
$$

With these there are nine equations with six unknown stresses and six unknown strains so additional equations are still required. These are given by the geometrical condition that, for small deformations, the strains are related to the displacements by

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} \qquad & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\epsilon_y &= \frac{\partial v}{\partial y} \qquad & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\epsilon_z &= \frac{\partial w}{\partial z} \qquad & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}
\end{aligned}
\tag{2.3}
$$

With these, the number of equations (fifteen) is equal to the number of unknowns (fifteen).

The foregoing Eqs. (2.1), (2.2), (2.3), are mathematical statements of the conditions of the problem.* These, together with the boundary conditions specify the problem. In the following sections, we shall examine the forms assumed by these equations in the case of plane stress and plane strain.

* These equations were first given by the French mathematician A. Cauchy (1789–1857) who published many papers on the theory of elasticity.

2-2 EQUATIONS OF ELASTICITY: PLANE STRESS AND PLANE STRAIN

Equilibrium Equations. A plane stress problem is defined as one in which σ_z equals zero throughout the body. Such problems are encountered in a number of engineering applications. For example, a plate loaded by forces in the xy-plane as shown in Fig. 2.2 has the stress components σ_z, τ_{yz}, and τ_{xz}

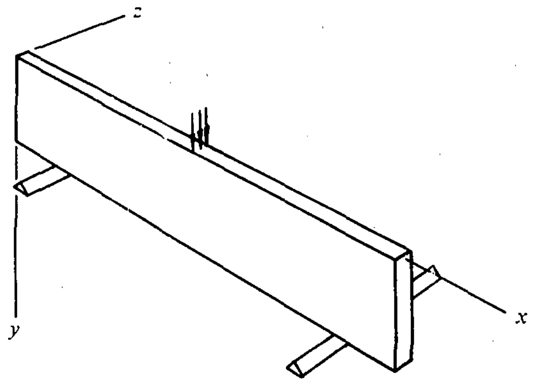


FIGURE 2.2

equal to zero on the surfaces. It can be assumed with negligible error that for a sufficiently thin plate these components are zero throughout when the applied forces are distributed uniformly across the width of the plate. Plane strain problems are also encountered in a number of practical applications. For example, a long pipe spinning about its axis is in a state of plane strain, except near its ends (plane stress prevails at the end surfaces). Adjacent sections with axial strain which varies through the thickness, are shown in Fig. 2.3 and it is seen that these cannot fit together to form a pipe without voids or cracks. In the actual pipe, however, the cohesion of the material prevents the formation of cracks in the pipe wall and thus holds the strain in the z-direction constant throughout so that plane sections perpendicular to the axis remain plane when the pipe is spinning.

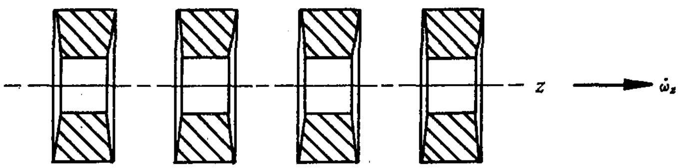


FIGURE 2.3

The equations of equilibrium for plane stress and plane strain are given by Eqs. (2.1) when the appropriate stresses are set equal to zero. However, for purposes of illustration we shall derive these equations from the element. There are three stresses (σ_x, σ_y, τ_{xy}), which may vary with the coordinates x and y, in plane stress and in plane strain problems. An element dx, dy, dz is shown in Fig. 2.4 with these stresses acting. The forces X and Y are body forces

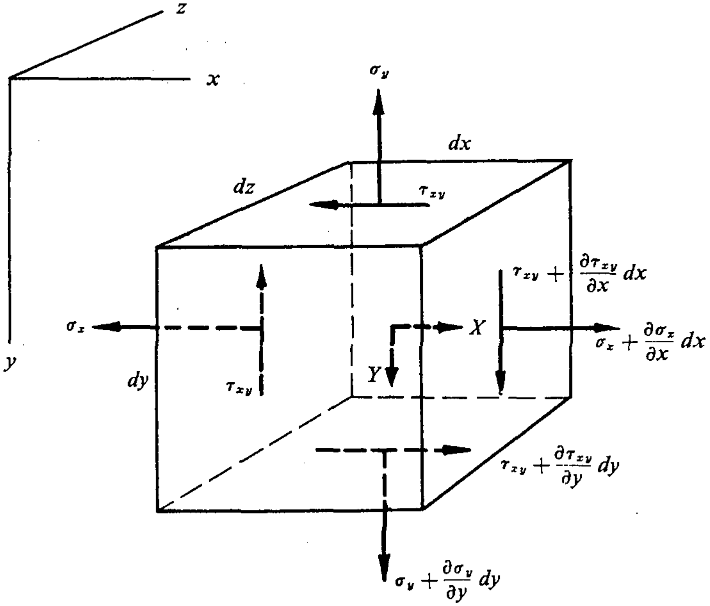


FIGURE 2.4

per unit volume which act in the x- and y-directions, respectively. The stresses, as shown, are all positive according to our sign convention. A stress component gives the intensity of stress acting on the face of the element and hence the force is given by multiplying the stress by the area of the face. Summation of the forces in the x-direction gives

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx - \sigma_x\right) dy\,dz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy - \tau_{xy}\right) dx\,dz + X\,dx\,dy\,dz = 0$$

A similar equation is obtained by summing the forces in the y-direction. When simplified, these equations reduce to

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad (2.4) \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0 \qquad (2.5) \end{aligned}\right\} \text{equilibrium equations}$$

Compatibility Equations for Plane Stress and Plane Strain. The three strains, ϵ_x, ϵ_y, and γ_{xy} which are of interest in the plane stress and plane strain problems, are functions of the x- and y-components of displacement, u and v

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.6}$$

The three strains are functions of the two displacements and, hence, are not independent. The following equation expresses this fact

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \qquad \text{compatibility condition} \tag{2.7}$$

Equation (2.7) is called the condition of compatibility and it must be satisfied in order that the strain components be related to the displacement functions as expressed in Eq. (2.6). The compatibility equation puts a condition on the stresses since the strains calculated from them must satisfy Eq. (2.7). Expressing the strains in terms of the stresses by means of Hooke's law for plane stress, substituting in Eq. (2.7) and eliminating τ_{xy} by means of the equilibrium equations, the following standard form is obtained for the compatibility equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1 + \nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \tag{2.8}$$

The compatibility equation for plane strain (ϵ_z = constant) can be shown to have a similar form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\left(\frac{1}{1 - \nu}\right)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \tag{2.9}$$

The compatibility equations for plane stress and plane strain are identical when the body forces are constant. The two equilibrium equations and the compatibility equation (2.8) or (2.9) for plane stress or plane strain are sufficient to determine the stresses when the boundary conditions are specified. See Probs. 2.8–2.13 for examples of solutions. The compatibility equations for the three dimensional problem are derived in Appendix I.

Boundary Conditions. The solution of Eqs. (2.4), (2.5), and (2.8) or (2.9) must also satisfy the boundary conditions. That is, the solution

$$\sigma_x = f_1(x, y), \quad \sigma_y = f_2(x, y), \quad \tau_{xy} = f_3(x, y)$$

must give stresses at points x, y on the boundary that are identical with the stresses produced by the surface loading, σ_n and τ_{sn}. From Mohr's circle these conditions require

$$\left.\begin{aligned} \sigma_n &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{sn} &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}\right\} \text{boundary conditions} \tag{2.10}$$

where θ is the angle between the x-axis and the unit vector $\mathbf{n}$ normal to the surface and directed out of the body as shown in Fig. 2.5. Equations (2.10) express the boundary conditions.

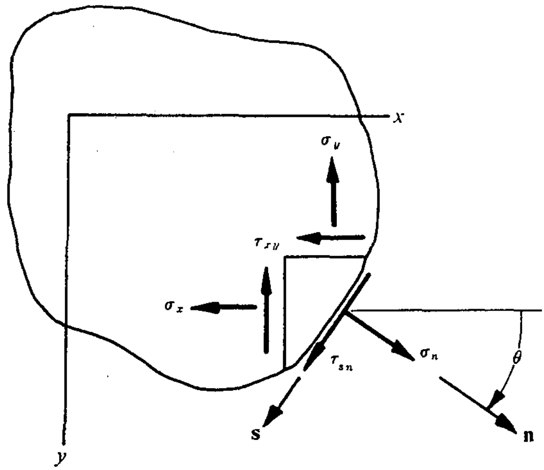


FIGURE 2.5

Summary of Equations for the Plane Stress and Plane Strain Problems. The equilibrium equations, the compatibility equation, and the boundary conditions completely describe the problem, and the solution of these equations gives the stress field in the body. In deriving the equations, it was assumed that the strains were small so that products of strains were negligible compared to the strains themselves, and that the displacements were finite and were continuous functions of position in the body. The equations of plane stress ($\sigma_z = \tau_{xy} = \tau_{yz} = 0$) are

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0 \end{aligned}\right\} \text{equilibrium}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1 + \nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \quad \text{compatibility} \qquad (2.11)$$

$$\left.\begin{aligned} \sigma_n &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{sn} &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}\right\} \text{boundary conditions}$$

For plane strain, only the compatibility equation is different

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$$

We note that if the derivatives of the body forces are zero, the equations are independent of the elastic constants. In this case, the stresses in a steel beam will be the same as in a lucite beam having the same dimensions and loading. This means that experimental models may be made of any elastic material that is convenient. Appendix III explains how this fact is utilized in the photoelastic method of measuring stresses and strains.

The Stress Function. A method commonly employed in the solution of problems in plane stress and plane strain introduces a stress function $\Phi(x, y)$. When there are no body forces acting, the stresses are related to the derivatives of Φ as follows:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x\, \partial y} \tag{2.12}$$

Equations (2.12) are such that the equilibrium equations are automatically satisfied when the body forces are zero. The compatibility equation written in terms of the stress function is*

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} \equiv \nabla^4 \Phi = 0 \tag{2.13}$$

The equations of elasticity for plane stress and plane strain with zero body forces are thus written as a single, fourth-order equation in Φ. Any solution of Eq. (2.13) gives stresses which satisfy equilibrium and compatibility. The problem is then to find that solution which represents the desired loading (satisfies the boundary conditions).

A uniaxial state of stress $\sigma_x = \sigma_1$, for example, is represented by the stress function $\Phi = (\sigma_1/2)y^2$, as may be readily verified using Eq. (2.12). This stress function satisfies compatibility, Eq. (2.13). Normal stresses which vary linearly or parabolically are represented by polynomial terms in the stress function such as c_3y^3, $c_3'x^3$, $c_3''xy^2$, or c_4y^4, $c_4'x^4$, $c_4''x^2y^2$ respectively. A stress function such as c_4y^4 does not satisfy compatibility, so additional terms must be added if Φ is to represent a state of stress in an isotropic elastic solid (see Prob. 2.8).

* This equation can be written as $\nabla^2(\nabla^2\Phi) = 0$ and is the well known bi-harmonic equation.

2-3 A UNIQUENESS THEOREM

The plane stress field is completely described by three functions

$$\sigma_x = f_1(x, y), \quad \sigma_y = f_2(x, y), \quad \tau_{xy} = f_3(x, y)$$

which must satisfy Eqs. (2.11). It is of interest to inquire whether there is only one set f_1, f_2, and f_3 that satisfies Eqs. (2.11) or whether there may be more than one set of internal stresses that satisfy the equations for a particular set of surface and body forces.

Consider a particular problem in which specified body forces and surface forces are acting on an elastic body. Let us assume that there are two solutions $\sigma_{x_1}, \sigma_{y_1}, \tau_{xy_1}$ and $\sigma_{x_2}, \sigma_{y_2}, \tau_{xy_2}$ that describe the state of stress in the body, both of which satisfy Eq. (2.11). We shall show that under certain conditions the two solutions must be identical, that is, the solution is unique.

The partial differential equations which these solutions satisfy and the boundary conditions are linear equations, for example,

$$\frac{\partial \sigma_{x_1}}{\partial x} + \frac{\partial \tau_{xy_1}}{\partial y} + X = 0$$

$$\frac{\partial \sigma_{x_2}}{\partial x} + \frac{\partial \tau_{xy_2}}{\partial y} + X = 0$$

Subtracting the second equation from the first gives

$$\frac{\partial(\sigma_{x_1} - \sigma_{x_2})}{\partial x} + \frac{\partial(\tau_{xy_1} - \tau_{xy_2})}{\partial y} = 0$$

If all of the equations are written in this difference form there is obtained the following set of equations where we have written σ_{x_3} for $(\sigma_{x_1} - \sigma_{x_2})$, σ_{y_3} for $(\sigma_{y_1} - \sigma_{y_2})$, and τ_{xy_3} for $(\tau_{xy_1} - \tau_{xy_2})$

$$\frac{\partial \sigma_{x_3}}{\partial x} + \frac{\partial \tau_{xy_3}}{\partial y} = 0$$

$$\frac{\partial \sigma_{y3}}{\partial y} + \frac{\partial \tau_{xy_3}}{\partial x} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{x_3} + \sigma_{y_3}) = 0 \tag{2.14}$$

$$0 = \frac{\sigma_{x_3} + \sigma_{y_3}}{2} + \frac{\sigma_{x_3} - \sigma_{y_3}}{2} \cos 2\theta + \tau_{xy_3} \sin 2\theta$$

$$0 = -\frac{\sigma_{x_3} - \sigma_{y_3}}{2} \sin 2\theta + \tau_{xy_3} \cos 2\theta$$

It is seen that Eqs. (2.12) describe the stress field σ_{x_3}, σ_{y_3}, and τ_{xy_3} in a body that has *no* surface forces and *no* body forces, that is, in an unloaded body. If the body is of a type that is free of internal stresses in the absence of external forces, it follows that $(\sigma_{x_1} - \sigma_{x_2}) = 0$, $(\sigma_{y_1} - \sigma_{y_2}) = 0$, and $(\tau_{xy_1} - \tau_{xy_2}) = 0$, that is, the stress fields (1) and (2) are the same. In this case, there is only one stress field that satisfies the equilibrium and compatibility equations and gives the prescribed stresses on the boundary. The uniqueness theorem is therefore proved.

The uniqueness theorem is not generally applicable when the strains are not sufficiently small, or when the stresses exceed the proportional limit, or when the displacements are not continuous functions of position in the body. Examples of discontinuities in displacements which give rise to internal stress in the absence of external loads are the dislocations in crystalline materials (see Section 8.1).

2-4 EQUATIONS OF EQUILIBRIUM IN TERMS OF DISPLACEMENTS

The three equilibrium equations that were derived in Section 2-1 involved six unknown stresses and, therefore, additional equations were required to describe the statically indeterminant problem. However, if we consider the three displacements u, v, and w to be the unknowns, the three equilibrium equations together with the boundary conditions suffice to describe the problem. For example, Hooke's law for isotropic material, as expressed by Eqs. (1.34) is

$$\begin{aligned} \sigma_x &= 2G\epsilon_x + \lambda(\epsilon_x + \epsilon_y + \epsilon_z) \qquad & \tau_{xy} &= G\gamma_{xy} \\ \sigma_y &= 2G\epsilon_y + \lambda(\epsilon_x + \epsilon_y + \epsilon_z) \qquad & \tau_{xz} &= G\gamma_{xz} \\ \sigma_z &= 2G\epsilon_z + \lambda(\epsilon_x + \epsilon_y + \epsilon_z) \qquad & \tau_{yz} &= G\gamma_{yz} \end{aligned} \tag{2.15}$$

The strains in these equations may be expressed in terms of the displacements and the resulting equations may be substituted into the equilibrium Eqs. (2.1) giving

$$\begin{aligned} (\lambda + G)\frac{\partial e}{\partial x} + G\,\nabla^2 u + X = 0 \\ (\lambda + G)\frac{\partial e}{\partial y} + G\,\nabla^2 v + Y = 0 \\ (\lambda + G)\frac{\partial e}{\partial z} + G\,\nabla^2 w + Z = 0 \end{aligned} \tag{2.16}$$

where

$$e = \epsilon_x + \epsilon_y + \epsilon_z; \quad \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

and X, Y, and Z are body forces.* Equations (2.16), together with the boundary conditions, completely describe the three functions u, v, and w. The strains and stresses may be calculated from the displacement functions, u, v, and w. Usually in engineering problems the boundary conditions are specified in terms of the surface forces rather than in terms of the surface displacements. In this case the equations in terms of stresses are more suitable than the equations in terms of displacements.

2-5 THE EQUATIONS OF HYDRODYNAMICS AND ELASTICITY

It is of some interest to compare the equations of elasticity with those of hydrodynamics. In the preceding sections we considered the state of stress in a solid body in which the stresses were related to the displacements. In viscous, incompressible fluids, the stresses are related to the velocities, and the equations which describe velocities in fluids are similar to those which describe displacements in elastic bodies. The fundamental equation of hydrodynamics is the Navier-Stokes equation,† and its scalar components may be written for steady state flow as

$$\begin{aligned} \tfrac{1}{3}\mu \frac{\partial e}{\partial x} + \mu \nabla^2 u + X - \frac{\partial p}{\partial x} &= 0 \\ \tfrac{1}{3}\mu \frac{\partial e}{\partial y} + \mu \nabla^2 v + Y - \frac{\partial p}{\partial y} &= 0 \\ \tfrac{1}{3}\mu \frac{\partial e}{\partial z} + \mu \nabla^2 w + Z - \frac{\partial p}{\partial z} &= 0 \end{aligned} \tag{2.17}$$

where μ is the absolute viscosity,

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and u, v, and w are the velocity components in the x-, y-, and z-directions, p is the mean pressure, and X, Y, and Z are body forces. The similarity between Eqs. (2.16) and (2.17) is apparent. The differences between the two sets of equations are indicative of the differences in material properties.

* L. Navier (1785–1836) is regarded as the founder of the modern theory of elastic solids. In his paper, *Memoirs sur les Lois de l'Equilibre et du Mouvement des Corps Solides Elastiques* (1827) he gave, for the first time, the three equations of equilibrium of a solid, Eqs. (2.16).

† See for example *Fluid Flow*, Acosta and Sabersky, Macmillan (1963).

2-6 ST. VENANT'S PRINCIPLE

St. Venant's principle states that the difference in stresses produced by two sets of statically equivalent forces acting on a surface area A diminishes with distance from A and becomes negligible at distances large compared to the linear dimensions of A. This principle is often used in the solution of problems in stress analysis. For example, a mathematical description of the basic problem in elasticity was formulated in the preceding sections, and such a purely mathematical approach may be used to obtain exact solutions which satisfy the boundary conditions that are prescribed in a particular problem. However, in practical problems the boundary conditions often cannot be specified exactly, as in the case of a beam which is built in at one end and loaded by the two equal forces shown in Fig. 2.6a. The forces produce a couple which must be balanced by actions at the built in end to maintain equilibrium of the beam. Two different reactions, each of which could balance the applied couple are shown in Fig. 2.6b and c. In a practical problem, the actual distribution of such reactions is seldom known, and an important question may be asked: How sensitive is the stress distribution in the beam to the detailed distribution of the reactions? St. Venant's principle,*

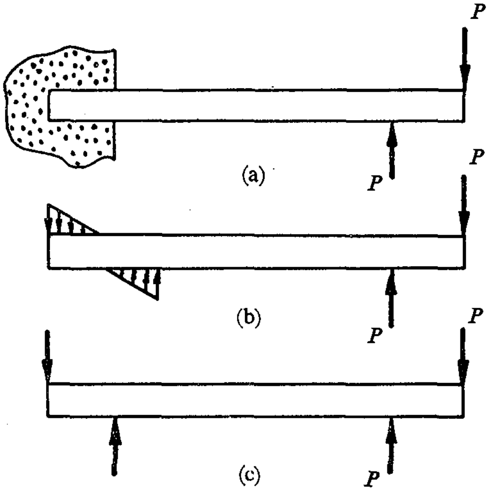


FIGURE 2.6

applied to the cantilever beam in Fig. 2.6 answers this question as follows: The detailed distribution of the reactions affects the stress distribution in the vicinity of the beam end, but at distances of several beam depths away from the reaction the stresses are essentially dependent only on the magnitude of the couple, and not on how the applied forces are distributed. St. Venant's

* Barré de St. Venant (1797–1886) is particularly known for his analysis of the torsion of shafts, given in Chapter 6.

principle thus states that only the stresses in the vicinity of applied loads are sensitive to details of how these loads are distributed. Several examples which demonstrate St. Venant's principle are given in Appendix III and in Section 5-6.

Another way of stating St. Venant's principle is as follows: If a set of self-equilibrating forces acts on a surface area A the internal stresses diminish with distance from A. The rate at which the stresses attenuate with distance may be influenced by the shape of the body and must be estimated in each case.

Problems

2.1 Obtain Eqs. (2.1) by first drawing a freebody diagram of the element which shows all of the stress components acting on it, and then writing the equations of equilibrium.

2.2 Verify Eq. (2.7) and (2.8).

2.3 Verify Eq. (2.9).

2.4 Verify that the stress function of Eqs. (2.12) gives stresses which satisfy equilibrium.

2.5 Derive Eqs. (2.16).

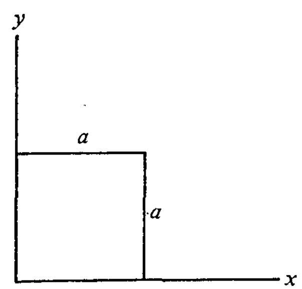
2.6 Explain how a stress field could exist in a body of uniform temperature when there are no body forces or surface forces acting. How would you describe the conditions that such a stress field must satisfy?

2.7 By using Eqs. (2.16), verify that the dilatation e satisfies $\nabla^2 e = 0$, if the body forces are constant.

2.8 Check whether the stress function $c(x^4/4!)$ satisfies compatibility and if it does not, add a term which is a function of y only so that it does. The resulting Φ represents a stress field for $-\infty < x < \infty$, $-\infty < y < \infty$. Draw a freebody diagram showing the surface loading of a 45° isosceles triangle whose legs lie along the x- and y-axes.

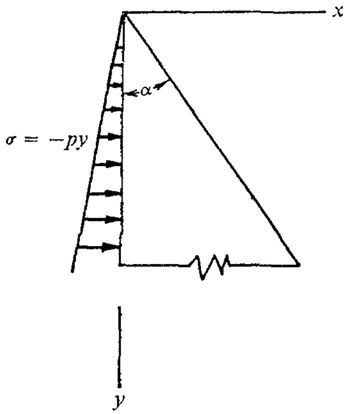
2.9 Determine the stresses acting on the boundary of the rectangular plate shown in the diagram and sketch the distribution of boundary stresses, for

$$\Phi = \frac{p}{a^2}\left(\tfrac{1}{2}x^2y^2 - \tfrac{1}{6}y^4\right)$$



2.10 Show that the stresses in the wedge-shaped plate that is loaded as shown in the diagram are represented by the stress function

$$\Phi = -\frac{p}{6}\left(y^3 + \frac{2x^3}{\tan^3\alpha} - \frac{3x^2y}{\tan^2\alpha}\right)$$



2.11 Show that the equations of equilibrium for plane stress in polar coordinates are

$$\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + X_r &= 0 \\ \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + X_\theta &= 0 \end{aligned} \tag{2.18}$$

where X_r and X_θ are the components of the body force in the r- and θ-directions.

2.12 Prove that the compatibility equation in polar coordinates can be written

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\sigma_r + \sigma_\theta) \equiv \nabla^2(\sigma_r + \sigma_\theta) = 0 \tag{2.19}$$

when X_r and X_θ are constant.

2.13 Show that the stress function Φ satisfies equilibrium in polar coordinates if the body forces are zero and

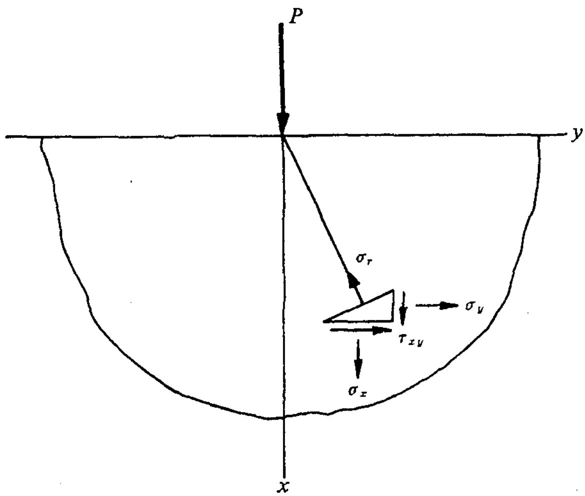
$$\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \\ \sigma_\theta &= \frac{\partial^2\Phi}{\partial r^2} \\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) \end{aligned} \tag{2.20}$$

2.14 Verify that the stress function $\Phi = -(P/\pi)y \tan^{-1} y/x$ gives the stress distribution in a semi-infinite plate with a force P per unit width applied normal to the boundary as shown in the diagram

$$\sigma_x = -\frac{2P}{\pi}\frac{x^3}{(x^2+y^2)^2}$$

$$\sigma_y = -\frac{2P}{\pi}\frac{xy^2}{(x^2+y^2)^2} \tag{2.21}$$

$$\tau_{xy} = -\frac{2P}{\pi}\frac{yx^2}{(x^2+y^2)^2}$$



2.15 Show that Prob. 2.14 is represented by the stress function

$$\Phi = -(P/\pi)\, r\theta \sin\theta$$

Deduce σ_r and show that σ_θ and $\tau_{r\theta}$ are zero throughout.

CHAPTER 3

APPLICATIONS TO BEAMS

3-1 INTRODUCTION

A variety of mathematical techniques are used to solve problems in elasticity. The more advanced mathematical methods will not be discussed here, but they can be found in books on the mathematical theory of elasticity.* Exact mathematical solutions of the equations of elasticity have been obtained for many interesting problems and we shall make use of certain of these solutions, and develop approximate solutions that will be compared with exact solutions. The suitability of the approximations that are made in solving problems in stress analysis may be checked by comparing the approximate solutions with exact solutions or with stresses and strains that are determined by experimental methods, such as those discussed in Appendix III.

In many practically important problems, the complex nature of the loading or the shape of the body may make an exact solution so difficult that approximations must be made in order to simplify the analysis. It is important to learn to make approximations in the analysis of real problems so that (1) a solution can be obtained by practicable mathematical techniques, and (2) the solution so obtained will have adequate accuracy for the use to be made of it.

* See, for example, *The Theory of Elasticity*, by S. Timoshenko and J. N. Goodier, McGraw-Hill (1951).

When it is impractical to work out an exact solution of a problem the following simplifications may be made: (1) the actual geometry of the body in the real problem may be approximated by one or more bodies of relatively simple form; (2) the actual loading in the real problem may be approximated by simplified loading (generally the static equivalent of the real loading); or (3) the imposed deflections in the real problem may be approximated by a set of simpler displacements. In the following paragraphs we shall first examine some exact solutions and shall then develop an approximate method of analyzing beam problems.

3-2 EXTENSION OF A BAR

The simplest stress distribution (plane stress) in a straight rectangular bar, whose axis is parallel to the x-axis, is:

$$\sigma_x = \sigma_0$$

$$\sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

This represents an axial extension of the bar by stresses σ_0 applied at each end as shown in Fig. 3.1a. It may be verified that the stress distribution satisfies the equations of equilibrium and compatibility. The resultant force in the bar is $P = \sigma_0 A$ and its line of action passes through the centroid of the cross section.*

The strain produced by the foregoing state of stress is

$$\epsilon_x = \frac{\sigma_0}{E}; \quad \epsilon_y = -\frac{\nu\sigma_0}{E}; \quad \epsilon_z = -\frac{\nu\sigma_0}{E}$$

$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$$

The length of the bar is increased by an amount

$$\Delta L = \epsilon_x L = \frac{\sigma_0 L}{E} = \frac{PL}{AE}$$

* By definition the centroidal z-axis is so located that the first moment of the area is zero, that is, $\int_A y\, dA = 0$. Similarly, the centroidal y-axis is so located that $\int_A z\, dA = 0$. The intersection of these two axes locates the *centroid* of the area. As may be verified, if the origin of the coordinate axes is located at the centroid, the following integral is zero, $\int_A r\, dA = 0$, where r is the radial distance to dA.

The foregoing uniform stress solution agrees with our usual assumption as to the behavior of a bar pulled in tension. Actually, when a real bar is pulled in tension it is necessary to grip the ends of the bar in some fashion in order to apply the forces, and because of this the stresses at the ends of the bar will not be a uniform σ_0. When the resultant of the gripping forces passes through the centroid of the cross section of the bar, the distribution will be substantially a uniform σ_0 at distances greater than about two bar diameters from the ends, as verified by use of photoelastic models such as those shown in Appendix III. This result is in accord with St. Venant's principle.

3-3 PURE BENDING OF PRISMATIC BARS

An inverse method of solution will be used to analyze the problem of pure bending of a prismatic bar, that is, we shall examine a simple solution of the equations of elasticity and show that it represents the stress field that exists in a bar subjected to pure bending. *Consider the stress field shown in Fig. 3.1b* which varies linearly with y

$$\begin{gathered} \sigma_x = c_1 y \\ \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \end{gathered} \tag{3.1}$$

The equations of equilibrium and compatibility are satisfied by Eqs. (3.1)

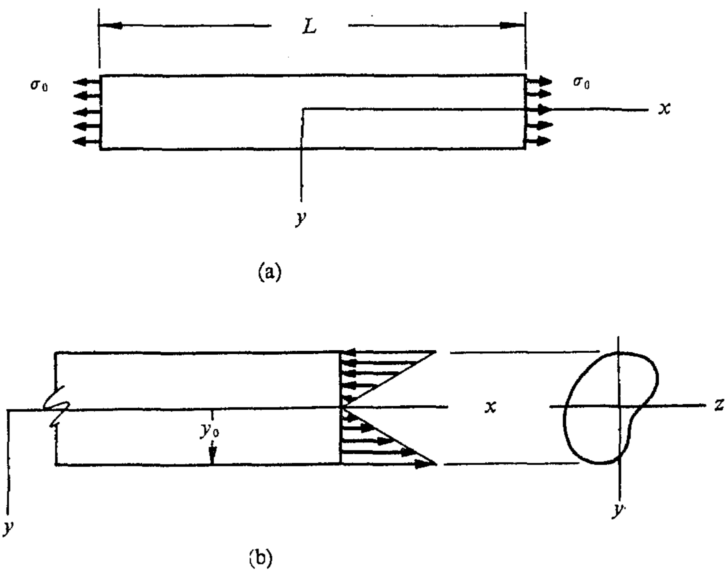


FIGURE 3.1

when the body forces X and Y are zero and, therefore, this represents an exact stress solution. Let us assume that this stress field exists in the initially *straight* prismatic bar of *constant* cross section shown in Fig. 3.1b, where the x-axis is the longitudinal axis of the bar. According to Eq. (3.1) the lateral surfaces of the bar are free of stress so that the internal stresses must be the result of forces applied at each end of the bar.* The resultant axial force on a typical cross section of the bar should be zero for pure bending, so

$$F_x = \int \sigma_x \, dA = c_1 \int_A y \, dA = c_1 y_c A = 0 \tag{3.2}$$

where y_c is the y-coordinate of the centroid of the cross section of area A. For y_c to be equal to zero, the origin of the coordinate system must be located at the centroid of the cross section. The stress as shown in Fig. 3.1b is then given by

$$\sigma_x = \sigma_0 \frac{y}{y_0} \tag{3.3}$$

where σ_0 is the stress at the point farthest from the origin. It is seen that on a cross section of the beam the bending stress is zero along the line $y = 0$ and is tension below and compression above. The line of zero stress is commonly called the *neutral axis* of the cross section.

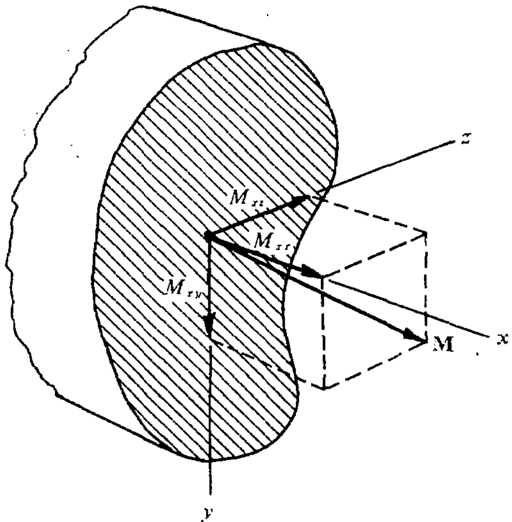
In general, the stresses distributed over the end of a bar will have a resultant moment $\mathbf{M}$ that has components M_{xx}, M_{xy}, and M_{xz}, as shown in Fig. 3.2a. The component M_{xx} is a twisting moment and M_{xy} and M_{xz} are bending moments.† Here the same subscripts and sign convention are used for bending moment vectors that are used for stress vectors; a positive moment M_{xy} acts on the $+x$-face in the $+y$-direction, or on a $-x$-face in the $-y$-direction, etc.

In the case under consideration there will be no twisting moment since there are no shearing stresses. The stress acting on an element of area dA in Fig. 3.2b contributes a negative $\sigma_x y \, dA$ about the z-axis and the integral over the area will give a moment M_{xz} about the z-axis

$$M_{xz} = -\int_A y\sigma_x \, dA = -c_1 \int_A y^2 \, dA = -c_1 I_z \tag{3.4}$$

* It may be noted that the stress fields $\sigma_x = c_3 y^3$, $\sigma_x = c_5 y^5$, etc., cannot represent the stresses in a bar loaded at its ends since they do not satisfy compatibility.

† The component of the resultant moment acting on the x-face and pointing in the y-direction is $\mathbf{M}_{xy}$. In general, the moment resulting from the stresses distributed over the cross section is given by $\mathbf{M} = \int_A \mathbf{r} \times \mathbf{s}_x \, dA$, where $\mathbf{r}$ is the radial vector from the origin to dA.



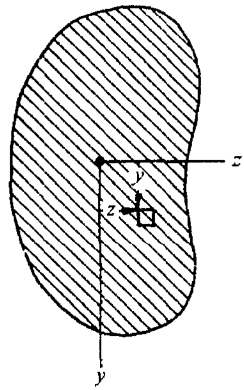


FIGURE 3.2

where I_z is the moment of inertia of the cross section about the neutral axis. Since the cross section is not symmetrical about the y axis, it is clear that there can also be a resultant moment about that axis

$$M_{xy} = + \int_A z\sigma_x \, dA = c_1 \int_A yz \, dA = c_1 I_{yz} \tag{3.5}$$

where I_{yz} is the product of inertia of the cross-sectional area,* the origin of the coordinate system being located at the centroid. Equation (3.4) determines the value of the constant to be $c_1 = -M_{xz}/I_z$. The assumed stress field of Eq. (3.1) can therefore be written

$$\sigma_x = -\frac{M_{xz}}{I_z} y$$

* If either the y- or the z-axis is an axis of symmetry, the product of inertia will be zero. Even for nonsymmetrical areas a certain orientation of axes can be found for which $I_{yz} = 0$. These are called principal axes of the area.

This equation relates the bending stress to the bending moment and the moment of inertia of the cross section. From Eqs. (3.4) and (3.5) it is seen that the moment M_{xy} is related to M_{xz} by

$$M_{xy} = -M_{xz}\frac{I_{yz}}{I_z}$$

When the neutral axis is a principal axis of the cross-sectional area, the product of inertia I_{yz} is equal to zero and the only nonzero moment is M_{xz}. In this case the stress distribution is $\sigma_x = -M_{xz}y/I_z$, which is equivalent to a moment M_{xz} about the neutral axis. Similarly, the stress distribution $\sigma_x = +M_{xy}z/I_y$ is equivalent to a moment M_{xy} about the principal y-axis.

Let us consider the bar shown in Fig. 3.3 that has moments M_{xz} applied to its ends, the z-axis being a *principal, centroidal axis* of the cross section. If

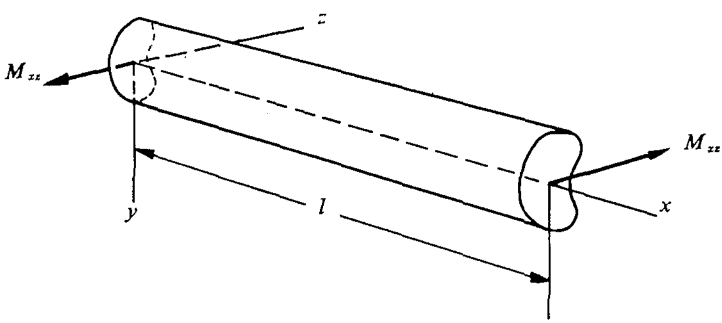


FIGURE 3.3

the moment is applied by distributed forces so that $\sigma_x = -M_{xz}y/I_z$ at $x = 0, l$, the exact solution for the stress field in the bar is

$$\begin{gathered}\sigma_x = -\frac{M_{xz}y}{I_z} \\ \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0\end{gathered} \tag{3.6}$$

It would be difficult to apply these σ_x stresses with a linear variation on the ends of the bar, but by St. Venant's principle, if the moment M_{xz} is applied by forces that are not distributed at $x = 0$ and $x = l$ according to $\sigma_x = -M_{xz}y/I_z$, the foregoing stress field is still a close approximation to the exact stress distribution everywhere except in the vicinity of the ends. For the usual type of bar in bending the stresses will be essentially described by $\sigma_x = -M_{xz}y/I_z$ at distances greater than about one bar depth h from the ends. Experiments must be made or exact solutions must be found to determine the exact stress distribution in the regions near the ends of the bar.

Some examples of stress fields which illustrate St. Venant's principle applied to bars subjected to pure bending are given in Appendix III.

In the particular case when both the y- and z-axes are principal axes and both bending moments M_{xy} and M_{xz} are applied at the ends with an appropriate linear stress distribution, the total stress field is given by

$$\sigma_x = -\frac{M_{xz}}{I_z} y + \frac{M_{xy}}{I_y} z \tag{3.7}$$

As illustrated in Fig. 3.4, a bending moment whose vector is not parallel to a principal axis of the section produces a linear stress distribution in which the neutral, or zero-stress, axis is not parallel to the direction of the moment vector. However, when the principal axes of the cross section are found, the bending moment M_{xn} can be resolved into components parallel to the principal axes and then Eq. (3.7) can be applied.

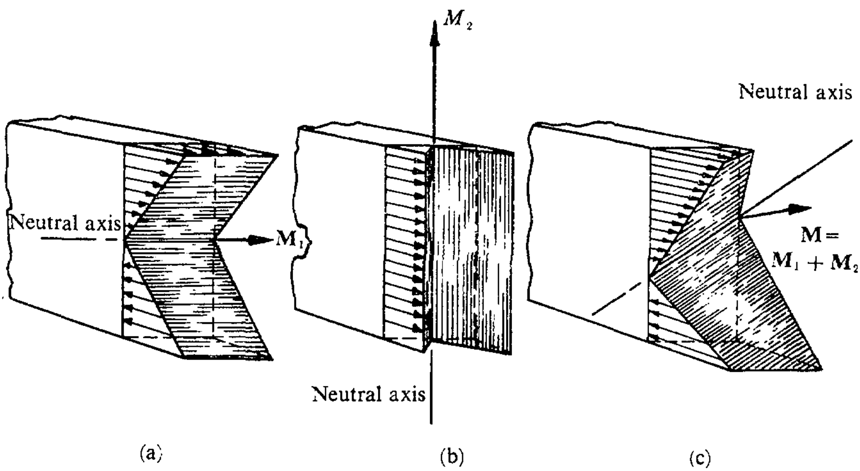


FIGURE 3.4

Displacements of a Bar in Pure Bending. According to Hooke's law the strains in a bar subjected to a bending moment M_{xz}, where the y, z-axes are principal axes, are given by

$$\begin{aligned} \epsilon_x &= -\frac{M_{xz}}{EI_z} y \\ \epsilon_y &= \nu \frac{M_{xz}}{EI_z} y \\ \epsilon_z &= \nu \frac{M_{xz}}{EI_z} y \\ \gamma_{xy} &= \gamma_{xz} = \gamma_{yz} = 0 \end{aligned} \tag{3.8}$$

There is no normal strain at the neutral axis and, hence, the axis of the beam at $y = z = 0$ has no axial stress or strain.

We shall examine the transverse deformation of this initially straight line. Since each element of length dx of the bar undergoes the same deformation, the curvature of the axis is constant throughout the length. The beam axis is deformed into a circular arc of radius R_z whose curvature is

$$\frac{1}{R_z} = \frac{d^2v_0/dx^2}{[1 + (dv_0/dx)^2]^{3/2}} = \text{constant} \tag{3.9}$$

where v_0 is the displacement of the beam axis in the y-direction. The radius of curvature R_z of the beam axis and the radius of transverse curvature R_x are related to the strains, as shown in Fig. 3.5, by

$$\begin{aligned} d\theta &= \frac{ds}{R_z} = -\frac{\epsilon_x\, ds}{y} \\ d\varphi &= \frac{dz}{R_x} = -\frac{\epsilon_z\, dz}{y} \end{aligned} \tag{3.10}$$

where ds is the element of arc length of the centroidal axis,

$$ds = [(dx)^2 + (dv_0)^2]^{1/2}$$

These equations may be written

$$\begin{aligned} \frac{d\theta}{ds} &= \frac{1}{R_z} = -\frac{\epsilon_x}{y} = \frac{M_{xz}}{EI_z} \\ \frac{d\varphi}{dz} &= \frac{1}{R_x} = -\frac{\epsilon_z}{y} = -\nu\frac{M_{xz}}{EI_z} \end{aligned} \tag{3.11}$$

The magnitude of the quantity EI_z determines how great a curvature will be produced by a given bending moment M_{xz} and, hence, EI_z is called the *flexural rigidity* of the beam. As shown in Fig. 3.5, the slope of the beam axis is given by $\tan\theta$ or dv_0/dx. It is also seen that planes in the bar that were initially parallel to the xz-plane are deformed by the bending into so-called anticlastic surfaces with curvature $1/R_x$. This is called the anticlastic curvature of the beam. Poisson's ratio for a material is sometimes determined by measuring the anticlastic curvature and using the second of Eqs. (3.11).

Combining Eqs. (3.9) and (3.11) gives the differential equation which describes the deformation of the centroidal axis

$$\frac{d^2v_0/dx^2}{[1 + (dv_0/dx)^2]^{3/2}} = \frac{M_{xz}}{EI_z} \tag{3.12}$$

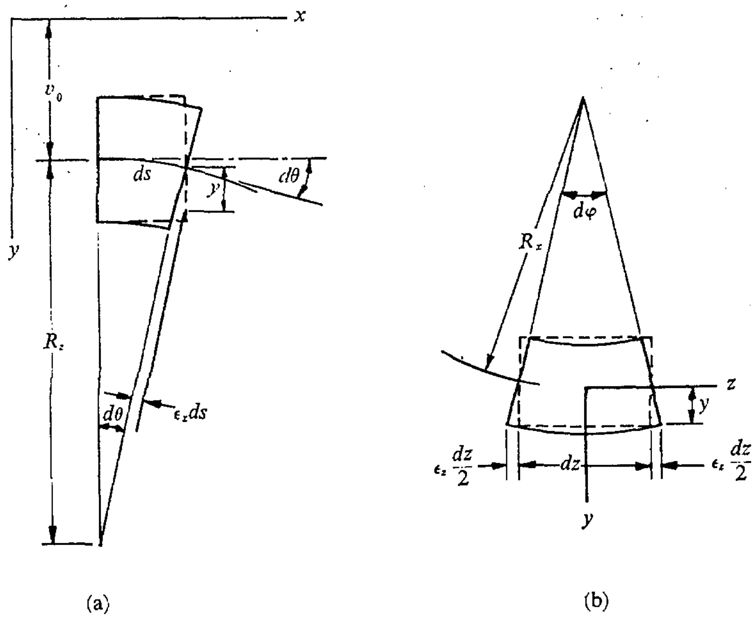


FIGURE 3.5

This is the exact equation relating M_{xz} and v_0. The deflections and slopes of beams that can be tolerated in engineering structures are usually very small $(dv_0/dx \ll 1)$ and the square of the slope is extremely small so that in these cases the differential equation can be written with satisfactory accuracy in the simplified form

$$\frac{d^2v_0}{dx^2} = \frac{M_{xz}}{EI_z} \tag{3.13}$$

Actually, Eq. (3.13) is the equation of a parabolic curve, but this closely approximates a circular arc when $(dv_0/dx)^2 \ll 1$.

The displacements u and v for any point in the beam can be found by integrating the strains. In the next section we shall illustrate the procedure by determining the displacements of a cantilever beam.

3-4 CANTILEVER BEAM CARRYING A CONCENTRATED LOAD

Many elements of machines or structures are relatively slender members which may be subjected to transverse loads as well as bending moments. An example is a cantilever beam with a transverse concentrated force at its end, as shown in Fig. 3.6. We shall determine the stress field in this beam.

We specify that the boundary conditions for the top and bottom surfaces and the loaded end of the beam are

$$
\left.\begin{aligned}
\sigma_y &= 0\\
\tau_{xy} &= 0
\end{aligned}\right\} \text{ at } y = \pm c
$$

$$
\left.\begin{aligned}
\int_{-c}^{c} b\tau_{xy}\,dy &= P\\
\int_{-c}^{c} b\sigma_x\,dy &= 0
\end{aligned}\right\} \text{ at } x = l
\tag{3.14}
$$

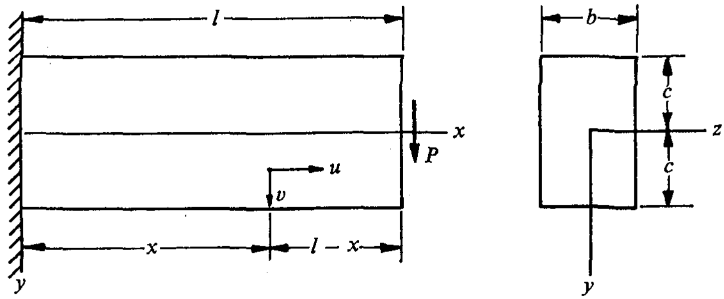


FIGURE 3.6

The first two of the above boundary conditions state that the top and bottom surfaces of the beam are to be free of stress. The last two boundary conditions, at $x = l$, are "relaxed" conditions in that the precise distribution of stress over the end of the beam is not specified, it being only required that the resultant of the stresses is a vertical force P. This gives more freedom in finding a satisfactory solution. The boundary condition at $x = 0$ will also be a "relaxed" condition as we shall be satisfied with any solution whose stresses give a resultant moment and force that holds P in equilibrium, and this condition will certainly be satisfied as the stress distribution will satisfy equilibrium.

To facilitate the analysis, we make the following assumptions: (1) that the bending stress σ_x is given by Eq. (3.6) as in the case of pure bending of a prismatic bar and that $\sigma_y = 0$; (2) the beam width, b, is sufficiently small compared to the beam depth, $2c$, so that the stress distribution is plane, that is $\sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$, and the nonzero stresses do not depend upon z; and (3) the body forces X and Y are zero.

As seen in Fig. 3.6, the bending moment at any cross section of the beam is given by $M_{xz} = P(l - x)$, so by assumption (1), the bending stress is

$$
\sigma_x = -\frac{P(l - x)y}{I_z}
\tag{3.15}
$$

With $\sigma_y = 0$ and $X = Y = 0$, the stress field satisfies the compatibility equation. The equilibrium equations to be satisfied by the stress field reduce to

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} = 0$$

The second of these equations indicates that τ_{xy} is a function of y only. Substituting for σ_x from Eq. (3.15) puts the first equilibrium equation in the form

$$\frac{d\tau_{xy}}{dy} = -\frac{Py}{I_z} \tag{3.16}$$

Integrating this equation gives for the shear stress

$$\tau_{xy} = -\frac{Py^2}{2I_z} + k \tag{3.17}$$

The constant of integration k can be determined from the fact that the stresses σ_x and τ_{xy} must satisfy the boundary conditions as stated by Eqs. (3.14). The condition $\tau_{xy} = 0$ at $y = \pm c$ gives

$$k = \frac{Pc^2}{2I_z}$$

and, therefore,

$$\tau_{xy} = \frac{P}{2I_z}(c^2 - y^2) \tag{3.18}$$

Thus, the shear stress has a parabolic distribution with a maximum at the neutral axis, as shown in Fig. 3.7. The other boundary condition involving the shear stress is that at $x = l$ it is required that

$$b\int_{-c}^{c} \tau_{xy}\, dy = P$$

This condition is automatically satisfied:

$$\int_{-c}^{c} b\tau_{xy}\, dy = P\int_{-c}^{c} \frac{b}{2I_z}(c^2 - y^2)\, dy = P\frac{b}{2I_z}\left(c^2 y - \frac{y^3}{3}\right)\Bigg|_{-c}^{+c}$$

$$= \frac{2bc^3}{3I_z} P = P \tag{3.19}$$

Therefore, the stress field

$$\sigma_x = -\frac{P(l - x)y}{I_z}$$

$$\sigma_y = 0 \tag{3.20}$$

$$\tau_{xy} = \frac{P}{2I_z}(c^2 - y^2)$$

satisfies the equilibrium equations and, as may be verified, it also satisfies the compatibility equation. It satisfies the boundary conditions if the load on the

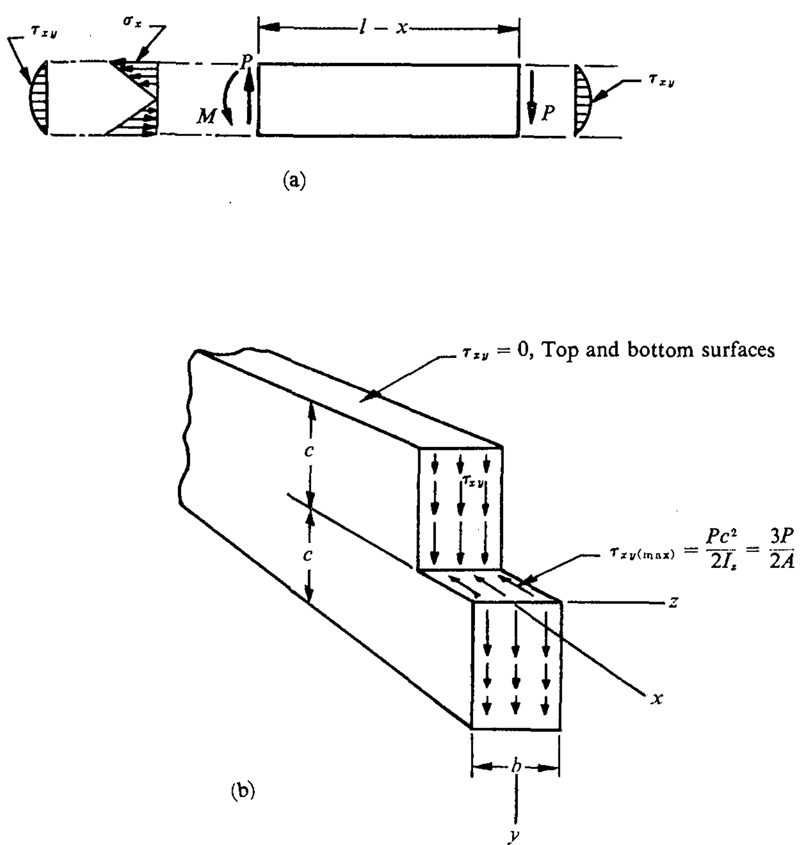


FIGURE 3.7

beam at $x = l$ and the reaction at $x = 0$ are distributed according to Eqs. (3.20). Hence, this stress field is the exact solution of the plane stress problem shown in Fig. 3.7a.

If the load were applied to the top of the beam instead of over the end, Eqs. (3.20) would not be applicable at $x = l$, however, by St. Venant's principle the stresses are essentially correct for $x \leqslant l - 2c$. Also, if the beam is built in, the support will restrain the deformations and this will affect the stress distribution in the neighborhood of $x = 0$.

Exact solutions of bending of beams that are too wide to be treated as plane stress problems are known for a few cases in which the cross sections are simple shapes such as rectangles, circles, or ellipses.* The exact solutions for these cross sections have a linear variation of bending stress as given by Eq. (3.20), but the distribution of shear stress is different. For example, the percentage error in maximum shear stress for rectangular beams of various depth-thickness ratios given by Eq. (3.20) is shown in Table 3.1.

Table 3.1 Percentage Error of Approximate Solution for Shear Stress as Given by Eq. (3.20)

Point	$b/2c = \frac{1}{2}$	1	2	4
$y = 0,\ z = 0$	+1.7%	+6.4%	+16.8%	+24.2%
$y = 0,\ z = b/2$	−3.2%	−11.2%	−28.4%	−49.7%

Equations (3.20) applied to a square cross section give a value of maximum shear stress which is about 11% too small. The exact solution also shows that shear stresses τ_{xz} are present in such beams; it is only as the width goes to zero that τ_{xz} goes to zero.

Computation of Displacements. The displacements of the cantilever beam with an end load may be determined by integration of the strains. Using Hooke's law, the strains as determined from the stresses of Eqs. (3.20) are

$$\epsilon_x = \frac{\partial u}{\partial x} = -\frac{P(l - x)y}{EI_z} \tag{3.21}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\nu P(l - x)y}{EI_z} \tag{3.22}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{P}{2GI_z}(c^2 - y^2) \tag{3.23}$$

We shall neglect the variation of displacement with z (anticlastic curvature) and take u and v to be functions of x and y only. Integration of Eq. (3.21) gives

$$u = -\frac{Plxy}{EI_z} + \frac{Px^2}{2EI_z} + u_1(y)$$

* S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, p. 326 (1951).

where $u_1(y)$ is an arbitrary function of y, since y is held constant in the integration. Integration of Eq. (3.22) gives

$$v = \frac{\nu P(l - x)y^2}{2EI_z} + v_1(x)$$

where $v_1(x)$ is a function of x. The functions u_1 and v_1 must be such that u and v satisfy Eq. (3.23), which requires

$$-\frac{Plx}{EI_z} + \frac{Px^2}{2EI_z} + \frac{du_1(y)}{dy} - \frac{\nu Py^2}{2EI_z} + \frac{dv_1(x)}{dx} = \frac{P}{2GI_z}(c^2 - y^2) \tag{3.24}$$

Separating functions of x and y in Eq. (3.24) gives

$$\frac{dv_1(x)}{dx} - \frac{Plx}{EI_z} + \frac{Px^2}{2EI_z} = -\frac{du_1(y)}{dy} + \frac{\nu Py^2}{2EI_z} + \frac{P}{2GI_z}(c^2 - y^2)$$

Since one side of the equation is a function of x only and the other side a function of y only, and since the equation must hold at all different values of x and y in the beam, it follows that each side is equal to the same constant, so that

$$\frac{dv_1(x)}{dx} - \frac{Plx}{EI_z} + \frac{Px^2}{2EI_z} = C_1 \tag{3.25}$$

$$-\frac{du_1(y)}{dy} + \frac{P}{2GI_z}(c^2 - y^2) + \frac{\nu Py^2}{2EI_z} = C_1 \tag{3.26}$$

Integration of Eqs. (3.25) and (3.26) gives

$$v_1(x) = \frac{Plx^2}{2EI_z} - \frac{Px^3}{6EI_z} + C_1x + C_2$$

and

$$u_1(y) = \frac{P}{2GI_z}\left(c^2 - \frac{y^2}{3}\right)y + \frac{\nu Py^3}{6EI_z} - C_1y + C_3 \tag{3.27}$$

where C_2 and C_3 are constants of integration. The displacement functions are thus

$$\begin{aligned} v &= \frac{Plx^2}{2EI_z} - \frac{Px^3}{6EI_z} + \frac{\nu P(l - x)y^2}{2EI_z} + C_1x + C_2 \\ u &= \frac{-P(l - x/2)xy}{EI_z} + \frac{P}{2GI_z}\left(c^2 - \frac{y^2}{3}\right)y + \frac{\nu Py^3}{6EI_z} - C_1y + C_3 \end{aligned} \tag{3.28}$$

The constant C_1 represents a rigid body rotation of the beam and C_2 and C_3 represent rigid body displacements of the beam. In other words, C_1, C_2, and C_3 locate the beam with respect to the coordinate system. These constants may be evaluated from the following conditions for the beam whose built-in end is as shown in Fig. 3.8

$$\left.\frac{\partial u}{\partial y}\right|_{x=y=z=0} = u|_{x=y=z=0} = v|_{x=y=z=0} = 0$$

These conditions give

$$C_1 = \frac{Pc^2}{2GI_z} \qquad C_2 = C_3 = 0$$

Therefore, the final form of the displacements is

$$v = \frac{P(3l - x)x^2}{6EI_z} + \frac{\nu P(l - x)y^2}{2EI_z} + \frac{Pc^2x}{2GI_z} \tag{3.29}$$

$$u = \frac{-P(l - x/2)xy}{EI_z} - \frac{Py^3}{6I_z}\left(\frac{1}{G} - \frac{\nu}{E}\right) \tag{3.30}$$

These expressions give the x- and y-displacements of every point in the beam. The beam axis ($y = 0$) has the displacement

$$v_0 = \frac{P(3l - x)x^2}{6EI_z} + \frac{Pc^2x}{2GI_z} \tag{3.31}$$

This is the deflection of the axis of a cantilever beam with end load P. The first term of Eq. (3.31) represents the deflection due to the bending strains,

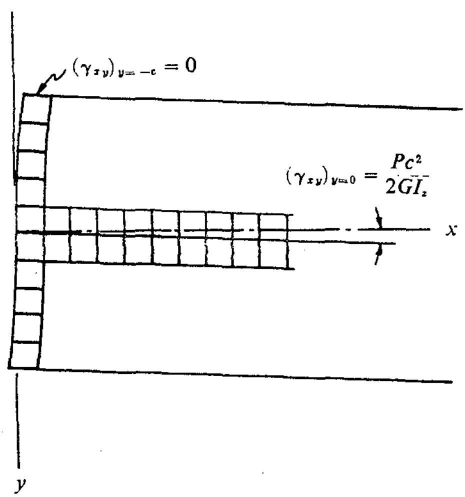


FIGURE 3.8

and the second term is the contribution from the shear strains (this term disappears when $G = \infty$). The effect of the shear strain may be visualized by noting that at the neutral axis of the beam the shear strain is

$$(\gamma_{xy})_{y=0} = \left(\frac{\tau_{xy}}{G}\right)_{y=0} = \frac{Pc^2}{2GI_z}$$

From Eq. (3.31) the slope of the beam axis is

$$\frac{dv_0}{dx} = \frac{P}{2EI_z}(2l - x)x + \frac{Pc^2}{2GI_z} \tag{3.32}$$

The slope of the axis thus has a shearing contribution that is equal to the shear strain at the axis. The displacement due to shear only is illustrated in Fig. 3.8 where *it also may be seen that the shear strain goes to zero at the top* and the bottom of the beam. The shear strain causes the cross section to warp out-of-plane.

According to Eq. (3.31) the maximum deflection of the beam occurs at $x = l$, where

$$(v_0)_{\max} = \frac{Pl^3}{3EI_z} + \frac{Pc^2l}{2GI_z} \tag{3.33}$$

and the ratio of the shear deflection to the bending deflection at this point is

$$\frac{Pc^2l/2GI_z}{Pl^2/3EI_z} = \frac{3}{2}\frac{c^2}{l^2}\frac{E}{G} = \frac{3}{4}(1 + \nu)\left(\frac{2c}{l}\right)^2 \approx \left(\frac{2c}{l}\right)^2 \tag{3.34}$$

When $(2c/l)$ is small, the shear contributes very little to the deflection and, hence, for slender beams the deflection may be taken to be that produced by the bending stress only. When the length is four times the depth the shearing displacement is only $\frac{1}{16}$ the bending displacement. In practice, most beams have length-to-depth ratios of 10 to 20.

It should be noted that the solution we found is the correct solution for a beam loaded on its ends in accordance with Eqs. (3.20). A beam loaded this way will have its left end deformed as shown in Fig. 3.8. A beam whose left end is built-in, as shown in Fig. 3.6, will have stresses different from Eqs. (3.20) in the vicinity of the built-in end but, according to St. Venant's principle, the stresses a few beam depths from the end will be represented satisfactorily by Eqs. (3.20).

3-5 THE TECHNICAL THEORY OF BENDING

The Basic Beam Equation. Because of the difficulty of computing the exact stress distributions and displacements in loaded beams there has been developed a so-called technical theory of bending which simplifies the analysis of the stresses and displacements without introducing serious inaccuracies except in the vicinity of concentrated forces. The reasoning behind this approximate theory is as follows. For slender beams, where the shearing strains have a very small effect on the deformation of the beam, it will be reasonable to take the shearing strains to be zero and to consider that the deformation of the beam is produced only by the bending strains.*

This last statement implies that the normal strains ϵ_y, produced by loads acting on the beam, are also sufficiently small to be taken to be zero. With reference to the straight, rectangular beam in Fig. 3.9, the technical theory of bending is based on the following simplifying assumptions

$$\epsilon_y = \frac{\partial v}{\partial y} = 0 \tag{3.35}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{3.36}$$

$$\epsilon_x = \frac{\sigma_x}{E} \quad \text{(independent of } z\text{)}$$

$$\epsilon_z = \gamma_{yz} = \gamma_{xz} = 0$$

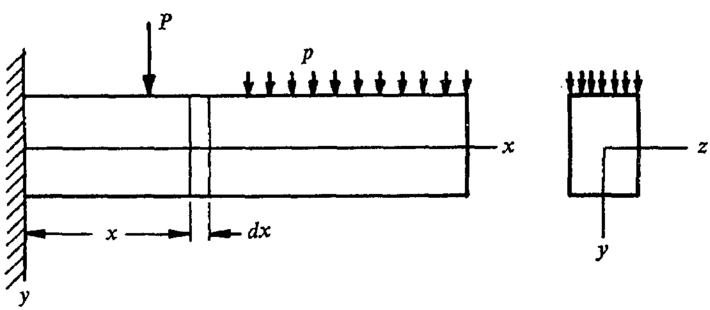


FIGURE 3.9

* The French engineer E. Mariotte (1620–1684), of Boyle-Mariotte gas law fame, in his book *Traite du Mouvement des Eaux* (1686) published results of bending tests on pipes and concluded that one side of the beam was compressed and the other extended. This pointed the way to a correct analysis of beams. The first completely rational explanation of bending, in which the neutral axis is placed in the correct position, was given by C. Coulomb (1736–1806) in his *Problemes de Statique, Relatif a la Architecture* (1856).

These equations specify how the idealized beam is to deform when it is stressed. Equation (3.35) states that v is independent of y and, hence, is a function of x only

$$v = f_1(x)$$

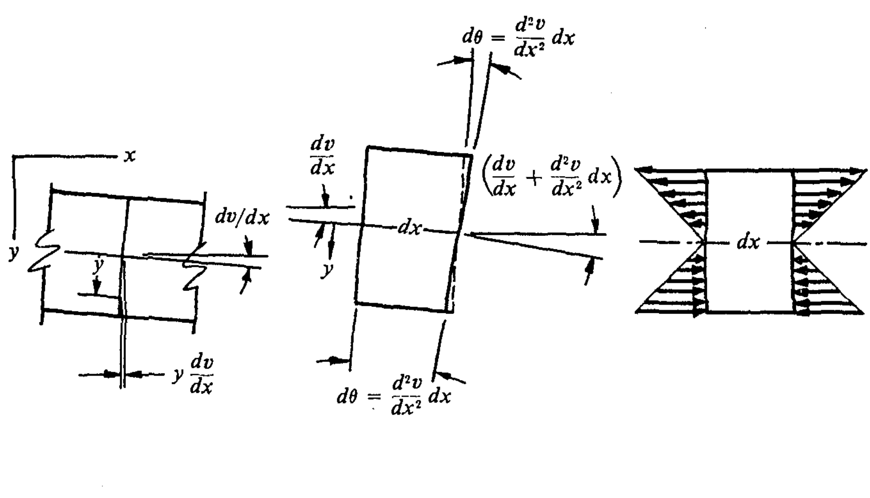
This equation states that all points in a cross section of the beam undergo the same transverse displacement. Substituting for v in Eq. (3.36) and integrating with respect to y, there is obtained

$$u = -y\frac{dv}{dx} + f_2(x) \tag{3.37}$$

In this analysis the x, y, z axes are principal, centroidal axes so that the z-axis is also the neutral axis and, hence, the term $f_2(x)$ represents a pure axial stretching or contracting of the beam and is not related to the bending. The first term in the equation represents the bending displacement and, therefore, the axial displacement u of a point in the beam produced by bending is a linear function of y

$$u = -y\frac{dv}{dx} \tag{3.38}$$

where dv/dx is the slope of the beam axis, which is assumed to be small. This equation states that *plane sections remain plane*, that is, during bending, a cross section of the beam remains planar, as shown in Fig. 3.10a. A typical



(a) (b) (c)

FIGURE 3.10

element of the beam of length dx will thus deform as shown in Fig. 3.10b. It follows from Eq. (3.38) that the bending strain and bending stress are also linear in y as shown in Fig. 3.10b, c

$$\epsilon_x = \frac{\partial u}{\partial x} = -y\frac{d^2v}{dx^2}$$
$$\sigma_x = E\epsilon_x = -Ey\frac{d^2v}{dx^2} \tag{3.39}$$

The σ_x stresses produce a bending moment which is given by the integral of $(y\sigma_x\, dA)$ over the area of the cross section*

$$M_{xz} = -\int_A y\sigma_x\, dA = \int_A Ey^2\frac{d^2v}{dx^2}\, dA = E\frac{d^2v}{dx^2}\int_A y^2\, dA$$

The integral in this expression is the moment of inertia of the cross-sectional area about the z-axis, so

$$M_{xz} = EI_z\frac{d^2v}{dx^2} \tag{3.40}$$

This is the basic equation of the bending of a beam having small slopes.

The foregoing analysis was for beam deformations sufficiently small so that the curvature could be approximated by d^2v/dx^2. The technical theory of bending can also be applied to beams whose slope is not small if the exact expression for the curvature is used (see Eqs. (3.9) and (3.11))

$$\frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}} = \frac{M_{xz}}{EI_z} \tag{3.40a}$$

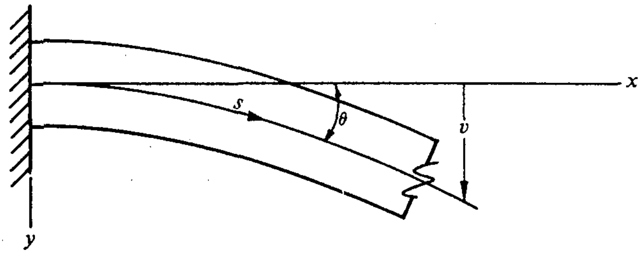
This equation can also be expressed in terms of the intrinsic coordinates s, θ shown in Fig. 3.11a, where s is a length measured along the axis of the beam and θ is the angle through which the beam axis has been rotated by the bending strains. In terms of these coordinates Eq. (3.40a) can be written

$$\frac{d\theta}{ds} = \frac{M}{EI} \tag{3.40b}$$

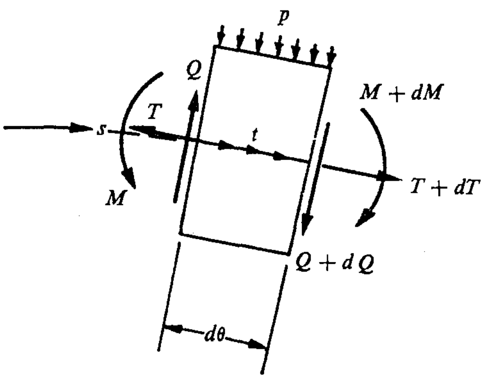
The bending moment at a point on the beam is a consequence of the external forces acting on the beam and it is often useful to express the equations of bending in terms of the transverse and axial loading on the beam.

* Note that with our sign convention for moments a positive M_{xz} is produced by compression in the part of the beam for which y is positive.

The differential relationship between the bending moment and the loading is found by the following equilibrium analysis. Consider a freebody diagram of an element of the beam as shown in Fig. 3.11b, where, at the point located by



(a)



(b)

FIGURE 3.11

s, a distributed load of p pounds per unit length is applied normal to the beam axis. A positive p acts in the direction shown. A distributed force t pounds per unit length is applied along the beam axis, and the tensile force carried by the beam at s is given by T. Positive shear force Q, and bending moment M are acting. Equilibrium of forces along the axis of the element requires

$$t\,ds + dT - Q\,d\theta = 0 \quad \text{or} \quad \frac{dT}{ds} = -t + Q\frac{d\theta}{ds} \tag{3.41}$$

Equilibrium of forces normal to the axis requires that

$$dQ + p\,ds + T\sin\frac{d\theta}{2} + (T + dT)\sin\frac{d\theta}{2} = 0$$

Setting $\sin d\theta/2 \cong d\theta/2$ and dropping the second-order term gives

$$\frac{dQ}{ds} = -p - T\frac{d\theta}{ds} \tag{3.42}$$

Equilibrium of moments, neglecting second-order infinitesimals, requires

$$dM + Q\,ds = 0$$

or

$$\frac{dM}{ds} = -Q \tag{3.43}$$

Equations (3.41), (3.42), and (3.43) are the *differential equations of equilibrium* of an element of the beam that has bending, shear, and axial tension. The last two can be combined to give

$$\frac{d^2M}{ds^2} = p + T\frac{d\theta}{ds} \tag{3.44}$$

If this equation is combined with Eq. (3.40b) there is obtained

$$\frac{d^2}{ds^2}\left(EI\frac{d\theta}{ds}\right) = p + T\frac{d\theta}{ds} \tag{3.45}$$

This is the basic equation of bending of a beam in terms of the applied loads.* When p and T are prescribed functions of s, integration of this equation gives θ as a function of s. A similar equation may be derived for curvature $1/R_y$, produced by transverse loading q in the z-direction and the tensile force T. Equation (3.45) may be used to describe the deformation of beams which have a slowly varying flexural rigidity EI along their length.

If the slopes of the beam are small, $d\theta/ds$ may be approximated by d^2v/dx^2, where v is the displacement of the beam axis, and the basic equation of bending of a beam in the y-direction in terms of the applied loads becomes

$$\frac{d^2}{dx^2}\left(EI_z\frac{d^2v}{dx^2}\right) = p + T\frac{d^2v}{dx^2} \qquad \text{(small slopes)} \tag{3.46}$$

and in the z-direction

$$\frac{d^2}{dx^2}\left(EI_y\frac{d^2w}{dx^2}\right) = q + T\frac{d^2w}{dx^2}$$

* Note that the term $T(d\theta/ds)$ is the equivalent lateral load produced by the axial tension.

The basic beam equation in terms of the shearing force is obtained by combining Eq. (3.40b) with Eq. (3.43)

$$\frac{d}{ds}\left(EI\frac{d\theta}{ds}\right) = -Q \tag{3.47}$$

A similar equation is obtained for the curvature $1/R_y$ produced by the shear force at 90° from Q.

In the remainder of this chapter we shall be concerned only with transverse loading of prismatic beams (I_z = constant) which produces small changes in slopes (axially loaded beams will be discussed in Chapter 4). The set of beam equations can be written

$$\begin{aligned} \frac{dQ}{dx} &= -p & \frac{d\theta}{dx} &= \frac{M_{xz}}{EI_z} \\ \frac{dM_{xz}}{dx} &= -Q & \frac{dv}{dx} &= \theta \\ \frac{d^2M_{xz}}{dx^2} &= p & \frac{d^2v}{dx^2} &= \frac{M_{xz}}{EI_z} \end{aligned} \tag{3.48}$$

These beam equations can also be written in the form*

$$\begin{aligned} EI_z\frac{d^4v}{dx^4} &= p \\ EI_z\frac{d^3v}{dx^3} &= -Q \\ EI_z\frac{d^2v}{dx^2} &= M_{xz} \\ \frac{dv}{dx} &= \theta \end{aligned} \tag{3.49}$$

The Stresses in a Beam. The bending stresses, from Eqs. (3.39) and (3.40) are given by

$$\sigma_x = -\frac{M_{xz}y}{I_z} \tag{3.50}$$

* The first of these equations was deduced by Daniel Bernoulli (1700–1782). The equation $EI\,\partial^4v/\partial x^4 = -\rho A\,\partial^2y/\partial t^2$ for a vibrating beam is called the Bernoulli-Euler equation, as L. Euler (1707–1783) also derived it and obtained solutions to it. Actually, Bernoulli and Euler were not clear as to the location of the neutral axis so they could not determine the correct value of EI.

where the y and z axes are principal centroidal axes of the cross section. The minus sign arises from the fact that a positive moment M_{xz} is such as to produce compressive stresses in the part of the beam where y is positive.

Although the shearing strains are assumed to be zero, this does not mean that the shearing stresses are zero, for they must be such as to satisfy equilibrium. If the bending stress σ_x varies with x there must be shearing stresses since, as was shown in Chapter 2, equilibrium requires

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

Because of the assumptions made in the technical theory of bending, the exact τ_{xy} cannot, in general, be determined. The average value τ_a across the section a-a can, however, be determined from equilibrium of the shaded portion of the beam shown in Fig. 3.12

$$\tau_a(b\,dx) = dF_x$$

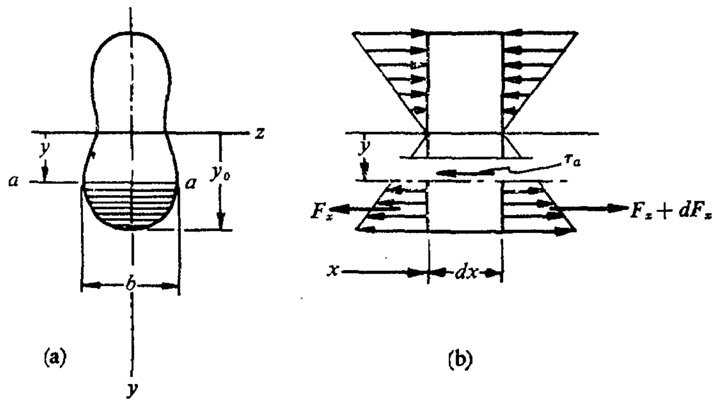


FIGURE 3.12

where b is the width of the beam at a distance y from the z-axis, and τ_a is the average shearing stress on the area $(b\,dx)$. The increment of force dF_x is given by

$$dF_x = \int_{-b/2}^{b/2} \int_{y}^{y_0} \left(\frac{\partial \sigma_x}{\partial x}\,dx\right) dy\,dz = \int_{A^*} \left(\frac{\partial \sigma_x}{\partial x}\,dx\right) dA$$

where y_0 is the ordinate of the boundary of the cross section, and A^* is the cross-sectional area between y and y_0. The equilibrium equation can therefore be written

$$b\tau_a = \int_{A^*} \frac{\partial \sigma_x}{\partial x}\,dA$$

Substituting $\sigma_x = -M_{xz}y/I_z$, there is obtained

$$b\tau_a = -\int_{A^*} \frac{dM_{xz}}{dx}\frac{y}{I_z}\,dA$$

Since $dM_{xz}/dx = -Q$ the equation can be written

$$\tau_a = \frac{Q}{bI_z}\int_{A^*} y\,dA \tag{3.51}$$

The integral in this equation is the moment of the shaded area A^* shown in Fig. 3.12, about the centroidal z-axis, and Q is the shear force. If Eq. (3.51) is applied to a beam of rectangular cross section of width b and depth h, the average shear stress at y is given by

$$\bar{\tau}_{xy} = \frac{Q}{I_z b}\int_{-b/2}^{b/2}\int_{y}^{h/2} y\,dy\,dz = \frac{Q}{2I_z}\left[\left(\frac{h}{2}\right)^2 - y^2\right]$$

$$= \frac{3}{2}\frac{Q}{bh}\left[1 - \left(\frac{y}{h/2}\right)^2\right] \tag{3.51a}$$

It is seen that the average shear stress $\bar{\tau}_{xy}$ is distributed parabolically from top to bottom of the cross section and the maximum shear stress occurs at the neutral axis, $\bar{\tau}_{\max} = \frac{3}{2}(Q/bh)$. For a thin rectangular beam, the average shear stress $\bar{\tau}_{xy}$ is essentially the same as the true shear stress τ_{xy}. In this case Eq. (3.51a) agrees with Eq. (3.18).

Shear Center. The shear stresses τ_a integrated over the symmetrical cross section shown in Fig. 3.12a are equivalent to the shear force Q acting through the center of symmetry. The stresses and deflections of a beam loaded with a shear force through the center of symmetry of the cross section are described by Eq. (3.50), (3.51), and (3.49) according to the technical theory. When the shear Q does not pass through the center of symmetry, a twisting moment $M_{xx} = Qd$ acts about the centroid and produces shear stresses shown schematically in Fig. 3.13a in addition to the shear stresses τ_a. The resulting shear strains cause the beam to twist so that there is both bending and twisting of the beam. The method of calculating the stresses produced by twist of the beam is discussed in Chapter 6.

Equation (3.51) may be used to find the shear stress in a beam of any cross section when the applied loading does not produce twisting of the beam. The average shear stress across any section a-a is equal to $Q/I_z b$ times the moment of area A^* about the z-axis. The average stress across section a'-a' of the angle beam in Fig. 3.13b can be found in the same way but will have a smaller value than the average stress on section a-a.

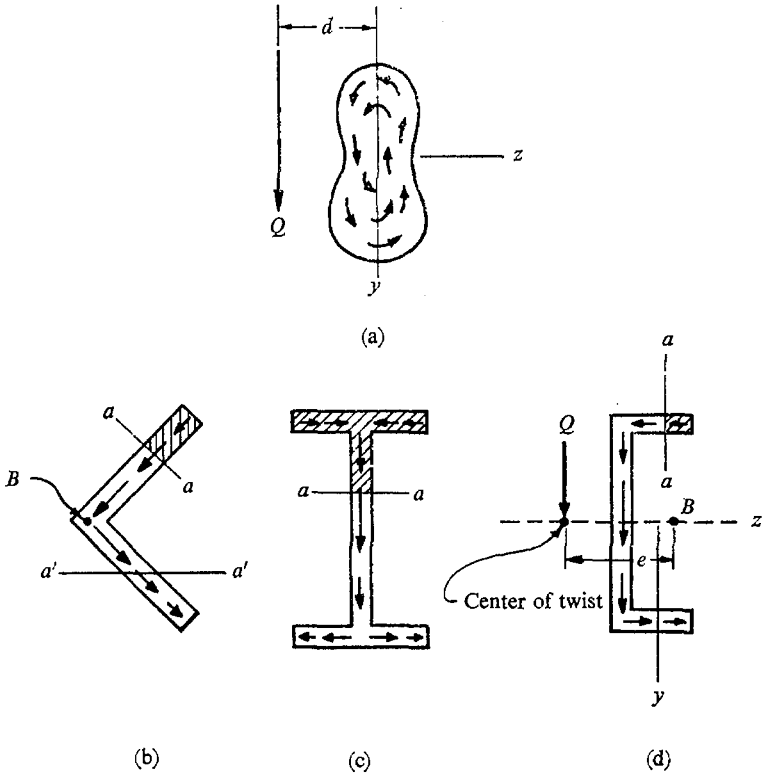


FIGURE 3.13

The line of action of the shear force which produces no twist is said to pass through the *center of twist* or *shear center*. The center of twist, which may be outside of the section, is located as follows. Assuming that the shear stress distribution is the no-twist distribution of Eq. (3.51), we calculate the twisting moment M_{xx} produced by these stresses about an arbitrary point B on the z-axis (see Fig. 3.13d). The shear force Q acting through the center of twist must produce the same moment M_{xx} about the arbitrary point, so that the distance from B to the center of twist is $e = M_{xx}/Q$. A similar line of action can be determined for a shear force perpendicular to Q and the intersection of the two lines of action locates the shear center. The calculation of M_{xx} can be simplified by proper selection of the arbitrary point. For example, M_{xx} is zero about point B on an axis of symmetry of the angle section in Fig. 3.13b so this point is the center twist of the section. If the beam section has an axis of symmetry (as have all the sections shown in Fig. 3.13) the center of twist lies on that axis. The shear stresses τ_a in a beam produced by a force Q acting through the center of twist actually produce some shear strain which is neglected in the technical theory. Beams with sections such as shown in Fig. 3.13b and d will therefore twist somewhat due to these shear strains, but the twisting deformation will be small compared to the bending deformation in long slender beams when the shear force acts through the center of twist.

Transverse Normal Stresses. If a beam carries a transverse load there will be transverse normal stresses. For example, a rectangular beam of width b carrying a lateral load of p pounds per unit length, as shown in Fig. 3.14,

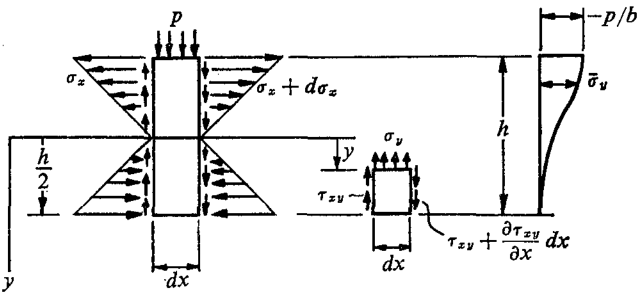


FIGURE 3.14

will have an average stress $-p/b$ on the upper surface. From the freebody diagram of the piece of the beam element it is seen that vertical equilibrium requires

$$\bar{\sigma}_y b\, dx = \int_{-b/2}^{b/2} \int_y^{h/2} \frac{\partial \tau_{xy}}{\partial x}\, dx\, dy\, dz = b \int_y^{h/2} \frac{\partial \bar{\tau}_{xy}}{\partial x}\, dx\, dy$$

where $\bar{\sigma}_y$ is the average stress over the width. For a rectangular beam the average shear stress is given by

$$\bar{\tau}_{xy} = \frac{3}{2}\frac{Q}{bh}\left[1 - \left(\frac{y}{h/2}\right)^2\right]$$

Recalling that $\partial Q/\partial x = -p$, we may write

$$\bar{\sigma}_y = -\int_y^{h/2} \frac{3}{2}\frac{p}{bh}\left[1 - \left(\frac{y}{h/2}\right)^2\right] dy$$

$$= -\frac{p}{b}\left[\frac{1}{2} - \frac{3}{2}\left(\frac{y}{h}\right) + 2\left(\frac{y}{h}\right)^3\right]$$

This stress varies from $-p/b$ at the top surface to 0 at the bottom surface as shown in Fig. 3.14.

The uniformly loaded cantilever beam of length l, shown in Fig. 3.9, will have stresses*

$$\sigma_x = \frac{-M_{xz}y}{I_z} = -\frac{p(l-x)^2}{2}\frac{y}{bh^3/12}$$

$$\bar{\tau}_{xy} = \frac{3}{2}\frac{Q}{bh}\left[1 - \left(\frac{y}{h/2}\right)^2\right] = \frac{3}{2}\frac{p(l-x)}{bh}\left[1 - \left(\frac{y}{h/2}\right)^2\right] \tag{3.52}$$

$$\bar{\sigma}_y = -\frac{p}{b}\left[\frac{1}{2} - \frac{3}{2}\left(\frac{y}{h}\right) + 2\left(\frac{y}{h}\right)^3\right]$$

The maximum values of these stresses are

$$(\sigma_x)_{\max} = \frac{3pl^2}{bh^2}$$

$$(\bar{\tau}_{xy})_{\max} = \frac{3}{2}\frac{pl}{bh} \tag{3.53}$$

$$(\bar{\sigma}_y)_{\max} = \frac{p}{b}$$

The ratios of maximum stresses are:

$$\frac{(\bar{\tau}_{xy})_{\max}}{(\sigma_x)_{\max}} = \frac{1}{3}\left(\frac{h}{l}\right)$$

$$\frac{(\bar{\sigma}_y)_{\max}}{(\sigma_x)_{\max}} = \frac{1}{3}\left(\frac{h}{l}\right)^2 \tag{3.54}$$

It is seen that if the depth-to-length ratio (h/l) is small, the stresses $(\bar{\tau}_{xy})_{\max}$ and $(\bar{\sigma}_y)_{\max}$ will be small compared to $(\sigma_x)_{\max}$. This is justification for assuming that the strains ϵ_y and γ_{xy} are so small that they can be set equal to zero in the technical theory of bending. It is seen that if the length is equal to or greater than three times the depth, the maximum shear stress will be equal to or less than $\frac{1}{9}$ of the maximum bending stress. If the length is less than about three times the depth, the shearing strains will not be negligibly small compared to the bending strains.

3-6 COMPOSITE BEAMS

Beams that are made of two or more materials having different moduli of elasticity can be analyzed in a manner similar to that used for homogeneous

* These are the stresses according to the technical theory. They do not satisfy the compatibility equation. See Prob. 3.6 for the exact solution.

beams. In practice, a composite beam usually will have the same cross section over its length as does the beam shown in Fig. 3.15 which is made of three materials having moduli E_1, E_2, and E_3. If, in addition, the y-axis is an axis of symmetry of the cross section, as for the beam in Fig. 3.15, a simple analysis

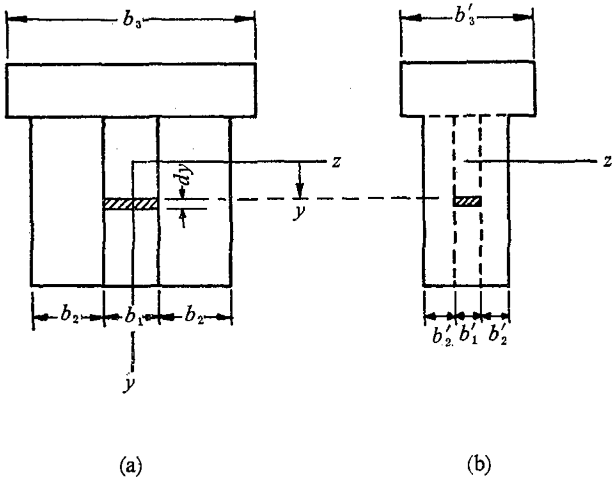


FIGURE 3.15

can be made for bending about the z-axis. The analysis of this beam is simplified by considering an equivalent homogeneous beam made of material having a modulus of elasticity E' and having the same depth as the real beam, but having a modified width such that the same bending moment produces the same curvature as in the composite beam.

If the equivalent beam has the same curvature as the composite beam, it will also have the same axial strain at the same distance y from the neutral axis, for example,

$$\frac{\sigma'}{E'} = \frac{\sigma_1}{E_1}$$

and therefore

$$\sigma' = \frac{E'}{E_1}\sigma_1 \tag{3.55a}$$

To have equivalence of forces it is necessary that the element of area $dA_1 = b_1\,dy$, shown in Fig. 3.15a, be transformed to an element of area having a modified width, $dA' = b_1'\,dy$, as shown in Fig. 3.15b, so that at the same distance from the neutral axis

$$\sigma'\,dA_1' = \sigma_1\,dA_1$$

or, substituting from Eq. (3.55a),

$$\frac{E'}{E_1}\,\sigma_1(b_1'\,dy) = \sigma_1(b_1\,dy)$$

from which

$$\frac{E'b_1'}{E_1} = b_1$$

It is thus seen that the modified width of the equivalent homogeneous beam of modulus E' is given by

$$b_1' = \frac{E_1}{E'}\,b_1$$

and, in the same way,

$$b_2' = \frac{E_2}{E'}\,b_2, \quad \text{etc.}$$

In general, the width b_i of an element in the real beam transforms according to

$$b_i' = \frac{E_i}{E'}\,b_i \tag{3.55b}$$

The bending stress in the equivalent homogeneous beam is given by

$$\sigma_x' = -\frac{M_{xz}y}{I_z'}$$

and from Eq. (3.55a) we see that the stress in the real beam where the modulus is E_i is given by

$$\sigma_x = -\frac{M_{xz}y}{I_z'}\,\frac{E_i}{E'} \tag{3.55c}$$

Only bending about an axis which is a principal axis of the equivalent beam can be treated in this manner. However, if the real beam has an axis of symmetry, the bending moment can be resolved into components parallel and perpendicular to the axis of symmetry. Equivalent beams can be constructed for bending about each of these axes which will be principal axes of the transformed sections.

3-7 DEFLECTION OF TRANSVERSELY LOADED BEAMS

In the technical theory of beams, the bending displacements can be calculated from a knowledge of the bending deformation at each point along the beam axis. For example, the initially curved cantilever beam shown in Fig. 3.16 is loaded by the force P.* The deformation of the beam is described by Eq. (3.40b) which is

$$\frac{d\theta}{ds} = \frac{M}{EI}$$

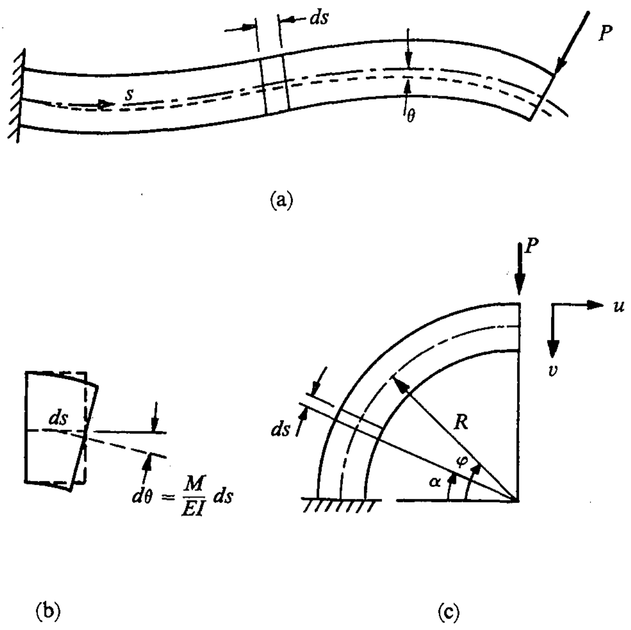


FIGURE 3.16

where s is measured along the axis of the beam and $d\theta$ is the change in slope of the beam axis produced by the deformation of the element ds, as shown in Fig. 3.16b.

As an example, we shall compute the vertical displacement of the slender curved beam shown in Fig. 3.16c. The element ds located by the angle α is deformed so that

$$d\theta = \frac{M}{EI}\,ds = \frac{PR}{EI}\cos\alpha\,ds$$

* The bending of an initially curved beam is considered in Sections 5-4 and 5-5 where it is shown that the basic beam equation for an initially straight beam is valid provided that the curved beam is sufficiently slender.

Because of this deformation, the point on the beam axis located by φ will move downward a distance

$$dv = \frac{PR}{EI}\, ds \cos\alpha \,(R\cos\alpha - R\cos\varphi)$$

It will also move to the right a distance

$$du = \frac{PR}{EI}\, ds \cos\alpha \,(R\sin\varphi - R\sin\alpha)$$

The total vertical displacement of the point is the sum of the contributions from all of the beam elements

$$\begin{aligned} v = \int dv &= \int_0^{\varphi} \frac{PR}{EI} \cos\alpha \,(R\cos\alpha - R\cos\varphi) R\, d\alpha \\ &= \frac{PR^3}{2EI}\left(\varphi - \frac{\sin 2\varphi}{2}\right) \end{aligned}$$

In many practical problems the beam is prismatic and has a straight axis as shown in Fig. 3.17. In this case the basic equations (3.49) may be integrated to find the deflection of the beam.

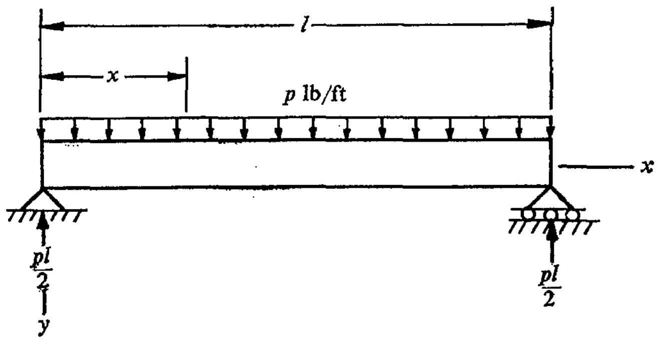


FIGURE 3.17

For example, the beam shown in Fig. 3.17, has a bending moment at point x given by

$$M_{xz} = -\frac{pl}{2}x + \frac{px^2}{2}$$

The basic equation of the beam can therefore be written

$$EI_z \frac{d^2v}{dx^2} = -\frac{plx}{2} + \frac{px^2}{2}$$

This can be integrated to give

$$EI_z \frac{dv}{dx} = -\frac{plx^2}{4} + \frac{px^3}{6} + C_1$$

$$EI_z v = -\frac{plx^3}{12} + \frac{px^4}{24} + C_1 x + C_2$$

The constants of integration are determined by the end conditions

$$v = 0 \qquad \text{at } x = 0, l$$

From these, we find

$$C_2 = 0; \quad C_1 = -\frac{pl^3}{24}$$

The deflection curve of a simply supported, uniformly loaded beam is thus given by

$$v = \frac{pl^4}{24EI}\left[\frac{x}{l} - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4\right]$$

The maximum displacement occurs at the midspan and is given by

$$v_{\max} = \frac{5}{384}\frac{pl^4}{EI}$$

Another method of computing the deflection would be to integrate the equation

$$EI_z \frac{d^4v}{dx^4} = p$$

If this equation is integrated four times there is obtained

$$EI_z v = \frac{px^4}{24} + \frac{a_1}{6}x^3 + \frac{a_2}{2}x^2 + a_3 x + a_4$$

The four constants of integration are determined by the end conditions

$$v = 0 \qquad \text{at } x = 0, l$$

$$M_{xz} = EI_z \frac{d^2v}{dx^2} = 0 \qquad \text{at } x = 0, l$$

This method avoids the necessity of calculating the bending moment but there are twice as many constants of integration to evaluate.

The foregoing example shows that the deflections produced by transverse loading of straight, prismatic beams can be determined by integrating either of the following two equations

$$EI_z \frac{d^2v}{dx^2} = M_{xz} \tag{3.56}$$

or

$$EI_z \frac{d^4v}{dx^4} = p \tag{3.56a}$$

For the type of loads usually encountered, these equations can be integrated without mathematical difficulties, but the integration may be very cumbersome if M, Q, and p are not continuous functions of x. For example, if a beam carries a distributed load p which is a continuous function along the entire beam length, Eq. (3.56a) integrated four successive times gives the deflection curve of the beam with four constants of integration. These constants are specified by the boundary conditions at the ends of the beam which describe the deflection, slope, shear force, or bending moment. If the distributed load is not continuous, or if concentrated loads or moments act along the beam, constants of integration are introduced at each point of discontinuity since the normal rules for integration do not permit integration over a discontinuity in the integrand. This difficulty can be overcome, and the calculation of beam deflections considerably simplified by the introduction of so-called half-range functions.*

A half-range function is defined to have a value only when the argument is positive, for example,

$$\{x - a\} = \begin{cases} 0 & \text{for } (x - a) < 0 \\ (x - a) & \text{for } (x - a) > 0 \end{cases} \tag{3.57}$$

The curly bracket signifies a half-range function. In general,

$$\begin{aligned}
&\frac{d}{dx}\{x - a\}^n = n\{x - a\}^{n-1} \qquad (n \neq 0) \\
&\frac{d}{dx}\{x - a\}^1 = \{x - a\}^0 = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x > a \end{cases} \\
&\frac{d}{dx}\{x - a\}^0 = \{x - a\}^{-1} = \begin{cases} 0 & \text{for } x < a \text{ and } x > a \\ \infty & \text{for } x = a \end{cases} \\
&\frac{d}{dx}\{x - a\}^{-1} = -\{x - a\}^{-2} = \begin{cases} 0 & \text{for } x < a \text{ and } x > a \\ \pm\infty & \text{for } x = a \end{cases}
\end{aligned} \tag{3.58}$$

* The application of half-range functions to beams is called Macauley's method. R. Macauley, *Messenger of Mathematics* (Jan. 1919).

Graphs of these functions are shown in Fig. 3.18 where it may be verified that relationships (3.58) hold. It follows, of course, that the integrals of the half-range functions are

$$
\begin{aligned}
&\int_0^x \{x - a\}^{n-1}\,dx = \frac{1}{n}\{x - a\}^n \qquad (n \neq 0)\\
&\int_0^x \{x - a\}^0\,dx = \{x - a\}^1\\
&\int_0^x \{x - a\}^{-1}\,dx = \{x - a\}^0\\
&\int_0^x \{x - a\}^{-2}\,dx = -\{x - a\}^{-1}
\end{aligned}
\tag{3.58a}
$$

It should be noted that, in terms of beam loads, $\{x - a\}^{-1}$ represents a positively directed concentrated force of unit magnitude applied at the point $x = a$. This function is called a delta function* and is written $\delta(a)$, where $\delta(a) = \{x - a\}^{-1}$. The loading $p = \{x - a\}^{-2}$ represents a concentrated unit bending moment **k** applied at $x = a$. The beam loadings which the half-range functions represent are shown in Fig. 3.18.

A simple example of the use of half-range functions is presented in computing the deflections of the cantilever beam carrying a uniform load over a portion of its length as shown in Fig. 3.19a. The differential equation of the beam is

$$EI_z \frac{d^4v}{dx^4} = p$$

and for this problem the load is $p_0\{x - a\}^0$ and, hence,

$$EI_z \frac{d^4v}{dx^4} = p_0\{x - a\}^0$$

$$EI_z \frac{d^3v}{dx^3} = p_0\{x - a\} + C_1$$

The shear force at $x = l$ must be zero so the constant of integration is $C_1 = -p_0(l - a)$. The bending moment is then given by

$$EI_z \frac{d^2v}{dx^2} = \frac{p_0}{2}\{x - a\}^2 - p_0(l - a)x + C_2$$

* Sometimes called the Dirac delta function.

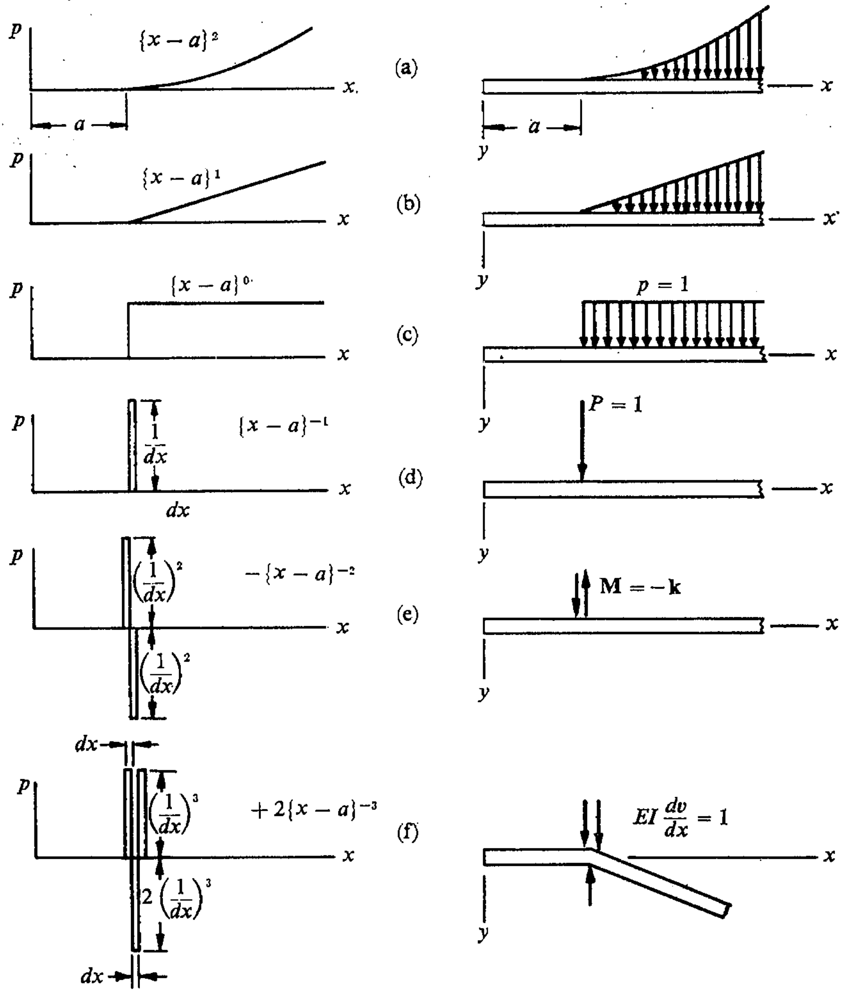
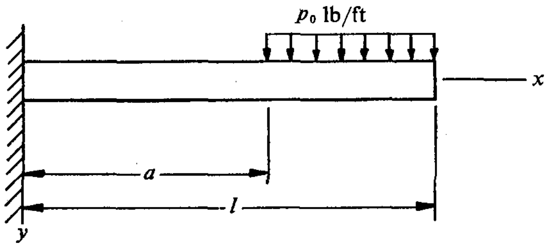


FIGURE 3.18

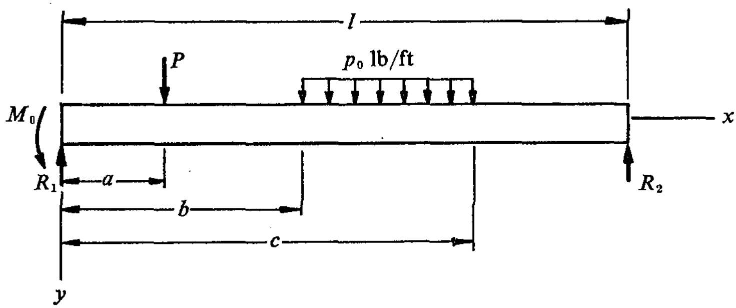
Since the bending moment is zero at $x = l$, this determines the constant of integration to be $C_2 = p_0/2(l^2 - a^2)$, hence, the slope and displacement will be given by:

$$EI_z \frac{dv}{dx} = \frac{p_0}{6}\{x-a\}^3 - \frac{p_0}{2}(l-a)x^2 + \frac{p_0}{2}(l^2-a^2)x + C_3$$

$$EI_z v = \frac{p_0}{24}\{x-a\}^4 - \frac{p_0}{6}(l-a)x^3 + \frac{p_0}{4}(l^2-a^2)x^2 + C_3 x + C_4$$



(a)



(b)

FIGURE 3.19

The conditions that the slope and displacement are zero at $x = 0$ determines the constants, and the displacement is given by

$$v = \frac{p_0}{24EI_z}\left[\{x - a\}^4 - 4(l - a)x^3 + 6(l^2 - a^2)x^2\right]$$

A somewhat more complicated example is the straight, prismatic beam loaded as shown in Fig. 3.19b. The total load on the beam, considering the reactions as applied loads, may be expressed as

$$EI_z \frac{d^4v}{dx^4} = p = -R_1\{x\}^{-1} - M_0\{x\}^{-2} + P\{x - a\}^{-1}$$
$$+ p_0[\{x - b\}^0 - \{x - c\}^0] \tag{3.59}$$

In this expression, R_2 has been omitted since it would appear in nonzero form only for $x > L$. Successive integrations of this equation give

$$EI_z \frac{d^3v}{dx^3} = -Q = -R_1\{x\}^0 + M_0\{x\}^{-1} + P\{x - a\}^0$$
$$+p_0[\{x - b\} - \{x - c\}] + C_1$$
$$EI_z \frac{d^2v}{dx^2} = M_{xz} = -R_1\{x\} + M_0\{x\}^0 + P\{x - a\}$$
$$+ \frac{p_0}{2} [\{x - b\}^2 - \{x - c\}^2] + C_1x + C_2$$
$$EI_z \frac{dv}{dx} = -\frac{R_1}{2}\{x\}^2 + M_0\{x\} + \frac{P}{2}\{x - a\}^2$$
$$+ \frac{p_0}{6} [\{x - b\}^3 - \{x - c\}^3]$$
$$+ \frac{C_1}{2} x^2 + C_2x + C_3$$
$$EI_zv = -\frac{R_1}{6}\{x^3\} + \frac{M_0}{2}\{x\}^2 + \frac{P}{6}\{x - a\}^3$$
$$+ \frac{p_0}{24} [\{x - b\}^4 - \{x - c\}^4]$$
$$+ \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3x + C_4$$

The reaction R_1 may be expressed in terms of the loading P, p_0, and M_0 by means of the equilibrium condition that the sum of moments about the point $x = L$ is equal to zero. The constants of integration can be found by applying the boundary conditions that at $x = 0$ it is required that $v = 0$, $M_{xz} = -M_0$, and at $x = L$, $v = M_{xz} = 0$. Actually, the way Eq. (3.59) was written, the boundary conditions on shear and moment are automatically satisfied so that $C_1 = C_2 = 0$. This is because all of the forces and moments (including reactions) were put in the expression for p, If the reactions are not included in p the constants of integration must also be evaluated to satisfy these boundary conditions.

Calculation of Beam Deflections by Castigliano's Theorem. The calculation of beam deflections by integrating the differential equation gives the complete deflection curve. In many problems only the deflections at discrete points on the beam are required and, when this is the case, the application of Castigliano's Theorem is often convenient. For a linear elastic system,* Castigliano's Theorem may be expressed as

$$e_P = \frac{\partial V}{\partial P}$$

where V is the strain energy in the system; P is a load on the system; and e_P

* The deflections of beams loaded by transverse forces are linear functions of the forces, as seen in the last section. Deflections of beams loaded by axial forces are not linear functions of the load as will be shown in Chapter 4.

is the displacement of the point of application of P in the direction of P, ($e_P = \mathbf{d}\cdot\mathbf{P}/|\mathbf{P}|$).

The strain energy in a beam may be evaluated by integrating the strain energy density V_0 given by Eq. (1.43), over the volume of the beam. Since, in the technical theory of bending

$$\epsilon_y = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = \sigma_z = 0$$

the strain energy density reduces to that of bending alone

$$V_0 = \tfrac{1}{2}\sigma_x\epsilon_x$$

Using Hooke's law for this case, $\epsilon_x = \sigma_x/E$ the strain energy in the beam is given by

$$\begin{aligned} V &= \int\!\!\int\!\!\int \frac{1}{2}\frac{\sigma_x^2}{E}\,dx\,dy\,dz \\ &= \int\!\!\int\!\!\int \frac{1}{2}\frac{M_{xz}^2 y^2}{EI_z^2}\,dx\,dy\,dz \\ &= \int_0^l \frac{1}{2}\frac{M_{xz}^2}{EI_z^2}\int\!\!\int y^2\,dy\,dz\,dx \end{aligned}$$

$$V = \int_0^l \frac{1}{2}\frac{M_{xz}^2}{EI_z}\,dx \tag{3.60}$$

Since $M_{xz} = EI_z\,d^2v/dx^2$, the strain energy in the beam can also be written

$$V = \int_0^l \frac{1}{2}EI_z\left(\frac{d^2v}{dx^2}\right)^2 dx \tag{3.60a}$$

When applied to Eq. (3.60), Castigliano's Theorem gives

$$e_P = \frac{1}{EI_z}\int_0^l M_{xz}\frac{\partial M_{xz}}{\partial P}\,dx \tag{3.61}$$

This, of course, gives the deflection due to flexural stresses only. The contribution of the shear stress can be included in the strain energy but for slender beams it will be negligibly small compared to the bending energy.

As an example of the application of Eq. (3.61) we shall calculate the bending deflection at the end of the cantilever beam shown in Fig. 3.6. The bending moment is $M_{xz} = P(l - x)$, so

$$e_P = \frac{1}{EI_z}\int_0^l P(l - x)^2\,dx = \frac{Pl^3}{3EI_z}$$

This agrees with the deflection obtained by integration of the strains in Section 3-4 when the deflection due to shear is neglected.

When it is necessary to find the deflection at a point where no load is acting, the following technique may be employed. A dummy load F is assumed to act at the point in the direction of the desired deflection. The deflection e_F is then found by use of Castigliano's theorem and the required deflection is given by e_F when F is set equal to zero. This procedure will also give the correct expression for the deflection curve if F is applied at a point x.

Deflection Due to Shear. As was shown in Fig. 3.8, the axis of a beam is given an increment of slope by the shearing strain and, hence, the total deflection of the beam is the sum of the deflection due to bending v_b, plus the deflection due to shear v_s. The total deformation of the beam is described by the following equations

$$v = v_b + v_s$$
$$EI_z \frac{d^2 v_b}{dx^2} = M_{xz} \tag{3.62}$$
$$\frac{dv_s}{dx} = \frac{\tau_0}{G} = \frac{kQ}{AG}$$

where τ_0 is the (maximum) shear stress at the neutral axis. The maximum shear stress τ_0 is equal to some constant k times the average shear stress Q/A. For a rectangular beam k is equal to 1.5 since $\tau_0 = \frac{3}{2}(Q/A)$, as given by Eq. (3.54). Since $dQ/dx = -p$, the last of the above equations can be written

$$\frac{d^2 v_s}{dx^2} = -\frac{k}{AG} p$$

Multiplying by EI_z and adding to the second of Eq. (3.62) gives

$$EI_z \frac{d^2 v}{dx^2} = M_{xz} - k \frac{EI_z}{AG} p \tag{3.63}$$

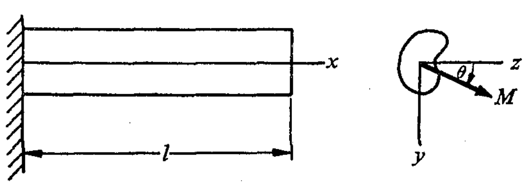
The integral of this equation gives the total deflection of the beam.

Problems

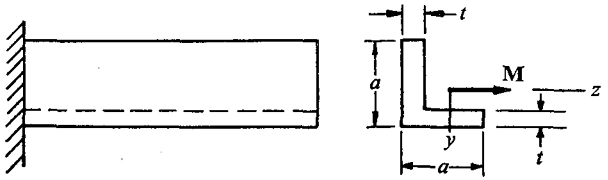
3.1 Verify that the equilibrium and compatibility equations are satisfied by the stress field of Eq. (3.1).

3.2 Verify that Eq. (3.1) satisfies the boundary conditions on the lateral surface of the prismatic bar.

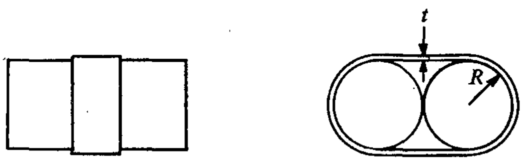
3.3 A bending moment **M** is applied to the end of a cantilever beam of the cross section shown. The z and y axes are the principal, centroidal axes. What are the expressions for the bending stresses produced in the beam? What is the orientation of the neutral axis? If the principal moments of inertia of the cross section are equal, what is the orientation of the neutral axis?



3.4 A cantilever beam made of a steel angle is subjected to bending moment **M** applied as shown. What is the orientation of the neutral axis? (Assume that $t \ll a$.)



3.5 Two drums are lashed together with an initially straight steel band of width b and thickness t ($t \ll R$). What is the maximum bending stress developed in the band?



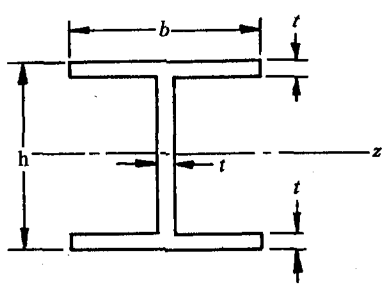
3.6 The cantilever beam of Fig. 3.6 is loaded with a uniformly distributed pressure q on the surface $y = -c$. Show that equilibrium and compatibility are satisfied by the following stresses, and sketch the corresponding boundary stresses on the beam. Note that the bending stress is not linearly distributed.

$$\sigma_x = -\frac{qb}{2I_z}\,[y(l-x)^2 - \tfrac{2}{3}y^3]$$

$$\sigma_y = -\frac{qb}{2I_z}\left[\frac{y^3}{3} - c^2 y + \tfrac{2}{3}c^3\right]$$

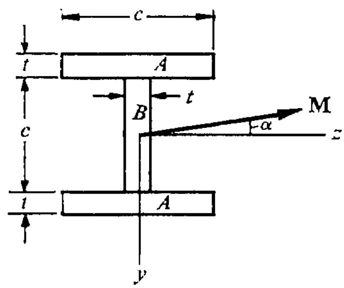
$$\tau_{xy} = \frac{qb}{2I_z}\,(l-x)(c^2 - y^2)$$

3.7 A steel I-beam has a cross section equivalent to that shown in the diagram. If the maximum allowable normal stress is $\sigma_0 = 20{,}000$ psi, what is the corresponding bending moment about the z-axis? $b = 6$ in., $h = 12$ in., $t = 0.375$ in.

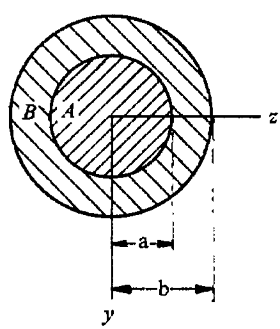


3.8 If the I-beam shown in Prob. 3.7 is subjected to a shearing force Q, it is a common procedure to calculate the maximum shearing stress as $\tau_{max} = Q/ht$. Justify this approximation.

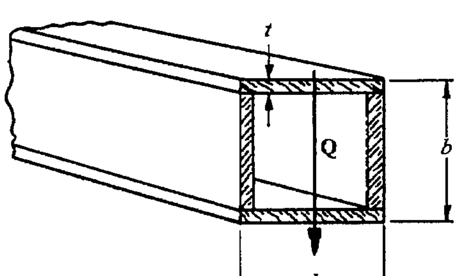
3.9 Consider the composite beam of two materials A and B as shown. A bending moment **M** is applied at angle α with the horizontal principal axis. What are the magnitudes and locations of maximum compressive and tensile stresses? The two materials have moduli E_A and E_B.



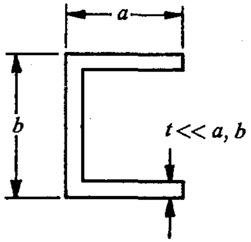
3.10 Consider the beam of circular cross section made of two materials subjected to a bending moment M. What is the maximum bending stress produced in the beam?



3.11 A box beam is made of wood glued together as shown below. What is the approximate required shear strength of the glue, if the shear force across the section is Q? ($t \ll b$).



3.12 Find an approximate expression for the location of the shear center of a beam whose cross section is a thin-walled channel. All calculations can be based on the center line of the walls.

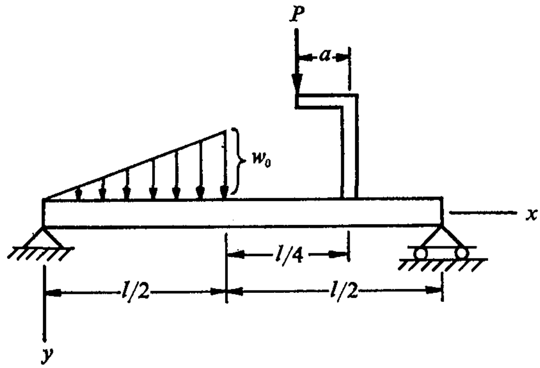


3.13 Show that the deflection of a straight, prismatic beam uniformly loaded and built in at each end is

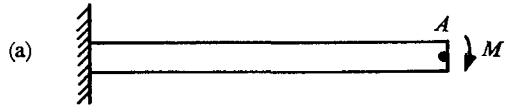
$$v = \frac{px^2}{24EI_z}(l - x)^2$$

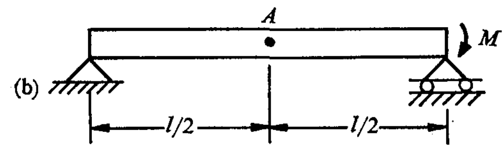
Sketch the bending moment-vs.-x diagram.

3.14 Find the deflection of the axis of the beam shown using half-range functions.

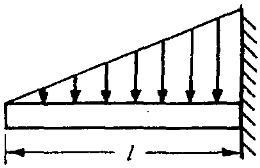


3.15 By Castigliano's Theorem, find the rotation at the point A for the two cases shown in the diagrams. Consider only strain energy due to bending.



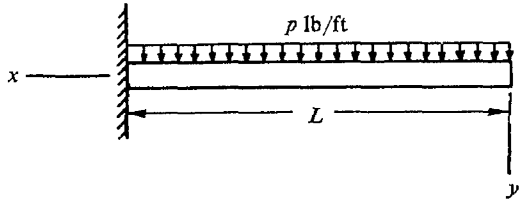


3.16 By energy methods, show that the deflection of the free end of a cantilever beam carrying a total load P distributed as shown is $v = Pl^3/15EI_z$.

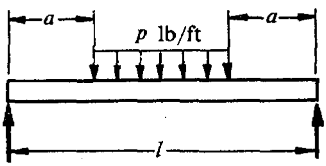


3.17 Show that the deflection of a straight, prismatic cantilever beam, shown in the diagram, that carries a uniform load of p pounds per foot, is

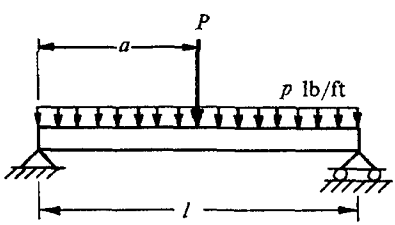
$$v = \frac{pl^4}{24EI_z}\left[3 - 4\frac{x}{l} + \left(\frac{x}{l}\right)^4\right] \tag{3.64}$$



3.18 Determine the deflection of the straight prismatic beam shown in the diagram.



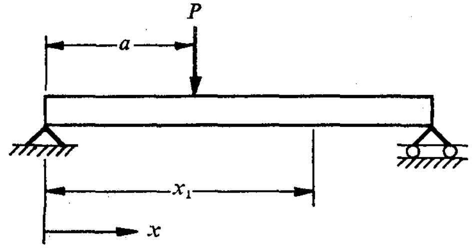
3.19 Determine the deflection curve for the beam shown in the diagram. Sketch the bending moment diagram and the shear force diagram.



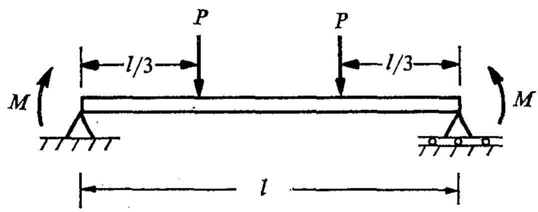
3.20 A straight, rectangular, cantilever beam carries a concentrated force P at its end. Using the energy method, compute the deflection at the end taking into consideration the shearing strain energy as well as the bending energy. Explain any difference between this result and Eq. (3.33).

3.21 Compute $v = f(x)$ for a simply supported beam with a uniform load p_0. Do this by the energy method, using a dummy load P.

3.22 Determine the deflection $v_1 = f(a)$ at point x_1 produced by the load P as it moves slowly from one end of the beam to the other. Use the reciprocal theorem.



3.23 By using the reciprocal theorem, determine the slope of the left end of the beam shown in the diagram.



3-8 STATICALLY INDETERMINATE SYSTEMS

The analysis of a system is strongly influenced by whether it is statically determinate or statically indeterminate. In the case of a statically determinate system the number of independent unknown quantities, such as forces, moments, and reactions is just equal to the number of independent equations of equilibrium that can be written. In a certain sense, a statically determinate system is a minimal system in that the number of members, parts, or supports, is just sufficient to withstand the applied forces; that is, the number is just sufficient to insure equilibrium.

In the case of statically indeterminate systems, the number of members, or parts, exceeds the number required for equilibrium. Thus the number of unknown forces and moments will exceed the number of independent equations of equilibrium that can be written. The equations of equilibrium, therefore, do not suffice to solve for the unknown forces. The additional equations that are required are obtained by expressing the condition that the strains in the members, or parts, must be such as to preserve continuity (compatibility). For example, the pinned framework shown in Fig. 3.20a is statically determinate and the addition of another diagonal member, as shown in Fig. 3.20b, makes it statically indeterminate. There are now six unknown member forces plus three reactions to be determined and only eight independent equations of equilibrium may be written (two at each joint). If the force in the sixth member is called F, the force in each of the other members will be a function of F and thus the strain energy V will be a function of F.

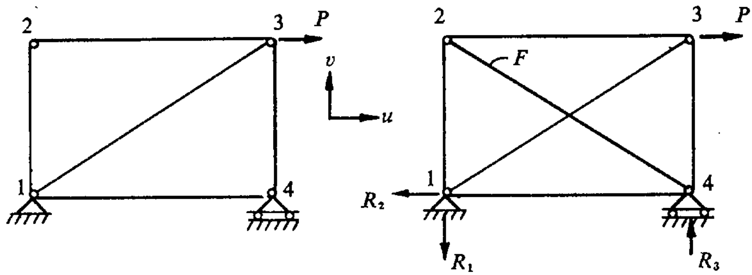


FIGURE 3.20

Therefore, the additional equation required to solve the problem is given by the principle of minimum strain energy

$$\frac{\partial V}{\partial F} = 0$$

This expresses the fact that in this problem the external work does not depend on F. If continuity were not preserved, the force in one or more of the members would appear as an external force that does work and in that case $\partial V/\partial F$ would not be zero.

Another way of looking at this statically indeterminate problem is in terms of the displacements. For example, if u_2, v_2 are the horizontal and vertical displacements of joint 2, etc., then the strain in each member can be expressed in terms of the u_n, v_n at each end. For example, the change in length of member 2-4 in Fig. 3.20 is

$$\Delta L = \sqrt{(u_2 - u_4)^2 + (v_2 - v_4)^2}$$

The force in the member is

$$F = AE\frac{\Delta L}{L}$$

The stresses and forces in the members can, therefore, be expressed in terms of the u_n, v_n. The forces meeting at a joint must be in equilibrium and, hence, the sum of the vertical components must equal zero, and the sum of the horizontal components must be zero for the forces meeting at a joint

$$\sum \mathbf{F} \cdot \mathbf{i} = 0$$
$$\sum \mathbf{F} \cdot \mathbf{j} = 0$$

If the number of joints is N, the total number of equilibrium equations that can be written is $2N$. Since each joint has two degrees of freedom u_n, v_n, the total number of unknown displacement components is also $2N$. Hence, if the forces are expressed in terms of the joint displacements the number of unknowns will just equal the number of equilibrium equations and the u_n and v_n can be determined. This is the counterpart of expressing the equilibrium equations of elasticity in terms of displacements as was done in Section 2-4. In this approach, compatibility (continuity) is assured by expressing the forces in terms of the displacements. The solution of some problems is easier if the forces are expressed in terms of the displacements but for most problems this is probably not the case.

Continuous Beams. If a uniform beam is continuous over three supports, as shown in Fig. 3.21a, there are three unknown vertical reactions and only two equations of equilibrium can be written for the beam. The third equation is given by expressing the condition that the calculated displacement at the middle support must be zero. This leads to the solution

$$R_c = \frac{Pa}{2L_1^2 L_2}(L_1^2 + 2L_1L_2 - a^2) \tag{3.65}$$

Another way of looking at the problem is to draw a freebody diagram of each span as shown in Fig. 3.21b. Now two equations of equilibrium can be written for each span and there are five unknowns to be determined, the four

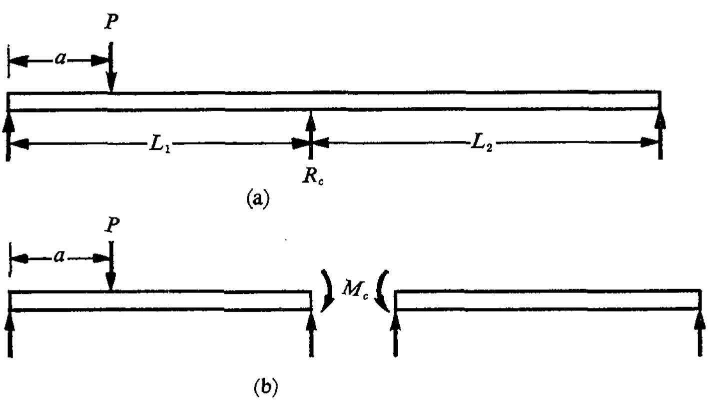


FIGURE 3.21

supporting forces and the moment M_c at the support. The additional equation is obtained by expressing the condition that the slope must be the same for each span at the middle support. This leads to the solution

$$M_c = \frac{Pa(L_1^2 - a^2)}{2L_1(L_1 + L_2)} \tag{3.66}$$

Knowing this value of M_c permits the reactive forces that support the span to be determined from the equilibrium equations.

If the foregoing method is applied to a beam continuous over many supports, two spans of which are shown in Fig. 3.22, there is obtained for the slope θ_n of the left span and the right span, respectively

$$(\theta_n)_L = \frac{M_n L_n}{3EI_n} + \frac{M_{n-1} L_n}{6EI_n} + \theta_L$$

$$(\theta_n)_R = \frac{-M_n L_{n+1}}{3EI_{n+1}} - \frac{M_{n+1} L_{n+1}}{6EI_{n+1}} + \theta_R$$

The terms on the right-hand sides represent the contributions of M_{n-1} and M_{n+1} to the slope of a simply supported beam. The angle θ_L is the contribution to the slope made by the applied loads that act on the nth span; that is, θ_L is the slope of a simply supported beam acted upon by the applied loads.

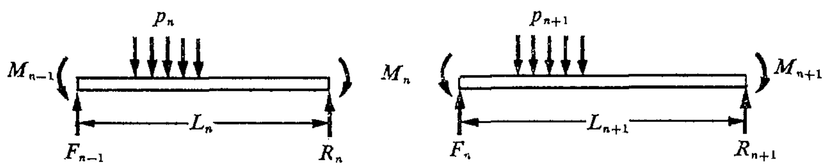


FIGURE 3.22

Similarly, θ_R is the contribution of the loads acting on the $(n + 1)$ span. Both θ_L and θ_R can be calculated readily. Since the beam is continuous over the nth support $(\theta_n)_L$ is equal to $(\theta_n)_R$ and, equating these two quantities, there is obtained the equation

$$M_{n-1}\frac{L_n}{EI_n} + 2M_n\left(\frac{L_n}{EI_n} + \frac{L_{n+1}}{EI_{n+1}}\right) + M_{n+1}\frac{L_{n+1}}{EI_{n+1}} = -6\theta_L + 6\theta_R \tag{3.67}$$

This is the well-known *equation of three moments.** Since one such equation can be written for each support, there will be just as many equations as there are statically indeterminate quantities M_n and, hence, all of the M_n can be determined.

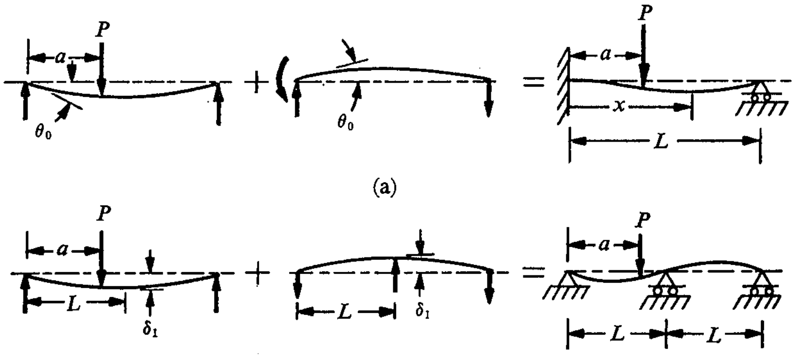
* Derived by B. Clapeyron (1799–1864).

Method of Superposition. Since, for small displacements, the stresses and deflections of beams are linear functions of the transverse load, the stresses and deflections may be superposed, that is

$$(\sigma)_{1+2} = \sigma_1 + \sigma_2$$

$$(v)_{1+2} = v_1 + v_2$$

where v_1 is the displacement produced by load 1, v_2 is the displacement produced by load 2, and v_{1+2} is the displacement produced by both sets of loads acting together. This is illustrated in Fig. 3.23 where it is shown how to build up the solution of complicated problems by superposing the solutions of simple problems.



(a)

(b)

FIGURE 3.23

The solutions for concentrated forces can be superposed to obtain solutions for distributed loads. For example, the displacement of the beam shown in Fig. 3.23a is a function of the point x where the displacement is measured and the point a where the force P is applied

$$v = Pf(x, a) \tag{3.68}$$

The increment of deflection produced by the increment of load $p\,da$ shown in Fig. 3.24 may therefore be written

$$dv = p(a)\,da\,f(x, a)$$

The deflection produced by a distribution of load over the span is then given by

$$v(x) = \int_0^l p(a)f(x, a)\,da \tag{3.69}$$

In Eqs. (3.68) and (3.69) the function $f(x, a)$ is the displacement produced by a unit force $[P = \delta(a)]$ acting at the point $x = a$. The function $f(x, a)$ is called the *Green's function* for the problem.*

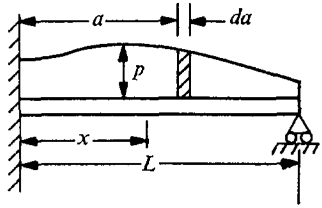


FIGURE 3.24

Statically Indeterminate Frameworks. A common type of structure is made of a continuous assemblage of beams and columns. An example is shown in Fig. 3.25a where a beam and two columns form a continuous frame. The conditions of compatibility are that the ends of all members framing into a joint must have the same rotation and the same displacement. The number of these equations that can be written is just equal to the number of statically indeterminate unknowns. For information on solving problems of statically indeterminate frames reference should be made to books on the theory of structures. One of the common methods of solving such problems is the method of "Moment Distribution." This numerical method of successive approximations is discussed in Chapter 10.

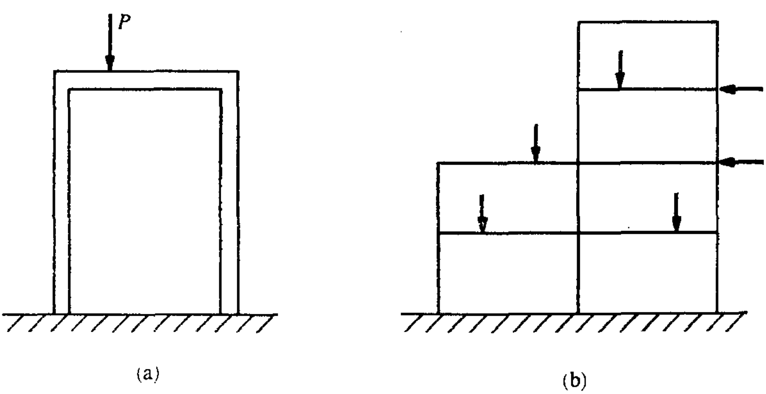


FIGURE 3.25

Energy Methods for Statically Indeterminate Problems. All of the equations expressing compatibility in the preceding sections can be derived by

* The British mathematician George Green (1793–1841), was the first to utilize this method of solving problems.

energy methods. For example, if the total strain energy V is written for the beam shown in Fig. 3.21a, treating the middle reaction R as an applied force, Castigliano's theorem gives

$$\frac{\partial V}{\partial R} = e_R = 0$$

where the displacement e_R at the point of application of R is equal to zero since the middle support is immovable. This will give Eq. (3.65). Similarly, if the total strain energy in the two spans of Fig. 3.21b is written treating M_c as an applied moment, the condition that no work is done by M_c is

$$\frac{\partial V}{\partial M_c} = 0$$

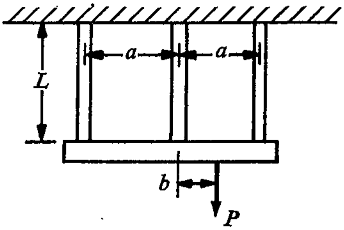
and this will give Eq. (3.66).

Problems

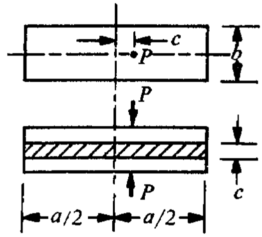
3.24 Draw a freebody diagram of the system shown in Fig. 3.20b with continuity not preserved in one of the diagonal members. Apply Castigliano's theorem to determine the magnitude of the discontinuity when the force in the cut member is zero.

3.25 Show that for the system of Fig. 3.20b the number of unknown joint displacements, u_n, v_n is equal to the number of independent equations of equilibrium.

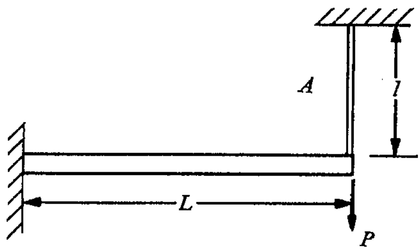
3.26 Three steel bars are attached to a rigid crosspiece which supports a force P as shown. The area of the middle bar is twice that of the outer bars. Determine the forces in the bars.



3.27 A rectangular block having dimensions a, b, c is pressed between two rigid plates as shown in the diagram. The elastic properties of the block are such that the pressure exerted on the plates is directly proportional to the vertical strain in the block. Determine the state of stress in the block. Explain why this is a statically indeterminate problem.



3.28 A cantilever beam is supported at its end by a rod as shown. What force is produced in the rod by the application of the force P? If the rod were not there the deflection of the end of the beam would be $\delta = PL^3/3EI$.



3.29 Deduce Eq. (3.65).

3.30 Deduce Eq. (3.66).

3.31 Solve for the end moments of a uniformly loaded beam that is built-in at each end.

3.32 Solve for the end moments of a beam built-in at each end and supporting a concentrated load P at its midpoint.

3.33 Solve for the bending moments at the built-in ends of a beam with a uniform load over its span. Use the energy method.

3.34 Solve for the bending moment at the built-in end of the beam shown in Fig. 3.23a. Use the energy method.

3.35 Solve for the middle reaction of the beam shown in Fig. 3.23b.

3.36 What is the slope at the right-hand end of the beam shown in Fig. 3.23a?

3.37 Derive the expression for the bending moment of a beam built-in at both ends and carrying a uniform load.

3.38 Derive the expression for the deflection of a uniform beam built-in at each end and carrying a uniform load over one-half of the span. Use half-range functions.

3.39 Derive the expression for the displacements of the beam shown in Fig. 3.23a carrying a uniform load over the left half of the span. Do by the method of Green's function.

3.40 Derive the expressions for the bending moment and the deflection at any point of the beam shown in Fig. 3.23a.

3.41 Derive the expressions for the bending moment and the deflection at any point of the beam shown in Fig. 3.23b.

3.42 Derive the expression for the deflection of a simply supported beam with a uniform load, using the method of Green's function. Make use of half-range functions.

3.43 Work out all of the details of deriving the equation of three moments.

3.44 A slender circular ring is stressed by two collinear forces P at opposite ends of a diameter. What is the bending moment in the ring at the point of application of P?

3.45 Prove that the Green's function in Eq. (3.69) is symmetrical with respect to x and a, that is, $f(x, a) = f(a, x)$.

3-9 BEAM ON AN ELASTIC FOUNDATION

Many interesting physical situations are described approximately by the so-called beam on an elastic foundation. By definition, the elastic foundation shown in Fig. 3.26 resists displacement with a force which is proportional to the displacement. This is a physical idealization which has been found to be a reasonable description for a number of practical problems.* If a beam, as

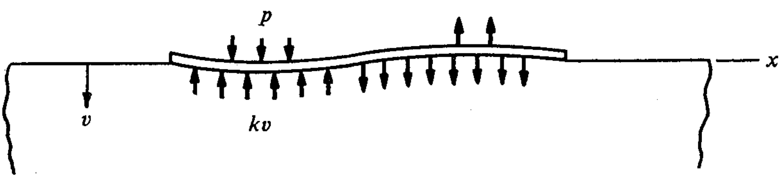


FIGURE 3.26

shown in Fig. 3.26, rests on an elastic foundation and is attached to it, the force per unit length resisting displacement of the beam is

$$q_f = -kv$$

where q_f = foundation loading per unit length of beam, force per unit length; k = foundation modulus, force per unit area; and v = displacement of the beam.

The deflection curve of an initially straight, prismatic beam on an elastic foundation must satisfy the beam equation (3.49) for small slopes, with the total load on the beam being $q_f + p$; that is,

* For a detailed presentation of such problems, see the book by M. Hetenyi. *Beams on Elastic Foundations*, University of Michigan Press (1946).

$$EI_z \frac{d^4v}{dx^4} = q_f + p = -kv + p$$

or

$$EI_z \frac{d^4v}{dx^4} + kv = p \tag{3.70}$$

where the x-axis is the centroidal axis of the beam, the y-axis is a principal axis of the beam cross section, and p is the applied load per unit length of beam. Equation (3.70) is the differential equation of a beam on an elastic foundation (small slopes).

Infinite Beam on an Elastic Foundation. An infinitely long beam on an elastic foundation carrying a concentrated load P at $x = 0$ is shown in Fig. 3.27. The shear force Q in the beam immediately to the left of the load is $+P/2$ and immediately to the right of the load is $-P/2$, so the boundary conditions at $x = 0 + \epsilon$ are

$$Q = -EI_z \frac{d^3v}{dx^3} = -\frac{P}{2}$$

$$\frac{dv}{dx} = 0$$

The other two boundary conditions for the right-hand half of the beam are deduced from the fact that the work done by the applied load is a finite quantity and, hence, the strain energy is also finite which requires:

$$v = \frac{dv}{dx} = 0 \qquad \text{as } x \to \infty$$

Because $p = 0$ for $|x| > 0$, the differential equation for all points except $x = 0$ reduces to

$$EI_z \frac{d^4v}{dx^4} + kv = 0 \tag{3.70a}$$

Using a trial solution, $v = e^{mx}$, it is found that Eq. (3.70a) is satisfied if

$$m^4 + \frac{k}{EI_z} = 0$$

The four roots of this equation are

$$m = \pm\beta(1 \pm i)$$

where

$$\beta = \sqrt[4]{\frac{k}{4EI_z}}$$

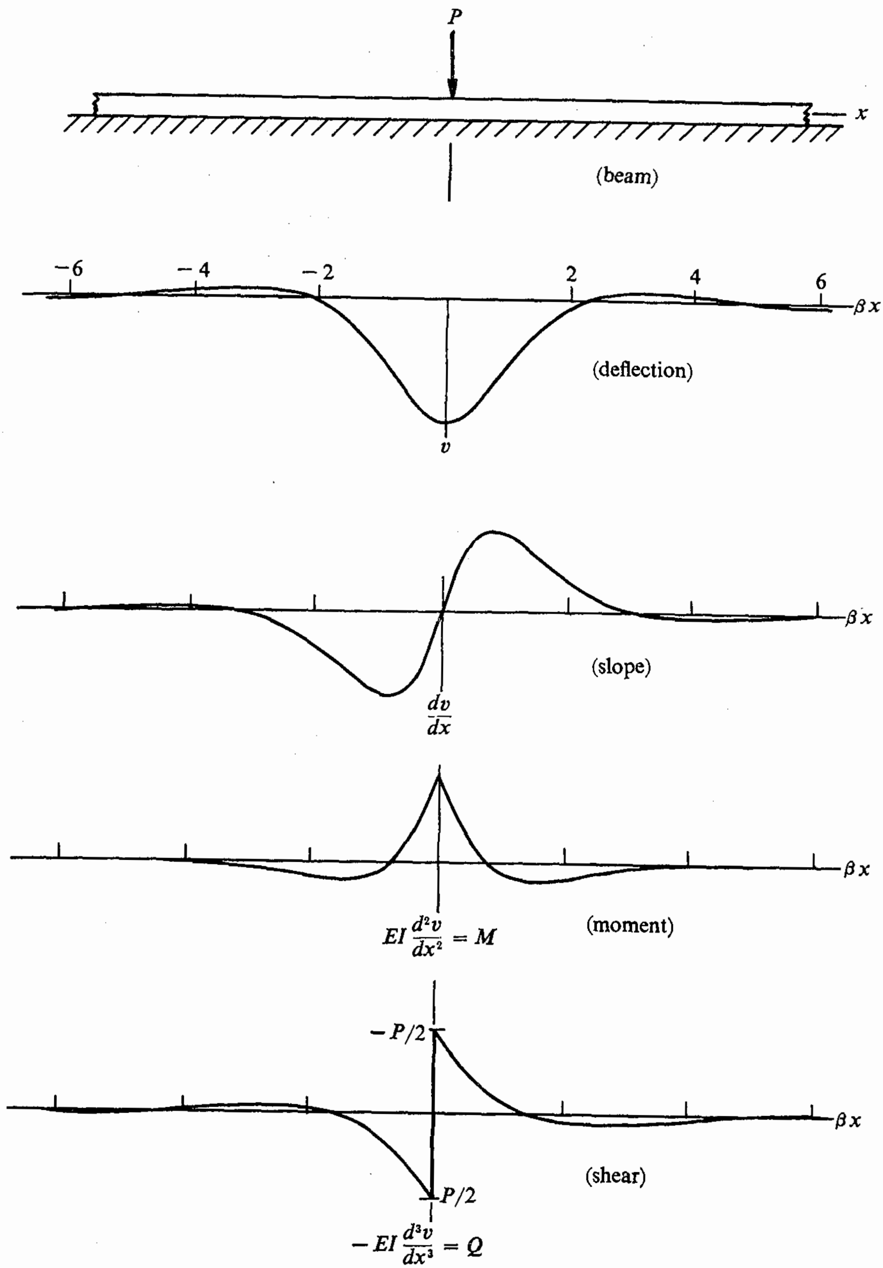

P
x
(beam)
−6
−4
−2
2
4
6
βx
(deflection)
v
βx
(slope)
dv/dx
βx
(moment)
EI d²v/dx² = M
−P/2
βx
(shear)
P/2
−EI d³v/dx³ = Q

FIGURE 3.27

The general solution of Eq. (3.70a) can, therefore, be written

$$v = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) \tag{3.71}$$

Evaluation of the constants by means of the boundary conditions gives, after some lengthy algebra, the solution for $x \geq 0$

$$\begin{aligned} v &= \frac{P\beta}{2k} e^{-\beta x}(\cos \beta x + \sin \beta x) \\ &= \frac{P\beta}{2k} e^{-\beta x} \left[\sqrt{2} \sin \left(\beta x + \frac{\pi}{4}\right)\right] \end{aligned} \tag{3.72}$$

The deflection, slope, bending moment, and shearing force are shown in Fig. 3.27.

It is seen from Eq. (3.72) that the essential characteristic of a beam on an elastic foundation is that any action applied at a point produces a displacement that attenuates with distance as an exponentially damped sine wave. The wavelength of the sine wave is

$$\lambda = \frac{2\pi}{\beta} = 2\pi \sqrt[4]{\frac{4EI_z}{k}}$$

If the beam carries a distributed load or has a short length the damped sinusoidal nature of the deflection may not be so apparent.

Semi-Infinite Beam on an Elastic Foundation. The semi-infinite beam with a concentrated force at the end shown in Fig. 3.28a has the following boundary conditions:

$$M = 0, \quad Q = -P \qquad \text{at } x = 0$$

$$v = \frac{dv}{dx} = 0 \qquad \text{at } x = \infty$$

The solution to Eq. (3.70a) with these end conditions is

$$v = \frac{2\beta P}{k} e^{-\beta x} \cos \beta x \qquad \text{(deflection)} \tag{3.73}$$

$$\frac{dv}{dx} = -\frac{2\beta^2 P}{k} e^{-\beta x}(\cos \beta x + \sin \beta x) \qquad \text{(slope)} \tag{3.74}$$

$$M = +EI_z \frac{d^2 v}{dx^2} = \frac{P}{\beta} e^{-\beta x} \sin \beta x \qquad \text{(bending moment)} \tag{3.75}$$

$$Q = -EI_z \frac{d^3 v}{dx^3} = -Pe^{-\beta x}(\cos \beta x - \sin \beta x) \qquad \text{(shearing force)} \tag{3.76}$$

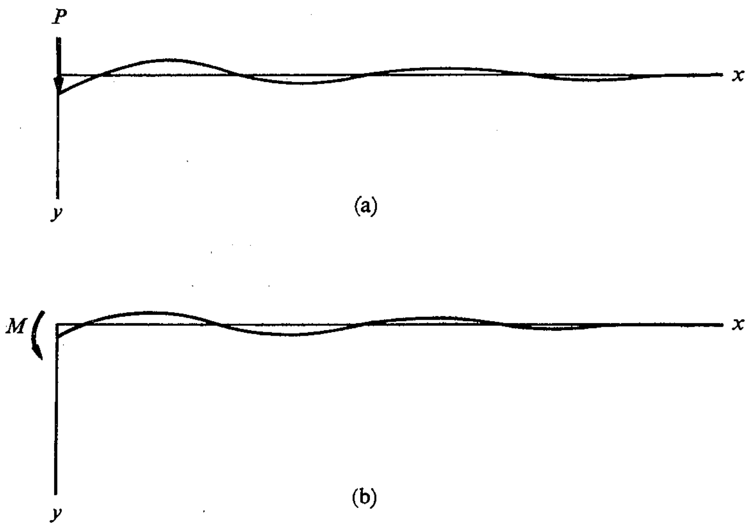


FIGURE 3.28

The semi-infinite beam with end moment, as shown in Fig. 3.28b, has the following boundary conditions

$$M = M_0, \quad Q = 0 \qquad \text{at } x = 0$$

$$v = \frac{dv}{dx} = 0 \qquad \text{at } x = \infty$$

The solution is

$$v = \frac{2\beta^2 M_0}{k} e^{-\beta x}(\cos \beta x - \sin \beta x) \qquad \text{(deflection)} \tag{3.77}$$

$$\frac{dv}{dx} = -\frac{4\beta^3}{k} M_0 e^{-\beta x} \cos \beta x \qquad \text{(slope)} \tag{3.78}$$

$$M = EI_z \frac{d^2v}{dx^2} = M_0 e^{-\beta x}(\cos \beta x + \sin \beta x) \qquad \text{(bending moment)} \tag{3.79}$$

$$Q = -EI_z \frac{d^3v}{dx^3} = 2\beta M_0 e^{-\beta x} \sin \beta x \qquad \text{(shearing force)} \tag{3.80}$$

3-10 FOOTING ON AN ELASTIC FOUNDATION

To support a building, it is sometimes necessary to build a footing that carries loads as shown in the sketches of Fig. 3.29. The ground is considered to be an elastic foundation and the footing is treated as a beam. To design the

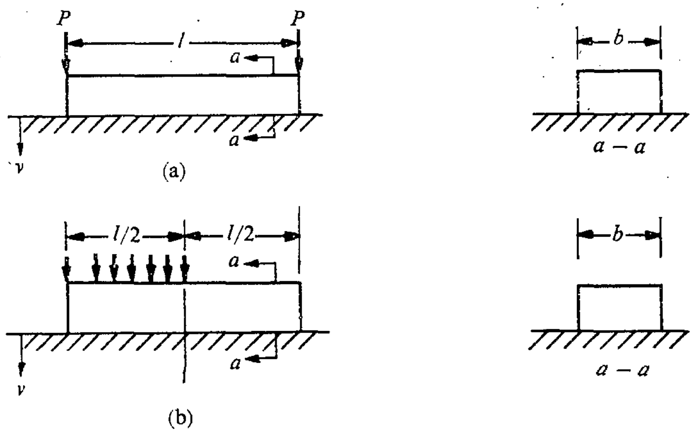


FIGURE 3.29

footing, it is necessary to know the bending moments and shear forces and also to know the maximum pressures exerted by the footing on the soil. The two problems illustrated in Fig. 3.29 can be formulated as follows:

Problem (a) $EI_z \dfrac{d^4v}{dx^4} + kv = 0$	$\left.\begin{matrix} M = 0 \\ Q = -P \end{matrix}\right\}$ at $x = -\dfrac{l}{2}$	$\left.\begin{matrix} M = 0 \\ Q = +P \end{matrix}\right\}$ at $x = +\dfrac{l}{2}$
Problem (b) $EI_z \dfrac{d^4v}{dx^4} + kv = p$ $= 0$	$-\dfrac{l}{2} < x < 0$ $\dfrac{l}{2} > x > 0$	$\left.\begin{matrix} M = 0 \\ Q = 0 \end{matrix}\right\}$ at $x = \pm\dfrac{l}{2}$

If the moment of inertia I_z of the footing is large and the foundation modulus k of the soil is small, the footing can be considered to be rigid and the soil pressures can be determined by static equilibrium, but if the flexibility of the footing must be taken into account, it is necessary to solve the differential equation.

3-11 THIN-WALLED TUBES

The stresses and displacements produced in a tube by axially symmetrical radial loading can be determined by an analysis based on that of the beam on an elastic foundation. The axially symmetrical ring loading on the tube shown in Fig. 3.30a produces a change in radius $v = f(x)$ along the tube. The average circumferential stress across the tube thickness t at any cross section is proportional to the circumferential strain.*

$$\sigma_\theta = \epsilon_\theta E = \frac{2\pi(r - v) - 2\pi r}{2\pi r} E = -\frac{E}{r} v$$

where r is the mean radius of the tube, and E is Young's modulus. Consider a typical strip of the tube whose width is $r\,d\theta$ as shown in Fig. 3.30a (all strips are the same because of symmetry). The slice may be considered to be a small beam with the side stresses σ_θ as shown in Fig. 3.30b. The forces per

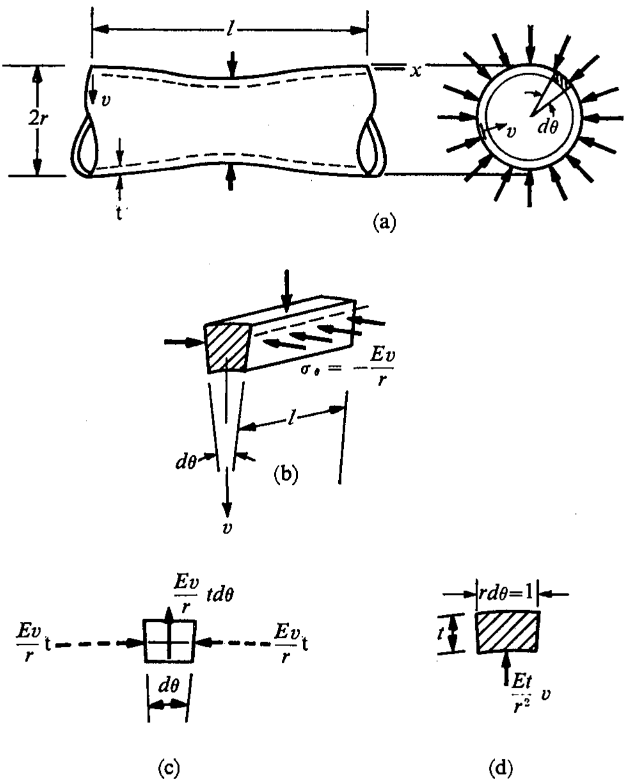


FIGURE 3.30

unit length on the sides are $t\sigma_\theta$. These forces balance in the circumferential direction but, as shown in Fig. 3.30c, have a net outward radial component per unit length along the beam given by

$$2\left(t\sigma_\theta \sin\frac{d\theta}{2}\right) \cong t\sigma_\theta\, d\theta = -\frac{Et}{r} v\, d\theta$$

* Hooke's law states that $E\epsilon_\theta = \sigma_\theta - \nu(\sigma_x + \sigma_r)$ but, since the average σ_x is assumed to be zero and the σ_r is relatively small, the effects of these stresses are neglected.

For convenience, we shall consider a beam of unit width ($r\,d\theta = 1$) as shown in Fig. 3.30d. The circumferential stress opposes radial displacement of the beam with a force per unit length equal to vEt/r^2. This is equivalent to the reaction from an elastic foundation having a modulus Et/r^2 per unit width of beam. The differential equation of this beam of unit width may thus be written

$$\frac{E't^3}{12}\frac{d^4v}{dx^4} = -\frac{Et}{r^2}v + p \tag{3.81}$$

which is the equation of a beam on an elastic foundation. The effective modulus $E' = E/(1 - \nu^2)$ must be used because the beam is an element of the tube and is not free to assume an anticlastic curvature (the sides of the beam cannot rotate). The following values of β and k must be used for this problem:

$$\beta = \sqrt[4]{\frac{k}{4E'I_z}} = \sqrt[4]{\frac{3(1 - \nu^2)}{r^2t^2}}$$

$$k = \frac{Et}{r^2}$$

Solutions of Eq. (3.81) can be used to analyze a number of practical problems. For example, a pipe built into a rigid wall and subjected to an internal pressure will expand except at the wall where it is prevented from doing so. To solve for the stresses and deflections, we note that if the pipe were completely free to expand under internal pressure p, the radius would increase by an amount e. At the rigid wall, the reactions Q_1 and M_1 on a unit width beam cut from the pipe must hold the slope at the wall zero, and produce a deflection equal to e with respect to the pipe at a large distance from the wall. This beam is shown in Fig. 3.31. It could be treated as one-half

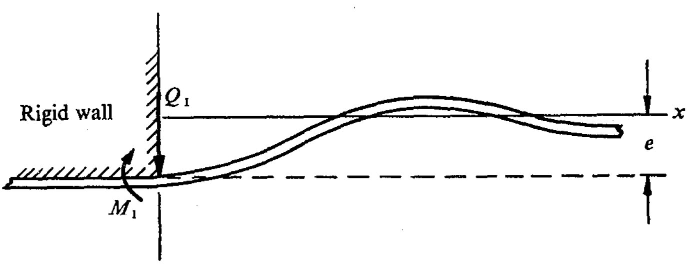


FIGURE 3.31

of an infinite beam carrying a concentrated force equal to $2Q_1$. Alternately, Eqs. (3.73), (3.74), (3.77), and (3.78) for semi-infinite beams with a concentrated load and a concentrated moment may be used to find the deflection and slope of the pipe element. Using Eqs. (3.73) and (3.77) for deflection at $x = 0$, we may write

$$e = \frac{2\beta Q_1}{k} - \frac{2\beta^2 M_1}{k}$$

and using Eqs. (3.74) and (3.78) for the slope at $x = 0$, we may write

$$\left.\frac{dv}{dx}\right|_{x=0} = -\frac{2\beta^2 Q_1}{k} + \frac{4\beta^3 M_1}{k} = 0$$

Solving for M_1 and Q_1, we obtain

$$M_1 = \frac{ke}{2\beta^2}$$

$$Q_1 = \frac{ke}{\beta}$$

All of the pertinent stresses and deflections in the pipe can be determined when the correct values of M_1 and Q_1 are known.

It should be noted that the exponential sine-wave displacements damp out quickly with x so that with small error the foregoing analysis can also be applied to a hemispherical shell with a restrained circumference.

3-12 MOVING LOAD ON A BEAM ON AN ELASTIC FOUNDATION

Several vehicle tracks have been constructed for high-speed testing on rocket-propelled sleds. Such a track is schematically shown in Fig. 3.32. The

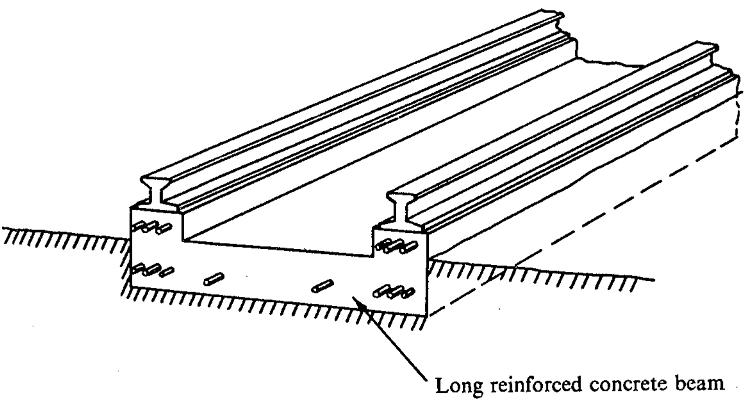


FIGURE 3.32

rocket-propelled sled slides on the track and it may reach velocities of 2000 to 3000 ft/sec. The total weight of the sled may be as much as 50,000 lb and the concrete beam must be designed to withstand the bending moments produced by the load. When the sled is at rest the beam stresses can be computed by means of Eq. (3.70); however, when the sled is traveling at high speed the deformation of the beam will be different and this case will be investigated. Consider then a beam of mass ρ per unit length resting on an elastic foundation and supporting a load P which is moving at a constant velocity. Since P imparts vertical accelerations to the beam, the inertia forces $-\rho(\partial^2 v/\partial t^2)$ per unit length must be included as part of the load so that the equation of the beam everywhere except under the load is

$$EI_z \frac{\partial^4 v}{dx^4} + kv = -\rho \frac{\partial^2 v}{\partial t^2}$$

In standard form, the equation is written

$$\frac{\partial^4 v}{\partial x^4} + \frac{\rho}{EI_z}\frac{\partial^2 v}{\partial t^2} + \frac{k}{EI_z} v = 0 \tag{3.82}$$

We shall assume that P has been moving with velocity c for a sufficient time so that a steady state has been reached, i.e., a deflection curve

$$V = f(x - ct) = f(\xi)$$

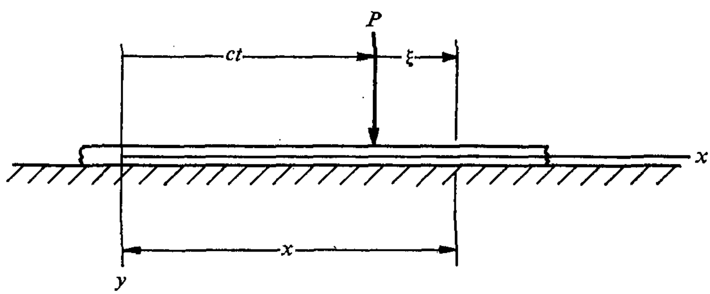


FIGURE 3.33

is moving with P as shown in Figs. 3.33 and 3.34. Changing to the independent variable $\xi = x - ct$ in Eq. (3.82), we obtain

$$\frac{d^4 v}{d\xi^4} + \frac{\rho c^2}{EI_z}\frac{d^2 v}{d\xi^2} + \frac{kv}{EI_z} = 0$$

This is a linear differential equation with constant coefficients whose solution for $\xi > 0$ and $c < c_c$ may be written

$$v = e^{-\alpha\xi}\left(A_1 \cos \beta\sqrt{1 + \left(\frac{c}{c_c}\right)^2}\,\xi + A_2 \sin \beta\sqrt{1 + \left(\frac{c}{c_c}\right)^2}\,\xi\right)$$

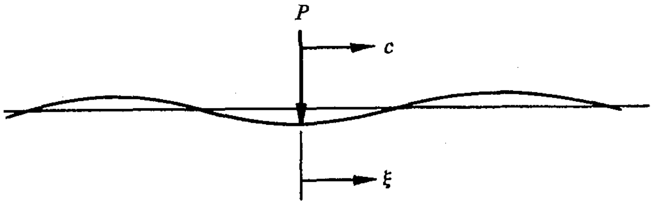


FIGURE 3.34

where

$$\alpha = \beta\sqrt{1 - \left(\frac{c}{c_c}\right)^2} \qquad c_c = \sqrt[4]{\frac{4EI_z k}{\rho^2}}$$

Terms containing the positive exponential do not appear in the solution since the deflection must be finite at $\xi = +\infty$. A solution of the same form may be found for $\xi < 0$. The conditions that the slope be continuous at $\xi = 0$ and that the load on the beam is equal to P are

$$\left.\frac{\partial v}{\partial \xi}\right|_{\xi = 0+\epsilon} = \left.\frac{\partial v}{\partial \xi}\right|_{\xi = 0-\epsilon}$$

$$\int_{-\infty}^{\infty} kv\, d\xi = P$$

After considerable algebra, these determine the constants A_1 and A_2 and the solution for $\xi > 0$, $c < c_c$ is

$$v = \frac{P\beta}{2k} e^{-\alpha\xi}\left(\frac{\cos \beta\sqrt{1 + (c/c_c)^2}\,\xi}{\sqrt{1 - (c/c_c)^2}} + \frac{\sin \beta\sqrt{1 + (c/c_c)^2}\,\xi}{\sqrt{1 + (c/c_c)^2}}\right)$$

Because of the term $\sqrt{1 - (c/c_c)^2}$ in the denominator, the deformation increases as c increases as illustrated in Fig. 3.35. Therefore, the maximum stresses in the beam depend upon the ratio c/c_c, where

$$c_c = \sqrt[4]{\frac{4EI_z k}{\rho^2}} = 2\beta\sqrt{\frac{EI_z}{\rho}}$$

This is called the critical velocity and it will be noted that when $c = c_c$ the

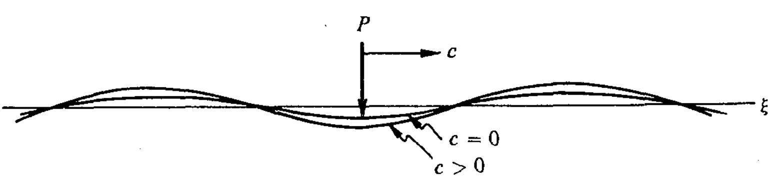


FIGURE 3.35

displacement becomes infinite, so it is not possible to have a steady state solution under this condition. This results from the fact that no damping was included in the analysis. Actually, when the beam deforms some energy is dissipated (heat) and some energy is also lost into the foundation. It is not unreasonable to assume that the energy loss affects the beam somewhat as if a damping force proportional to the vertical velocity ($\partial v/\partial t$) were acting. In this case, the differential equation is

$$\frac{\partial^4 v}{\partial x^4} + \frac{\rho}{EI_z}\frac{\partial^2 v}{\partial t^2} + \frac{D}{EI_z}\frac{\partial v}{\partial t} + \frac{kv}{EI_z} = 0$$

where D is the damping constant. The solution of this equation remains finite for all c.*

The appropriate solution for the undamped beam when P travels at supercritical velocity is, for $\xi > 0$:

$$v = -\frac{P\beta}{2k}\left[\frac{2\sin\beta(\sqrt{(c/c_c)^2 + 1} + \sqrt{(c/c_c)^2 - 1})\xi}{\sqrt{(c/c_c)^4 - 1}\,(\sqrt{(c/c_c)^2 + 1} + \sqrt{(c/c_c)^2 - 1})}\right]$$

and, for $\xi < 0$

$$v = -\frac{P\beta}{2k}\left[\frac{2\sin\beta(\sqrt{(c/c_c)^2 + 1} - \sqrt{(c/c_c)^2 - 1})\xi}{\sqrt{(c/c_c)^4 - 1}\,(\sqrt{(c/c_c)^2 + 1} - \sqrt{(c/c_c)^2 - 1})}\right]$$

For c very much larger than c_c these reduce to

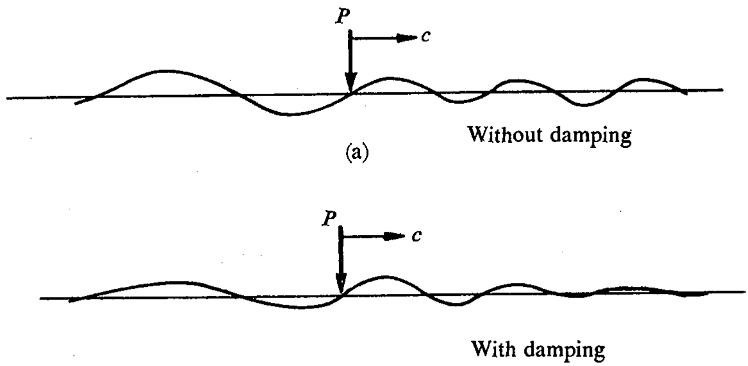
$$v = -\frac{P\beta}{2k}\left(\frac{c_c}{c}\right)^3 \sin\frac{2c}{c_c}\beta\xi \qquad (\xi > 0)$$

and

$$v = -\frac{P\beta}{k}\left(\frac{c_c}{c}\right)^3 \sin\frac{c_c}{c}\beta\xi \qquad (\xi < 0)$$

The deformations are shown in Fig. 3.36a. It will be noted that v is zero at $\xi = 0$ but $dv/d\xi$ is not zero at $\xi = 0$. This means that P is, in effect, always running uphill and, therefore, it requires a horizontal force $P\,dv/d\xi$ to keep pushing it along. The horizontal force does work at a rate $P(dv/d\xi)c$. This work goes into increasing the lengths of the fore and aft wave trains at $\pm\infty$. If damping is included, the waves decrease exponentially with $\pm\xi$, as shown in Fig. 3.36b, and $(P\,dv/d\xi)c$ is the power that is dissipated in damping.

* See J. Kenney, "Steady state vibrations of beam on elastic foundation for moving load," *J. App. Mech.*, 1954. The solution is straightforward but involves lengthy algebraic manipulations.



(a)

(b)

FIGURE 3.36

It is seen from the foregoing equations that as c becomes larger, the amplitude of the deformation becomes smaller and in the limit P is just skating over the beam and barely disturbing it. This is the explanation of the fact that a thin sheet of ice on the surface of a pond that would not support a stationary boy's weight will hold him if he is sliding at a sufficiently high velocity, $c \gg c_c$; the same is true of an aeroplane landing on the ice. On the other hand, an automobile on ice that can support its weight when $c = 0$ may not be able to support the car when c approaches c_c; the same problem may occur in a pontoon bridge. This behavior was also encountered on one of the first railroad lines constructed in England by George Stephenson in 1830. The line ran across a bog which had a fluid substratum covered by a dense mat of vegetation. It was observed that when the train traveled across the bog it was accompanied by a wave.

Problems

3.46 Assume a solution to Eq. (3.70a) in the form $v = Ae^{mx}$ and derive Eq. (3.71).

3.47 Obtain Eq. (3.72) by applying the boundary conditions to the general solution.

3.48 Obtain Eqs. (3.73)–(3.76) by applying the boundary conditions to the general solution.

3.49 A continuous steel rail has $I = 73.6$ in.^{4} and measurements show that the tie-roadbed system has a stiffness approximately the same as a continuous elastic foundation having $k = 500$ lb/in.^{2}. A locomotive wheel applies a force of 10,000 lb to the rail. (a) What is the maximum deflection produced by the wheel? (b) What is the wavelength of the deflection oscillations? (c) Rail has $h = 6$ in. and centroid is at midheight. What is the maximum bending stress produced by the wheel? (d) The optimum road bed is one that gives the smoothest ride without excessive cost. Local variations in k are objectionable because they give a jolting ride at high speeds. However, it is not practicable to construct a roadbed with absolutely uniform k. In view of this, what sort of k is desirable? Large? Small? Why? What would be limiting factors in trying to achieve this?

3.50 Show that for rails of the same shape loaded by a force P, the maximum bending stress is proportional to P/A, where A is the cross-sectional area of the rail.

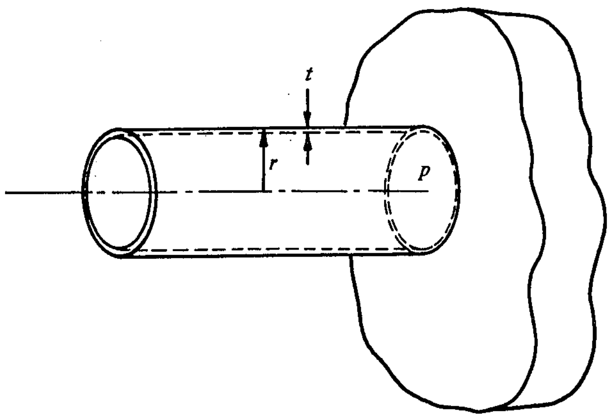
3.51 Determine the maximum soil pressure for Fig. 3.29a, treating the footing as a flexible beam on an elastic foundation.

3.52 Determine the maximum soil pressure and the maximum bending moment and shear force in the footing of Fig. 3.29b, assuming that the footing is rigid, and the soil is an elastic foundation.

3.53 Verify Eq. (3.81) by showing that $EI(1 - \nu^2)^{-1}\, d^2v/dx^2 = M_{xz}$ for pure bending when anticlastic curvature is prevented ($\epsilon_z = 0$).

3.54 A narrow collar is given a shrink fit into a long pipe of radius r and thickness t. The midpoint radius of the collar is R and its cross-sectional area is A. If the outer radius of the pipe is greater by e than the inner radius of the collar, what is the shrink fit force per unit length of collar?

3.55 If the steel pipe in the figure has $r = 12$ in., $t = \frac{1}{2}$ in., and the internal pressure is 1000 psi, what is the maximum bending stress developed? What is the wavelength of oscillations in the pipe wall? How rapidly does the bending stress die out with distance from the heavy plate?



3.56 A pressure vessel of radius r and wall thickness t has hemispherical ends of thickness t. A cylinder with internal pressure p has $\sigma_t = pr/t$, whereas a spherical shell has $\sigma_t = pr/2t$. The cylinder tends to expand more than the sphere by a radial distance e. The decrease in bending is so rapid with x that we may treat the cylinder as being infinitely long, and we may treat the hemisphere as another cylinder. For practical purposes we may assume the radius of each cylinder to be r and ask what M_0 and Q_0 are required to give $v_0 = e/2$ and to give a continuous slope at $x = 0$. Find the maximum bending stress as a function of p, r, and t.

3.57 What is the approximate critical velocity c_c for a one-foot thick sheet of ice on a lake? ($E = 1.5 \times 10^6$ psi.)

3.58 An infinitely long cable of mass ρ per unit length is stretched with a tension force H and is bonded to an elastic foundation of modulus k. Determine the solution for a concentrated transverse force applied to the cable at $x = 0$.

3.59 Do Prob. 3.58 for the case of P moving with constant velocity c.

CHAPTER 4

ELASTIC INSTABILITY

4-1 AN EXAMPLE OF ELASTIC INSTABILITY

Under certain conditions of loading a body may be stable but if the magnitude of the load is increased by a small amount the body may undergo large displacements or it may even collapse. In such cases the instability can come even when the stresses are quite small, far from the yield point or failure stress of the material, so that the body still behaves elastically. The essential features of this instability are exhibited by the simple system shown in Fig. 4.1, which is made of two rigid members connected by a hinge, the assembly being kept straight by springs that produce a straightening moment $M = k\theta$ on each member. In Fig. 4.1b the system is held in a deflected position by the collinear forces P whose magnitude must be such that the moment about the hinge is equal to the spring moment $k\theta$, that is

$$Pl \sin \theta = k\theta$$

The magnitude of the load P that is required to hold the deflected angle θ, is thus*

$$P = \frac{k}{l} \frac{\theta}{\sin \theta} \tag{4.1}$$

* A nonlinear relationship between load and deflection results from the fact that the hinge moment in Fig. 4.1b cannot be calculated by using the initial geometry of Fig. 4.1a, but depend on both P and θ. This is analogous to the system of Fig. 1.36.

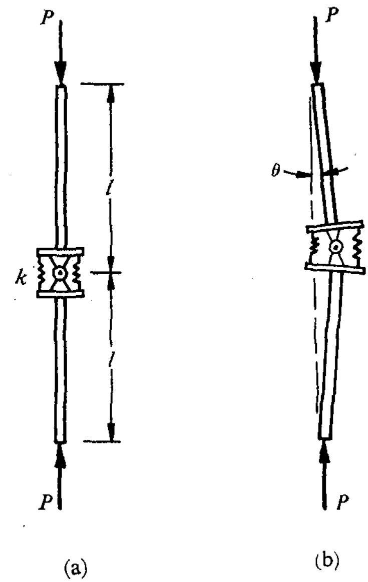


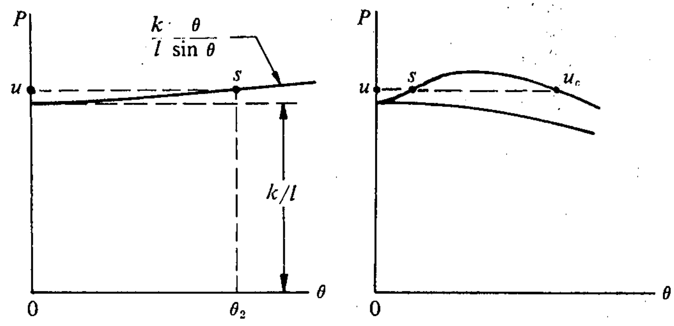
FIGURE 4.1

This relation is plotted in Fig. 4.2a where it is seen that for $P < k/l$ the system remains straight, that is, it has only one equilibrium position ($\theta = 0$) and this is stable. However, when $P > k/l$ the system has two equilibrium positions, $\theta_1 = 0$ (unstable), and θ_2 satisfying Eq. (4.1) (stable). In Fig. 4.2b there are shown two other types of P-θ diagrams that can be obtained with different spring systems. These are characterized by having negative slopes which represent unstable positions leading to collapse. If the system, for example, is at the point marked u it is unstable and it will move to the deflected point marked s which is stable. If it is moved over to the point marked u_c it will be unstable and will collapse (see Prob. 4.4).

If the deflection is small ($\theta \ll 1$), Eq. (4.1) can be linearized by expanding $\sin\theta$ in a series and neglecting terms higher than the first

$$P = \frac{k}{l}\frac{\theta}{\theta - (\theta^3/3!) + \cdots} \cong \frac{k}{l}$$

This is called the critical buckling load, P_{cr}. As seen in Fig. 4.2a, according to the linearized solution, the load P_{cr} will hold the system in equilibrium anywhere in the range $0 < \theta < \infty$. This would imply that θ is indeterminate when the load is P_{cr}. However, the nonlinearized solution shows that θ is actually completely determinate. In fact, if Eq. (4.1) is expanded in series and



(a) (b)

FIGURE 4.2

only the first two terms are retained, we obtain

$$P = P_{cr}\left(1 + \frac{\theta^2}{6} + \cdots\right)$$

or

$$\theta = \sqrt{6\left(\frac{P}{P_{cr}} - 1\right)}$$

Setting $P - P_{cr} = \delta P$ gives

$$\theta = \left(6\,\frac{\delta P}{P_{cr}}\right)^{1/2}$$

If the load P exceeds P_{cr} by $\delta P = 0.01P_{cr}$, the deflection is $\theta = 14°$ which is a relatively large angle in view of the small increment of load.

Another way of looking at elastic instability is from the work-energy viewpoint ($\delta V = \delta W$). When a constant force P acts on the initially straight system of Fig. 4.1a it will do an increment of work δW if θ is given a virtual displacement $\delta\theta$. At the same time an increment of strain energy δV will be stored in the springs. If $\delta V > \delta W$ the column is stable since the load does not do enough work to supply the δV required to make the displacement. If $\delta W > \delta V$ the column is unstable since the load can supply the energy required to further displace the system. At buckling the load P is such that $\delta V = \delta W$. A work-energy analysis will usually make clear the nature of a problem in elastic stability which may otherwise be difficult to understand.

4-2 BUCKLING OF A SIMPLY SUPPORTED COLUMN *

The straight, prismatic column shown in Fig. 4.3a is loaded by an axial force P and is supported in such a way that the bending moment at each end is zero (pin-ended). If the column is slender it can be held in a bent position by a load P of a certain magnitude without exceeding the elastic limit. We shall calculate this load. In Fig. 4.3b the column is shown held in a buckled form by collinear forces P applied at the centroid of the cross section. A freebody diagram of the lower portion of the column (Fig. 4.3c) shows the

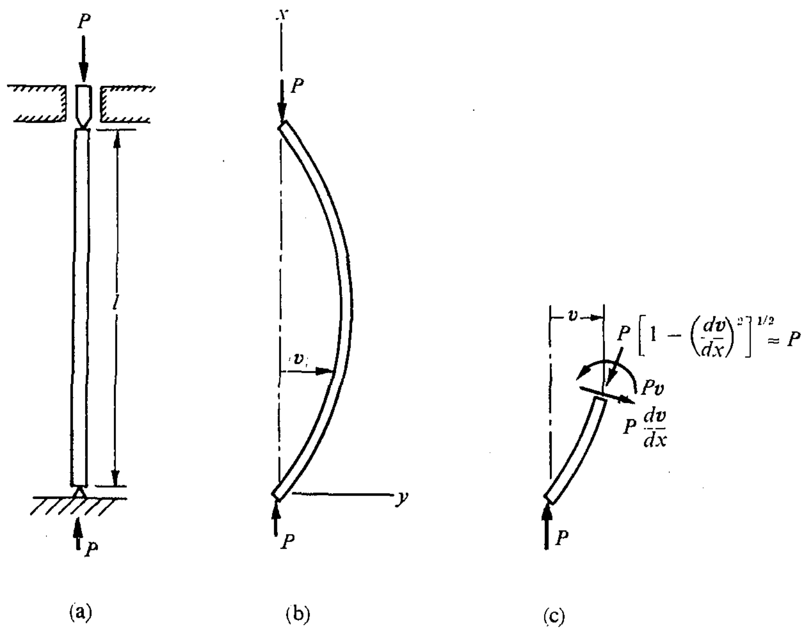


FIGURE 4.3

axial force, the shear force, and the bending moment acting on a section normal to the axis of the beam. The bending moment M_{xz} is equal to $-Pv$ so the beam equation, $EI\, d^2v/dx^2 = M_{xz}$, can be written†

$$EI_z \frac{d^2v}{dx^2} + Pv = 0 \tag{4.2}$$

* The first study of the elastic instability of a column was made by John Bernoulli (1654–1705) in his paper "Curvature laminae elasticae" (1694). He assumed, correctly, that the curvature was proportional to the bending moment.

† See also Prob. 4.7 for an alternate way of looking at the buckling equation.

The end conditions that must be satisfied are $v = 0$ at $x = 0, l$. The solution of Eq. (4.2) can be written

$$v = C_1 \sin \sqrt{\frac{P}{EI_z}}\, x + C_2 \cos \sqrt{\frac{P}{EI_z}}\, x$$

The end condition at $x = 0$ requires $C_2 = 0$; and the condition at $x = l$ requires

$$C_1 \sin \sqrt{\frac{P}{EI_z}}\, l = 0$$

Two possible solutions exist which satisfy this equation, namely, $C_1 = 0$ in which case the column remains straight; and $\sin \sqrt{(P/EI_z)}\, l = 0$ in which case the column has a deflected shape. The sine can be zero when its argument has any of the following values

$$\sqrt{\frac{P}{EI_z}}\, l = n\pi \qquad (n = 1, 2, 3, \ldots)$$

or, in other words, the column can be held in a deflected shape only when P has a critical value

$$P_{\text{cr}} = \frac{n^2\pi^2 EI_z}{l^2} \tag{4.3}$$

This value of P is called the nth Euler buckling load.* The deflected shape corresponding to the nth load is

$$v = C_1 \sin \frac{n\pi x}{l} \tag{4.4}$$

According to this solution the value of P required to hold the column in a deflected position is independent of the amplitude C_1 of the deflection. This is because we used the linearized form of the beam equation in which the curvature was approximated by d^2v/dx^2 rather than using the exact expression (Eq. 3.12). If the exact differential equation is used it is found that the magnitude of P does depend on the amplitude C_1 as shown in Fig. 4.4.†

* L. Euler (1707–1783), in the Appendix of his book *Methodus Inveniendi Lineas Curvas* (1744), gave the first detailed analysis of buckling of columns. However, since at this time slender metal columns were not used (only heavy wood and stone columns were used), Euler's work found no practical application for over one hundred years.

† The nonlinearized differential equation is $EI_z\, d^2\theta/ds^2 + P \sin \theta = 0$, where θ is the angle between the tangent to the column axis and the x-axis, $P \sin \theta$ is the shear force, and s is measured along the beam axis. The solution of this equation can be expressed in terms of elliptic functions.

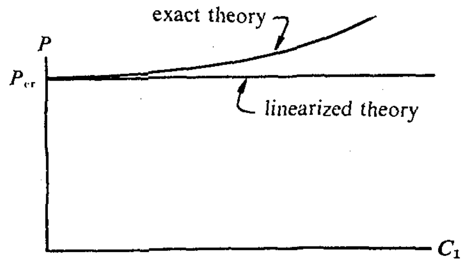


FIGURE 4.4

The shapes of the buckling modes corresponding to the first three critical loads are shown in Fig. 4.5. Although all modes are mathematically possible, in practice the higher modes cannot be maintained without restraining the nodal points against moving laterally, otherwise the column will always deflect into the first mode.

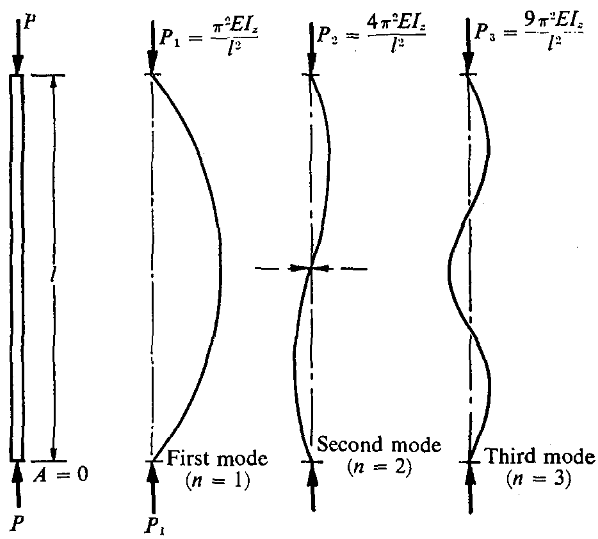


FIGURE 4.5

In addition to the critical loads P_n, where $n = 1, 2, 3, \ldots$, the analysis indicates that the column can support an arbitrary load, $0 < P < \infty$, so long as $C_1 = 0$. To examine the meaning of this result, it will be illuminating to look at the buckling problem from the work-energy viewpoint. In this case, the critical buckling load is that for which $\Delta V = \Delta W$. As the column deflects, the point of application of P moves downward a distance e, as shown in Fig. 4.6, where e has the value

$$e = \int_0^{l-e} (ds - dx) = \int_0^{l-e} [(dx^2 + dv^2)^{1/2} - dx]$$
$$= \int_0^{l-e} \left\{\left[1 + \left(\frac{dv}{dx}\right)^2\right]^{1/2} - 1\right\} dx$$

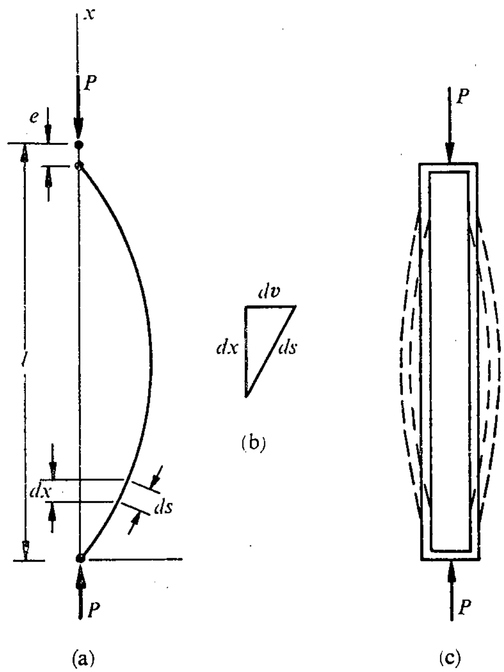


FIGURE 4.6

For small displacements (linearized theory) the limits of integration are taken to be 0 and l, and the integrand can be expanded in a series and only the first term retained,* so that

$$e = \int_0^l \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

* Our analysis of bending assumes that v and dv/dx are very small. We see that the quantity e is a second-order term in dv/dx and, since P is finite, the product $P(dv/dx)^2$ and, hence, the work is also of second order. The induced bending stress is of first order in d^2v/dx^2 so the strain energy is of second order, $(d^2v/dx^2)^2$. This shows that e must be of second order if δV is to be equal to δW. It is thus seen that in the buckling problem second-order strains cannot be neglected.

During buckling P remains constant, so the work done is

$$\Delta W = P\int_0^l \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx \tag{4.5}$$

The bending stresses increase the strain energy during buckling. For columns of usual proportions the contribution to the strain energy from the shearing stresses is so small compared to that from the bending stresses that it is customarily neglected. The total strain energy is then*

$$V = \int_0^l \frac{1}{2}\,EI_z\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{P^2l}{2AE}$$

The increase in strain energy when the initially straight column buckles is then†

$$\Delta V = \int_0^l \frac{1}{2}\,EI_z\left(\frac{d^2v}{dx^2}\right)^2 dx$$

The criterion for buckling is, therefore

$$\Delta W = \Delta V$$

$$P\int_0^l \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx = \int_0^l \frac{1}{2}\,EI_z\left(\frac{d^2v}{dx^2}\right)^2 dx$$

If $v = C\sin n\pi x/l$ is substituted in this equation, there is obtained for the buckling load

$$P_n = \frac{n^2\pi^2EI_z}{l^2}$$

When the applied load P is greater than P_n the system is unstable since the work done by P is greater than the strain energy stored. From this it may be concluded that the straight column is unstable when P is greater than the lowest buckling load P_1. If P is equal to P_1 the column is in neutral equilibrium for small displacements (linearized theory). If P is smaller than P_1 the straight column is in stable equilibrium. The solution $C = 0$ thus represents a stable or an unstable configuration according to whether or not P is smaller than or greater than P_1.

* There is no cross-product term in the expression for strain energy because the P/AE strain is constant over the section and the bending strain has an average value of zero.

† During buckling, the strain energy associated with the axial compression does not change so there is no contribution from this.

The fact that multiple solutions, $v = 0$ and $v = C \sin n\pi x/l$ were found in this problem for the same load P is not in violation of the uniqueness theorem. Actually, the straight column and the buckled column do not have the same boundary conditions since the buckled configuration will have both σ_x and τ_{xy} stresses on the ends, whereas the unbuckled configuration will have only σ_x stresses. However, the double column system shown in Fig. 4.6c will have the same boundary conditions for both buckled and unbuckled configurations. Since the axial force in a column is given in terms of the axial strain as $P = EA\,\partial u/\partial x$, we see that to have a buckling problem it is necessary for the axial strain to be equal to or greater than some value. The uniqueness theorem was derived for the linearized theory of elasticity which supposes that the strains could be infinitesimally small. The buckling problem, therefore, does not satisfy the premises on which the uniqueness theorem is based.

Critical Buckling Loads for Other End Conditions. As may be seen in Fig. 4.7, the solution for the pin-ended column can be used to construct the

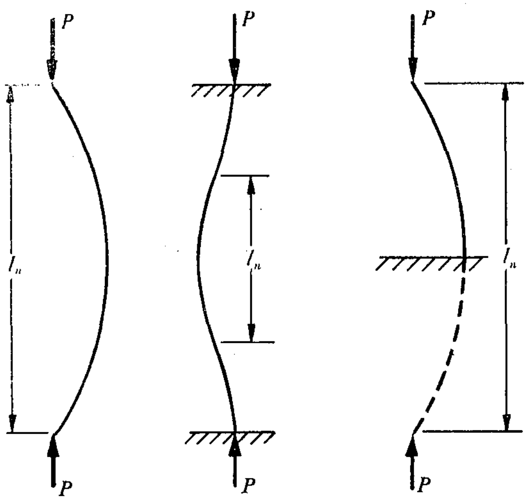


FIGURE 4.7

solutions for buckled columns having other end conditions. These solutions could, of course, be derived from the differential equation by imposing the appropriate boundary conditions. It may be noted that the buckling load for the various end conditions and modes may be expressed as

$$P_n = \frac{\pi^2 EI}{l_n^2} \tag{4.6}$$

where l_n is the length of beam comprising one-half of a sine wave in the buckled shape.

More generally, the complete differential equation of a buckled, nonuniform column is

$$\frac{d^2}{dx^2}\left(EI_z \frac{d^2v}{dx^2}\right) + P\frac{d^2v}{dx^2} = 0 \tag{4.7}$$

The solution of this equation has four constants of integration which will be adequate to satisfy any boundary conditions. This differential equation is of special type in that it has a nonzero solution only for certain discrete values of P; that is, $v_n = f_n(P_n, x)$. The functions v_n are called the *characteristic functions* of Eq. (4.7) and the values of P_n are called the *characteristic values.* The German names are sometimes used with the v_n and P_n being called *eigenfunctions* and *eigenvalues* of the equation. Equations of this type are frequently encountered in engineering and physics.

Failure of Columns. The foregoing analysis was based on the supposition that the material of which the column is made remains linearly elastic. In the case of a real column, however, if I_z/l^2 is not small the axial stress will exceed the elastic limit of the material before the buckling load is reached; in particular, the stresses in a steel column will reach the yield point stress, σ_0. For a column of cross-sectional area A, an axial load $A\sigma_0$ is an upper bound for P since at this load the entire column is deformed plastically and will certainly fail and this possibility of failure must be considered as well as the possibility of elastic buckling. In the case of elastic buckling the critical stress at which buckling of a straight pin-ended column occurs is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI_z}{l^2 A} = \frac{\pi^2 E}{(l/r)^2}$$

where r is the radius of gyration of the cross section ($I_z = Ar^2$). The ratio (l/r) is called the slenderness ratio, and the column will buckle in the direction in which the cross section has the largest value of (l/r), that is, in the direction of the smallest value of r. For steel having $E = 30 \times 10^6$ psi and $\sigma_0 = 36{,}000$ psi, a ratio $l/r = 90$ makes $\sigma_{cr} = \sigma_0$. This is indicated in Fig. 4.8 where there is shown a diagram of σ_{cr} vs. l/r. Three regions are shown on the diagram; one in which the column is stable, one in which the column is elastically unstable, and one in which the column is plastically unstable.

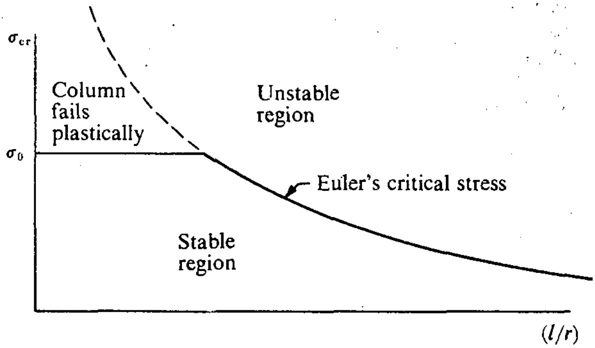


FIGURE 4.8

4-3 COLUMN WITH INITIAL CURVATURE

In the derivation of P_{cr} it was assumed that the column was initially straight and loaded concentrically, in which case for $P < P_{cr}$ the compressive stress is uniformly distributed over the cross section. This is obviously an idealized case as in practice columns are never perfectly straight and are seldom loaded perfectly concentrically. If a column is not straight but has a slightly curved axis its load carrying capacity is impaired. For example consider a pin-ended column whose initial, unloaded shape is given by

$$v_1 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

As may be seen in Fig. 4.9, a concentric load P will produce a bending

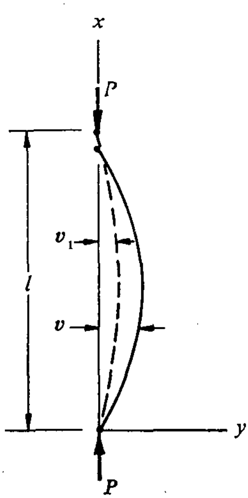


FIGURE 4.9

moment $M_{xz} = -Pv$ and will increase the displacement by an amount $(v - v_1)$. The linearized differential equation of the column is, therefore,

$$EI_z \frac{d^2}{dx^2}(v - v_1) = -Pv \tag{4.8}$$

Substituting for v_1, the equation may be written

$$\frac{d^2v}{dx^2} + \frac{P}{EI_z} v = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l}$$

The solution of this equation is

$$v = C_1 \sin \sqrt{\frac{P}{EI_z}}\, x + C_2 \cos \sqrt{\frac{P}{EI_z}}\, x + \sum_{n=1}^{\infty} \frac{a_n}{1 - P/P_n} \sin \frac{n\pi x}{l}$$

where $P_n = (n\pi)^2 EI/l^2$. The end conditions are $v = 0$ at $x = 0$, l and these determine the constants to be

$$C_1 \sin \sqrt{\frac{P}{EI_z}}\, l = 0$$

$$C_2 = 0$$

Thus, for $P < \pi^2 EI/l^2$, both C_1 and C_2 must be zero, and the deflection of the column is

$$v = \sum_{n=1}^{\infty} \frac{1}{1 - (P/P_n)} a_n \sin \frac{n\pi x}{l} \tag{4.9}$$

We shall now consider the behavior of a pin-ended column whose initial curvature is given by $v_1 = a_1 \sin \pi x/l$, the remainder of the coefficients $a_n = 0$. The effect of the axial load is then to amplify the initial displacement by the nonlinear factor $1/(1 - P/P_{cr})$, where $P_{cr} = P_1 = \pi^2 EI/l^2$. The behavior of the column for various values of initial curvature is shown in Fig. 4.10 where it is seen that the deflection becomes increasingly large as $P \rightarrow P_{cr}$. The stress distribution in the column is given by

$$\sigma_x = -\frac{P}{A} - \frac{My}{I_z}$$

where M is the bending moment ($M = -Pv$). The maximum compressive stress occurs at $x = l/2$, and is given by

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{a_1 A}{Z} \frac{1}{1 - P/P_{cr}}\right] \tag{4.10}$$

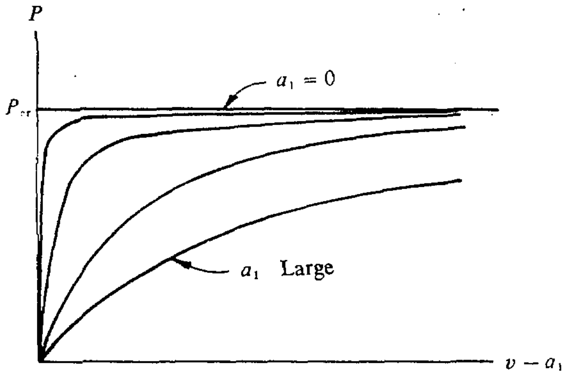


FIGURE 4.10

where Z is the section modulus of the column ($Z = I_z/c$, where c is the distance from the neutral axis to the point of maximum stress). Writing $P/A = \sigma_{av}$ and $P/P_{cr} = \sigma_{av}/\sigma_{cr}$, this equation can be written

$$\sigma_{max} = \sigma_{av}\left[1 + \frac{a_1 A}{Z}\frac{1}{1 - \sigma_{av}/\sigma_{cr}}\right]$$

where $\sigma_{cr} = \pi^2 E/(l/r)^2$. The load P_L, for which σ_{max} is equal to the yield stress σ_0, is the limit load for which the column remains elastic. This load will produce an average stress, $\bar{\sigma}_L = P_L/A$ and, using this notation, the foregoing equation can be written

$$\sigma_0 = \bar{\sigma}_L\left[1 + \frac{a_1 A}{Z}\frac{1}{1 - \bar{\sigma}_L/\sigma_{cr}}\right]$$

which can be written in the form

$$\left(\frac{\bar{\sigma}_L}{\sigma_0}\right)^2 - \left[1 + \frac{\sigma_{cr}}{\sigma_0}\left(1 + \frac{a_1 A}{Z}\right)\right]\frac{\bar{\sigma}_L}{\sigma_0} + \frac{\sigma_{cr}}{\sigma_0} = 0 \qquad (4.11)$$

This equation, which is quadratic in $(\bar{\sigma}_L/\sigma_0)$, is plotted in Fig. 4.11 in a form that gives $(\bar{\sigma}_L/\sigma_0)$ as a function of (l/r) with E and A/Z held constant. If such a column were to have a factor of safety* of N it should be designed so that the actual load P is equal to $\sigma_L A/N$.

* The column will, of course, not collapse when its point of highest stress reaches σ_0. The yielding must progress across the width of the column at the midpoint when the column collapses. This requires a slightly larger load than P_L.

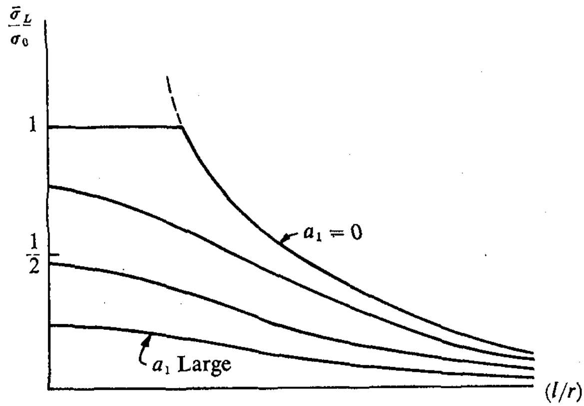


FIGURE 4.11

4-4 COLUMN WITH ECCENTRIC LOADING

Another case of practical importance is that of an initially straight pin-ended column subjected to eccentric loading with eccentricity e, as shown in Fig. 4.12. The bending moment at any cross section of the column is Pv and it is assumed that this moment acts about a principal axis (z-axis) of the

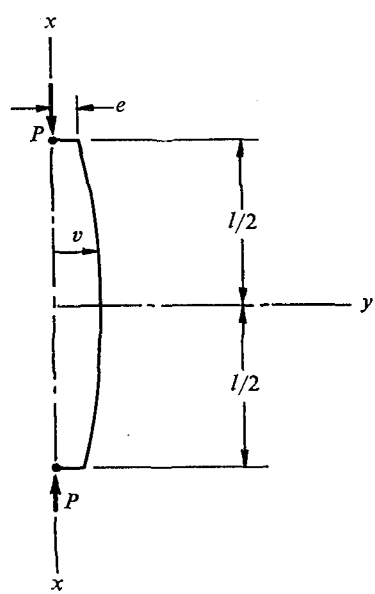


FIGURE 4.12

cross section. The differential equation for the column deflection is

$$EI_z \frac{d^2v}{dx^2} + Pv = 0$$

The solution of this equation is

$$v = C_1 \sin \sqrt{\frac{P}{EI_z}}\, x + C_2 \cos \sqrt{\frac{P}{EI_z}}\, x$$

The constants C_1 and C_2 are determined from the end conditions, $v = e$ at $x = \pm l/2$ which give

$$C_1 = 0$$

$$C_2 = \frac{e}{\cos\left[\sqrt{(P/EI_z)}(l/2)\right]}$$

The deflection of the column is thus given by

$$v = \frac{e}{\cos\left[\sqrt{(P/EI_z)}(l/2)\right]} \cos\left(\sqrt{\frac{P}{EI_z}}\, x\right)$$

At midheight ($x = 0$), the displacement is a maximum

$$v_{\max} = e \sec\left(\sqrt{\frac{P}{EI_z}}\, \frac{l}{2}\right)$$

Since $P_{cr} = \pi^2 EI_z/l^2$, this equation can be written

$$v_{\max} = e \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) \tag{4.12}$$

It is seen that the maximum deflection becomes increasingly large as P approaches P_{cr}. As in the case of the initially curved column, sudden buckling does not occur with an eccentrically loaded column but the elastic limit will be reached at the highest stressed point when

$$\frac{P}{A} + \frac{Mc}{I_z} = \sigma_0$$

or, when

$$\sigma_0 = \frac{P}{A} + \frac{Pec}{I_z} \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

This equation can be written

$$\sigma_0 = \bar{\sigma}_L \left(1 + \frac{ec}{r^2} \sec \frac{l}{2r} \sqrt{\frac{\bar{\sigma}_L}{E}}\right) \tag{4.13}$$

where $\bar{\sigma}_L = P_L/A$ is the nominal stress which causes yielding. Equation (4.13) is called the secant formula. It may be plotted in the form of $(\bar{\sigma}_L/\sigma_0)$ vs. (l/r) for a particular material (σ_0 and E are then fixed) with different values of (ec/r^2), as shown in Fig. 4.13. It is seen that the general form of this diagram

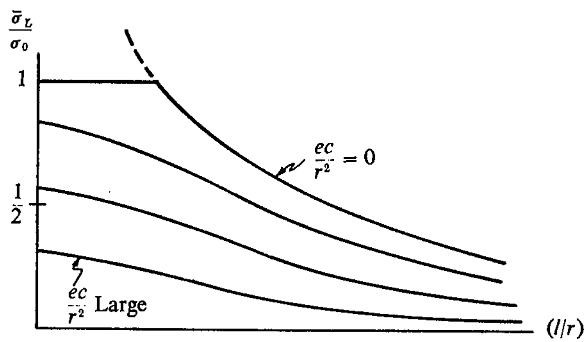


FIGURE 4.13

is similar to that of Fig. 4.11, that is, initial curvature and eccentricity of loading have similar effects.

It should be noted that the radius of gyration to be used in Eq. (4.13) is that which is in the direction of the eccentricity e which is also the direction of deflection. An analysis for possible failure of the column by buckling in the perpendicular plane must also be made, for the radius of gyration in that direction may be so small that the column will fail by buckling in that plane.

4-5 CONSIDERATIONS IN THE DESIGN OF COLUMNS

In the design of a column it would generally not be known what initial curvature or accidental eccentricity of loading the actual column might have. Since a small initial curvature or a small accidental eccentricity may cause a large stress, it is necessary to take into account the imperfections that the column might have. The problem is largely one of estimating the magnitudes of accidental curvatures and eccentricities that are probable in practice. Extensive experimental investigations of buckling have led to a number of different criteria for various structural shapes and materials. These criteria are usually given in the form of an empirical curve which has been fitted to the experimental data of $(\bar{\sigma}_L/\sigma_0)$ vs. (l/r) and which incorporates a given factor of safety. For example, the American Institute of Steel Construction Specification (AISC 1961) gives for main load carrying members, made of structural steel having $\sigma_0 = 36{,}000$ psi, the following allowable P/A stresses*:

$$\sigma_a = 21{,}560\left[1 - \frac{(l/r)^2}{31{,}500}\right] \qquad \text{for } \frac{l}{r} < 120$$

$$\sigma_a = \frac{149{,}000{,}000}{(l/r)^2} \qquad \text{for } 120 < \frac{l}{r} < 200$$

For secondary members the above values of σ_a are multiplied by the factor $1/(1.6 - l/200r)$. These empirical equations represent curves that are similar to those for the secant formula plotted in Fig. 4.13. It should be noted that the secant formula gives the actual stress in the column under precisely specified conditions, whereas, the AISC formulas give the allowable axial stress P/A under average conditions. From these formulas it would be possible to estimate the curvatures and eccentricities that AISC thinks steel columns have on the average.

4-6 COMBINED AXIAL AND LATERAL LOADING OF SLENDER MEMBERS

Frequently beams in bending are also subject to axial forces and in this case the deflection results from the combined action of the loads. Since the deflections due to axial forces are nonlinear functions of the forces, the superposition principle does not apply. A useful approximation which considerably simplifies the calculation of the deflections of pin-ended members with combined axial and lateral loads will be developed in this section. We begin by considering the beam shown in Fig. 4.14 which is subjected to a

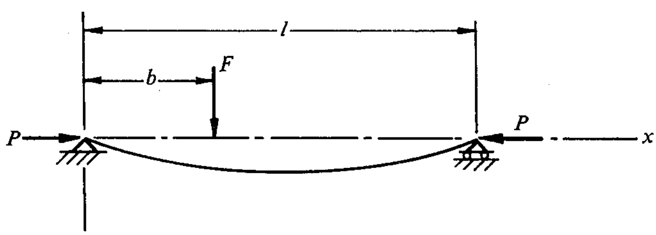


FIGURE 4.14

* The 1963 edition of this specification gives a more complicated equation, whose values are not greatly different.

transverse load F and an axial load P. The deflection curve can be represented by a Fourier series:

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x \tag{4.14}$$

Each term of this series satisfies the end conditions $v = 0$, $d^2v/dx^2 = 0$ at $x = 0, l$. We shall evaluate the coefficients by the method of virtual work; $\delta V = \delta W$.

The work done by P and F is

$$\begin{aligned} W &= P \int_0^l (ds - dx) + \frac{1}{2} F v_b \\ &= P \int_0^l \frac{1}{2} \left(\frac{dv}{dx}\right)^2 dx + \frac{1}{2} F v_b \end{aligned}$$

and

$$\delta W = P\delta \int_0^l \frac{1}{2} \left(\frac{dv}{dx}\right)^2 dx + F\, \delta v_b \tag{4.15a}$$

The strain energy of bending and axial compression is

$$V = \int_0^l \frac{1}{2} EI_z \left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{1}{2} \frac{P^2 l}{AE} \tag{4.15b}$$

Substituting Eq. (4.14) for v into the Eqs. (4.15a) and (4.15b) and integrating,* gives

$$\delta W = \frac{P\pi^2}{4l} \sum_{n=1}^{\infty} n^2\, \delta(a_n^2) + F \sum_{n=1}^{\infty} \delta(a_n) \sin \frac{n\pi b}{l}$$

$$V = \frac{\pi^4 EI_z}{4l^3} \sum_{n=1}^{\infty} n^4 a_n^2 + \frac{P^2 l}{2AE}$$

Noting that $\delta V = \sum_{n=1}^{\infty} (\partial V/\partial a_n)\delta a_n$, the equation $\delta V = \delta W$ is

* Note that the functions $\sin \pi x/l$, $\sin 2\pi x/l$, etc., are orthogonal functions, that is,

$$\int_0^l \left(\sin \frac{n\pi}{l} x\right)\left(\sin \frac{m\pi}{l} x\right) dx = 0, \qquad m \neq n$$

$$\int_0^l \sin^2 \frac{n\pi}{l} x\, dx = \frac{l}{2}$$

$$\frac{\pi^4 EI_z}{2l^3}\sum_{n=1}^{\infty} n^4 a_n\,\delta a_n = \frac{P\pi^2}{2l}\sum_{n=1}^{\infty} n^2 a_n\,\delta a_n + F\sum_{n=1}^{\infty}\sin\frac{n\pi b}{l}\,\delta a_n$$

or

$$\sum_{n=1}^{\infty}\left\{\left(\frac{\pi^4 EI_z}{2l^3}n^4 - \frac{P\pi^2}{2l}n^2\right)a_n - F\sin\frac{n\pi b}{l}\right\}\delta a_n = 0$$

Solving for a_n, there is obtained*

$$a_n = \frac{2F\sin n\pi b/l}{n^2(n^2\pi^4 EI_z/l^3 - P\pi^2/l)} = \frac{2Fl^3\sin n\pi b/l}{\pi^4 EI_z}\,\frac{1}{n^4(1 - P/P_n)}$$

where $P_n = n^2\pi^2\,EI_z/l^2$ as before. The deflection of the beam is thus given by

$$v = \frac{2Fl^3}{\pi^4 EI_z}\sum_{n=1}^{\infty}\frac{1}{n^4(1 - P/P_n)}\sin\frac{n\pi b}{l}\sin\frac{n\pi x}{l} \tag{4.16}$$

It is seen that as $P \to P_n$ the deflection becomes increasingly large which indicates that P_n is a buckling load.

If the load F is at or near the center of the beam a reasonable approximation to the deflection will be given by the first term of the solution only

$$v = \frac{2Fl^3}{\pi^4 EI_z}\,\frac{1}{1 - P/P_{cr}}\sin\frac{\pi b}{l}\sin\frac{\pi x}{l} \tag{4.17}$$

The accuracy of this approximate solution can be checked by comparing with the known solution for the case $b = l/2$ and $P = 0$; that is, for a beam with a concentrated force at the midpoint, and with no axial load. The exact solution gives

$$v_b = \frac{Fl^3}{48EI_z}$$

The approximate solution gives

$$v_b = \frac{2Fl^3}{\pi^4 EI_z} = \frac{Fl^3}{48.7EI_z}$$

and it is seen that for this case the approximate solution is in error by less than 1.5%.

Equation (4.17) shows that the effect of the axial compressive load is to increase the deflection produced by the transverse load by a factor $1/(1 - P/P_{cr})$. Since the deflection is a linear function of *transverse loads*, say F_1 and F_2, the

* The fact that the terms of the Fourier series are orthogonal is the reason why a_n is found directly without the necessity of solving simultaneous equations.

principle of superposition can be used and the deflection produced by only the transverse load F_1 can be added to the deflection produced by F_2 acting alone, and when the sum is multiplied by the factor $1/(1 - P/P_{cr})$ the effect of the axial load is accounted for. This procedure can be used for any combination of lateral forces, distributed loads, and end moments. The same effect is seen in the case of the initially curved column. This leads to the following approximate rule: The effect of an axial compressive force is to increase the displacement by the factor $1/(1 - P/P_{cr})$, and the effect of an axial tensile force is to decrease the displacement by the factor $1/(1 + P/P_{cr})$. This procedure will, of course, give good results only when the deflection produced by the combined action of the transverse and axial forces is similar in shape to the deflection produced by the transverse loads alone. When this condition is not satisfied, the method of solution must be modified as described in the following section.

4-7 RAYLEIGH-RITZ METHOD

The method of solution used in the preceding section depended upon several very special circumstances. First, each term in the Fourier series satisfied the end conditions of the beam; second, the terms of the series were orthogonal functions, and third, the load on the beam was such that the first term only of the series was a reasonable approximation to the true deflected shape. The first and third of these conditions were essential to the analysis and the second condition was a great convenience but was not essential. The following generalization will enable the energy method to be used for beams with arbitrary end conditions and arbitrary loading when appropriate orthogonal functions are not known.

The Rayleigh-Ritz method* may be explained as follows. Instead of a Fourier series let us take the deflection to be expressible in a power series

$$v = \sum_{n=1}^{\infty} a_n x^n \tag{4.18}$$

Suppose that the beam is built-in at $x = 0$ and is simply supported at $x = l$; the end conditions are then

$$\left.\begin{aligned} v &= 0 \\ \frac{dv}{dx} &= 0 \end{aligned}\right\} x = 0 \qquad \left.\begin{aligned} v &= 0 \\ \frac{d^2v}{dx^2} &= 0 \end{aligned}\right\} x = l$$

* The method was first used by Lord Rayleigh and was subsequently generalized by W. Ritz.

These equations will permit elimination of four of the unknown a_n's. The condition $\delta V = \delta W$, written in terms of the a_n and δa_n will give as many equations as there are unknown a_n's. Solution of these equations permits calculation of the correct deflection v.

The foregoing procedure is not practical when a very large number of unknown a_n's are involved. A workable approximate solution could be obtained by taking only the first six terms of the power series, that is,

$$v_a = \sum_{n=1}^{6} a_n x^n$$

After imposing the four end conditions there will be only two unknown a_n's to determine by use of the equation $\delta V = \delta W$. This requires only the simultaneous solution of a pair of equations. The result will be the best fit to the exact deflection curve that can be obtained with that v_a. Whether or not the solution is reasonably accurate depends upon whether or not the expression for v_a is capable of being a reasonable approximation to the true deflection curve.

In practice, it is customary to take for v_a an approximate expression that does not have a large number of unknown coefficients, say

$$v_a = a_1 f_1(x) + a_2 f_2(x) \tag{4.19}$$

where both f_1 and f_2 are functions of x that satisfy the end conditions. They might, for example, each be polynomials. The functions f_1 and f_2 are selected so that the approximate expression will fit closely the exact deflection. An idea can be obtained of how good the approximation is by noting that v_a is the exact solution for the beam carrying a transverse load p_a as determined from the differential equation

$$p_a = EI_z \frac{d^4 v_a}{dx^4} + P \frac{d^2 v_a}{dx^2}$$

The v_a will fit the true v much better than p_a will fit the true p. In fact, each successive derivative of v_a gives a poorer fit to the corresponding derivative of v. As a consequence of this, it is usually easy to pick a v_a that gives a very good fit to the true v but it is much more difficult to pick a v_a that will give a good fit to d^3v/dx^3. The accuracy can, of course, always be improved by adding more suitable terms, $a_n f_n(x)$, to the assumed v_a.

As a simple illustration of the Rayleigh-Ritz method, we shall compute the deflection of the simply supported beam shown in Fig. 4.15a when it is subjected to a uniform load p_0. We take as the approximate deflection curve an expression which gives $v = 0$ at $x = 0, l$*

$$v_a = Cx(l - x)$$

The variations of strain energy and work are

$$\delta V = \delta \int_0^l \frac{1}{2} EI_z \left(\frac{d^2 v_a}{dx^2}\right)^2 dx = \delta(2EI_z C^2 l) = 4EI_z Cl\,\delta C$$

$$\delta W = \int_0^l p_0 \delta v_a\, dx = \tfrac{1}{6} p_0 l^3\,\delta C$$

The principle of virtual work requires $\delta V = \delta W$, or

$$4EI_z Cl\,\delta C = \tfrac{1}{6} p_0 l^3\,\delta C$$

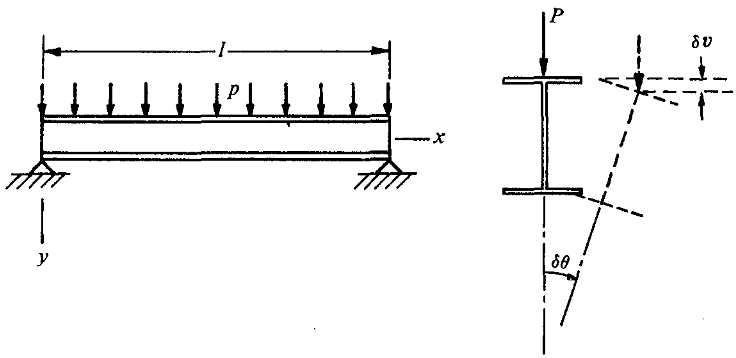
from which

$$C = \frac{1}{24}\frac{p_0 l^2}{EI_z}$$

and the approximate displacement is

$$v_a = \frac{p_0 l^4}{24EI_z}\left(\frac{x}{l} - \frac{x^2}{l^2}\right)$$

This gives for the deflection at midspan, $v_{\max} = p_0 l^4/96EI_z$, which is 83% of the exact value. This is surprisingly good in view of the crude deflection curve we assumed which cannot give reasonable approximations to the correct bending moments and shear forces. Also, the assumed curve satisfied only



(a) (b)

FIGURE 4.15

* A better expression would be the symmetrical displacement

$$v_a = C_1 x(l - x) + C_2 x^2(l - x)^2$$

the kinematic boundary conditions (those imposed on the displacement or slope) but did not satisfy the moment-force boundary conditions, since the assumed curve has at each end a curvature $d^2v/dx^2 = -2C$ and a shear force $-EI_z\, d^3v/dx^3 = 0$. A better approximation would have been obtained had the assumed deflection curve also satisfied the conditions $EI_z\, d^2v/dx^2 = 0$ at $x = \pm l$. If the beam has a variable moment of inertia, it is usually very difficult to integrate the differential equation and in such cases the Rayleigh-Ritz method is particularly useful. See Probs. 4.25–4.32 for examples of the method.

Another way of looking at the Rayleigh-Ritz method is as follows. If we were to attach a mechanism made of rigid members to the beam of Fig. 4.15a which would constrain it to deflect in the shape $v = Cx(l - x)$, no matter what load was applied, we could determine the exact deflection by means of the equation

$$\delta V = \delta W$$

As there is no strain energy stored in the constraining mechanism, the deflection will be precisely as determined by the Rayleigh-Ritz method. Since the constraint increases the effective stiffness of the beam, the deflections will be smaller than for the unconstrained beam. We can, therefore, conclude that when applying the Rayleigh-Ritz method the effective stiffness will be increased and, hence, the computed approximate deflections will be *smaller* than the exact deflections and the buckling load computed by means of the Rayleigh-Ritz method will be *greater* than the exact buckling load (see Prob. 4.25).

4-8 OTHER TYPES OF BUCKLING PROBLEMS

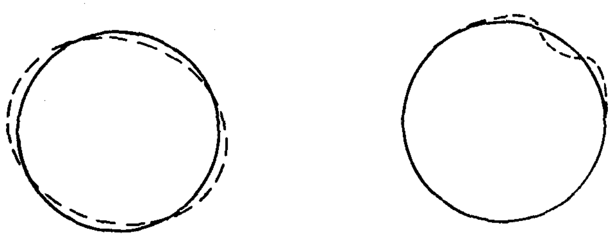
There are many different types of buckling problems, some of which are very subtle and difficult to analyze.* Any loaded system that can deflect in such a way that the work done by the applied loads may exceed the strain energy stored is a problem in elastic stability. A number of examples will be mentioned here to illustrate the types of problems that may be encountered.

The simply supported steel I-beam shown in Fig. 4.15b carries a load P at its midpoint. The bottom flange is in tension but the top flange is in compression and therefore tends to be unstable if the beam is too long. The mode

* The solution to many interesting and practical problems of buckling are presented in the following books: Timoshenko and Gere, *Theory of Elastic Stability*, McGraw-Hill (1961); and Bleich, *Buckling Strength of Metal Structures*, McGraw-Hill (1952).

of lateral deflection, shown in Fig. 4.15b is such that $\delta\theta = f(x)$ with $\delta\theta = 0$ at $x = 0, l$. It is seen that P does work during this deflection, $\delta W = P(\delta v) = Pf_1(\delta\theta)$. The strain energy increases because of the lateral bending of the flanges of the beam and because of the twisting of the beam, and it is a function of $\delta\theta$ only and not an explicit function of P, that is, $\delta V = f_2(\delta\theta)$. It is, thus, only necessary to make P large enough to insure that the work done will equal the strain energy stored and for this load the beam will buckle laterally. This θ-type of torsional twisting can also be involved in the buckling of an axially loaded column if the torsional rigidity is low.

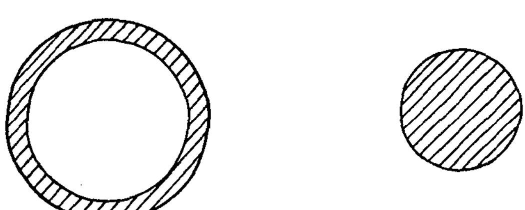
If the air is expelled from a long cylindrical tank the external air pressure may cause the cylinder to buckle as shown in Fig. 4.16a. Ring buckling of this type is discussed in Section 5-5. If a spherical tank is subjected to external pressure, it may buckle with a dimple, as shown in Fig. 4.16b. Short cylindrical tanks will also buckle with dimples. A flat plate compressed in its plane may also buckle.



(a) (b)

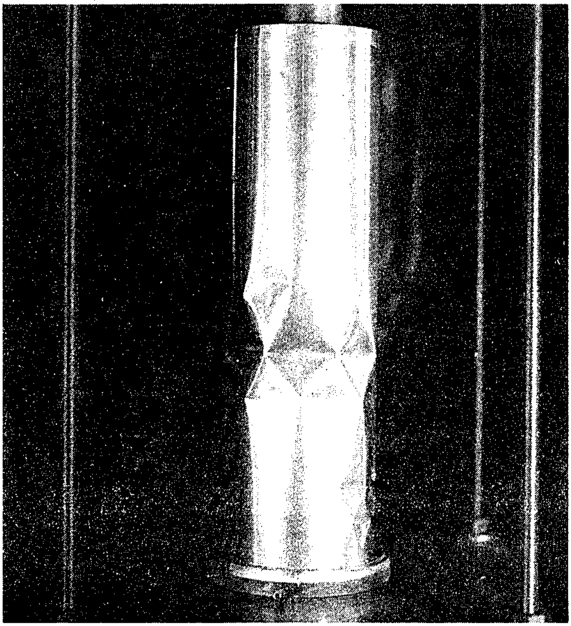
FIGURE 4.16

We have seen that, in general, the carrying capacity of a column is improved by making the l/r ratio small.. For example, the tubular column shown in Fig. 4.17a will have a much larger radius of gyration than the solid circular column (Fig. 4.17b) of the same cross-sectional area. There is, however, a

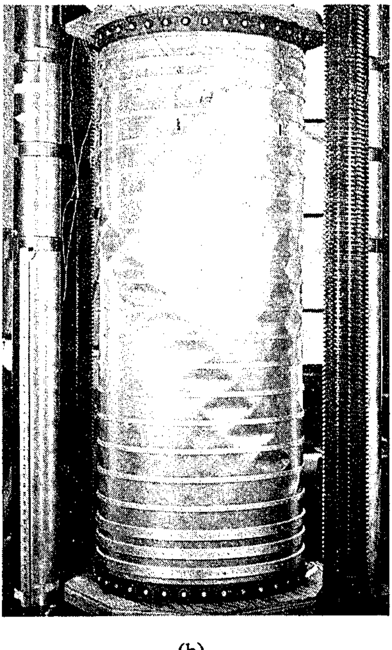


(a) (b)

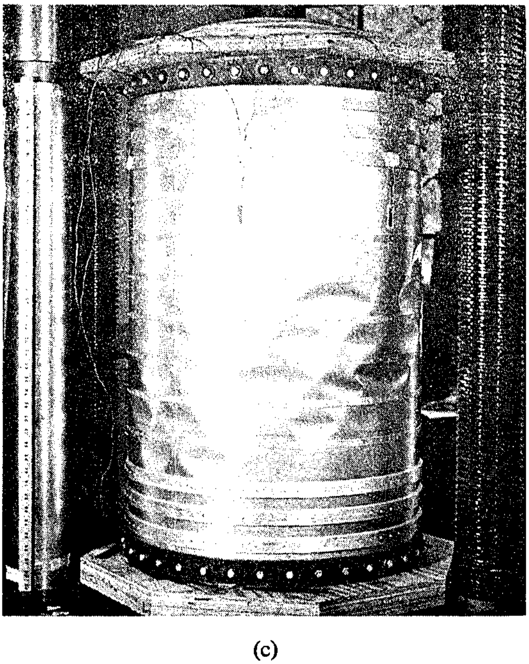
FIGURE 4.17



(a)



(b)



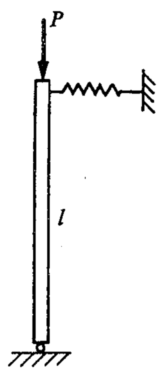
(c)

FIGURE 4.18 Local buckling of thin-walled tubes produced by axial compression. Tests made at the GALCIT Laboratory, California Institute of Technology.

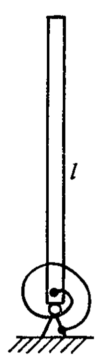
limit beyond which the buckling load cannot be increased. By increasing the radius of the tube the radius of gyration is increased and the wall thickness is decreased, but if the wall thickness becomes too small the wall will buckle with local dimples before the Euler buckling load of the column is reached. Examples of local buckling of thin-walled tubes produced by axial compression are shown in Fig. 4.18. Similarly, if the flanges of an I-beam column are too thin they will deflect in local buckling before the Euler buckling load of the I-beam is reached.

Problems

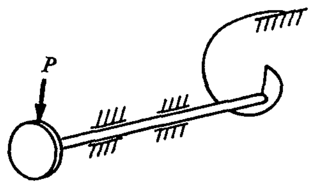
4.1 A rigid bar is pinned at its base and a linear spring is attached to its top. The spring force is zero when the bar is vertical. A vertical load P is applied as shown. For what value of P will the system become unstable?



4.2 A rigid bar of density γ and cross-sectional area A is pinned at its base where a torsional spring is attached as shown. The spring produces a moment $k\theta$ where θ is the angle between a vertical axis and the axis of the bar. What is the maximum length of bar which can be used if the system is to be stable at $\theta = 0$?

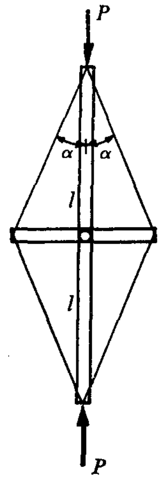


4.3 A shaft with a wheel of radius R at one end has at the other end a linear spring that resists rotation. For what value of P will the system become unstable?

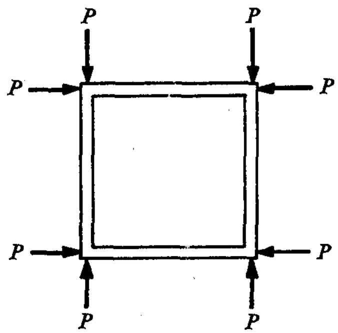


4.4 The system of Fig. 4.1 has a spring whose moment-rotation characteristics are given by $M = k(\theta - \theta^2)$. Make a P vs θ plot noting the stable and unstable regions.

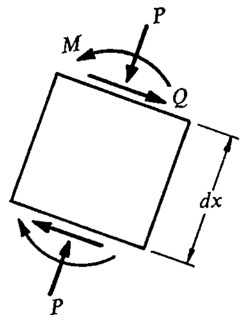
4.5 Two rigid bars of length l are hinged together. By adding a crosspiece and stringing wire with high initial tension, as shown, a stable system is formed. What is the critical buckling force P in the plane of the wires? Note, the only place that strain energy can be stored is in the linearly elastic wires. These are fastened to the ends of the bars and crosspieces.



4.6 The square frame shown in the diagram is made of square steel bars. What is the largest value of P that the frame can support without becoming unstable?

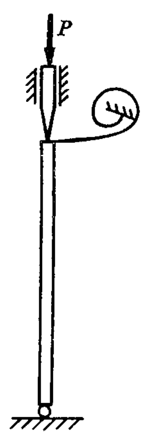


4.7 Comparing Eq. (4.2) with the equation of a laterally loaded beam, $EI\, d^4v/dx^4 = p$, it is seen that the term $-P\, d^2v/dx^2$ may be interpreted as a lateral load. With reference to a freebody diagram of an increment dx of the column shown in the accompanying figure, give a physical explanation of this effective lateral load.



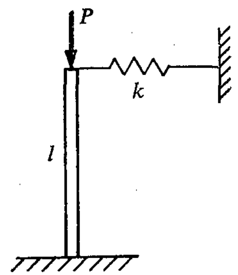
4.8 A straight prismatic column is simply supported at the top and built-in at the bottom. Determine the Euler buckling load.

4.9 Determine the Euler buckling load of the pin-ended column shown in the diagram, whose top is restrained elastically against rotation by a linear spring such that $M = k\theta$ is applied at the top.



4.10 A simply supported beam of length l is attached to an elastic foundation and is subject to an axial load P. Can the column buckle into a sinusoidal shape? If so, what is the buckling load?

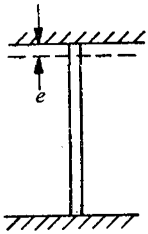
4.11 Determine the buckling load for the initially straight column, shown in the diagram, whose base is built-in and whose top is restrained elastically by a spring such that the force there is k times the deflection.



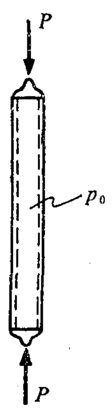
4.12 Derive the exact equation of buckling of a straight rod $EI\, d^2\theta/ds^2 + P \sin\theta = 0$, where θ is the angle between the tangent to the column axis and the x-axis, and s is measured along the column axis. Show that the equation of motion of a simple pendulum has the same form as the exact column equation.

4.13 A square, simply supported column carries an axial load P. Determine the buckling load by the energy method ($\delta V = \delta W$) including the shearing strain energy in addition to the bending energy.

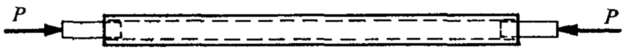
4.14 A straight, prismatic column is rigidly built-in at each end. The upper support moves downward a distance e. Determine e_{cr} by means of energy relations, assuming $\delta v = \delta A(1 - \cos 2\pi x/l)$.



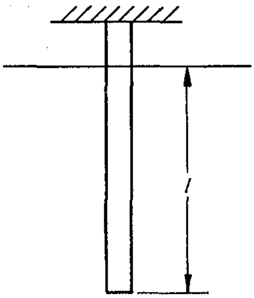
4.15 A steel pipe with closed ends is used as a column and the Euler buckling stress is $\sigma_{cr} = 10{,}000$ psi. The column is then given an internal air pressure p_0 which in addition to the circumferential stress produces an axial tension stress $\sigma_x = 10{,}000$ psi. How much has the buckling load been increased by this? What is σ_{cr}?



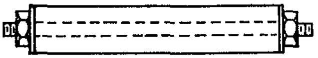
4.16 A steel pipe is filled with water that is put under pressure by two pistons as shown in the diagram. What is the buckling load? What is σ_{cr} in the pipe? (Neglect weight of pipe and water.)



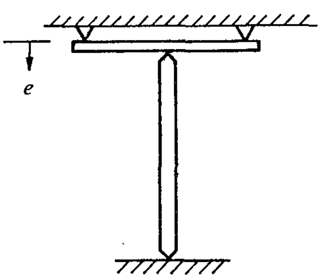
4.17 A square cantilever column is submerged in water as shown. Will it buckle?



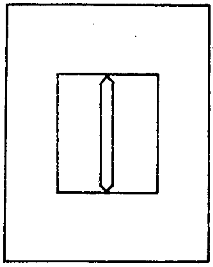
4.18 A round wood column has a steel rod threaded through an axial hole with a close fit. By tightening the nuts at the ends of the rod an axial force P is developed in the column. Can the column be buckled this way?



4.19 A simply supported column is loaded through a beam of moment of inertia I_b. The ends of the beam are pressed down a distance e. Will the column buckle when the axial force reaches P_{cr}? What is the amplitude of the displacement of the buckled column as a function of e? This is analogous to a testing machine in which strain energy is stored in the machine as well as in the column being tested.

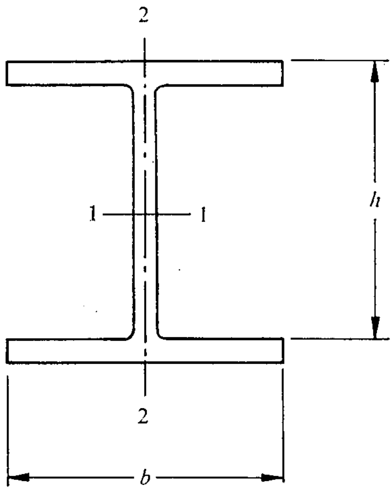


4.20 A heavy steel frame supports a thin metal strip so that at temperature T_0 there is no stress in the strip. The coefficient of temperature expansion for the strip is two-thirds that of steel. What is the critical buckling temperature for the system?

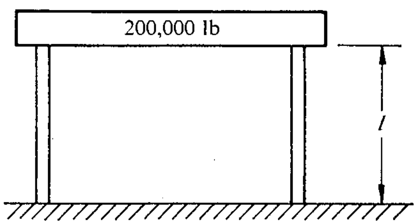


4.21 A pin-ended, rectangular, steel column (2 in. $\times$ 3 in.) supports an axial load of 10,000 lb. What is the critical length of column?

4.22 According to the American Institute of Steel Construction Handbook, an 8 WF 28 steel column is 8 in. deep and weighs 28 lb/ft and has wider flanges than does an I-beam. The properties of this column are $I_{22} = 21.6$ in.^{4}; $I_{11} = 97.8$ in.^{4}; $A = 8.23$ in.^{2}; $h = 8.06$ in.; $b = 6.54$ in. A compressive load of 200,000 lb is applied on the 2-2 axis 1 in. eccentric. If the yield point of the steel is 40,000 lb/in.^{2}, will the column fail? The column is 16 ft long and is pin-ended.



4.23 A weight of 200,000 lb is to be carried equally by four square steel bars whose yield point stress is 40,000 lb/in.^{2}. The bars may be considered to be built-in at each end. What is the minimum size of steel column that will give a factor of safety of 2? The bars are 10 ft long.



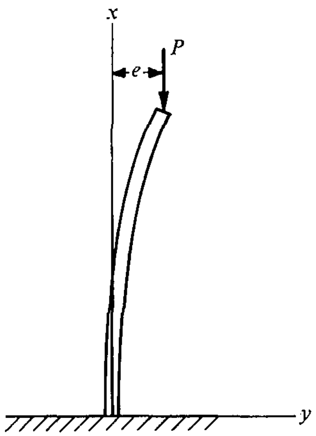
4.24 The principle of minimum potential energy states that $\delta U = 0$, where $U = V + U_a$ and U_a is the potential energy of the applied loads. The potential energy of a force P_i is taken as $P_i e_i$, where e_i is the deflection of the point of application of P_i in the direction of P_i, analogous to the potential energy of a load in a gravity field. Show that $\delta U = 0$ is equivalent to $\delta V = \delta W$ when a loaded system is given an infinitesimal virtual displacement.

4.25 Use the Rayleigh-Ritz method to compute the buckling load of a prismatic cantilever column carrying a vertical load as shown in the diagram. Assume that the buckled shape of the column is $v = e(x/l)^2$. Noting that the bending moment at any point is $M = P(e - v)$, we have

$$\delta V = \delta \int_0^l \frac{1}{2} \frac{M^2}{EI_z}\, dx$$

$$\delta W = P\,\delta \int_0^l \frac{1}{2} \left(\frac{dv}{dx}\right)^2 dx$$

Show that the approximate buckling load is about 2% too high.



4.26 Do Prob. 4.25 *using the following expression for strain energy:*

$$V = \int_0^l \frac{1}{2}\, EI_z \left(\frac{d^2v}{dx^2}\right)^2 dx$$

Show that the error in the approximate buckling load is much greater than was obtained in Prob. 4.25. Explain why this should be so.

4.27 Use the Rayleigh-Ritz method to calculate the approximate buckling load of a cantilever column. Use polynomials for the assumed v such that the approximate buckling load is more accurate than that found in Prob. 4.25.

4.28 The cantilever column shown in Prob. 4.25 has a varying moment of inertia

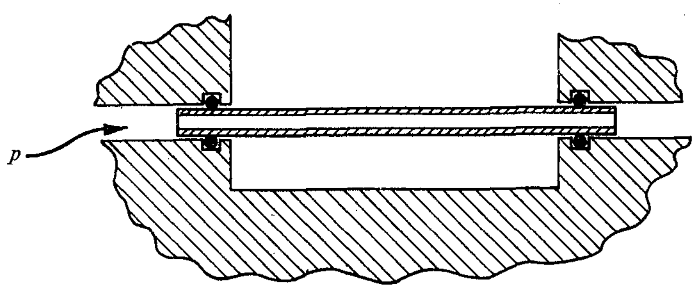
$$I = I_0\left(1 - \frac{1}{2}\frac{x}{l}\right)$$

Use the Rayleigh-Ritz method to compute the buckling load, choosing a simple expression, with one unknown constant, for v_a.

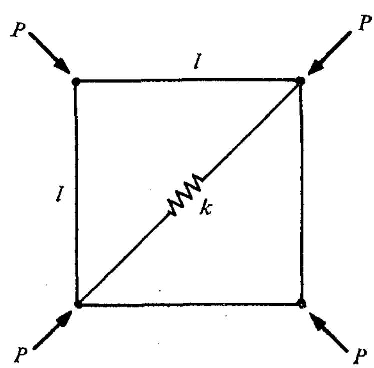
4.29 Use the Rayleigh-Ritz method to find the approximate buckling load of a pin-ended column assuming $v_a = C_1[1 - (2x/l)^2]$. Compare this buckling load with the correct value; also compare bending moment diagrams. Modify v_a to obtain an improved value of the buckling load.

4.30 A straight, pin-ended column carries an axial load P and its weight is w pounds per unit length. Determine approximately the critical load (P_{cr}) taking into account the effect of w. Use an energy approach and assume $v_a = C \sin \pi x/l$.

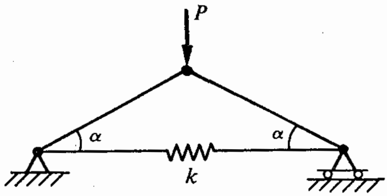
4.31 A pipe fits into two rigid walls with end seals as shown, and a constant internal pressure p is maintained. What is the critical pressure if there is no friction between the pipe and the walls?



4.32 A rigid four-bar linkage is held square by a diagonal spring as shown. When loaded by the collinear, diagonal forces, for what value of P will the system become unstable?



4.33 A rigid two-bar linkage is held at angle α_0 by a spring, as shown. A gradually increasing force P is then applied. For what value of P does the system become unstable?



CHAPTER 5

APPLICATIONS TO AXIALLY SYMMETRICAL PROBLEMS, CURVED BEAMS AND STRESS CONCENTRATIONS

5-1 AXIALLY SYMMETRICAL PROBLEMS

A number of interesting problems of practical significance have stress distributions that are symmetrical about an axis. We shall discuss some of these problems without working out the solutions; others will be analyzed in detail. One example is the case of a force applied normal to the plane face of a semi-infinite solid, as shown in Fig. 5.1. This is the so-called Boussinesq problem, which is named after the man who first deduced the correct distribution of internal stresses. The stress distribution can be shown to be*

$$\sigma_z = -\frac{3P}{2\pi}\frac{z^3}{R^5} \qquad \sigma_r = \frac{P}{2\pi}\left[(1-2\nu)\left(\frac{1}{r^2} - \frac{z}{r^2R}\right) - \frac{3r^2z}{R^5}\right]$$

$$\tau_{rz} = -\frac{3P}{2\pi}\frac{rz^2}{R^5} \qquad \sigma_\theta = \frac{P}{2\pi}(1-2\nu)\left(-\frac{1}{r^2} + \frac{z}{r^2R} + \frac{z}{R^3}\right) \tag{5.1}$$

$$\tau_{\theta z} = 0 \qquad \tau_{r\theta} = 0 \qquad (R = \sqrt{r^2 + z^2})$$

* See, for example, Timoshenko and Goodier, *Theory of Elasticity*, McGraw-Hill (1951).

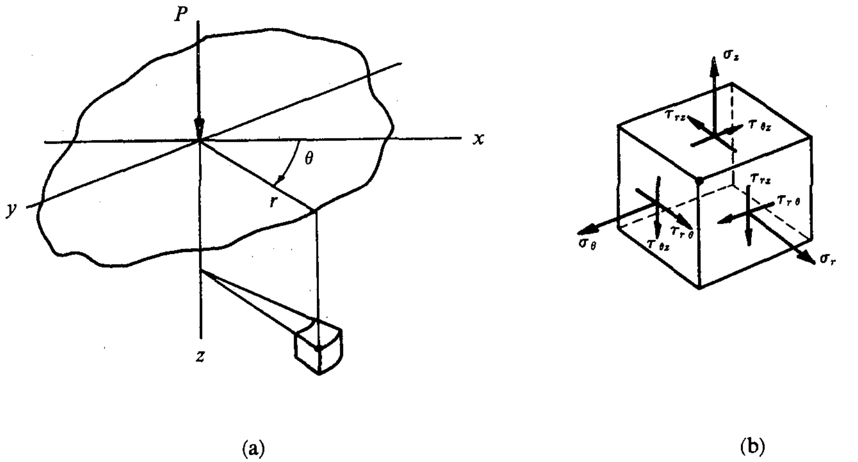


FIGURE 5.1

These stresses act on the element shown in Fig. 5.1. The displacements produced by the point load are

$$u = (1 - 2\nu)(1 + \nu)\frac{P}{2\pi E}\left(\frac{z}{rR} - \frac{1}{r} + \frac{1}{1 - 2\nu}\frac{rz}{R^3}\right)$$

$$w = \frac{P}{2\pi E}\left[(1 + \nu)\frac{z^2}{R^3} + 2(1 - \nu^2)\frac{1}{R}\right]$$

The mathematics for axially symmetrical problems in plane stress or plane strain are simpler than for those problems involving three dimensions since fewer coordinates are involved. For example, in Fig. 5.2 there is shown a hole in a flat plate of unit thickness that extends to $\pm\infty$ in the x- and y-directions. The edges of the circular hole are loaded with a uniform normal pressure p. Using polar coordinates a typical element, together with the stresses σ_r, σ_θ, $\tau_{r\theta}$, is shown. The element must be in equilibrium in the r- and θ-directions under the action of the forces acting upon its sides. In Fig. 5.3 a freebody of the element shows all of the *forces* acting upon it; note that a force is equal to a stress times the area on which it acts, and that the plate has unit thickness. The equilibrium equations in the r- and θ-directions respectively are

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{r\theta}}{r\,\partial \theta} + X_r &= 0 \\ \frac{\partial \sigma_\theta}{r\,\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + X_\theta &= 0 \end{aligned} \tag{5.2}$$

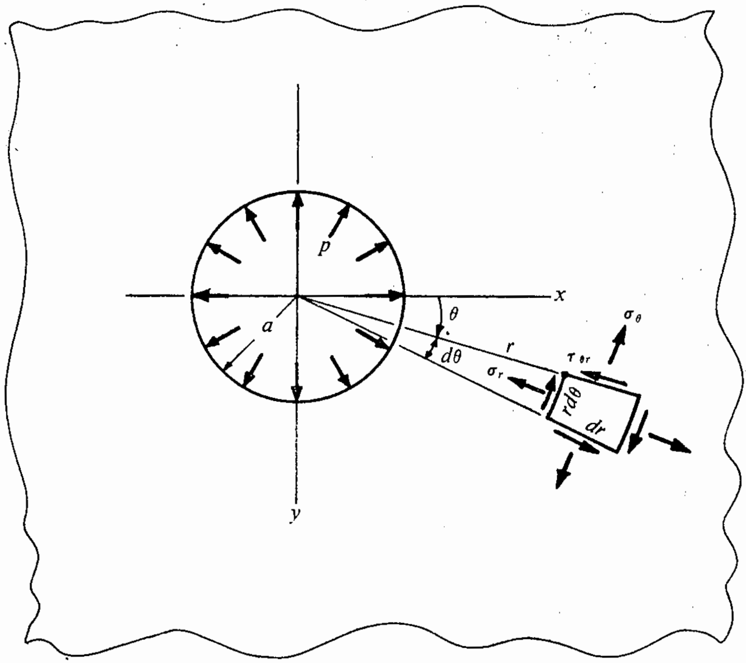


FIGURE 5.2

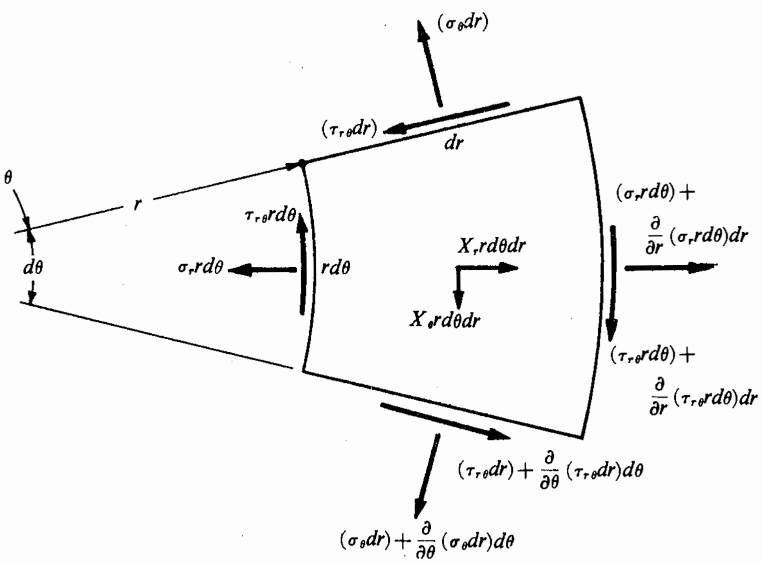


FIGURE 5.3

where X_r and X_θ are the body forces in the r- and θ-directions. These are the general equations of equilibrium for planar stress in polar coordinates. For an axially symmetrical problem, such as that shown in Fig. 5.2, it is obvious that the stresses are symmetrical about the z-axis; that is, σ_r, σ_θ, $\tau_{r\theta}$ are independent of θ. Therefore, in Eqs. (5.2) the derivatives with respect to θ are equal to zero and the equations of equilibrium for an axially symmetrical problem in planar stress are

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = 0 \tag{5.3}$$

$$\frac{d\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} + X_\theta = 0 \tag{5.4}$$

For the case $X_\theta = 0$, the solution of Eq. (5.4) is

$$\tau_{r\theta} = \frac{C}{r^2} \tag{5.4a}$$

All other axially symmetrical problems having $X_\theta = 0$ must have $\tau_{r\theta} = 0$ since any other distribution of shear stress would not satisfy Eq. (5.4).

5-2 THICK-WALLED CYLINDERS

A long, thick-walled cylinder subjected to a uniform internal or external pressure is a problem in plane strain. A typical section through the cylinder is shown in Fig. 5.4a. A point P as shown in the figure has radial displacement u and tangential displacement v. In this problem, because of the symmetry, $v = 0$. In Fig. 5.4b is shown a typical element having sides dr and $r\,d\theta$, and the dotted lines show the displaced position of the element. As can be seen in Fig. 5.4b the strains are:

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r}, \quad \gamma_{r\theta} = 0 \tag{5.5}$$

These equations specify the nature of the deformations (compatability conditions) and together with Hooke's law and the equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

they are sufficient to solve the problem.

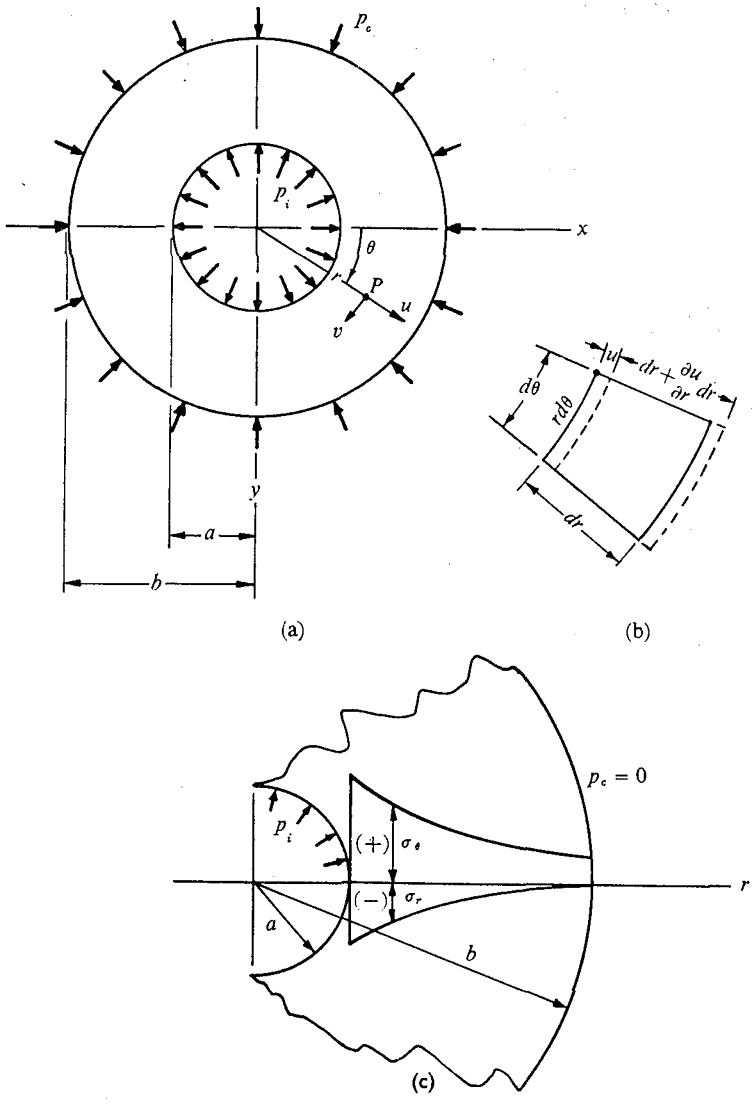


FIGURE 5.4

Hooke's law giving the stresses in terms of the strains (Eq. 1.34) is

$$\sigma_r = (\lambda + 2G)\epsilon_r + \lambda\epsilon_\theta + \lambda\epsilon_z$$

$$\sigma_\theta = (\lambda + 2G)\epsilon_\theta + \lambda\epsilon_z + \lambda\epsilon_r$$

$$\sigma_z = (\lambda + 2G)\epsilon_z + \lambda\epsilon_r + \lambda\epsilon_\theta$$

Using Eq. (5.5), Hooke's law for this problem can be written

$$\begin{aligned} \sigma_r &= (\lambda + 2G)\frac{du}{dr} + \frac{\lambda u}{r} + \lambda\epsilon_z \\ \sigma_\theta &= (\lambda + 2G)\frac{u}{r} + \lambda\frac{du}{dr} + \lambda\epsilon_z \end{aligned} \tag{5.6}$$

where ϵ_z is a constant (plane strain). These equations give the stresses as determined by the displacement. When these expressions for σ_r and σ_θ are substituted into the equilibrium equation, it assumes the form

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{5.7}$$

We have thus put the equilibrium equation into a form involving only the displacement u, and the integral of this equation gives the displacement of the thick-walled cylinder. The solution of Eq. (5.7) is

$$u = C_1 r + \frac{C_2}{r}$$

where C_1 and C_2 are constants of integration. Substituting into Eqs. (5.6), the following expressions are obtained for the stresses

$$\sigma_r = C_3 - \frac{C_4}{r^2}$$

$$\sigma_\theta = C_3 + \frac{C_4}{r^2}$$

where C_3 and C_4 are arbitrary constants. As may be verified, these equations are applicable both to plane strain problems and plane stress problems. The boundary conditions as shown in Fig. 5.4a, are

$$(\sigma_r)_{r=b} = -p_e, \qquad (\sigma_r)_{r=a} = -p_i$$

These conditions determine the constants to be:

$$C_3 = \frac{a^2p_i - b^2p_e}{b^2 - a^2}, \quad C_4 = \frac{a^2b^2(p_i - p_e)}{b^2 - a^2}$$

With these, the stresses produced by the internal and external pressures can be written

$$\sigma_r = -p_i\frac{1 - b^2/r^2}{1 - b^2/a^2} - p_e\frac{1 - a^2/r^2}{1 - a^2/b^2} \tag{5.8}$$

$$\sigma_\theta = -p_i\frac{1 + b^2/r^2}{1 - b^2/a^2} - p_e\frac{1 + a^2/r^2}{1 - a^2/b^2} \tag{5.9}$$

It should be noted that these stresses are independent of the elastic constants and are also independent of the particular value of ϵ_z. The same equations apply to the plane stress problem. A graph of the stress distribution for internal pressure only is shown in Fig. 5.4c, where it is seen that σ_θ is tension with the maximum value at the inner surface, and σ_r is compression with the maximum stress also at the inner surface. It is seen that the outer portion of the thick-walled cylinder is relatively ineffective in containing the pressure. Even if the thickness of the cylinder becomes infinite the maximum σ_θ is reduced only to $(\sigma_\theta)_{\max} = +p_i$ at $r = a$.

The displacements within the cylinder can be determined from Hooke's law since in this problem $\epsilon_\theta = u/r$. Therefore,

$$\frac{u}{r} = \frac{\sigma_\theta}{E} - \frac{\nu}{E}(\sigma_r + \sigma_z)$$

Substituting for the stresses, there is obtained

$$u = \frac{r}{E}\left[\frac{a^2p_i - b^2p_e}{b^2 - a^2}(1 - \nu) + \frac{(p_i - p_e)a^2b^2}{b^2 - a^2}\frac{(1 + \nu)}{r^2} - \nu\sigma_z\right] \tag{5.10}$$

The stress σ_z can be determined from Hooke's law,

$$E\epsilon_z = (\sigma_z - \nu\sigma_r - \nu\sigma_\theta)$$

For the particular case of $\epsilon_z = 0$, it is found that σ_z is uniform over the cross section.

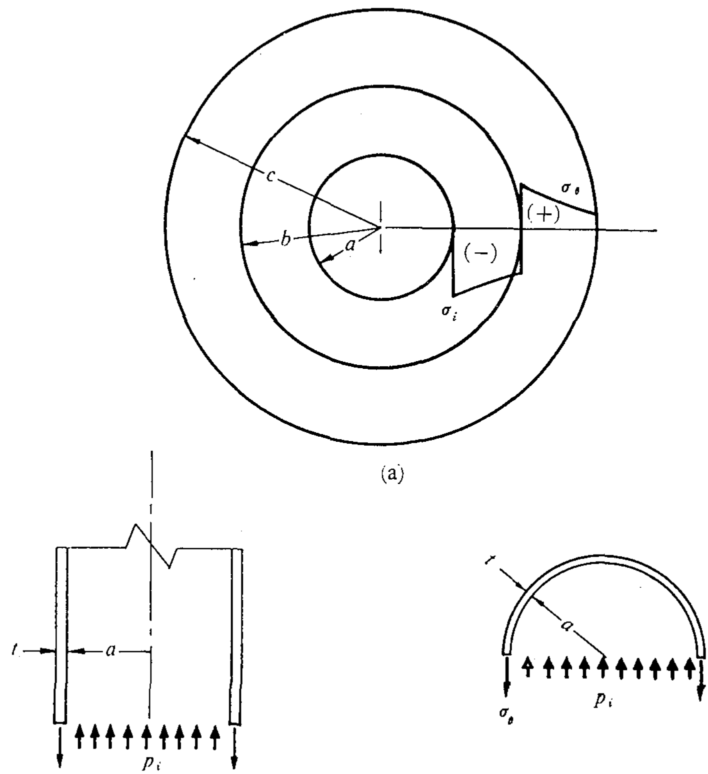
$$\sigma_z = \nu(\sigma_r + \sigma_\theta) = 2\nu\frac{a^2p_i - b^2p_e}{b^2 - a^2} \tag{5.11}$$

For a cylinder having $\sigma_z = 0$, the elongation is described by the strain

$$\epsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = -\frac{2\nu}{E}\frac{a^2p_i - b^2p_e}{b^2 - a^2}$$

Note that ϵ_z is also uniform over the cross section.

Shrink-Fit of Cylinders. In the use of thick-walled cylinders to contain high pressures a common procedure is to shrink-fit two or more cylinders together to obtain a more efficient use of the material. With high internal pressures it is desirable to limit the maximum σ_θ stress, which normally would occur at the inner face of the cylinder. If two cylinders having inner and outer radii of a, b, and $(b - \delta)$, c are shrink-fitted together, the resulting σ_θ stress *will* be distributed as shown in Fig. 5.5a. The σ_θ stress at the inner surface is now a compressive stress σ_i, whose value depends upon the misfit δ and upon the thickness of each cylinder, and also upon the elastic constants. In this condition the assembly can withstand a higher internal pressure than if it were a single cylinder with no shrink-fit stresses. Each of the cylinders in Fig. 5.5a will have a stress distribution as given by Eqs. (5.8, 5.9).



(b)

FIGURE 5.5

Thin-Wall Pressure Vessel. If the walls of a cylinder are sufficiently thin the circumferential stress σ_θ and the axial stress σ_z can be taken to be constant over the wall thickness. In this case, as shown in Fig. 5.5b for a cylinder with closed ends, equilibrium requires

$$\sigma_\theta = \frac{p_i a}{t}$$

$$\sigma_z = \frac{p_i a}{2t} \tag{5.12}$$

where p_i is the internal pressure, a is the internal radius, and t is the wall thickness. The first of these equations is the limiting form of Eq. (5.8) as the wall thickness approaches zero.

For a spherical pressure vessel, equilibrium requires that the wall be in a state of biaxial tension, $\sigma = p_i a/2t$.

5-3 ROTATING DISC OF UNIFORM THICKNESS

A circular disc rotating about its axis of symmetry with constant angular velocity will be subjected to σ_r and σ_θ stresses which, if too large, will cause the disc to rupture. It is thus a problem of considerable practical importance to know the maximum stresses that will be produced by an angular velocity $\dot{\omega}$. This is an axially symmetrical problem of plane stress and, by D'Alembert's principle, each element of density ρ can be considered to be acted upon by a

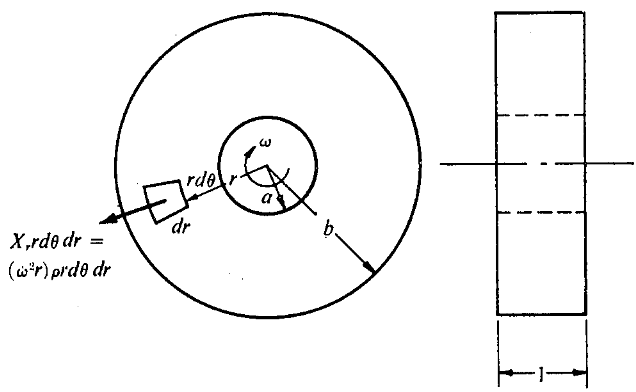


FIGURE 5.6

radially directed inertia force as shown in Fig. 5.6. The equation of equilibrium (5.3) is then

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\dot{\omega}^2 r = 0 \tag{5.13}$$

Expressing the stresses in terms of the displacement by means of Hooke's law with $\sigma_z = 0$, the equilibrium equation (5.7) in this case has the form

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\dot{\omega}^2 r = 0 \tag{5.14}$$

The solution of this equation is

$$u = C_1 r + \frac{C_2}{r} - \frac{1 - \nu^2}{E} \rho \dot{\omega}^2 \frac{r^3}{8}$$

As in the case of the thick-walled cylinder the strains can be calculated from this equation and the stresses can then be determined from Hooke's law.

If the disc is solid, the boundary conditions to be satisfied are

$$(u)_{r=0} = 0$$

$$(\sigma_r)_{r=b} = 0$$

These conditions give

$$C_1 = \frac{(3 + \nu)(1 - \nu)}{8E} \rho \dot{\omega}^2 b^2$$

$$C_2 = 0$$

and the resulting stresses are

$$\begin{aligned} \sigma_r &= \rho b^2 \dot{\omega}^2 \frac{(3 + \nu)}{8} \left(1 - \frac{r^2}{b^2}\right) \\ \sigma_\theta &= \rho b^2 \dot{\omega}^2 \frac{(3 + \nu)}{8} \left(1 - \frac{1 + 3\nu}{3 + \nu} \frac{r^2}{b^2}\right) \end{aligned} \qquad (5.15)$$

The maximum stresses occur at the center of the solid disc where,

$$(\sigma_r)_{r=0} = (\sigma_\theta)_{r=0} = \rho b^2 \dot{\omega}^2 \frac{(3 + \nu)}{8}$$

If the disc has a central hole of radius a, the boundary conditions to be satisfied by the stresses are

$$(\sigma_r)_{r=a} = (\sigma_r)_{r=b} = 0$$

These conditions give the following values for the constants of integration:

$$C_1 = \frac{(3 + \nu)(1 - \nu)}{8E} (a^2 + b^2) \rho \dot{\omega}^2$$

$$C_2 = \frac{(3 + \nu)(1 + \nu)}{8E} a^2 b^2 \rho \dot{\omega}^2$$

and the stresses are

$$\sigma_r = \frac{3+\nu}{8}\left[\frac{a^2}{b^2} + 1 - \frac{r^2}{b^2} - \frac{a^2}{r^2}\right]\rho b^2\omega^2$$
$$\sigma_\theta = \frac{3+\nu}{8}\left[\frac{a^2}{b^2} + 1 - \left(\frac{1+3\nu}{3+\nu}\right)\frac{r^2}{b^2} + \frac{a^2}{r^2}\right]\rho b^2\omega^2 \tag{5.16}$$

The maximum values of σ_r and σ_θ are:

$$(\sigma_r)_{r=\sqrt{ab}} = \frac{3+\nu}{8}\left(1 - \frac{a}{b}\right)^2\rho b^2\omega^2$$
$$(\sigma_\theta)_{r=a} = \frac{3+\nu}{4}\left[1 + \left(\frac{1-\nu}{3+\nu}\right)\frac{a^2}{b^2}\right]\rho b^2\omega^2 \tag{5.17}$$

When the hole in the disc becomes extremely small ($a \to 0$) the maximum tangential stress becomes

$$(\sigma_\theta)_{r=a} \to \frac{3+\nu}{4}\rho b^2\omega^2 \tag{5.18}$$

Both of the examples discussed above were idealized cases. In practical cases the disc would be keyed or shrink-fitted to a shaft or would be made an integral part of the shaft. In these cases the shaft introduces ϵ_z and γ_{zr} constraints so that σ_z and τ_{zr} will be different from zero in the vicinity of the shaft and this requires special analysis. The shrink-fit pressures would produce stresses in the disc according to Eqs. (5.8) and (5.9).

5-4 BENDING OF CURVED BEAMS

Another axially symmetrical stress problem of practical interest is that of a curved beam subjected to bending. We shall examine the stress distribution in a curved beam with a circular axis and a thin rectangular cross section, shown in Fig. 5.7, that is subjected to pure bending. We may assume that this thin beam is in a state of axially symmetric plane stress with $\sigma_z = 0$, so the pertinent equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

In addition to equilibrium, the stresses must also satisfy compatibility. The beam may be restrained so that the displacements are not axially symmetric.

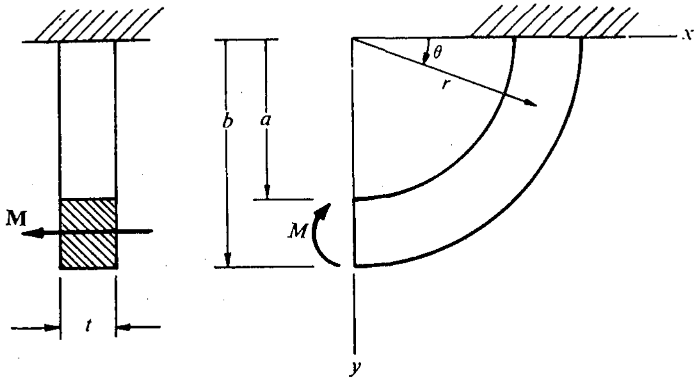


FIGURE 5.7

In the problems of the thick-walled cylinder and rotating disc, the strains were axially symmetrical, and we could write the strains in terms of only one displacement component. In this problem we must use the more general compatibility equation. For plane stress the compatibility equation (2.8) can be written for constant body forces*

$$\nabla^2(\sigma_r + \sigma_\theta) = 0$$

In polar coordinates the compatibility equation for the axially symmetric case has the form

$$\frac{d^2(\sigma_r + \sigma_\theta)}{dr^2} + \frac{1}{r}\frac{d(\sigma_r + \sigma_\theta)}{dr} = 0$$

This equation can be integrated to give $\sigma_r + \sigma_\theta = C' + C_1' \ln r$ which may be written as $\sigma_r + \sigma_\theta = C_2' + C_1' \ln (r/a)$. This equation, together with the equilibrium equation determines the stresses to be

$$\sigma_r = C_1 + C_2 \ln \frac{r}{a} + \frac{C_3}{r^2}$$

$$\sigma_\theta = C_1 + C_2\left(1 + \ln \frac{r}{a}\right) - \frac{C_3}{r^2} \qquad (5.19)$$

where the C's are constants of integration. Note that when $C_2 = 0$ the solution describes the stresses in a thick-walled cylinder. We shall now show that this solution describes the stresses in the curved beam in pure bending.

* Note that the sum of the normal stress components is invariant.

The boundary conditions for the curved beam are that on the curved surfaces the normal stresses are zero

$$(\sigma_r)_{r=a} = (\sigma_r)_{r=b} = 0$$

Also on the ends of the beam the normal stresses should have a resultant force equal to zero and a resultant moment equal to M

$$t\int_a^b \sigma_\theta \, dr = 0$$

$$t\int_a^b r\sigma_\theta \, dr = M$$

In addition, the shear stresses must be zero on all boundaries. This is automatically satisfied since our solution has $\tau_{r\theta}$ identically zero everywhere. The first two boundary conditions give

$$C_3 = -a^2 C_1$$

and

$$C_1\left(\frac{a^2}{b^2} - 1\right) = C_2 \ln \frac{b}{a}$$

from which

$$C_1 = \frac{\ln b/a}{a^2/b^2 - 1} C_2; \quad C_3 = \frac{a^2 \ln b/a}{1 - a^2/b^2} C_2$$

The stresses are then

$$\sigma_r = C_2\left(\frac{\ln b/a}{a^2/b^2 - 1} + \ln \frac{r}{a} + \frac{\ln b/a}{1 - a^2/b^2}\frac{a^2}{r^2}\right)$$

$$\sigma_\theta = C_2\left(1 + \frac{\ln b/a}{a^2/b^2 - 1} + \ln \frac{r}{a} - \frac{\ln b/a}{1 - a^2/b^2}\frac{a^2}{r^2}\right)$$

These stress distributions give zero resultant force on the ends of the beam since there can be no axial force if there are no pressures on the curved surfaces. The constant C_2 must be such that the moment of the σ_θ stresses over the end is equal to M,

$$M = \frac{C_2 t b^2}{4(1 - a^2/b^2)}\left[\left(1 - \frac{a^2}{b^2}\right)^2 - 4\frac{a^2}{b^2}\ln^2 \frac{b}{a}\right]$$

The stresses in a curved beam in pure bending are, therefore

$$\begin{aligned}\sigma_r &= \frac{4M}{tb^2K}\left[\left(1-\frac{a^2}{b^2}\right)\ln\frac{r}{a}-\left(1-\frac{a^2}{r^2}\right)\ln\frac{b}{a}\right]\\ \sigma_\theta &= \frac{4M}{tb^2K}\left[\left(1-\frac{a^2}{b^2}\right)\left(1+\ln\frac{r}{a}\right)-\left(1+\frac{a^2}{r^2}\right)\ln\frac{b}{a}\right]\end{aligned} \tag{5.20}$$

where,

$$K = \left[\left(1-\frac{a^2}{b^2}\right)^2 - 4\frac{a^2}{b^2}\ln^2\frac{b}{a}\right]$$

The variation of the stresses σ_r and σ_θ over the depth of the beam are shown in Fig. 5.8.

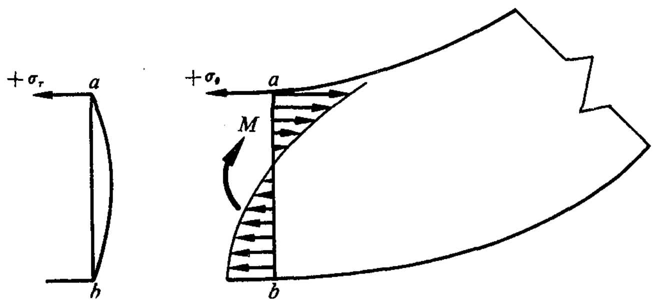


FIGURE 5.8

The stress σ_r is a compressive stress when the bending moment is directed as shown in Fig. 5.8, and is always much less than the maximum value of σ_θ. The bending stress σ_θ is a maximum on the inner surface of the curved beam at $r = a$, and the axis of zero stress is closer to the inner curved surface than it is to the outer surface.

When the beam depth h is small compared to the inner radius a, Eqs. (5.20) reduce to

$$\begin{aligned}\sigma_r &= \frac{3M}{2tbh^3}(4y^2-h^2) = \frac{Ma^2}{8bI_z}\left[\left(\frac{2y}{h}\right)^2-1\right]\left(\frac{h}{a}\right)^2\\ \sigma_\theta &= \frac{12}{th^3}My = \frac{My}{I_z}\end{aligned} \tag{5.21}$$

where y is measured from the centroid of the cross section. For $h/a \to 0$ this reduces to the stress distribution in a straight beam, Eq. (3.6).

The displacements of the curved beam can be determined by means of the general expressions, in polar coordinates, that define the strains in terms of the displacements. As can be seen in Fig. 5.9, these are

$$\epsilon_r = \frac{\partial u}{\partial r}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{5.22}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

The radial displacement and strain are shown in Fig. 5.9a where it should also be noted that the element has strained an amount u/r in the tangential direction. The tangential displacement and strain are shown in Fig. 5.9b. The shearing strain is shown in Fig. 5.9c. The total displacement and strain are shown in Fig. 5.9d.

Since the stresses are known functions of r and θ, the strains ϵ_r, ϵ_θ, $\gamma_{r\theta}$ are also known functions of r and θ. The foregoing equations can thus be integrated to give u and v.

5-5 THE TECHNICAL THEORY OF BENDING FOR CURVED BARS

If the curvature of a beam is relatively small it may be treated as a straight beam* as far as the bending stress is concerned ($\sigma_\theta = My/I$) and so far as its bending rigidity is concerned ($d^2u_0/ds^2 = M/EI$). However, if the curvature is not small its effect upon the stress and rigidity must be taken into account. As in the case of straight beams, the technical theory of bending for a curved beam, an element of which is shown in Fig. 5.10, assumes that the strain normal to the beam axis and the shearing strain are so small that they may be neglected. The strains, according to Eqs. (5.22), are therefore,

$$\epsilon_r = \frac{\partial u}{\partial r} = 0$$

$$\gamma_{r\theta} = \frac{\partial u}{r\,\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0$$

$$\epsilon_\theta = \frac{\sigma_\theta}{E}$$

* The condition is that the ratio of thickness h to inner radius a is small ($h/a \ll 1$).

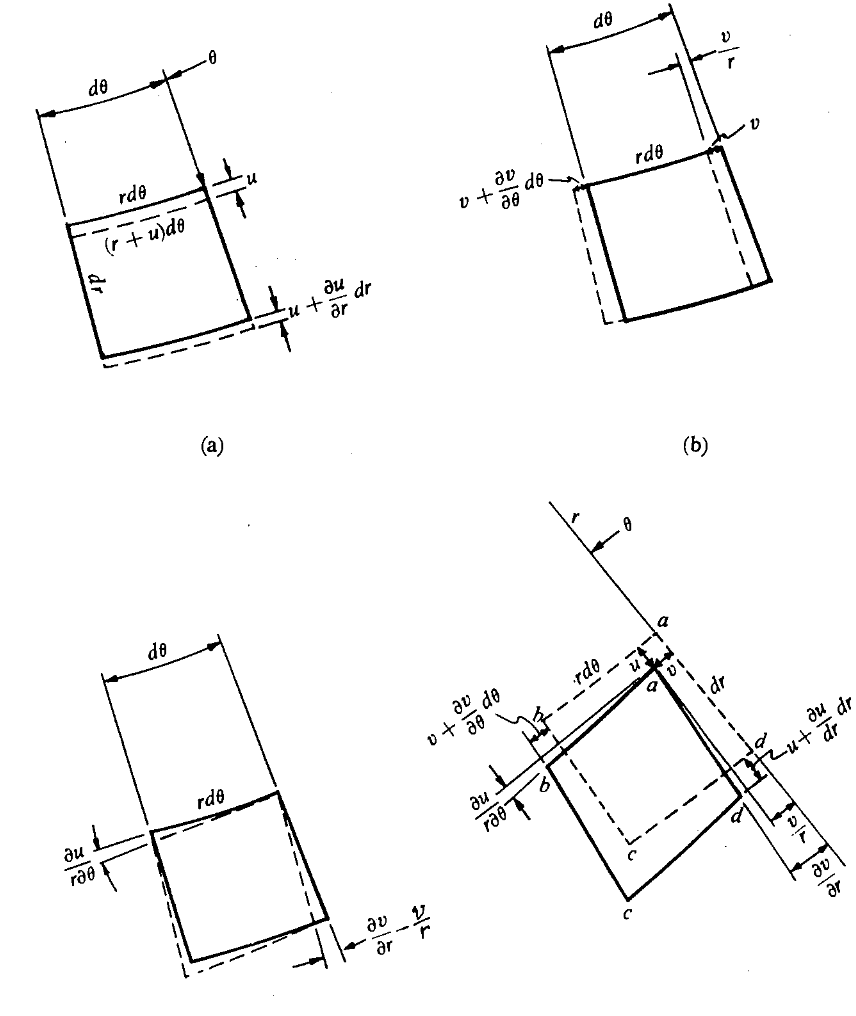


FIGURE 5.9

From the first of these equations it is seen that the radial displacement is a function of θ only; $u = f_1(\theta)$. Substituting in the second equation, there is obtained

$$\frac{\partial f_1(\theta)}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0$$

The solution of this equation is

$$v = rf_2(\theta) + \frac{\partial f_1(\theta)}{\partial\theta} = rf_2(\theta) + \frac{\partial u}{\partial\theta} \tag{5.23}$$

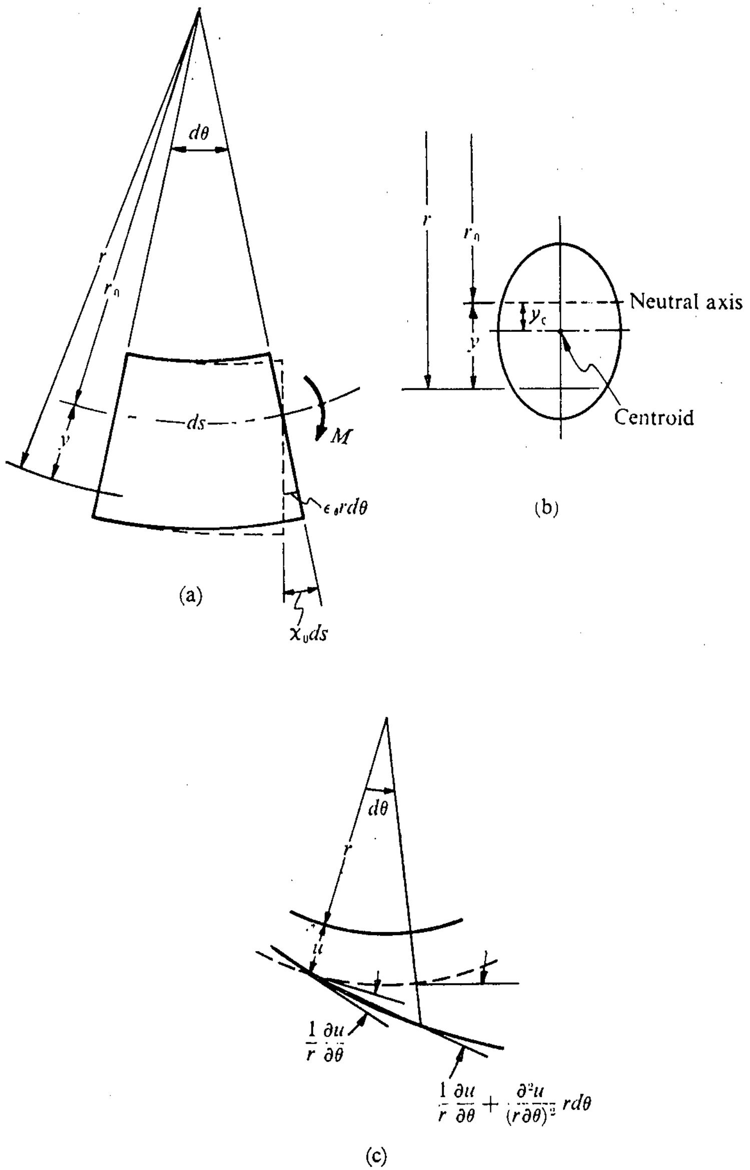


FIGURE 5.10

Since this equation is linear in r it states that plane sections remain plane during bending, as shown in Fig. 5.10a, where χ_0 is the change in curvature of the neutral axis. The change in curvature of the neutral axis can be expressed in terms of its displacement as follows. When the neutral axis is given a uniform radial displacement $u_0 =$ constant, the radius increases from r_0 to

$r_0 + u_0$ and, hence, for small displacements the curvature changes by an amount

$$\frac{1}{r_0} - \frac{1}{r_0 + u_0} = \frac{u_0}{r_0^2 + r_0 u_0} \approx \frac{u_0}{r_0^2}$$

In addition, if the displacement u_0 is a variable there is a change of curvature $\partial^2 u_0 / r^2 \partial\theta^2$, as shown in Fig. 5.10c. The total change in curvature of the neutral axis is, therefore,

$$\chi_0 = \frac{\partial^2 u_0}{r_0^2\, \partial\theta^2} + \frac{u_0}{r_0^2} \tag{5.24}$$

As can be seen in Fig. 5.10, at a distance y from the neutral axis, the change in curvature produces a shortening $y\chi_0\, ds$. This produces a strain

$$\epsilon_\theta = -\frac{y\chi_0\, ds}{r\, d\theta} = -\chi_0 r_0 \frac{y}{r}$$

Since $r = r_0 + y$, this may be written

$$\epsilon_\theta = -\chi_0 \left(\frac{y}{1 + y/r_0}\right)$$

The bending stress can, therefore, be written

$$\sigma_\theta = E\epsilon_\theta = -E\chi_0 \left(\frac{y}{1 + y/r_0}\right) \tag{5.25}$$

This is a hyperbolic stress distribution, as shown in Fig. 5.11a. The radius r_0

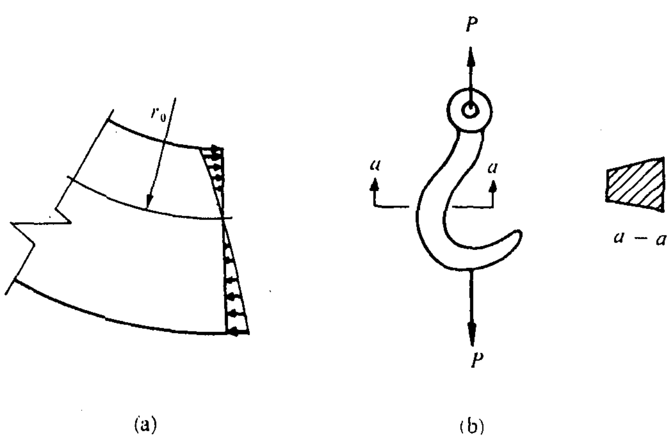


FIGURE 5.11

locates the neutral axis and it can be evaluated from the condition that the integral of σ_θ over the cross section is zero. Hence,

$$\int_A \frac{y}{r_0 + y}\, dA = \int_A \frac{r - r_0}{r}\, dA = \int_A dA - \int_A \frac{r_0}{r}\, dA = 0 \qquad (5.25a)$$

From this equation there is obtained the following expression for r_0

$$r_0 = \frac{A}{\displaystyle\int_A dA/r} \qquad (5.26)$$

To relate the stress to the bending moment M we impose the condition

$$M = -\int_A y\sigma_\theta\, dA = E\chi_0 \int_A \left(\frac{y^2}{1 + y/r_0}\right) dA$$

Designating the integral by

$$I' = \int_A \left(\frac{y^2}{1 + y/r_0}\right) dA$$

we can write

$$M = EI'\chi_0 = EI'\left(\frac{\partial^2 u_0}{r_0^2\, \partial\theta^2} + \frac{u_0}{r_0^2}\right) \qquad (5.27)$$

Equation (5.27) is the differential equation of bending of a circular beam. The bending stress is given by Eq. (5.25) after substituting $\chi_0 = M/EI'$,

$$\sigma_\theta = -\frac{M}{I'}\left(\frac{y}{1 + y/r_0}\right) \qquad (5.28)$$

For a beam of rectangular cross section this stress distribution agrees very closely with the more exact stress distribution of Eq. (5.20).

The pseudo moment of inertia I' can be evaluated easily by noting that

$$I' = \int_A \frac{y^2}{1 + y/r_0}\, dA = r_0 \int_A \left(y - \frac{y}{1 + y/r_0}\right) dA$$

$$= r_0 \int_A y\, dA - r_0 \int_A \frac{y}{1 + y/r_0}\, dA$$

The second of these integrals is zero since σ_θ integrated over the cross-section is zero (see Eq. 5.25a). The first integral is just the moment of the cross-sectional area about the neutral axis, which can be expressed by Ay_c. The pseudo moment of inertia can therefore be written:

$$I' = Ar_0 y_c$$

where A is the cross-sectional area of the beam and y_c is the centroidal distance as measured from the neutral axis. If the depth h of the beam is small compared to the radius ($h/r_0 \ll 1$), then $(1 + y/r_0)$ is very close to unity, the stress distribution is practically linear, and I' is practically equal to I.

The foregoing equations can be applied with satisfactory accuracy to curved beams whose radius of curvature varies along the axis and whose cross section is not rectangular. For example, the cross-section of the hook shown in Fig. 5.11b can be proportioned so that according to the technical theory of bending the maximum compressive stress is equal to the maximum tension stress.

Bending of Slender Circular Beams. Equation (5.27) developed in the preceding section can be applied to a variety of problems of bending of circular beams, for example, the slender circular beam loaded by diametrically opposed forces, shown in Fig. 5.12. This beam is statically

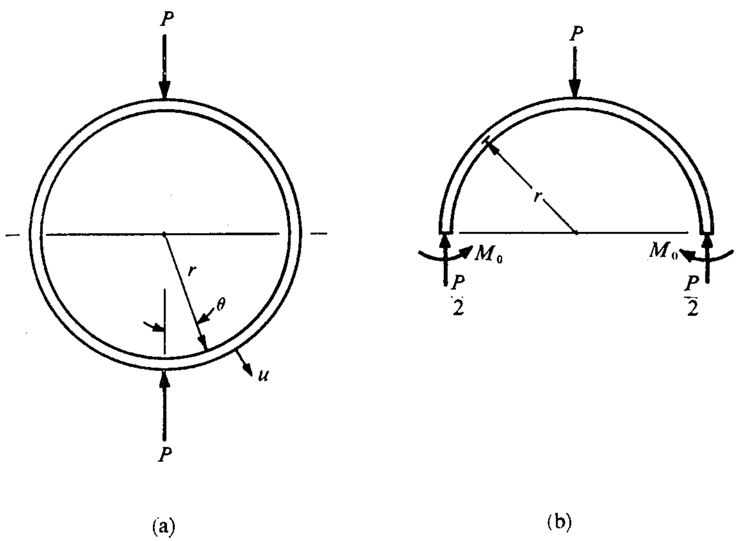


FIGURE 5.12

indeterminate to the first order. As can be seen in Fig. 5.12b, the bending moment, shear force, and axial force at any point on the beam can be expressed in terms of $P/2$ and M_0. Therefore, if M_0 is known, the bending moment, stresses and displacements can all be computed. The value of M_0 could be determined by means of Castigliano's Theorem. However, we shall do this problem by an alternate method which will be useful in later applications. We shall use the principle of virtual work ($\delta V = \delta W$), expressing the displacement in a Fourier series. From symmetry it is seen that the radial displacement can be expressed by

$$u = \sum_{n=1}^{\infty} a_n \cos n\theta$$

The strain energy of bending, making use of Eq. (5.27), can be written

$$V = \int_0^{2\pi} \frac{1}{2} \frac{M^2}{EI}\, r\, d\theta = \int_0^{2\pi} \frac{1}{2} \frac{EI}{r^4} \left(\frac{\partial^2 u}{\partial \theta^2} + u\right)^2 r\, d\theta$$

and substituting for u, there is obtained

$$V = \int_0^{2\pi} \frac{1}{2} \frac{EI}{r^3} \left[\sum_{n=1}^{\infty} a_n(-n^2 + 1) \cos n\theta\right]^2 d\theta = \frac{\pi EI}{2r^3} \sum_{n=1}^{\infty} (n^2 - 1)^2 a_n^2$$

The variation of V results from varying the coefficients a_n, that is*

$$\delta V = \sum_{n=1}^{\infty} \frac{\partial V}{\partial a_n}\, \delta a_n = \frac{\pi EI}{r^3} \sum_{n=1}^{\infty} (n^2 - 1)^2 a_n\, \delta a_n$$

The increment of work done by the forces P during the displacement δu is

$$\delta W = -P\, \delta u|_{\theta=0} - P\, \delta u|_{\theta=\pi}$$

Noting that $\delta u = \sum_n (\partial u/\partial a_n)\, \delta a_n$, the increment of work may be written

$$\delta W = -P \sum_n (\cos n0 + \cos n\pi)\, \delta a_n$$

$$\delta W = -2P \sum_n \delta a_n \qquad (n = 2, 4, 6, \ldots)$$

$$\delta W = 0 \qquad (n = 1, 3, 5, \ldots)$$

Writing $\delta V - \delta W = 0$ gives

$$\sum_n \left[\frac{\pi EI}{r^3} (n^2 - 1)^2 a_n + 2P\right] \delta a_n = 0 \qquad (n = 2, 4, 6, \ldots)$$

$$\sum_n \left[\frac{\pi EI}{r^3} (n^2 - 1)^2 a_n\right] \delta a_n = 0 \qquad (n = 1, 3, 5, \ldots)$$

* See Appendix IV for a more complete description of variational methods.

Since the δq_n are arbitrary the expressions in the brackets must be zero for the sums to be zero and hence

$$a_n = -\frac{2Pr^3}{\pi(n^2 - 1)^2 EI} \qquad (n = 2, 4, 6, \ldots)$$

$$a_n = 0 \qquad (n = 1, 3, 5, \ldots)$$

The displacement is thus given by

$$u = -\frac{2Pr^3}{\pi EI} \sum_n \frac{\cos n\theta}{(n^2 - 1)^2} \qquad (n = 2, 4, 6, \ldots)$$

From this expression the bending moment and shear force can be calculated. The relative displacement of the two forces P is given by*

$$u_0 + u_\pi = -0.149 \frac{Pr^3}{EI}$$

Instability of a Circular Ring. If a circular ring is subjected to a uniform external pressure the internal circumferential or hoop force is pr, as determined by equilibrium of an element, Fig. 5.13b. If the pressure exceeds a critical value p_{cr} the ring will become unstable and will buckle, as shown in Fig. 5.13a. We shall calculate the critical load by the principle of virtual work ($\delta V = \delta W$). Assuming negligible hoop strain, the strain energy is that of bending only and this is easily calculated from Eq. (5.27),

$$V = \int_0^{2\pi} \frac{1}{2} \frac{M^2}{EI} r\, d\theta = \int_0^{2\pi} \frac{1}{2} EI\chi^2 r\, d\theta$$

where χ is the change in curvature

$$\chi = \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + u\right)$$

This expression for V is the bending strain energy when the ring has displacement u. The increment change in strain energy when the displacement is varied an amount δu is

$$\delta V = \delta \int_0^{2\pi} \frac{1}{2} \frac{EI}{r^4} \left(\frac{\partial^2 u}{\partial \theta^2} + u\right)^2 r\, d\theta$$

* $\sum_n \frac{1}{(n^2 - 1)^2} = \left(\frac{\pi}{4}\right)^2 - \frac{1}{2} \qquad (n = 2, 4, 6, \ldots)$

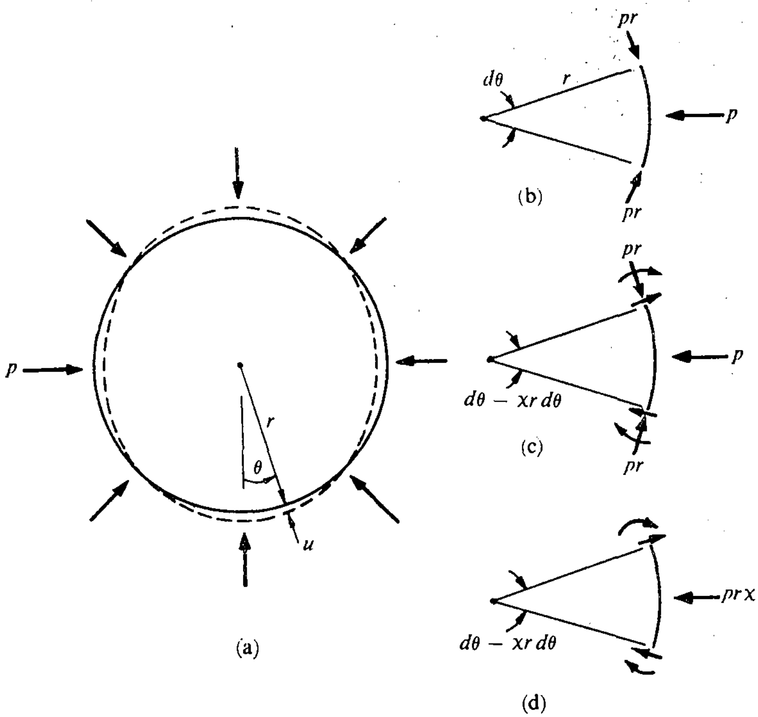


FIGURE 5.13

The corresponding increment of work done by p is given by $p\ \delta A$, where δA is the change of enclosed area of the ring resulting from an increment of displacement δu. Since δA is a bit lengthy to calculate (see Prob. 5.29), we shall determine the increment of work in another way. The effect of a circumferential or hoop force P on the displacement is the same as the effect of a lateral load of intensity P times the curvature as was discussed in problem 4.7. For example, in Fig. 5.13b the curvature is $1/r$ and, hence, the effect of the hoop force pr is the same as an outwardly directed pressure $(pr)\cdot 1/r = p$, which effectively holds in equilibrium the external pressure p. The deformed element with bending moments and shear forces acting on it, shown in Fig. 5.13c, has a curvature $1/r - \chi$ and, hence, the axial force is equivalent to an outwardly directed pressure given by

$$(pr)\cdot\left(\frac{1}{r} - \chi\right) = p - pr\chi$$

The net effect of this plus the external pressure p is an inwardly directed effective pressure $pr\chi$ and the equivalent loaded element is shown in Fig.

5.13d. For this equivalent problem we calculate the work done by the load $-pr\chi$ during an increment of displacement δu*

$$\delta W = \int_0^{2\pi} (-pr\chi)(\delta u)r\, d\theta = -\int_0^{2\pi} p\left(\frac{\partial^2 u}{\partial \theta^2} + u\right) \delta u\, d\theta$$

The displacement may be expressed as a Fourier series

$$u = \sum_n a_n \cos n\theta$$

$$\delta u = \sum_n \delta a_n \cos n\theta$$

The increment of strain energy is given by

$$\delta V = \sum_n \frac{\partial V}{\partial a_n} \delta a_n$$

Substituting and integrating there is obtained for the equation $\delta V = \delta W$

$$\sum_n \frac{EI}{r^3} \pi(n^2 - 1)^2 a_n \,\delta a_n = -\sum_n p\pi(-n^2 + 1)a_n \,\delta a_n$$

or, collecting terms,

$$\sum_n \left[\frac{EI}{r^2}(n^2 - 1) - pr\right] \frac{\pi}{r}(n^2 - 1)a_n \,\delta a_n = 0$$

Since δa_n is arbitrary this expression can be zero only if the term in the brackets is zero, which gives

$$p_{cr} = \frac{EI}{r^3}(n^2 - 1) \qquad (n = 2, 3, 4, \ldots) \tag{5.29}$$

The lowest buckling load is for $n = 2$

$$p_{cr} = \frac{3EI}{r^3} \tag{5.29a}$$

The term $a_1 \cos \theta$ represents a rigid body displacement of the ring and hence does not figure in the buckling problem. Equation 5.29a can be used to calculate the critical pressure for a pipe. In this case, however, the effective modulus of rigidity is $EI(1 - \nu^2)$ since the pipe is in a state of plane strain

* It is assumed that the change in magnitude of the circumferential force pr during the displacement δu is negligibly small.

(see Section 3-11). If the ends of the pipe are rigidly supported in such a way as to prevent the pipe from going out of round the foregoing analysis is not applicable. In this case the pipe must be analyzed as a shell instead of a ring (see, for example, Section 7-7).

Flexibility of a Circular Tube. If a straight, steel pipe is anchored firmly at each end and its temperature is then raised 100°F, there will be produced an axial compressive stress of approximately 20,000 psi. It can be seen from this that pipes that are exposed to the sun or that transport hot fluids may exert excessively large forces on their end supports or may fail in buckling unless precautions are taken to keep the temperature stresses suitably low. The axial force can be kept low by putting a bend in an otherwise straight pipe, as shown in Fig. 5.14. Actually, this loop derives most of its flexibility

FIGURE 5.14

from the four circular bends at the corners and relatively little from the straight segments between them. The large flexibility of the corner bends can be explained as follows.

The relation between bending moment and change of curvature of a bent tube in pure bending, shown in Fig. 5.15a, when the tube is rigid enough so that its cross section remains circular (no deforming of cross section) is

$$\chi_\varphi = \frac{M_\varphi}{EI_z}$$

It is assumed that r is small compared to R. If the tube is thin, however, it deforms out of round under the action of the bending moment as shown in Fig. 5.15 and because of this a smaller moment is required to produce the same change in curvature. To determine the relation between M_φ and χ_φ it is necessary to observe the exact displacements of the ring so that the bending strain ϵ_φ and stress σ_φ can be computed. We shall follow the analysis of von Karman, who made the first solution of this problem.* There are two contributions to the bending strain ϵ_φ: the effect of the change in curvature χ_φ, and the effect of the flattening of the tube shown in Fig. 5.15b. The bending strain may be deduced from Fig. 5.15c to be

$$\epsilon_\varphi = \frac{a'a' - aa}{aa} = \frac{[(R + y - \Delta y)\,d\varphi - (y - \Delta y)\chi_\varphi R\,d\varphi] - (R + y)\,d\varphi}{(R + y)\,d\varphi}$$

* T. von Karman, *Collected Papers*, Butterworth (1956).

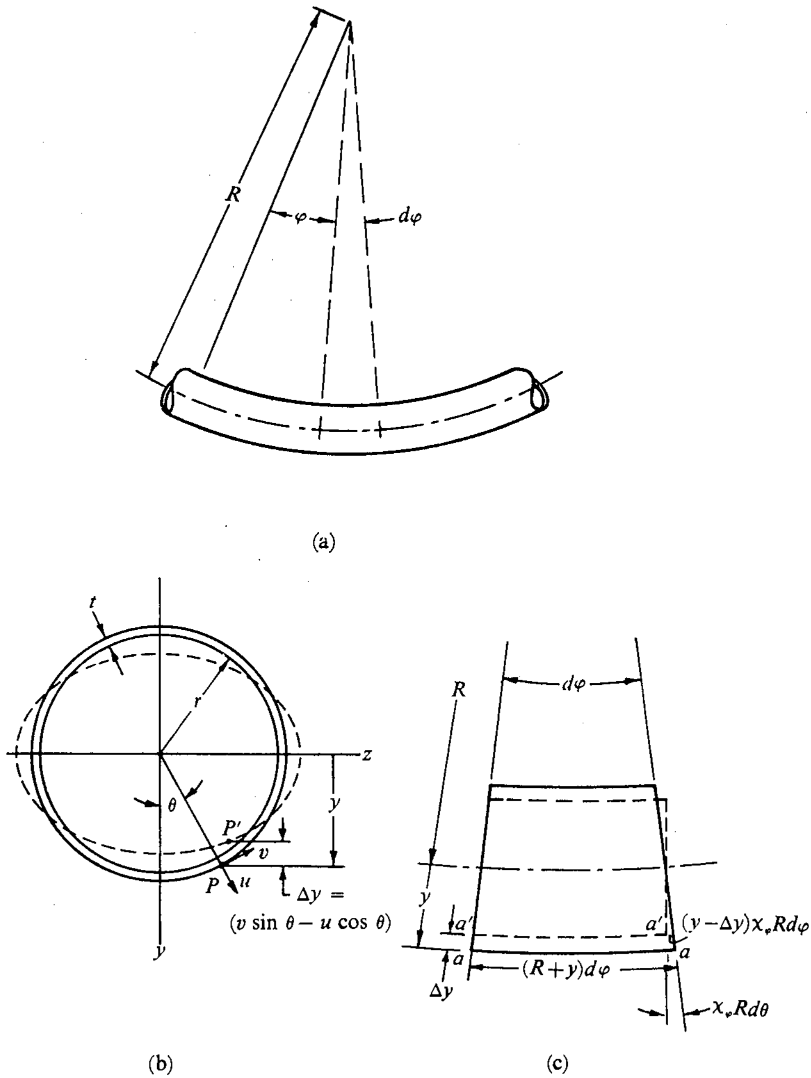


FIGURE 5.15

Noting that $y \ll R$, $\Delta y \ll y$, $y = r \cos \theta$, and $\Delta y = v \sin \theta - u \cos \theta$:

$$\epsilon_\varphi = -\chi_\varphi r \cos \theta - \frac{v \sin \theta - u \cos \theta}{R}$$

During the deformation out of round, the average circumferential strain at the middle of the tube wall $(\epsilon_\theta)_0$ is so small that it can be taken equal to zero,

$$(\epsilon_\theta)_0 = \frac{u}{r} + \frac{\partial v}{r\,\partial\theta} = 0$$

from which

$$u = -\frac{\partial v}{\partial\theta}$$

This expresses the condition of inextensibility of the ring. We shall solve the problem by using the principle of minimum strain energy, assuming the displacements to be*

$$u = b\cos 2\theta$$

$$v = -c\sin 2\theta$$

Making use of the relation $u = -\partial v/\partial\theta$, there is obtained

$$v = -c\sin 2\theta = -2c\sin\theta\cos\theta$$

$$u = 2c\cos 2\theta = 2c(\cos^2\theta - \sin^2\theta)$$

With these displacements, the strain ϵ_φ is

$$\epsilon_\varphi = -r\chi_\varphi\cos\theta + \frac{2c}{R}\cos^3\theta$$

The total strain energy per unit length of the curved tube is that of bending out of round plus that due to ϵ_φ,

$$V = \int_0^{2\pi} \tfrac{1}{2}EI_\theta\chi_\theta^2 r\,d\theta + \int_0^{2\pi} (\tfrac{1}{2}E\epsilon_\varphi^2)tr\,d\theta$$

where $I_\theta = t^3/12$ for the section of unit length.† Substituting for ϵ_φ and applying the condition $\partial V/\partial c = 0$, there is obtained

$$c = \frac{(Rr/2)\chi_\varphi}{\frac{5}{6} + (Rt/r^2)^2}$$

from which

$$\epsilon_\varphi = -\chi_\varphi r\cos\theta\left[1 - \frac{\cos^2\theta}{\frac{5}{6} + (Rt/r^2)^2}\right]$$

* These assumed displacements are the first significant terms in the Fourier series expressing u and v. They represent a displacement similar to that shown in Fig. 5.15b.

† In the first integral the effective modulus $E/(1 - \nu^2)$ should properly be used. A strain energy term involving the cross product $\epsilon_\theta\epsilon_\varphi$ does not appear because ϵ_φ is constant over the thickness of an element $r\,d\theta$ and the average value of ϵ_θ across the thickness is zero.

Noting that $r \cos \theta = y$, this strain can be written

$$\epsilon_\varphi = -\chi_\varphi y\left(1 - \beta \frac{y^2}{r^2}\right) \tag{5.30}$$

where

$$\beta = \frac{6}{5 + 6(Rt/r^2)^2} \tag{5.30a}$$

The bending stress, $\sigma_\varphi = E\epsilon_\varphi$, must satisfy the condition

$$\int_0^{2\pi} \sigma_\varphi yt \, d\theta = -M$$

This gives

$$\chi_\varphi = \frac{M}{EI_z(1 - \frac{3}{4}\beta)} \tag{5.30b}$$

where

$$I_z = \int_A y^2 \, dA = \pi r^3 t$$

for the tube. Thus

$$\sigma_\varphi = -\frac{My}{I_z(1 - \frac{3}{4}\beta)}\left[1 - \beta\left(\frac{y}{r}\right)^2\right] \tag{5.30c}$$

It is seen that the bending stiffness is $EI_z\,(1 - \frac{3}{4}\beta)$ which can be appreciably smaller than the stiffness of a straight ($R = \infty$) tube. For small Rt/r^2 Eq. (5.30c) does not give satisfactory values.*

5-6 STRESS CONCENTRATIONS

It may happen that an irregularity in the shape of a loaded body will induce large stresses in a very localized region. This concentrated stress may exceed the strength of the material and may cause a crack to form, thus leading to failure. An example of such a stress concentration is the plane stress problem in which a large plate is stressed in simple tension so that

$$\sigma_x = \sigma;\ \sigma_y = 0;\ \tau_{xy} = 0.$$

* A more exact analysis of this problem is given by R. A. Clark and E. Reissner, "Bending of curved tubes," *Advances in Applied Mechanics*, Vol. 2, Academic Press (1951). Many problems of the stresses and deformations of pipes are discussed in the book *Design of Piping Systems*, by M. W. Kellogg Company, John Wiley & Sons, Inc. (1956). For $Rt/r^2 < 1$ the Clark-Reissner solution should be used. This solution has the expression $Rt/1.65\, r^2$ in Eq. (5.30b) in place of $(1 - \frac{3}{4}\beta)$.

If, then, a hole of radius a is drilled in the plate, as shown in Fig. 5.16, the distribution of stress will be modified so that the boundary conditions on the edge of the hole are satisfied

$$(\sigma_r)_{r=a} = (\tau_{r\theta})_{r=a} = 0$$

As may be verified,* the correct solution for the stresses is

$$\begin{aligned}
\sigma_r &= \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta \\
\sigma_\theta &= \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \\
\tau_{r\theta} &= -\frac{\sigma}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta
\end{aligned} \tag{5.31}$$

The tension stress σ_x along the y-axis is given by σ_θ when $\theta = \pi/2$

$$(\sigma_\theta)_{\theta=\pi/2} = \sigma\left(1 + \frac{a^2}{2r^2} + \frac{3a^4}{2r^4}\right) \tag{5.31a}$$

This stress distribution is shown in Fig. 5.16 where it is seen that the maximum

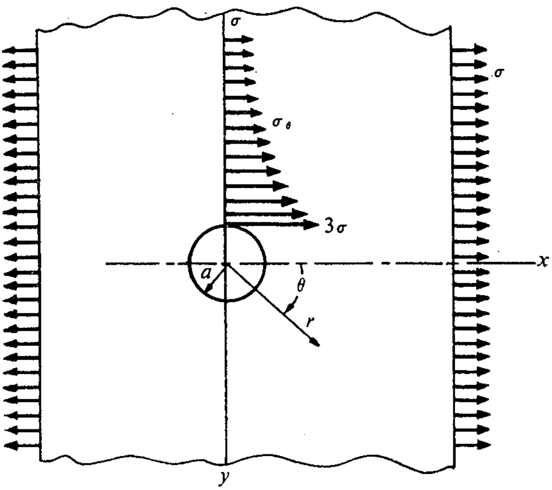


FIGURE 5.16

* The stresses satisfy the equilibrium and compatibility equations, and the boundary conditions. The uniqueness theorem tells us that this is the only elastic solution if the unloaded body is free of stress.

stress is at the edge of the hole and has a value $(\sigma_\theta)_{\max} = 3\sigma$. *It is thus seen* that because of the hole there is a stress concentration factor* of 3. If the plate has an elliptical hole with major semi-axis a in the direction of the tension as shown in Fig. 5.17a, the maximum tension stress at the edge of the hole is given by

$$(\sigma)_{\max} = \sigma\left(1 + 2\frac{b}{a}\right)$$

As the ellipse becomes flatter the ratio b/a becomes larger and the stress concentration factor becomes larger. In the limit as $a \to 0$, the ellipse becomes a crack of length $2b$ and the stress concentration factor tends towards infinity. Actually, of course, as the stress concentration becomes larger, either the material will yield plastically around the ends of the crack or the material will fail in tension and the crack will propagate. The effect of plastic yielding in a small region around the tip of the crack has an effect similar to increasing the radius at the crack tip and thus reducing the stress concentration factor. However, if the applied stress is oscillatory, the continual plastic working at the tip of the crack will cause the material to fail in fatigue and the length of the crack will increase. To prevent such a crack from growing, a hole may be drilled at each end as shown in Fig. 5.17b which effectively increases the radius to correspond to a smaller b/a.

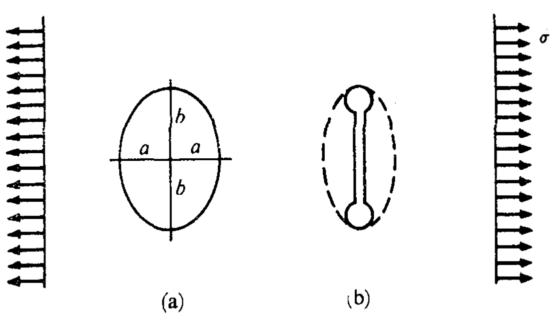


FIGURE 5.17

The plate with a circular hole gives another verification of St. Venant's principle. A uniform tension in the x-direction is expressed in polar coordinates by

* The stress concentration factor is the ratio of the actual stress to the stress that would be at the point if there were no stress concentration.

$$\sigma_r = \frac{\sigma}{2}(1 + \cos 2\theta)$$

$$\sigma_\theta = \frac{\sigma}{2}(1 - \cos 2\theta) \tag{5.32}$$

$$\tau_{r\theta} = \frac{\sigma}{2}\sin 2\theta$$

Therefore, if the foregoing σ_r and $\tau_{r\theta}$ are applied to the boundary of the hole, the effect of these stresses plus the uniform tension σ applied to the plate at $x = \pm\infty$ is to produce a uniform tension $\sigma_x = \sigma$ throughout the plate. We deduce, therefore, from Eq. (5.31a) that the effect of these σ_r and $\tau_{r\theta}$ stresses applied to the boundary of the hole is to produce

$$(\sigma_\theta)_{\theta=\pi/2} = -\sigma\left(\frac{a^2}{2r^2} + \frac{3a^4}{2r^4}\right)$$

The numerical values of the ratio σ_θ/σ at $\theta = \pi/2$ are given in the following table:

a/r	1	0.5	0.25
$(\sigma_\theta/\sigma)_{\theta=\pi/2}$	2	0.23	0.03

Since the stresses applied to the boundary of the hole have a zero resultant, St. Venant's principle would indicate that the stress produced should be essentially zero at a distance of about twice the diameter, that is $a/r = 0.25$. The table shows that at this distance the stress has indeed decreased to a very small fraction of σ.

In general, any notches, abrupt changes in cross section, reentrant corners, etc., will produce stress concentrations. Examples of these are shown in Fig. 5.18. The smaller the radius of curvature of the notch the larger will be the stress concentration factor. If a body is made of ductile material and is loaded statically, plastic yielding around a stress concentration may be acceptable. However, if the material is brittle, or the load is oscillatory, the stress concentration factor must be held to an appropriately small value. The stress concentration factors for a number of shapes have been determined analytically.* For other shapes, it may be necessary to determine the stress concentration factors experimentally (see Appendix III). A tabulation of stress concentration factors has been made by R. E. Peterson.†

* H. Neuber, *Theory of Notch Stresses*, J. W. Edwards and Company (1946).

† R. E. Peterson, *Stress Concentration Design Factors*, John Wiley and Sons (1953).

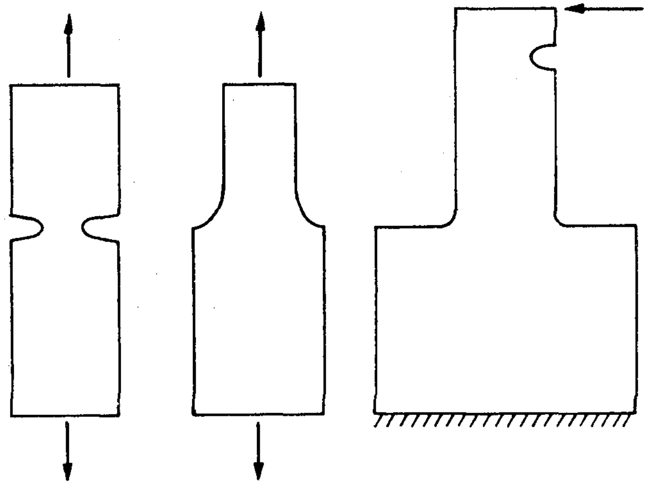


FIGURE 5.18

5-7 CONTACT STRESSES

The stresses and deformations produced by the force that is transmitted by two bodies in contact is of practical importance for ball bearings, roller bearings, impacts between bodies, etc. This is the so-called Hertz* problem of contact stresses. We shall not solve the problem but shall discuss its significant features. Consider, for example, two steel spheres of radius R_1 and R_2 with point contact as shown in Fig. 5.19. If the centers of the spheres

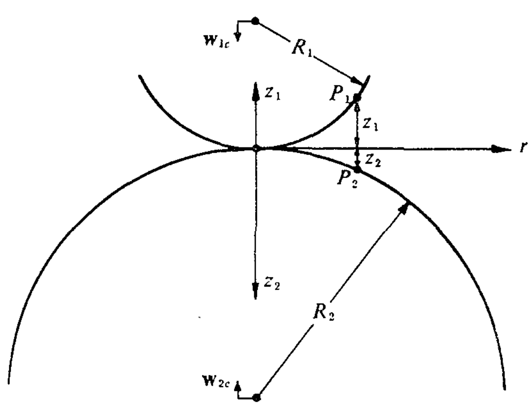


FIGURE 5.19

* H. Hertz, *J. Math.* (*Crelle's J.*), Vol. 92 (1881). See also A. E. H. Love, *Mathematical Theory of Elasticity*, Dover, p. 193.

are displaced towards each other by an amount $\mathbf{w} = \mathbf{w}_{1c} + \mathbf{w}_{2c}$, the two surface points P_1 and P_2 shown in Fig. 5.19 will be moved towards each other by an amount $\mathbf{w} - \mathbf{w}_1 - \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are the respective displacements, of the points P_1 and P_2 relative to the centers, produced by the contact forces. If the displacement is sufficient to bring points P_1 and P_2 into contact, the net relative displacement will be

$$|\mathbf{w} - \mathbf{w}_1 - \mathbf{w}_2| = z_1 + z_2$$

For small values of r the geometry of a circle is such that we may write $z_1 = r^2/2R_1$, $z_2 = r^2/2R_2$ and, hence,

$$|\mathbf{w} - \mathbf{w}_1 - \mathbf{w}_2| = \frac{r^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

The deformation of sphere 2 will be as indicated in Fig. 5.20. So far as the stresses in the vicinity of the contact surface are concerned, the problem can

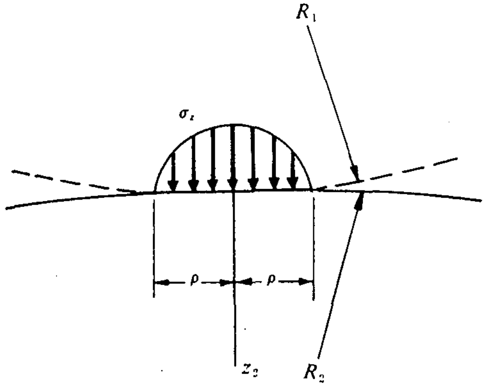


FIGURE 5.20

be analyzed by superposition, using the Boussinesq solution for displacements produced by a point load on a semi-infinite body. This is permissible since the radius ρ of the contact surface is very small compared to R_2. The pressure distribution σ_z must be such that the foregoing equation is satisfied. Hertz found that the contact pressures were given by

$$\sigma_z = -\sigma_0\left(1 - \frac{r^2}{\rho^2}\right)^{1/2} \tag{5.33}$$

where, for $\nu = 0.3$, the radius of the contact surface is given by

$$\rho = 1.109\left[\frac{P}{E(1/R_1 + 1/R_2)}\right]^{1/3} \tag{5.34}$$

The maximum contact pressure, from equilibrium, is

$$\sigma_0 = 1.5\sigma_{\text{ave}} = -\frac{1.5P}{\pi\rho^2} = -0.388P^{1/3}\left[E\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]^{2/3} \tag{5.35}$$

The analysis shows that the maximum stress in the spheres is the σ_0 at the center of the contact area. However, at this point the other two principal stresses are both equal to $\sigma_0(1 + 2\nu)/2$ and, hence, the maximum shear stress at the surface is only equal to

$$\tau = \tfrac{1}{2}(\sigma_z - \sigma_r) = \tfrac{1}{4}(1 - 2\nu)\sigma_0 \approx 0.1\sigma_0 \tag{5.36}$$

The largest shear stress is found to be at a point on the z-axis approximately $\rho/2$ from the contact surface. The maximum shear stress here is

$$\tau_{\max} \approx 0.31\sigma_0 \tag{5.37}$$

Yielding would thus start at this point in a material which followed the Tresca yield criterion. The maximum tension stress σ_r occurs at $r = \rho$, $z = 0$, and it has a value

$$\sigma_r = \frac{1 - 2\nu}{3}\sigma_0 \approx 0.2\sigma_0 \tag{5.38}$$

A brittle material, weak in tension, would thus fail at this point if the contact force is too large.

The relation between the contact force P and the relative displacement of the centers of two elastic spheres ($\nu = 0.3$) given by the analysis is

$$|\mathbf{w}| = 1.23\left(\frac{P}{E}\right)^{2/3}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{1/3} \tag{5.39}$$

If the two steel spheres make a direct impact the duration of contact is long compared to the longest period of vibration of either sphere and, hence, the static contact force and deformation can be used to analyze the impact. The centers of gravity of the spheres thus move as if impact were being made through a nonlinear spring having the force-displacement relation of Eq. (5.39).*

Hertz solved the more general problem of contact between bodies of any curvature and different elastic constants. The initial curvature of each body at the point of contact is specified by the two principal radii of curvature. His solution shows the linear dimensions of the contact area to be proportional to the cube root of the force, and the relative displacement of the bodies to be proportional to the two-thirds power of the force. The proportionality constants are determined by the elastic constants and the principal radii of curvature of the bodies.†

* W. Goldsmith, *Impact*, Arnold (1960).

† A useful summary of contact stress solutions is given by A. B. Jones, "New departure engineering data—Analysis of stresses and deflections," Vols. I and II, *General Motors Sales Corporation, New Departure Division* (1946).

Problems

5.1 Derive the equations of equilibrium (5.2) in polar coordinates from the freebody diagram of an element.

5.2 What is the physical problem described by Eq. (5.4a)?

5.3 Derive Eq. (5.7) from the equilibrium equation and Hooke's law.

5.4 Show that Eq. (5.7) is also correct for a plane stress problem.

5.5 Equation (5.7) can be put in a form having constant coefficients by making a change of variable $r = e^s$. Use this method to integrate Eq. (5.7).

5.6 Carry through the steps of evaluating constants C_3 and C_4 for the thick-walled cylinder to obtain Eqs. (5.8).

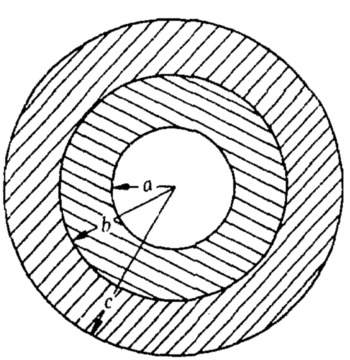
5.7 What is the stress distribution for the infinite plate with a hole shown in Fig. 5.2? At how many hole diameters from the edge of the hole are the stresses reduced to 1% of the maximum value? Is this in accordance with St. Venant's principle?

5.8 A circular plate (plane stress) is subjected to an external pressure p_e. What are the internal stresses if there is no hole in the plate? What are the internal stresses if an exceedingly small hole of radius a is drilled through the center?

5.9 The elementary solution for σ_θ in a thin-walled pressure cylinder assumes that σ_θ and σ_z are uniform across the thickness. Deduce σ_θ and σ_z from equilibrium. For what wall thickness will this elementary solution give σ_θ with less than 10% error?

5.10 A long, cylindrical steel pressure vessel with hemispherical ends is subjected to an internal pressure $p = 1000$ psi. The vessel has radii $a = 2.4$ ft, $b = 2.5$ ft. What are the magnitudes of the three principal stresses at $r = a$ at the midpoint of the cylinder? What is the change in outside diameter produced by this p?

5.11 Two thick-walled cylinders are shrink-fitted together as shown in the diagram. Before heating the outer cylinder the radial interference was δ. Determine the contact pressure for $\delta \ll b$.



5.12 Derive the differential equation (5.14) for the rotating disc from the equilibrium equation, compatibility conditions, and Hooke's law.

5.13 Derive the integral of Eq. (5.14).

5.14 Apply the boundary conditions and derive Eqs. (5.15) for the solid spinning disc.

5.15 Apply the boundary conditions to derive Eqs. (5.16) for the annular spinning disc.

5.16 Sketch the stress distributions of Eqs. (5.15) and (5.16) for the rotating discs.

5.17 Deduce that the radial stress in a rotating annular disc is maximum at $r = \sqrt{ab}$.

5.18 For the rotating disc shown in Fig. 5.6 find the value of the stress gradient $d\sigma_r/dr$ at $r = a$ and show that it approaches ∞ as $a \to 0$. Does this seem reasonable?

5.19 Derive Eqs. (5.20) for the stresses σ_r and σ_θ in a curved beam from the equilibrium and compatibility equations.

5.20 Verify that Eqs. (5.20) give zero axial force on a cross section of the curved beam.

5.21 What is the error in the maximum bending stress given by Eqs. (5.21) when $b/a = 1.2$?

5.22 Deduce Eqs. (5.22) from Fig. 5.9.

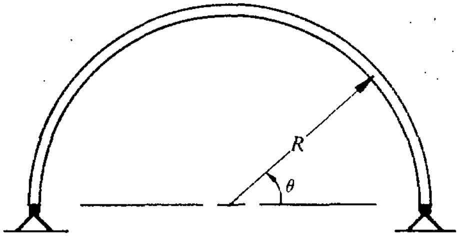
5.23 Find the general solution for the u and v displacements in the axially symmetric stress problem by integrating Eq. (5.22) when the stresses are given by Eq. (5.19) with $\sigma_z = \tau_{r\theta} = 0$. Show that the displacement v is not single valued when θ is increased by 2π. Show how this relates to the curved beam in pure bending.

5.24 A bar of rectangular cross section of width b is formed so that the outside surface has a radius c and the inside surface has a radius a. Show that in the technical theory of bending the radius of curvature of the neutral axis is given by $r_0 = (c - a)/(\ln c/a)$. What is σ_r according to the technical theory?

5.25 Show that in the technical theory of bending the strain energy in a curved beam may be written as

$$V = \int_0^{\theta_0} \frac{M^2\, d\theta}{2Ay_cE}$$

5.26 A thin, semicircular beam acts as a pin-ended arch to carry its own weight, as shown in the diagram. Derive an expression for the bending moments in the beam as a function of θ. Is it justifiable to neglect the axial deformation in such a problem?



5.27 Derive an expression for the displacements of a thin circular ring subjected to an external pressure

$$p = p_0 + p_2 \cos 2\theta + p_3 \cos 3\theta + \cdots$$

Take the internal circumferential force to be $F = p_0 r$. Is it reasonable to neglect the effects of p_2, p_3, etc. on F?

5.28 A circular ring beam is subjected to an external pressure $p = p_2 \cos 2\theta$. What is the circumferential force in the beam?

5.29 For the buckling of the ring under external pressure p, as shown in Fig. 5.13, calculate the increment change of the enclosed area δA produced by a displacement

$$u = c - u_0 \cos n\theta$$

First, determine the value of c from the condition that the circumferential length of the ring does not change. According to the accompanying diagram, this condition is

$$0 = \int_0^{2\pi} \{[(r + u)^2 \, d\theta^2 + du^2]^{1/2} - r \, d\theta\}$$

$$= \int_0^{2\pi} \left\{\left[\frac{(r + u)^2}{r^2} + \left(\frac{du}{r \, d\theta}\right)^2\right]^{1/2} - 1\right\} r \, d\theta$$

Show that this can be expanded to give

$$0 = \int_0^{2\pi} \left[\frac{u}{r} + \frac{1}{2}\left(\frac{du}{r \, d\theta}\right)^2 + \cdots\right] r \, d\theta$$

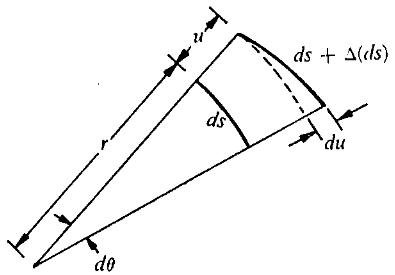
Show that this equation requires

$$u = -\frac{n^2}{4r} u_0^2 + u_0 \cos n\theta$$

and show that

$$\delta A = \int_0^{2\pi} \tfrac{1}{2}(r + u)^2 \, d\theta - \pi r^2 \approx \frac{\pi}{2} u_0^2(1 - n^2)$$

Show that using $p \, \delta A$ for the work done leads to the same expression for the critical load as given by Eq. (5.29).



5.30 A large circular ring of concrete having E_c, I_c, A_c, has a steel rod E_s, A_s, with negligible I_s, around it. As the steel rod is put into tension the concrete ring will undergo compression. Is it possible to buckle the assembly?

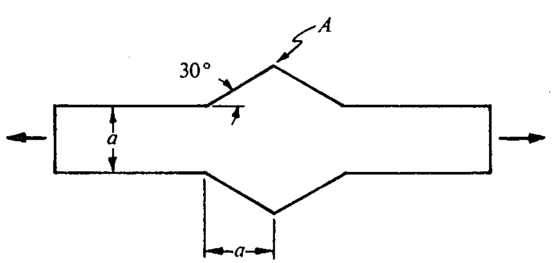
5.31 Write out a complete derivation of the flexibility of a curved tube.

5.32 Show that the stress field $\sigma_x = \sigma$, $\sigma_y = \tau_{xy} = 0$ is expressed in polar coordinates by Eq. (5.32).

5.33 Show that the stresses of Eqs. (5.31) satisfy the boundary conditions at $r = a$ and at $r = \infty$.

5.34 What is the stress concentration factor produced in a rotating disc by a small hole at its center?

5.35 What is the stress concentration factor at the sharp corner A shown in the diagram?



5.36 Estimate how wide a plate in simple tension must be to justify the use of a stress concentration factor of 3 ± 5% for a hole of radius a in the center of the plate.

5.37 Make a sketch of the principal stress trajectories for a plate with a circular hole subjected to uniform tension. Space the lines so that each line corresponds approximately to the same force, $F = \sigma a$. Also make a sketch of the principal shear stress trajectories.

5.38 By appealing to St. Venant's principle deduce that a notch has only a localized effect on the stress distribution.

CHAPTER 6

APPLICATIONS TO TORSION PROBLEMS

6-1 TORSION OF PRISMATICAL BARS

The evaluation of the stresses and strains produced by the twisting of beams and shafts is a problem of practical importance. St. Venant was the first to make a comprehensive analysis of this problem using a so-called semi-inverse method of solution.* The direct method of solution is: given the applied forces to find the stresses, strains and displacements; the inverse method is: given the displacements to find the applied forces. St. Venant assumed, on the basis of physical intuition, that when subjected to torsion the straight bar shown in Fig. 6.1a would have only the stresses τ_{xz} and τ_{xy}. The total state of stress in the bar of uniform cross section would thus be given by

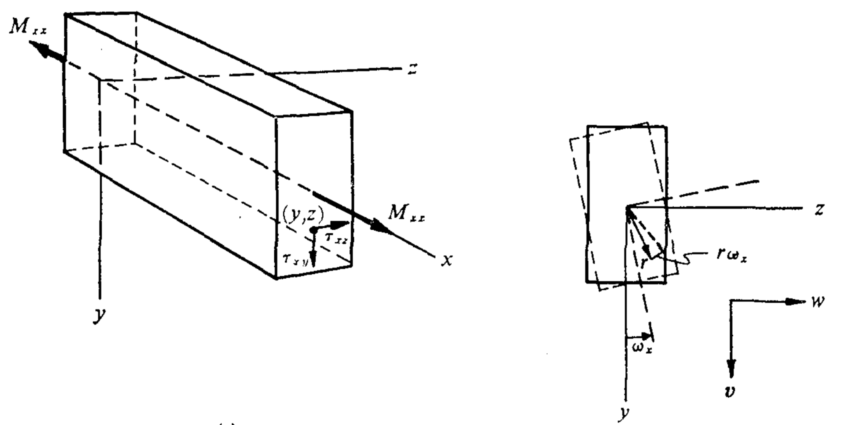
$$\begin{aligned} \tau_{xz} &= F_1(y, z) \\ \tau_{xy} &= F_2(y, z) \\ \sigma_x &= \sigma_y = \sigma_z = \tau_{yz} = 0 \end{aligned} \tag{6.1}$$

As may be verified, this is equivalent to assuming that the displacements in the bar are given by

$$\begin{aligned} u &= \theta f(y, z) \\ v &= -\theta zx \\ w &= \theta yx \end{aligned} \tag{6.2}$$

* Barre de St. Venant, *Memoires des Savants Étrangers*, Vol. 14 (1855).

where θ is a constant whose value depends upon the amount that the bar is twisted. These expressions for v and w specify that a cross section of the bar undergoes a rigid-body rotation through an angle $\omega_x = \theta x$, thus θ is the twist per unit length. During the rotation the cross section warps out-of-plane as described by $u = \theta f(y, z)$. The function $f(y, z)$ is called the warping function. According to Eqs. (6.1) and (6.2) an element on the surface of the bar will be deformed in shearing strain and, because of the warping, will be displaced in the x-direction and will rotate, as indicated in Fig. 6.1b. If the strains in the



(a)

(b)

FIGURE 6.1

rod are calculated from Eqs. (6.2) and the stresses are then calculated from the strains, they will agree with Eqs. (6.1). For example, from Eqs. (6.2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$$

hence,

$$\sigma_x = \sigma_y = \sigma_z = 0$$

The shear stress τ_{xz} is given by

$$\tau_{xz} = G\gamma_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{xz} = \left(y + \frac{\partial f(y, z)}{\partial z}\right)\theta G = F_1(y, z)$$

It is seen from the foregoing expression that the warping function $f(y, z)$ is related to the shear strains γ_{xz} and γ_{xy} by

$$\gamma_{xz} = y\theta + \frac{\partial f(y, z)}{\partial z}\,\theta$$

$$\gamma_{xy} = -z\theta + \frac{\partial f(y, z)}{\partial y}$$

The last terms in these expressions represent the effect of the warping on the torsional shear strains. If the warping were zero, the shear strains would be given by

$$\gamma_{xz} = y\theta$$

$$\gamma_{xy} = -z\theta$$

As seen in Eqs. (6.1) and (6.2) a portion of the solution has been assumed so that to obtain the complete solution it is only necessary to determine τ_{xz} and τ_{xy}, or what is equivalent, to determine $f(y, z)$. If the stresses τ_{xz}, τ_{xy} satisfy the equations of elasticity and the prescribed boundary conditions, they will represent the correct solution. When the stresses of Eq. (6.1) are substituted in the general equations of equilibrium, Eqs. (2.1), these reduce to

$$\begin{aligned} \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \qquad (x\text{-direction}) \\ \frac{\partial \tau_{xy}}{\partial x} &= 0 \qquad (y\text{-direction}) \\ \frac{\partial \tau_{xz}}{\partial x} &= 0 \qquad (z\text{-direction}) \end{aligned} \tag{6.3}$$

The first of these equations states the requirement for equilibrium in the x-direction; the last two equations state that the stresses do not vary along the length of the bar.

The equation of compatibility is derived from Hooke's law and the strain-displacement relations, which for this problem can be written

$$\tau_{xy} = G\gamma_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xz} = G\gamma_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

The displacement u can be eliminated by differentiating the equations respectively by z and y and subtracting the second from the first

$$\frac{\partial \tau_{xy}}{\partial z} - \frac{\partial \tau_{xz}}{\partial y} = G\frac{\partial^2 v}{\partial x\,\partial z} - G\frac{\partial^2 w}{\partial x\,\partial y} = -G\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

The expression within the parentheses is $2\omega_x$, that is, twice the rotation about the x-axis. Since, in this problem, $\omega_x = \theta x$, the compatibility equation can be written

$$\frac{\partial \tau_{xy}}{\partial z} - \frac{\partial \tau_{xz}}{\partial y} = -2G\theta \tag{6.4}$$

Equations (6.3) and (6.4), together with the boundary conditions, are sufficient to solve the torsion problem.

The Stress Function. A common procedure for solving torsion problems is to use a stress function Φ that identically satisfies equilibrium, Eq. (6.3)

$$\frac{\partial \Phi}{\partial z} = \tau_{xy}, \quad \frac{\partial \Phi}{\partial y} = -\tau_{xz} \tag{6.5}$$

When these expressions for the stresses are substituted in the compatibility equation there is obtained

$$\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = -2G\theta \tag{6.6}$$

The compatibility equation thus takes the form of Poisson's equation in terms of the stress function.

All solutions of Eq. (6.6) are solutions of torsion problems but, to solve for the torsion of a particular bar, it is necessary that the solution of Eq. (6.6) satisfy the boundary conditions for the problem under consideration. One boundary condition for the torsion problem is that the lateral surface of the bar is free of stress. It is seen from Eqs. (6.5) that the derivative of Φ in the s-direction gives the stress τ_{xn} in the perpendicular direction, as shown in Fig. 6.2a and b. Therefore, the condition that the surface of the bar be stress free, $\tau_{xn} = 0$ on the boundary, is expressed by $\partial\Phi/\partial s = 0$, where s is measured along the boundary. Therefore, in terms of the stress function, the boundary condition is:

$$\Phi = \text{constant on boundary} \tag{6.7}$$

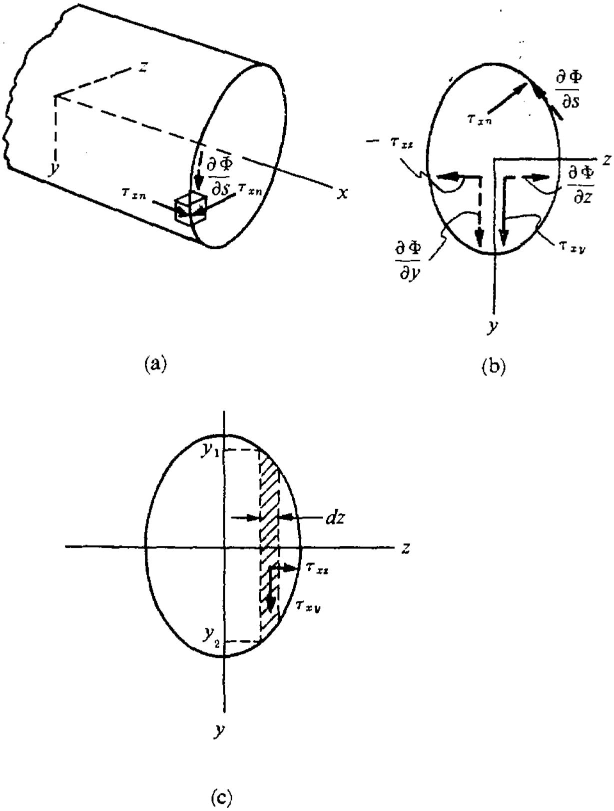


FIGURE 6.2

The twisting moment can be expressed in terms of Φ by noting that the twisting moment acting on the cross section (see Fig. 6.2c) is the integral over the area of the moment of the stresses

$$M_{xx} = \iint_A (y\tau_{xz} - z\tau_{xy})\, dy\, dz$$

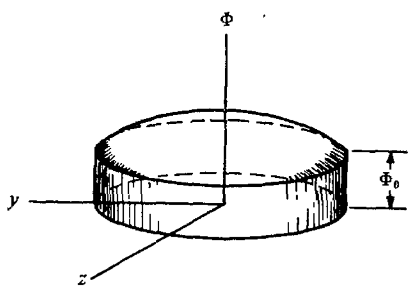


FIGURE 6.3

In terms of the stress function, this is

$$M_{xx} = -\iint_A \left(y\frac{\partial\Phi}{\partial y} + z\frac{\partial\Phi}{\partial z}\right) dy\, dz$$

The first term on the right-hand side can be integrated by parts as follows:

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2} y\frac{\partial\Phi}{\partial y}\, dy\, dz = \int_{z_1}^{z_2} y\Phi\Big|_{y_1}^{y_2} dz - \int_{z_1}^{z_2}\int_{y_1}^{y_2} \Phi\, dy\, dz$$

$$= \Phi_0 \int_{z_1}^{z_2} (y_2 - y_1)\, dz - \int_{z_1}^{z_2}\int_{y_1}^{y_2} \Phi\, dy\, dz$$

$$= \Phi_0 A - \int_{z_1}^{z_2}\int_{y_1}^{y_2} \Phi\, dy\, dz$$

where Φ_0 is the constant value of Φ on the boundary, and A is the area of the cross section. The second term in the expression for M_{xx} can also be integrated and when these results are substituted in the equation there is obtained

$$M_{xx} = 2\iint_A \Phi\, dy\, dz - 2\Phi_0 A \tag{6.8}$$

This can be interpreted in terms of Fig. 6.3, where Φ is plotted over the cross section. The integral of Φ over the area is the volume of the domed figure shown in Fig. 6.3. The quantity $\Phi_0 A$ represents that portion of the volume lying below Φ_0. The right-hand side of Eq. (6.8) thus represents the volume above Φ_0. We may set Φ_0 equal to zero on the boundary and Eq. (6.8) will then state*

$$M_{xx} = \text{two times the volume under } \Phi \tag{6.9}$$

with

$$\Phi_0 = 0$$

6-2 SOLUTION FOR A CIRCULAR BAR

In the case of a circular bar of radius R the stress function Φ must be symmetrical about the axis of the bar; it must be zero on the boundary, and,

* Since the stresses are related to the derivatives of Φ, any value of Φ_0 will give the same stresses. When the bar has more than one boundary, as in a tube, the value of Φ on only one of the boundaries is arbitrary (see Section 6-6).

according to Eq. (6.6), the sum of the second derivatives of Φ must be a constant. An obvious function that satisfies these conditions is the equation of a circle $y^2 + z^2 - R^2 = 0$, from which

$$\Phi = B(R^2 - y^2 - z^2) = B(R^2 - r^2)$$

From the uniqueness theorem, this is also the only solution that will satisfy Eq. (6.6) and the boundary conditions. Substituting in Eq. (6.6), it is found that

$$\Phi = \tfrac{1}{2}G\theta(R^2 - y^2 - z^2) \tag{6.10}$$

The twisting moment corresponding to this Φ is

$$\begin{aligned} M_{xx} &= 2\iint_A \tfrac{1}{2}G\theta(R^2 - y^2 - z^2)\,dy\,dz \\ &= G\theta\int_0^R 2\pi r(R^2 - r^2)\,dr \\ &= G\theta\,\frac{\pi R^4}{2} \end{aligned}$$

This relation between twisting moment and twist per unit length can be written

$$\theta = \frac{M_{xx}}{GI_p} = \frac{M_{xx}}{C_t} \tag{6.11}$$

where I_p is the polar moment of inertia of A about the x-axis.* The quantity $C_t = M_{xx}/\theta$ is called the torsional rigidity of the bar. With this relation the stress function can be written

$$\Phi = \frac{M_{xx}}{2I_p}(R^2 - y^2 - z^2)$$

The shearing stresses are given by

$$\begin{aligned} \tau_{xy} &= \frac{\partial\Phi}{\partial z} = -\frac{M_{xx}}{I_p}z \\ \tau_{xz} &= -\frac{\partial\Phi}{\partial y} = \frac{M_{xx}}{I_p}y \end{aligned} \tag{6.12}$$

* $I_p = \int_0^R r^2\,dA = I_y + I_z$

As shown in Fig. 6.4, the resultant of τ_{xy} and τ_{xz} is a tangentially-directed stress perpendicular to the radius vector r

$$\tau_{x\alpha} = \frac{M_{xx}}{I_p} r$$

There is no warping of cross sections in this problem, and the shear strain is given by

$$\gamma_{x\alpha} = r\theta = \frac{\tau_{x\alpha}}{G}$$

The bar will experience tension stress at 45° to the axis as shown in Fig. 6.4b. If the bar is made of a brittle material weak in tension, such as chalk, it will fail at 45°.

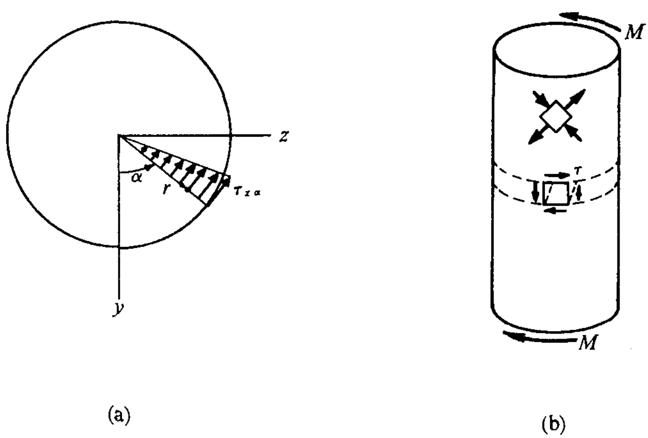
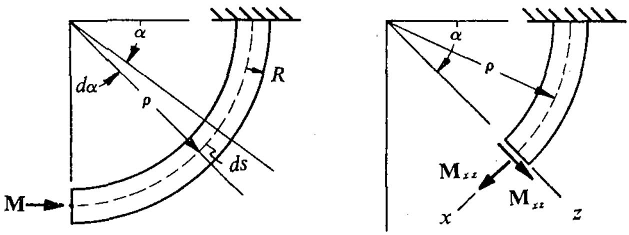


FIGURE 6.4

If twisting moments M_{xx} are applied at the ends of a long bar by forces acting on the lateral surfaces or by shear stresses other than those described by Eq. (6.12), then according to St. Venant's principle the actual stress distribution closely approaches that given by Eq. (6.12) everywhere except near the ends.

6-3 CURVED CIRCULAR BARS

The foregoing solution can also be applied to curved, circular bars, providing that the radius of curvature ρ of the bar is large compared to the radius R of the bar. It cannot be applied if the value of R/ρ is not small for then the difference in length of the surface elements must be taken into account and there is a stress concentration at the inner point of the bar. For example, if $R/\rho = \frac{1}{5}$ the maximum stress calculated by Eqs. (6.12) is 10% too small; when $R/\rho = \frac{1}{12}$ the maximum stress calculated by Eqs. (6.12) is 5% too small.* A slender curved bar will, in general, be subjected to both torsion and bending and in this case it is convenient to use Castigliano's theorem to compute the displacements of the bar.



(a) (b)

FIGURE 6.5

The strain energy in an element ds along a slender rod having $R/\rho \ll 1$, such as shown in Fig. 6.5, is

$$dV = \left[\int_A \left(\frac{\sigma_x^2}{2E} + \frac{\tau_{r\alpha}^2}{2G}\right) dA\right] ds$$

$$= \left\{\int_A \left[\frac{1}{2E}\left(\frac{M_{xz}y}{I_z}\right)^2 + \frac{1}{2G}\left(\frac{M_{xx}r}{I_p}\right)^2\right] dA\right\} ds$$

where M_{xz} is the bending moment and M_{xx} is the twisting moment acting on the element. Since the integral of $(y^2\, dA)$ and $(r^2\, dA)$ over the cross section gives respectively I_z and I_p, the increment of strain energy can be written

$$dV = \left(\frac{1}{2}\frac{M_{xz}^2}{EI_z} + \frac{1}{2}\frac{M_{xx}^2}{GI_p}\right) ds \tag{6.13}$$

This expression includes the strain energy of bending and the strain energy of torsion.

As an example, we shall calculate the twist produced by a torque **M** applied to the end of a rod whose centroidal axis is a circular arc, as shown

* A. M. Wahl, *Mechanical Springs*, Second Edition, McGraw-Hill (1963).

in Fig. 6.5. The bending and twisting moments in the rod are $M_{xz} = M \cos \alpha$ and $M_{xx} = -M \sin \alpha$. Substituting these in Eq. (6.13) gives

$$V = \int_0^{\pi/2} \left(\frac{M^2 \cos^2 \alpha}{2EI_z} + \frac{M^2 \sin^2 \alpha}{2GI_p} \right) \rho \, d\alpha = \frac{M^2 \rho}{2} \int_0^{\pi/2} \left(\frac{\cos^2 \alpha}{EI_z} + \frac{\sin^2 \alpha}{GI_p} \right) d\alpha$$

or

$$V = \frac{(2 + \nu)\rho}{2R^4 E} M^2$$

The rotation at the point of application of **M**, by Castigliano's theorem, is

$$\Theta = \frac{\partial V}{\partial M} = \frac{(2 + \nu)\rho M}{R^4 E} \tag{6.14}$$

The foregoing solution can be applied to the stretching of a coil spring if account is taken of the pitch angle of the coils, see Probs. 6.8 and 6.9.

6-4 BARS OF NONCIRCULAR CROSS SECTION

The torsion of a circular bar is a particularly simple problem which, because of the axial symmetry, has no warping of cross sections. Bars that do not have axial symmetry will undergo warping of the cross sections and the solutions of such problems are, in general, more difficult to work out because it is not easy to find solutions of Eq. (6.6) which have $\Phi = 0$ on a prescribed boundary. Solutions are known for certain special shapes, though these are not all of practical interest. An example of one of these is the torsion of a shaft whose cross-sectional shape is an ellipse

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

where a and b are the semi-axes of the ellipse. It is obvious that the following stress function will satisfy all of the requirements

$$\Phi = B\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right) \tag{6.15}$$

Similarly, a rectangular bar with sides $2a$ and $2b$ will have a stress function expressible in the form of a double Fourier series

$$\Phi = \sum_n \sum_m B_{nm} \cos \frac{n\pi}{2a} y \cos \frac{m\pi}{2b} z \tag{6.16}$$

The torsion of a square bar is a good illustration of the warping of the cross section. In Fig. 6.6a is shown the cross section of the bar, and in (b) is shown a side view of a piece of the bar of length ΔL which has been deformed by the cross section rotating *without* warping. The shear strains developed by this deformation are

$$\gamma_{xz} = y\theta$$
$$\gamma_{xy} = -z\theta$$

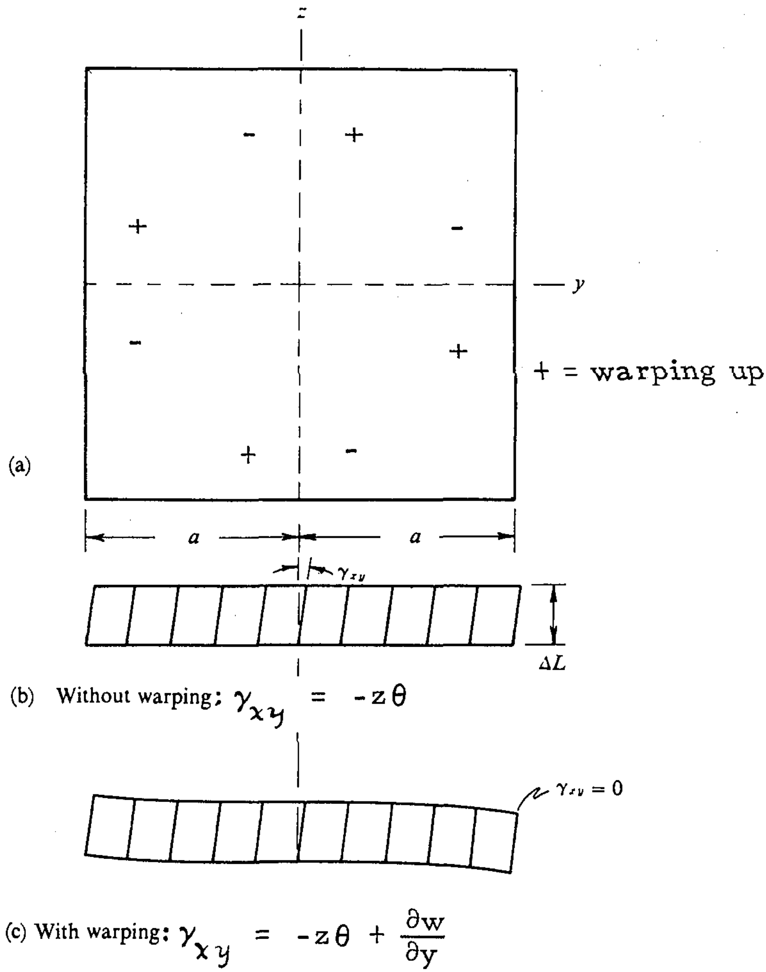


FIGURE 6.6

On the surface at $z = a$ the shear strain is $\gamma_{xy} = -a\theta$ which is constant over the width of the bar. According to this there must be shear stresses $\tau_{xy} = -Ga\theta$ on the surfaces of the bar at $y = \pm a$. However, the boundary conditions for the bar require the shear stresses and strains to be zero on the surfaces.

This condition is satisfied by letting the cross sections warp out of plane as shown in Fig. 6.6c where it is seen that the warping is such that the shear strain is zero at $y = \pm a$. It is seen in (c) that the elements at $y = \pm a$ have rotated without shear strain whereas the element at $y = 0$ has strained without rotating. Since each face of the bar behaves in the same way the element (c) must, in addition, rotate so as to make $u = 0$ at the corners. The warping is thus associated with rotations ω_y, ω_z which are required to satisfy the boundary conditions. Such rotation and warping is also produced by the transverse shear in a beam, as shown in Fig. 3.8.

6-5 MEMBRANE ANALOGY

Equation (6.6) is the so-called Poisson's equation and it appears in many different fields of study. A differential equation of this form describes the

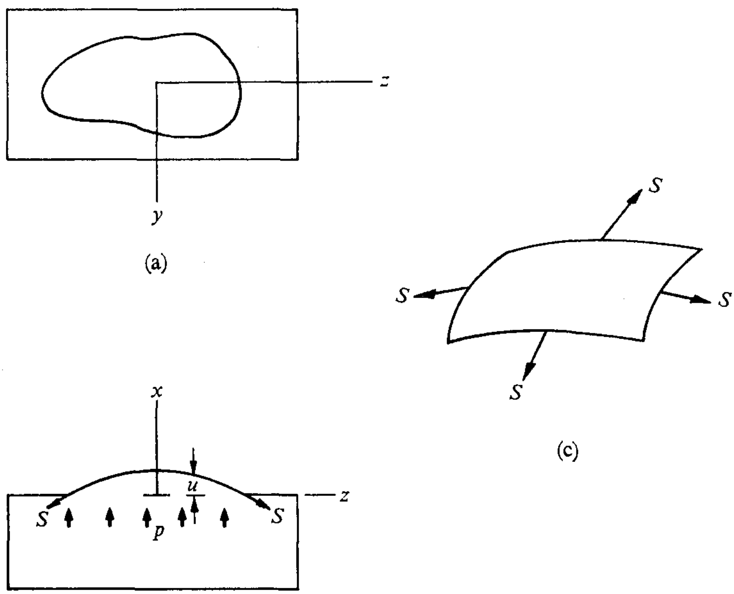


FIGURE 6.7

potential produced by a distribution of electric charge; it describes certain types of fluid flow; and it also describes the deflection of a soap film which is subjected to pressure. Hence, solutions of these problems can also be interpreted as solutions of torsion problems. We shall examine the soap-film problem from this point of view. Consider a hole of certain shape that is cut in a thin plate as indicated in Fig. 6.7. A membrane (soap film) is stretched across the hole and a pressure p deflects it through small displacements u as shown. The membrane has the property that it has a uniform, biaxial tension of magnitude S pounds per foot, which remains constant during the displacement. There are no shearing stresses in this membrane. For *small* slopes the y and z curvatures of the membrane may be written

$$\chi_y = \frac{\partial^2 u}{\partial y^2}, \quad \chi_z = \frac{\partial^2 u}{\partial z^2}$$

The equation of equilibrium of an element, $dy\,dz$ shown in Fig. 6.7c, of the membrane is

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{p}{S} \tag{6.17}$$

This is Poisson's equation and, comparing it with Eqs. (6.5) and (6.6), we see that the following quantities are analogous:

Torsion	Membrane
Φ	u
G	$\frac{1}{S}$
2θ	p
τ_{xn}	$\frac{\partial u}{\partial s}$
M_{xx}	2 (volume under membrane)
C_t	$\frac{4(\text{volume})}{p}$

The membrane is thus a physical representation of the stress function Φ, if its slope is small. The slope of the membrane in the s-direction is analogous to the shear stress in the perpendicular n-direction.*

An experimental method of solving torsion problems that is sometimes used is as follows. Two stiff plates, say, plywood sheets, are bolted together and a hole the same shape as the cross section of the bar is cut through them. Adjacent to this a circular hole is also cut. A large, very thin sheet of rubber is stretched uniformly and equally in both directions and the two plates are placed one on each side of the sheet and bolted together. This gives essentially a plate with two holes across each of which is stretched a membrane with the

* The possibility of using a membrane to obtain experimental solutions of torsion problems was pointed out by L. Prandtl (1875–1950).

same tension S. If an overpressure p is exerted on one side of the membranes, they will deflect into representations of the stress function for the bar under consideration and for a circular bar. The slopes of the membranes and the volumes under them can be measured. Since the solution for the circular bar is known, it affords an automatic calibration for determining τ, θ, and M_{xx} for the bar under consideration without the necessity of knowing the exact magnitudes of p and S.

One of the most useful aspects of the membrane analogy is the aid it renders in visualizing the approximate deflected shape of a membrane, and hence, of the stress function Φ. It is thus possible to judge where the stresses are greatest, and it is also possible to judge whether an approximation has a significant effect on the maximum slope of the membrane or upon the volume under it.

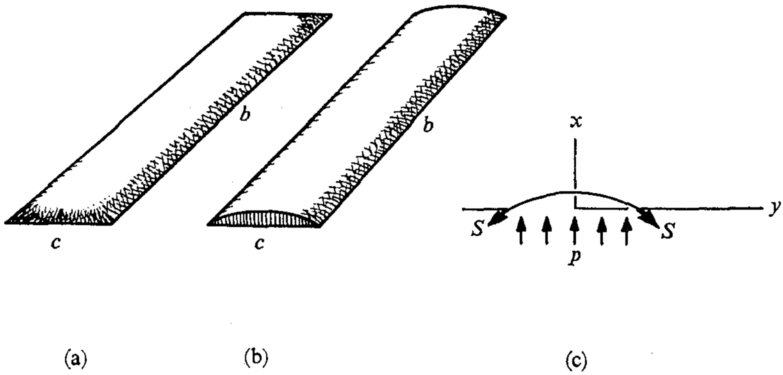


FIGURE 6.8

An example of the use of the membrane analogy is its application to the problem of the torsion of a bar with a narrow rectangular cross section. The membrane analogy is shown in Fig. 6.8 where a membrane spans a rectangular hole. The true deflected shape of the membrane is depicted in Fig. 6.8a, where it is seen that the maximum slope is near the center of the long side. If the side b is long compared to c, it will be a good approximation to assume that the membrane has the same shape over its entire length as if it were part of an infinitely long membrane, as shown in Fig. 6.8b. The maximum slope alongside b is not affected by this approximation, and although the volume under the membrane is increased by the approximation, the error thus introduced is small. Accordingly, for the approximate membrane the equilibrium Eq. (6.16) becomes

$$\frac{d^2u}{dy^2} = -\frac{p}{S}$$

The solution of this equation that satisfies the boundary condition $u = 0$ at $y = \pm c/2$, is the parabola

$$u = \frac{1}{2}\frac{p}{S}\left[\left(\frac{c}{2}\right)^2 - y^2\right]$$

which has the properties

$$\frac{du}{dy} = -\frac{p}{S}y$$

$$\text{Volume} = \tfrac{2}{3}\,c\left(\frac{p}{S}\frac{c^2}{8}\right)b = \tfrac{1}{12}\,\frac{p}{S}\,c^3b$$

From the analogy $\left(\frac{p}{S}\right) = 2G\theta$ hence

$$\tau_{xz} = -2G\theta y$$

$$M_{xx} = 2(\tfrac{1}{12}2G\theta c^3 b) = \tfrac{1}{3}G\theta c^3 b$$

We thus have

$$\theta = \frac{1}{\frac{1}{3}Gc^3b}\,M_{xx} \tag{6.18}$$

$$\tau_{\max} = G\theta c = \frac{M_{xx}}{\frac{1}{3}bc^2} \tag{6.19}$$

The torsional rigidity of the cross section ($C_t = M_{xx}/\theta$) is

$$C_t = \tfrac{1}{3}Gc^3b$$

From the true deflected shape of the membrane shown in Fig. 6.8a it can be deduced that the true torsional stress distribution is as shown in Fig. 6.9a. From the approximate shape of the membrane shown in Fig. 6.8b it can be deduced that the corresponding stress distribution is as shown in Fig. 6.9b which has forces F corresponding to the infinite slope of the membrane at each end. The forces F are equivalent to the τ_{xy} stress components shown in Fig. 6.9a so far as contributing to the twisting moment; it may be verified that the τ_{xz} stresses shown in Fig. 6.9b produce a twisting moment equal to $\frac{1}{2}M_{xx}$ and the F forces produce an equal moment. Equation (6.19) for torsional stresses can be used to compute approximately the torsional stresses in thin-wall sections such as shown in Fig. 6.10. The equation is only applicable at points not close to corners, that is, it can only be applied where the shape of the membrane is similar to that shown in Fig. 6.8. However, the

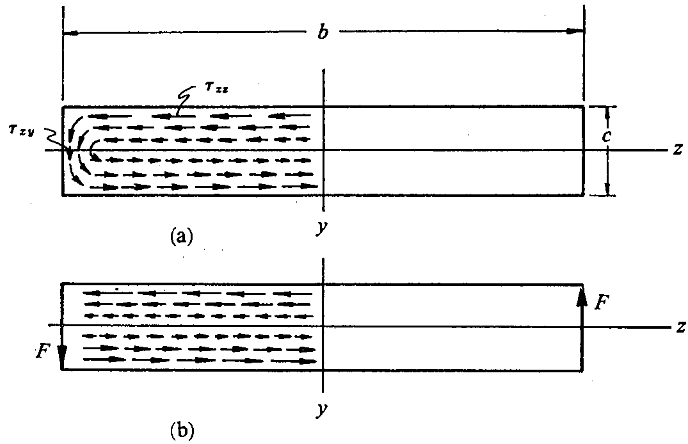


FIGURE 6.9

maximum stresses will occur at reentrant corners where the slope of the membrane is large. To minimize the stress concentration at a reentrant corner, a well-rounded fillet rather than a sharp corner is used.

The actual volume under the membrane for sections such as those shown in Fig. 6.10 will be closely approximated by assuming a uniform parabolic shape, as shown in Fig. 6.8. Equation (6.18) can, therefore, be used with good accuracy. For example, when applied to the I-beam section of Fig. 6.10c there is obtained

$$\begin{aligned} M_{xx} &= 2(\tfrac{1}{3}Gc_1^3b\theta) + \tfrac{1}{3}Gc_2^3h\theta \\ C_t &= \tfrac{1}{3}G(2c_1^3b + c_2^3h) \end{aligned} \tag{6.20}$$

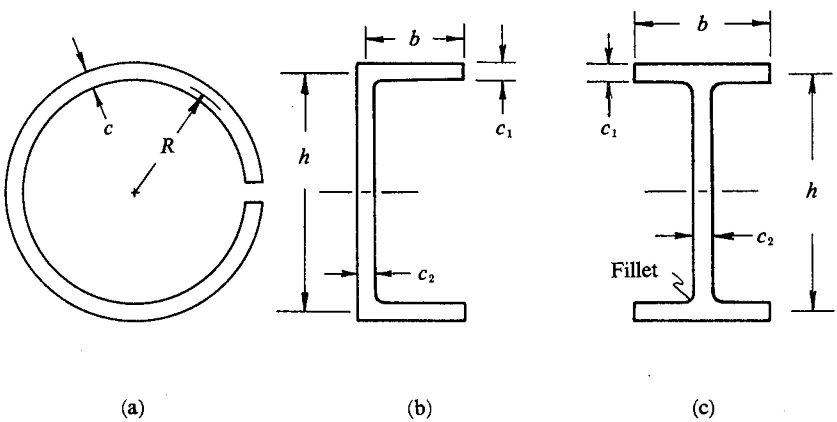


FIGURE 6.10

6-6 TORSION OF TUBULAR SECTIONS

The membrane analogy can also be applied to thin-walled tubes having closed sections, such as that shown in Fig. 6.11a. The corresponding membrane problem is shown in Fig. 6.11b. The membrane spans the thickness t and is attached to the fixed plate which has a hole the same shape as the outer boundary of the bar. It is also attached to a rigid, weightless plate whose outline corresponds to the inner boundary of the bar. This plate is free to move vertically, but is constrained against tilting or sideways displacement.*

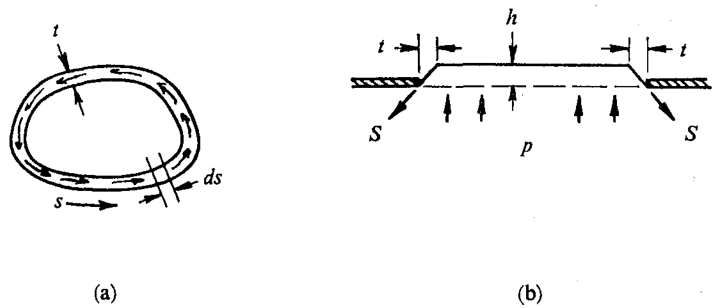


FIGURE 6.11

Although the membrane, because of the pressure p on it, is actually slightly curved the approximation is made that it is straight, as shown in the diagram. In this case, the slope is h/t (shown exaggerated in the diagram since $h/t \ll 1$) and the volume under the membrane is essentially hA, therefore all the pertinent quantities are known when h is known. The height is determined from equilibrium. The total upward force on the plate is pA and the vertical component of S along the edge is hS/t, therefore, equilibrium requires

$$\oint S \frac{h}{t}\, ds = pA$$

where the integral is taken around the boundary. The height h is therefore given by

$$h = \frac{p}{S} A \frac{1}{\oint ds/t}$$

* This analogy is deduced by deriving Eq. (6.8) for the tube, taking proper account of the limits of integration, and by examining the work-energy relations, $\delta V = \delta W$, along the lines of Appendix IV, for the tube and corresponding membrane. In the case of a tube the stress function Φ may be taken equal to zero on the external boundary but on the internal boundary $\Phi = \Phi_i =$ constant. The value of Φ_i can be determined from the height h of the plate shown in Fig. 6.11.

The height is also given by

$$h = \frac{\text{volume}}{A} = \frac{M_{xx}}{2A}$$

From these the enclosed volume and the slope of the membrane are given by

$$\text{volume} = Ah = \frac{p}{S} A^2 \frac{1}{\oint ds/t}$$

$$\text{slope} = \frac{h}{t} = \frac{p}{S}\frac{A}{t}\frac{1}{\oint ds/t}$$

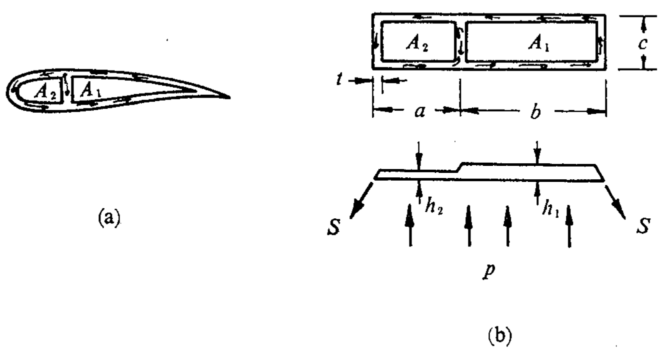


FIGURE 6.12

In the analogy p/S corresponds to $2G\theta$, and the volume corresponds to $M_{xx}/2$, and h/t corresponds to τ, so that the two preceding equations give

$$\theta = \frac{M_{xx}}{4GA^2}\oint \frac{ds}{t} \tag{6.21}$$

$$\tau = \frac{2GA}{t}\frac{\theta}{\oint ds/t} = \frac{M_{xx}}{2At} \tag{6.22}$$

In the case of a tube of uniform wall thickness t and perimeter length s_0, these equations give

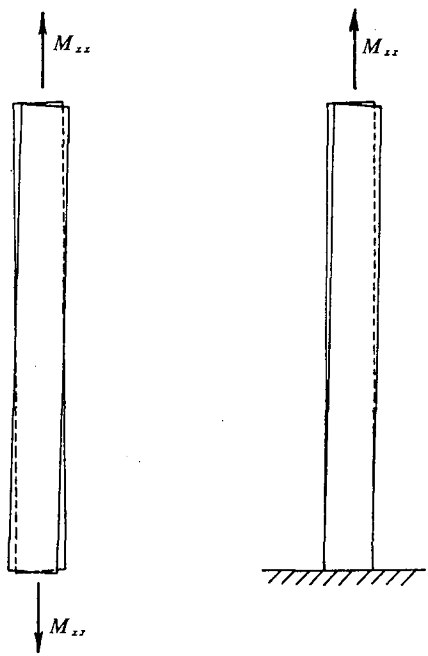
$$\theta = \frac{M_{xx}s_0}{4GA^2t} \tag{6.23a}$$

$$\tau = \frac{M_{xx}}{2At} \tag{6.23b}$$

The approximation that is made in assuming the membrane to be straight is the same as that made in assuming the shear stress τ to be constant across the wall thickness. For a thin wall this is, of course, reasonable. The same approach can be used to analyze the torsion of multiply connected sections as shown in Fig. 6.12a, b. An equation of equilibrium can be written for each area A_1, A_2, etc., and these will determine the heights h_1, h_2, etc.

6-7 RESTRAINT OF WARPING

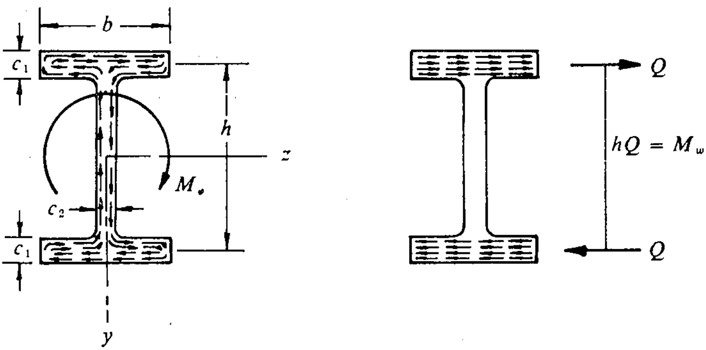
Bars whose shapes are other than circular will undergo a warping of cross sections when twisted. If the bar is rigidly built-in at one end this will inhibit



(a) Free (b) Built-in

FIGURE 6.13

the warping of cross sections in the vicinity of the end and because of this the resistance to twist will be greater than would be predicted on the basis of pure St. Venant torsion. The nature of this warping restraint is especially well illustrated by the twisting of an I-beam. The twist of the unrestrained beam will be as shown in Fig. 6.13a where it is seen that the end cross sections are warped out of plane. If the lower end is built in, as shown in Fig. 6.13b, there can be no warping out of plane, but in this case the flanges of the beam will bend as cantilever beams. Because of this bending there will be developed beam-type σ_x and τ_{xz} stresses in addition to the ordinary torsional stresses. The deformations associated with these stresses are analyzed below.



(a) (b)

FIGURE 6.14

Torsional stresses in an I-beam are distributed as shown in Fig. 6.14a when the beam is subjected to a pure twisting moment M_φ. The moment is related to the twist per unit length θ by the equation

$$M_\varphi = C_t\theta = C_t\frac{d\varphi}{dx}$$

where C_t is the torsional rigidity of the I-beam and φ is the angle of twist at any x. The built-in I-beam is loaded with a moment M_{xx}. We designate the part of this moment carried by the torsional stresses as $M_\varphi = C_t(d\varphi/dx)$, and the remainder which results from restraint of warping is designated as M_w. The cantilever bending action of the flanges produces shear stresses τ_{xz}, illustrated in Fig. 6.14b, which have a resultant of Q in each flange. The shear force is related to the deformation of the flange by the beam equation

$$EI_f\frac{d^3w}{dx^3} = -Q$$

where I_f is the moment of inertia of a flange about its centroidal y-axis, and w is the displacement of the flange axis in the z-direction. As shown in Fig. 6.14b, the two flange shear forces form a couple that has a twisting moment

$$M_w = hQ = -EI_fh\frac{d^3w}{dx^3}$$

For small deformations,

$$\frac{dw}{dx} = \frac{h}{2}\frac{d\varphi}{dx}$$

so

$$M_w = -\frac{EI_fh^2}{2}\frac{d^3\varphi}{dx^3}$$

Setting the total twisting moment equal to $M_\varphi + M_w$ gives the equation

$$C_w\frac{d^3\varphi}{dx^3} - C_t\frac{d\varphi}{dx} = -M_{xx} \tag{6.24}$$

where $C_w = EI_fh^2/2$ is called the warping stiffness. For the beam shown in the Fig. 6.13b the end conditions are

$$\varphi|_{x=0} = 0 \qquad \text{(no rotation)}$$

$$\left.\frac{d\varphi}{dx}\right|_{x=0} = 0 \qquad \text{(flanges built-in so } dw/dx = 0\text{)}$$

$$\left.\frac{d^2\varphi}{dx^2}\right|_{x=l} = 0 \qquad \text{(flange moment is zero)}$$

The corresponding solution is

$$\frac{d\varphi}{dx} = \frac{M_{xx}}{C_t}\left[1 - \frac{\cosh k(l-x)}{\cosh kl}\right] \tag{6.25}$$

where

$$k = \sqrt{\frac{C_t}{C_w}}$$

From this there are obtained the following expressions for M_φ and M_w:

$$M_\varphi = M_{xx}\left[1 - \frac{\cosh k(l-x)}{\cosh kl}\right]$$

$$M_w = M_{xx}\frac{\cosh k(l-x)}{\cosh kl}$$

It is seen that at the built-in end ($x = 0$) the torsional moment is zero, and all of the twisting moment is resisted by the flange bending ($M_w = M_{xx}$, $M_\varphi = 0$). The torsional moment M_φ increases with distance from the built-in end. The importance of the warping resistance depends on the stiffness ratio

$C_t/C_w = k^2$. To further illustrate the effect of the stiffness ratio we consider a beam of infinite length ($l = \infty$). In this case the solution reduces to:

$$\frac{d\varphi}{dx} = \frac{M_{xx}}{C_t}(1 - e^{-kx})$$

$$M_\varphi = M_{xx}(1 - e^{-kx})$$

$$M_w = M_{xx}e^{-kx}$$

The moment M_w thus decreases exponentially with x. It is seen that the restraint of warping at the built-in end increases the twisting stiffness. The stresses σ_x and τ_{xz} produced because of the bending and shearing of the flanges may be computed by use of Eq. (6.25) (see Prob. 6.23). A solid square shaft, built-in at one end would also develop such stresses but the effect of the restraint of warping would not be so pronounced as the stiffness ratio C_t/C_w is not as small as for the I-beam. The effect of restraint of warping is most pronounced in beams of open thin-walled section as for these the torsional stiffness C_t is relatively small. Methods of calculating C_w for such beams have been developed.*

Problems

6.1 Show that the displacements $u = \theta f(y, z)$, $v = -zx\theta$, $w = +yx\theta$ insure that $\sigma_x = \sigma_y = \sigma_z = \tau_{yz} = 0$.

6.2 Sketch an infinitesimal element of a bar in torsion and deduce Eq. (6.3).

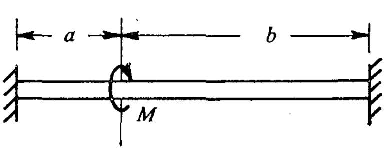
6.3 Show that

$$\int y\Phi\Big|_{y_1}^{y_2} dz = \Phi_0 A$$

where y_1 and y_2 are on the boundary of the area A as shown in Fig. 6.2c.

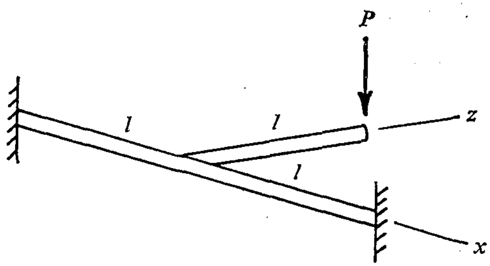
6.4 Pick some function that satisfies Poisson's equation and deduce a torsion problem that it can represent.

6.5 A round shaft of radius r_0 is fixed at both ends and a torque is applied as shown. What is the maximum stress in the shaft, and what is the angle of rotation where the torque is applied?

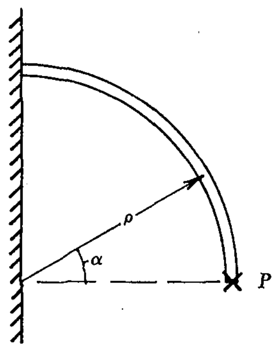


* S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, McGraw-Hill (1961).

6.6 Two round bars of radius r_0 are assembled and loaded as shown. What is the deflection at the point of application of the load? What is the change in slope at that point?



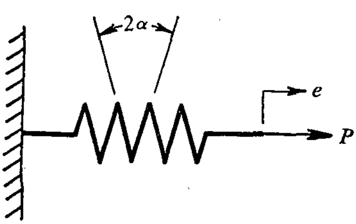
6.7 A curved bar of circular cross section is loaded with a concentrated force normal to the plane of the bar, as shown. What is the deflection at the loaded end?



6.8 A close-coiled spring wound from wire of diameter d with pitch angle α, and n coils of diameter D is extended by an axial load P. Show that the extension is

$$e = \frac{8nD^3}{Gd^4} P \cos\alpha$$

Consider only torsional strains and assume d/D is small. What rotation of the loaded end occurs due to the bending strains?



6.9 Compute the maximum shearing stress in the spring of Prob. 6.8. This is the sum of the torsional plus transverse shear stresses. Assume that the transverse shear is uniformly distributed over the cross section and show that

$$\tau_{\max} = \frac{16PR}{\pi d^3}\left(1 + \frac{d}{4R}\right)$$

6.10 Determine the torsional stresses in a bar of elliptical cross section. Show that the point of maximum shear stress is on the boundary at the point nearest to the axis of the bar. Why should the maximum stress be here rather than at the point farthest from the axis?

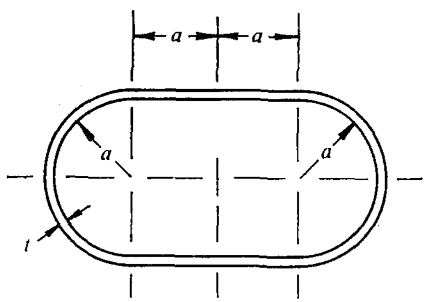
6.11 Determine the warping function for a bar of elliptical cross section and make a contour line sketch of the warping.

6.12 Derive expressions for the torsional stresses in a square bar.

6.13 Derive Eq. (6.17) for a pressure-loaded membrane with small slopes.

6.14 Show that Eq. (6.12) applied to a cylindrical tube reduces to Eq. (6.23b) when the wall thickness is very small.

6.15 What is the shear stress in a thin-walled tube of uniform wall thickness with cross section shown in the diagram, which is subjected to a twisting moment M?

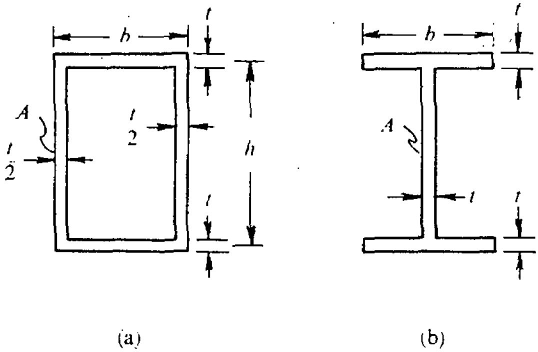


6.16 Determine the torsional rigidity of a thin-walled, square, tubular member of uniform wall thickness.

6.17 Determine the torsional rigidity of a thin-walled circular tube that has been cut through along its length on one side (Fig. 6.10a). Compare this with the torsional rigidity of an uncut tube.

6.18 Determine the total twist produced in a thin-walled tube of uniform wall thickness and length l, with moment M. The cross section is shown in Fig. 6.12b. Determine the maximum shear stress.

6.19 Compare the torsional rigidities of the two cross sections of equal area shown in the diagram. Compare the maximum shear stresses at the points marked A.



6.20 A steel drilling stem in an oil well is 10,000 ft long and is 6 in. in diameter. The drilling torque produces a maximum shear stress of 5000 psi. What is the rotation of the top end with respect to the bottom?

6.21 A straight circular bar of radius $R = 1$ in. is subjected to an axial tension force $P = 5000$ lb, a bending moment $M_{xz} = 5000$ in.-lb, and a twisting moment $M_{xx} = 10{,}000$ in.-lb. What is the maximum tension stress in the bar and on what plane does it act?

6.22 Verify that Eq. (6.25) satisfies Eq. (6.24) and the boundary conditions. Obtain an expression for the twist as a function of x.

6.23 Show that the stresses at any x due to the restraint of warping in the I-beam are given by

$$|\sigma_x|_{\max} = \frac{Eh^2}{4}\frac{d^2\varphi}{dx^2}$$

$$|\tau_{xz}|_{\max} = \frac{3}{4}\frac{EI_f h}{A}\frac{d^3\varphi}{dx^3}$$

where A is the cross-sectional area of a flange.

CHAPTER 7

APPLICATIONS TO PLATES AND SHELLS

7-1 THE BENDING OF PLATES

When a thin plate, initially flat, is subjected to forces acting perpendicular to it, the plate will be deformed into a curved shape as shown in Fig. 7.1. The deformation is mainly the result of bending strains although a small amount is caused by shearing deformation normal to the plane of the plate. There are also local compressive deformations in the vicinity of the points of application of the forces. It is assumed that the plate is not deformed in such a way as to produce appreciable stretching or contracting in the plane of the plate. This will always be true if the deformations out of the plane are small compared to the thickness of the plate. If the displacement are not small an analysis must take into account both the bending and stretching.

In many practical problems the plate is thin, that is, the ratio of plate thickness to span length is small; and the method of loading and supporting the plate is such that the shearing deformations, the local compressive deformations and the in-plane extension and contraction are all relatively small. The foregoing statements are precisely those that are cited to justify the use of the technical theory of bending of beams and, in fact, the statements are equivalent to saying that under these conditions the plate behaves like a two-dimensional beam in bending.

The plate shown in Fig. 7.1a and 7.1c has its middle surface lying in the x-y plane and the thickness of the plate is h, so that the bottom and top faces of the undeformed plate correspond to $z = \pm h/2$. The technical theory of bending is based upon the following simplified deformation conditions (a), and stress-strain relations (b), which are assumed to govern the straining of the plate:

$$\left.\begin{aligned}\epsilon_z &= \frac{\partial w}{\partial z} = 0\\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0\\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0\end{aligned}\right\}\text{(a)}\qquad \left.\begin{aligned}\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y)\\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x)\\ \gamma_{xy} &= \frac{1}{G}\tau_{xy}\end{aligned}\right\}\text{(b)} \tag{7.1}$$

These relations may be thought of as being approximate descriptions of a thin plate made of isotropic material. Alternatively, they may be thought of

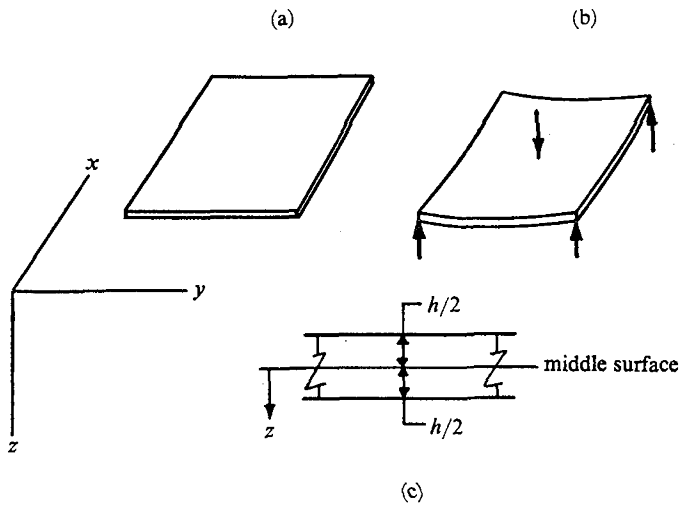


FIGURE 7.1

as exact descriptions of an idealized plate made of nonisotropic material, having E_z, G_{xz}, G_{yz} infinite, and ν_{xz}, ν_{yz} equal to zero. In this case the stresses, strains and displacements calculated on the basis of Eq. (7.1) will be correct for the idealized nonisotropic plate. It is also found, in general, that these stresses, strains and displacements are close approximations to those that would be produced in a thin, isotropic plate having the same loading and boundary conditions. The chief discrepancies occur when the isotropic plate is loaded, or supported, in such a way that relatively large γ_{xz}, γ_{yz} shearing strains are produced.

Equations (7.1) describe the type of straining that the plate can undergo.

They represent a generalization of the equations of the technical theory of beams. The form of these equations was purposely chosen to be such that all of the strains of the deformed plate could be described in terms of the transverse displacement w of the middle surface. For example, the first equation in (7.1) states that

$$\epsilon_z = \frac{\partial w}{\partial z} = 0$$

hence

$$w = f(x, y) \tag{7.2}$$

This states that w is independent of z, or, all points through the thickness of the plate have the same $w = f(x, y)$. The second and third equations in (7.1) state that

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

Since w is independent of z these equations can be integrated to give

$$v = -z\frac{\partial w}{\partial y} + f_1(x, y) \qquad u = -z\frac{\partial w}{\partial x} + f_2(x, y)$$

The second terms in these two expressions represent stretching and shearing of the middle surface ($z = 0$) of the plate and since our analysis deals only with deformations of plates in which the stretching and shearing of the middle surface is negligibly small, the displacements u and v can be written

$$\begin{aligned} u &= -z\frac{\partial w}{\partial x} \\ v &= -z\frac{\partial w}{\partial y} \end{aligned} \tag{7.3}$$

It is seen in Fig. 7.2 that these equations state, in essence, that a line (a-b) which is originally straight and perpendicular to the middle surface remains straight and perpendicular to the middle surface after bending. The equations also imply that the extensional strains of the middle surface are zero. The strains at any point in the plate may be calculated from Eqs. (7.3) to give

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned} \tag{7.4}$$

The second derivatives $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, and $\partial^2 w/\partial x\,\partial y$ are approximate expressions for the curvatures of the middle surface. Using the Greek letter Chi (χ) for curvatures, the foregoing equations can be written

$$\begin{aligned} \epsilon_x &= -z\chi_x \\ \epsilon_y &= -z\chi_y \\ \gamma_{xy} &= -2z\chi_{xy} \end{aligned} \tag{7.4a}$$

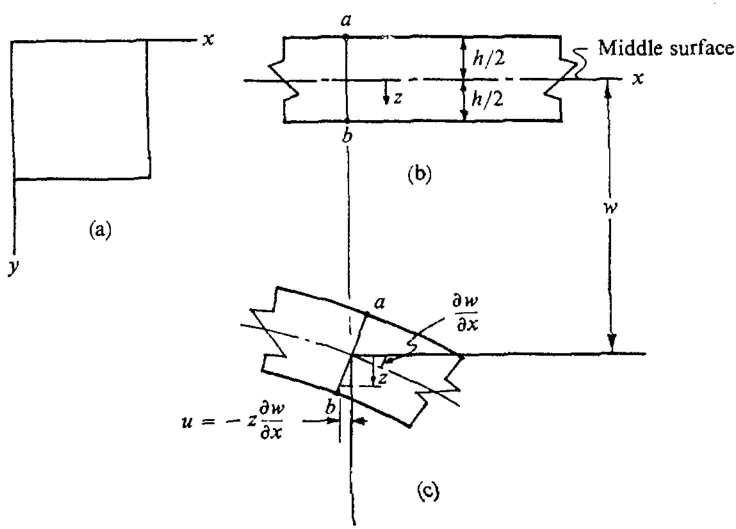


FIGURE 7.2

The physical significance of the curvatures is made more apparent by writing the expressions as follows:

$$\chi_x = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) \qquad \chi_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$

$$\chi_y = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) \qquad \chi_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$$

These expressions are thus the rates at which slopes are changing. In Fig. 7.3 there is shown how a twisted strip of plate has the slope $\partial w/\partial y$ changing along x. χ_{xy} is thus the twist of the plate as measured along a line parallel to the x-axis. χ_{yx} is the twist measured along a line parallel to the y-axis, and it is seen that at any location (x, y) the two curvatures are the same, $\chi_{xy} = \chi_{yx}$. It is seen from Eqs. (7.4a) that the curvatures χ_x, χ_y, χ_{xy} transform in the same way as do the strains ϵ_x, ϵ_y, ϵ_{xy}, that is, Mohr's circle can be used to transform the curvatures. It follows from Mohr's circle that $\chi_x + \chi_y = \nabla^2 w$

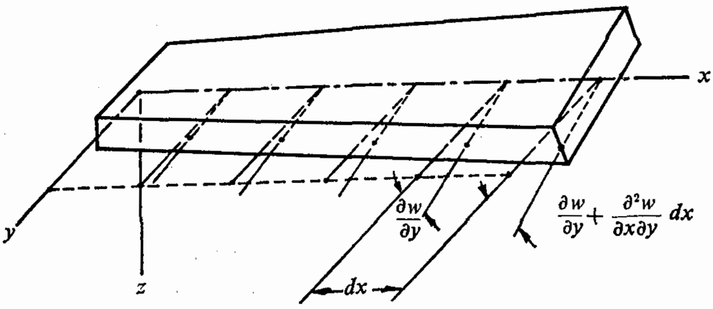


FIGURE 7.3

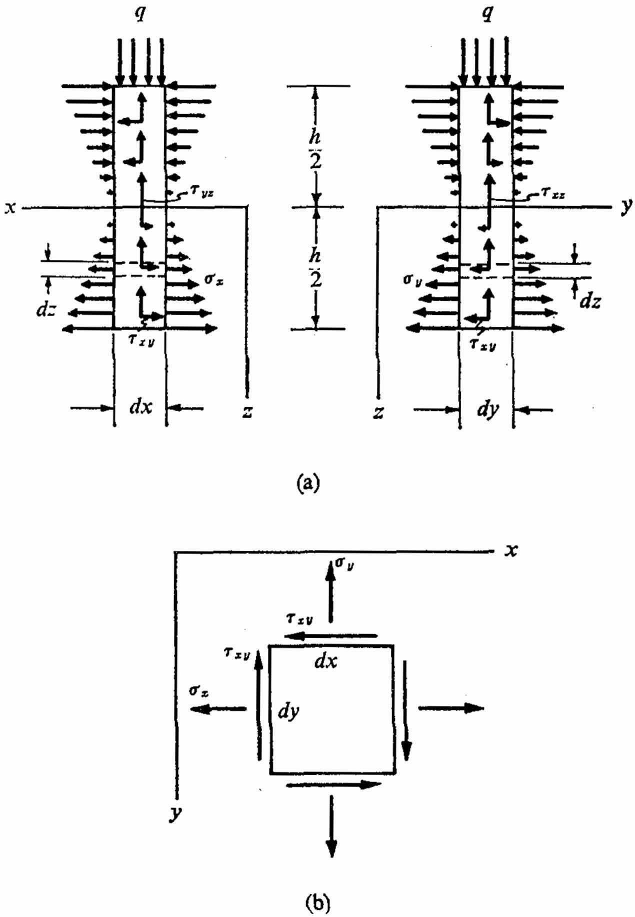


FIGURE 7.4

is invariant with respect to rotation of coordinate axes. It is seen that when the curvatures are known the ϵ_x, ϵ_y, ϵ_{xy} strains are determined and the σ_x, σ_y and τ_{xy} stresses can be computed by means of Hooke's law, Eqs. (7.1). According to Eqs. (7.1), the stresses σ_z, τ_{xz}, τ_{yz} are not related to strains so they cannot be determined from Hooke's law. They can be determined, as shown in the following section, by means of the general equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

Stresses in a Plate. If Eqs. (7.1b) are inverted, the stress-strain relations for a thin plate are

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \\ \tau_{xy} &= G\gamma_{xy} \end{aligned} \tag{7.5}$$

In terms of the curvatures of the middle surface these equations are

$$\begin{aligned} \sigma_x &= -z\frac{E}{1-\nu^2}(\chi_x + \nu\chi_y) \\ \sigma_y &= -z\frac{E}{1-\nu^2}(\chi_y + \nu\chi_x) \\ \tau_{xy} &= -z\frac{E}{1-\nu^2}(1-\nu)\chi_{xy} \end{aligned} \tag{7.6}$$

In the last equation, use has been made of the relation $G = E/2(1+\nu)$. According to Eqs. (7.6) the stresses σ_x, σ_y and τ_{xy} vary linearly with z, as shown in Fig. 7.4a. A thin slice of thickness dz of the element shown in Fig. 7.4a has a freebody diagram as shown in Fig. 7.4b. According to Eq. (7.5) this slice behaves essentially as if it were in a state of plane stress. It may be inferred from this that the solution of plate problems must be similar to the solution of plane stress problems.

From the general equilibrium equations and Eqs. (7.6) the stresses τ_{xz} and τ_{yz} are determined by integration (see Fig. 7.5a)

$$\tau_{xz} = \int_z^{h/2} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) dz = -\frac{E}{2(1-\nu^2)}\left(\frac{h^2}{4} - z^2\right)\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right]$$

$$\tau_{yz} = \int_z^{h/2} \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}\right) dz = -\frac{E}{2(1-\nu^2)}\left(\frac{h^2}{4} - z^2\right)\left[\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right] \tag{7.7}$$

It is seen from these equations that the vertical shear stresses, τ_{xz} and τ_{yz} vary parabolically over the thickness of the slab.

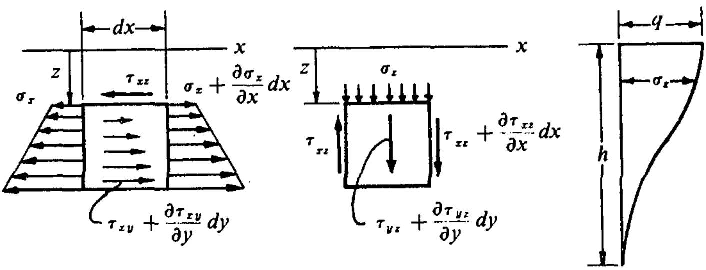


FIGURE 7.5

The stress σ_z is determined by integrating the third equilibrium equation:

$$\sigma_z = \int_z^{h/2} \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right) dz$$

which expresses equilibrium of the element shown in Fig. 7.5b. Substituting for τ_{xz} and τ_{yz}, there is obtained

$$\begin{aligned}\sigma_z &= -\int_z^{h/2} \frac{E}{2(1-\nu^2)}\left(\frac{h^2}{4} - z^2\right)\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right] dz \\ &= -\frac{E}{2(1-\nu^2)}\left(\frac{h^3}{12} - \frac{h^2 z}{4} + \frac{z^3}{3}\right)(\nabla^4 w)\end{aligned} \tag{7.8}$$

In writing this expression use has been made of the notation

$$\nabla^2 w = \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$\nabla^4 w = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w$$

The stress σ_z thus varies as a cubic parabola over the depth of the slab, as shown in Fig. 7.5c. At the upper surface of the slab at $z = -h/2$, the stress σ_z must equal the intensity of the applied compressive load, that is $\sigma_z = -q$. For this condition Eq. (7.8) gives

$$\frac{E}{1-\nu^2}\frac{h^3}{12}\nabla^4 w = q \tag{7.9}$$

Equation (7.9) is a condition of equilibrium of the slab. It corresponds to the equation $EI\, d^4w/dx^4 = q$ in the theory of bending of beams. The flexural rigidity D of the slab is defined to be

$$D = \frac{E}{1-\nu^2}\frac{h^3}{12}$$

With this notation the equation of the slab is

$$D\nabla^4 w = q \tag{7.10}$$

or, in expanded form,

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q \tag{7.10a}$$

Equation (7.10) relates the deformation of the slab to the applied load.*

7-2 BENDING MOMENTS AND TWISTING MOMENTS

The stresses σ_x have a resultant moment but no resultant force. Referring to the element (dx, dy, h) shown in Fig. 7.4a, it is seen that the stresses σ_x acting on the right side of the element have a resultant moment given by

$$\int_{-h/2}^{+h/2} z\sigma_x(dy)\,dz = (dy)\int_{-h/2}^{+h/2} z\sigma_x\,dz = (dy)M_x$$

where M_x is the bending moment per unit length.† The bending moment M_y

* This equation was first derived by LaGrange, but the first satisfactory theory of bending of plates was given by Navier.

† It should be noted that with plate moments we do not use the vector convention for signs, but use the same sign conventions as we do for stresses; that is, a positive moment produces positive stresses in the *positive* half of the plate.

per unit length and the twisting moment M_{xy} per unit length are given by similar expressions. The corresponding stresses are found using Eq. (7.6),

$$
\begin{aligned}
M_x &= \int_{-h/2}^{+h/2} z\sigma_x \, dz & \sigma_x &= \frac{M_x z}{I} \\
M_y &= \int_{-h/2}^{+h/2} z\sigma_y \, dz & \sigma_y &= \frac{M_y z}{I} \\
M_{xy} &= \int_{-h/2}^{+h/2} z\tau_{xy} \, dz & \tau_{xy} &= \frac{M_{xy} z}{I}
\end{aligned}
\tag{7.11}
$$

where I is the moment of inertia of the cross section per unit length ($I = h^3/12$). It is seen from these equations that the moments have a one-to-one correspondence with the stresses and, hence, they transform as the stresses do and Mohr's circle can be used to transform the moments.

Substituting in Eq. (7.11) for the stresses in terms of the curvatures as given by Eq. (7.6), and integrating, there is obtained

$$
\begin{aligned}
M_x &= -D(\chi_x + \nu\chi_y) \\
M_y &= -D(\chi_y + \nu\chi_x) \\
M_{xy} &= -D(1 - \nu)\chi_{xy}
\end{aligned}
\tag{7.12}
$$

where D is the flexural rigidity of the plate per unit length

$$D = \frac{EI}{1 - \nu^2} = \frac{Eh^3}{12(1 - \nu^2)}$$

If the foregoing equations are inverted, there is obtained

$$
\begin{aligned}
\chi_x &= -\frac{1}{D(1 - \nu^2)}(M_x - \nu M_y) \\
\chi_y &= -\frac{1}{D(1 - \nu^2)}(M_y - \nu M_x) \\
\chi_{xy} &= -\frac{M_{xy}}{D(1 - \nu)}
\end{aligned}
\tag{7.13}
$$

Equations (7.12) and (7.13) represent the stress-strain relations for the plate expressed in terms of the moments and curvatures instead of the stresses and strains.

7-3 TRANSVERSE SHEAR FORCES

The vertical shearing stresses τ_{xz} and τ_{yz} give resultant vertical shearing forces, as shown in Fig. 7.6a. The shearing forces per unit length are

$$Q_x = \int_{-h/2}^{+h/2} \tau_{xz}\, dz$$
$$Q_y = \int_{-h/2}^{+h/2} \tau_{yz}\, dz \tag{7.14}$$

The subscript on the Q denotes the face on which the shearing force acts. The sign convention for Q_x and Q_y is the same as for the shearing stresses τ_{xz} and τ_{yz}, namely, that a positive shearing force acts on a positive face in a positive z-direction, or on a negative face in a negative z-direction.

7-4 EQUATIONS OF EQUILIBRIUM

In Fig. 7.6 there are shown all the forces and moments that act upon an element of the plate. In (a) is shown the intensity q per unit area of the load applied to the top surface of the plate and the shearing forces, Q_x and Q_y,

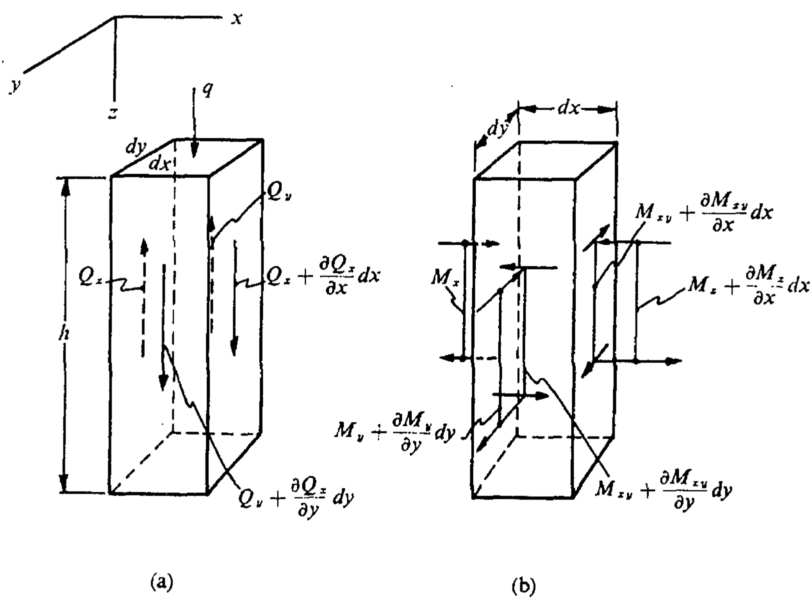


FIGURE 7.6

on the sides of the element. These forces, and also the moments in (b) are shown as positive quantities. In (b) there are shown the couples that represent the bending moments M_x and M_y, and the couples that represent the twisting moments M_{xy}. The moments have been shown as couples to make clear the relation between the moments M_x, M_y, M_{xy} and stresses σ_x, σ_y, τ_{xy} (compare Fig. 7.6b with Fig. 7.4a). The element of the plate must be in equilibrium for forces in the z-direction, for moments about the x-axis and for moments about the y-axis. The resulting equations of equilibrium are respectively

$$
\begin{aligned}
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} &= -q \\
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} &= Q_x \\
\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} &= Q_y
\end{aligned}
\tag{7.15}
$$

Substituting Q_x and Q_y in the first equation from the second and third equations, gives the following equation of vertical equilibrium expressed in terms of the moments:

$$
\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \tag{7.16}
$$

Substituting into this equation the expressions for M_x, M_y, M_{xy} given by Eqs. (7.12) and canceling terms, the equation takes the form

$$
\frac{\partial^2 \chi_x}{\partial x^2} + 2\,\frac{\partial^2 \chi_{xy}}{\partial x\,\partial y} + \frac{\partial^2 \chi_y}{\partial y^2} = \frac{q}{D} \tag{7.17}
$$

This expresses vertical equilibrium in terms of the curvatures. From the definition of the curvatures, $\chi_x = \partial^2 w/\partial x^2$, $\chi_y = \partial^2 w/\partial y^2$, and $\chi_{xy} = \partial^2 w/\partial x\,\partial y$, this equation can be written

$$
\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{7.18}
$$

This agrees with Eq. (7.10) which was derived in a different way. Equation (7.18) is the basic plate equation and any expression for w that satisfies this equation is a solution of a plate problem.* For example, as may be verified

* Many solutions are given in the book by Timoshenko and Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill Book Company (1959).

by substitution in Eq. (7.18), the displacement of a loaded rectangular plate that is simply supported along all four edges (no rotational restraint at the support) can be expressed by

$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{(m^2/a^2 + n^2/b^2)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where a and b are the lateral dimensions of the plate and the constants C_{mn} are the coefficients of the double Fourier series expansion of the load

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Two edges of the plate coincide with the x- and y-axes.

7-5 BOUNDARY CONDITIONS FOR A PLATE

The boundary conditions for a plate are similar to those for a beam. Figure 7.7 shows the reactions of a plate that is simply supported along its edges.

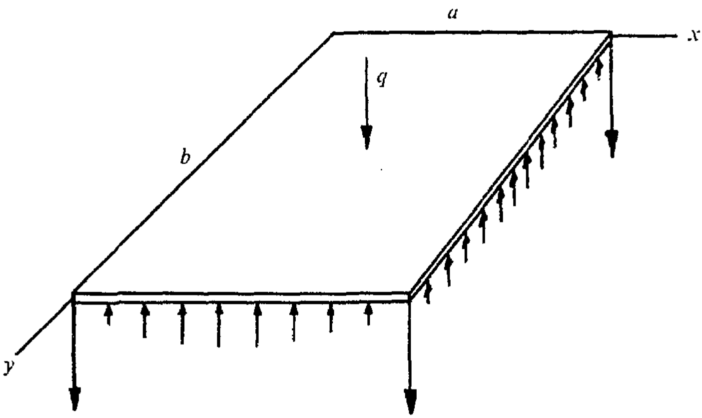


FIGURE 7.7

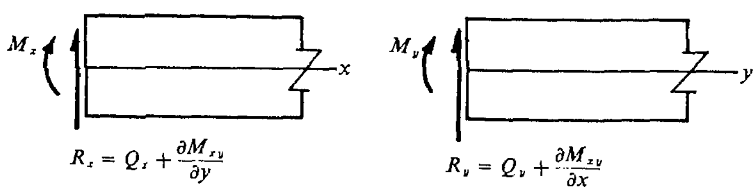
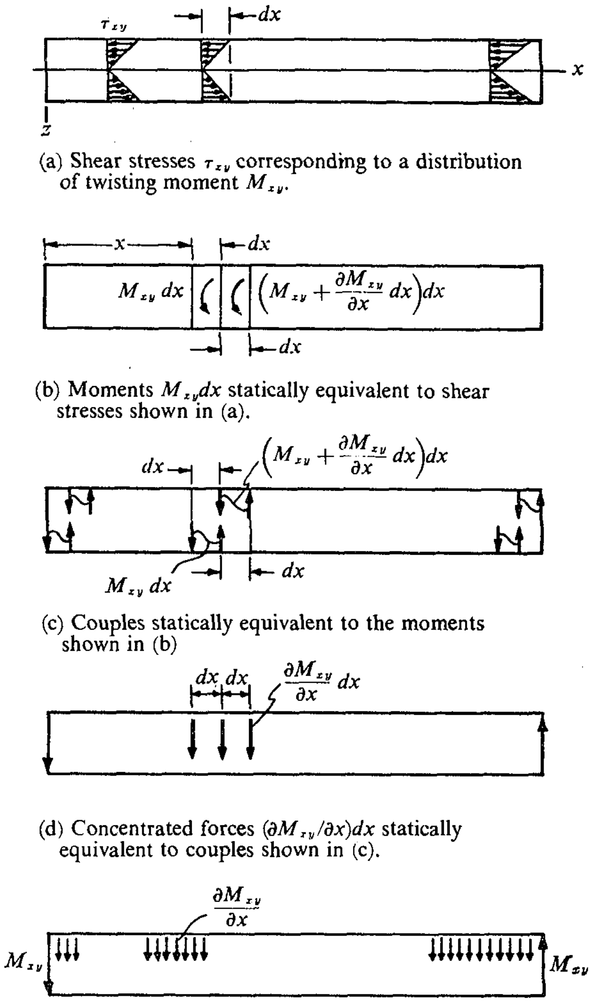


FIGURE 7.8

The displacement at the edge of the plate may be specified, the slope may be specified, the vertical reaction may be specified, or the bending moment may be specified. The twisting moment M_{xy} along the edge, however, cannot



(a) Shear stresses τ_{xy} corresponding to a distribution of twisting moment M_{xy}.

(b) Moments $M_{xy}dx$ statically equivalent to shear stresses shown in (a).

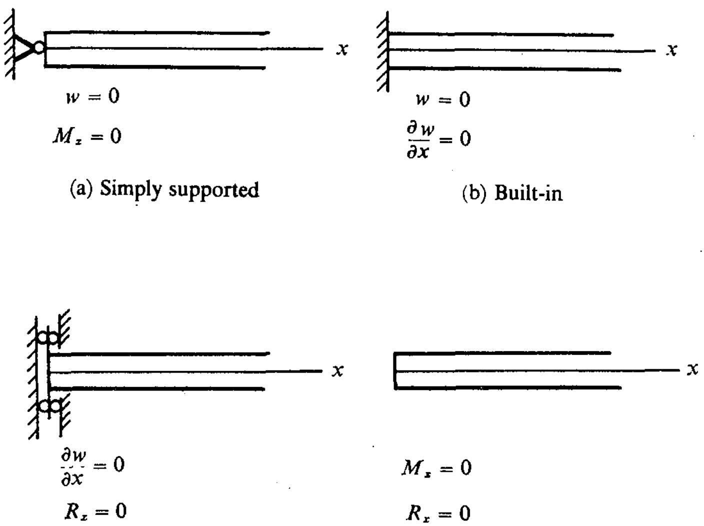
(c) Couples statically equivalent to the moments shown in (b)

(d) Concentrated forces $(\partial M_{xy}/\partial x)dx$ statically equivalent to couples shown in (c).

(e) Distribution of vertical shear of intensity $\partial M_{xy}/\partial x$ per unit length statically equivalent to forces shown in (d).

FIGURE 7.9

be specified independently. This is a consequence of the approximations that were made in assuming the shearing strains γ_{xz} and γ_{yz} to be zero. The approximations were made in order that all of the strains, including those associated with M_{xy} could be expressed in terms of a single dependent variable w and, hence, we cannot specify anything about the strains that is independent of the bending displacement w and its derivatives. If the shear strains γ_{xz}, γ_{yz} are included in the analysis, the total deflection of the plate is $w = w_b + w_s$, where w_b is the deflection produced by the bending deformation and w_s is the deflection produced by the shearing deformation, and in this case the extra boundary conditions on M_{xy} can be specified. We can see, by analogy with the beam equation, that Eq. (7.18) will permit only two boundary conditions to be imposed at each edge. A more detailed analysis shows that the twisting moment along the edge is associated with the vertical reaction



(a) Simply supported

(b) Built-in

(c) Sliding edge

(d) Free edge

FIGURE 7.10

as shown in Fig. 7.8, where $R_x = Q_x + \partial M_{xy}/\partial y$ is the vertical reaction at the edge. In effect, this states that a distribution of M_{xy} along an edge is statically equivalent to a distribution of vertical shears.* This equivalence is illustrated in Fig. 7.9. See also, Probs. 7.17, 7.18, 7.28 and IV.6.

Various possible boundary conditions for a plate are shown in Fig. 7.10. In addition to the boundary conditions shown it is, of course, possible to have the displacement and rotation of an edge restrained by structural members whose constraint may be represented by springs.

* These boundary conditions for the plate were first correctly stated by G. Kirchhoff (1824–1887). He used the method of Appendix IV (Prob. 6) to derive the correct boundary conditions.

7-6 CIRCULAR PLATES

The solutions of problems of circular plates that support axially symmetric loads may be readily worked out and many of these solutions are of

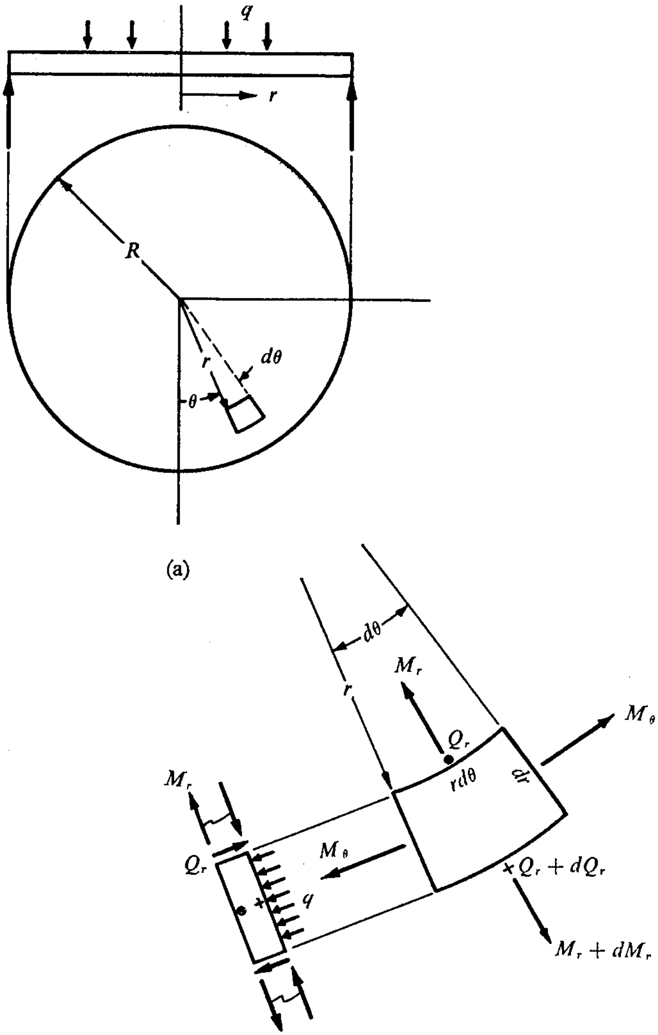


FIGURE 7.11

great practical importance. In Fig. 7.11 there is shown a freebody diagram of an element of such a plate. The element is acted upon by the radial bending moment M_r, the circumferential bending moment M_θ, the vertical shear Q_r, and the applied load q. From symmetry of the plate and load it is concluded that the shear Q_θ and the twisting moment $M_{r\theta}$ are zero and the moment M_θ is independent of θ. This plate element in polar coordinates can be compared with the general plate element in rectangular coordinates shown in Fig. 7.4. Note that in Fig. 7.11b only the bottom forces of the moment couples are shown.

Vertical equilibrium of the element in Fig. 7.11 requires the shear forces to equilibrate the load q. It should be noted that the r-sides of the element are not the same length and the θ-sides are not parallel. The total upward shear on the negative face of the element is $Q_r(rd\theta)$. The difference in value of the total shear force at r (the negative face) and at $r + dr$ (the positive face) is

$$\frac{d}{dr}(Q_r r)\,d\theta\,dr \tag{7.19}$$

The total downward force exerted by the load is $q(rd\theta\,dr)$. Hence, equilibrium requires

$$\frac{d}{dr}(rQ_r) + rq = 0$$

The equilibrium equation for moments can be written in a similar manner. The total moment on the negative face of the element is $M_r(rd\theta)$. The difference in value of this moment at the negative face and that at the positive face is

$$\frac{d}{dr}(M_r r)\,d\theta\,dr$$

Since the moments per unit length M_θ are not in the same plane they will have a component $-M_\theta\,d\theta$ per unit length parallel to and in the direction of M_r. The component produced by the two M_θ's is

$$-M_\theta\,d\theta\,dr$$

The shear forces acting on the element produce a couple with the moment arm dr, and the moment produced is

$$(Q_r r d\theta)\,dr$$

The equation of equilibrium is

$$\frac{d}{dr}(rM_r) - M_\theta = rQ_r \tag{7.20}$$

The dQ_r and q do not appear in the equilibrium equation because they produce higher order moment terms. If Eq. (7.20) is differentiated with respect to r and Eq. (7.19) is used to eliminate the term $d(rQ_r)/dr$ there is obtained the general equilibrium equation in terms of moments

$$\frac{d^2}{dr^2}(rM_r) - \frac{dM_\theta}{dr} = -rq \tag{7.21}$$

This corresponds to Eq. (7.16) in rectangular coordinates.

Referring to Eqs. (7.12), we see that

$$M_r = -D(\chi_r + \nu\chi_\theta)$$

$$M_\theta = -D(\chi_\theta + \nu\chi_r)$$

The curvature χ_r is calculated along a radius line and χ_θ is calculated along a line that is perpendicular to the radius line*

$$\chi_r = \frac{d^2w}{dr^2}$$

$$\chi_\theta = \frac{1}{r}\frac{dw}{dr}$$

If we substitute in Eq. (7.21) for M_r and M_θ in terms of w, there is obtained

$$\frac{d^2}{dr^2}\left[r\left(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}\right)\right] - \frac{d}{dr}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) = \frac{rq}{D}$$

The terms containing Poisson's ratio cancel so that the equation reduces to

$$\frac{d^2}{dr^2}\left(r\frac{d^2w}{dr^2}\right) - \frac{d}{dr}\left(\frac{1}{r}\frac{dw}{dr}\right) = \frac{rq}{D}$$

As may be verified, the foregoing equation can be put in the form

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{q}{D} \tag{7.22}$$

The displacement w can thus be determined by four successive integrations

$$w = \int\frac{1}{r}\int r\int\frac{1}{r}\int\frac{rq}{D}\,dr\,dr\,dr\,dr \tag{7.22a}$$

* The Laplacian is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\,\partial\theta^2}\right) = \nabla^2$$

For example, if the plate is uniformly loaded, q is a constant, and the integrations give

$$w = \frac{q}{D}\left\{\frac{r^4}{64} + c_1r^2 + c_2 + c_3 \ln r + c_4r^2(1 - \ln r)\right\} \tag{7.23}$$

The constants of integration are determined by the boundary conditions. For example, if the plate is built-in around the edge the conditions are

$$w\Big|_{r=R} = 0, \quad \frac{dw}{dr}\Big|_{r=R} = 0$$

It is found that the logarithmic terms give an infinite displacement and an infinite moment at $r = 0$, and therefore we take $c_3 = c_4 = 0$. The boundary conditions then give

$$c_1 = -R^2/16, \quad c_2 = R^4/64$$

and, hence

$$w = \frac{q}{64D}(R^2 - r^2)^2 \tag{7.24}$$

This is the displacement of a uniformly loaded, circular, built-in plate. The bending moments, the shear force, and other quantities can be calculated from this expression for w. The maximum displacement is

$$w_{\max} = \frac{qR^4}{64D}$$

The bending moments are

$$M_r = \frac{q}{16}[(1 + \nu)R^2 - (3 + \nu)r^2]$$

$$M_\theta = \frac{q}{16}[(1 + \nu)R^2 - (1 + 3\nu)r^2]$$

At the center of the plate the moments are

$$(M_r)_{r=0} = (M_\theta)_{r=0} = (1 + \nu)\frac{qR^2}{16}$$

At the edge of the plate the moments are

$$(M_r)_{r=R} = -\frac{qR^2}{8}$$

$$(M_\theta)_{r=R} = -\frac{\nu qR^2}{8}$$

The maximum stress occurs at the edge of the plate and is equal to

$$(\sigma_r)_{r=R} = \frac{3}{4}\frac{qR^2}{h^2}$$

A uniformly loaded, circular plate that is simply supported along its edges has boundary conditions

$$w = 0, \quad M_r = 0 \qquad \text{at } r = R$$

$$w = \text{finite}, \quad \frac{dw}{dr} = 0 \qquad \text{at } r = 0$$

The displacement of the plate is found to be

$$w = \frac{q}{64D}(R^2 - r^2)\left(\frac{5+\nu}{1+\nu}R^2 - r^2\right) \tag{7.25}$$

Problems

7.1 A simply supported beam of length L, depth h, and unit width is deflected so that $w = A \sin \pi x/L$ with the two ends restrained against any axial movement. Show that the average axial strain is $\epsilon_a = (\frac{1}{2}\pi\, A/L)^2$. Compute the axial force F_a' and the total vertical load carried by it, F_a. Compare this with the total vertical load carried by the beam action ($F_b = -2EI\, d^3w/dx^3|_{x=0}$) and show that for F_a/F_b to be a small quantity it is necessary that $3(A/h)^2$ be small and hence that the maximum displacement A must be small compared to the thickness of the beam or plate.

7.2 By means of suitable sketches show the deformations produced in an element dx, dy, dz, at a point $z = -c$, by $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, and $\partial^2 w/\partial x\, \partial y$, respectively.

7.3 Show by sketching a strip of plate parallel to the x-axis that $\partial^2 w/\partial x\, \partial y$ represents a twist along the x-direction. Show that the term also represents the twist of a strip that is parallel to the y-axis.

7.4 Show that the moments M_x, M_y, and M_{xy} and the curvatures χ_x, χ_y, χ_{xy} at a point in a plate can be transformed to new coordinates x', y' by Mohr's circle. What is meant by the principal curvatures of a plate?

7.5 Show that the stresses can be computed by $\sigma_x = M_x z/I$, $\tau_{xy} = M_{xy}z/I$.

7.6 Derive Eqs. (7.7) and (7.8) for τ_{xz}, τ_{yz}, and σ_z.

7.7 Show that Eqs. (7.12) and (7.13) are correct expressions of Hooke's law for a plate.

7.8 A square plate is acted on by uniform bending moments M along all four edges and these produce internal moments $M_x = M$, $M_y = M$, $M_{xy} = 0$. What is the resultant deflection w? Show that the plate is bent to a spherical surface.

7.9 A square plate is acted on by a uniform twisting moment M along all four edges and this produces internal moments $M_x = 0$, $M_y = 0$, $M_{xy} = M$. Compute the displacement w.

7.10 A square plate has moments $M_x = M$, $M_y = -M$, $M_{xy} = 0$. Compute the displacement w. Show that, so far as internal moments and deflections are concerned, Probs. 7.9 and 7.10 are the same.

7.11 Show that if $M = (M_x + M_y)/(1 + \nu)$ the expressions for shear forces can be written

$$Q_x = \frac{\partial M}{\partial x} \qquad Q_y = \frac{\partial M}{\partial y}$$

show that the plate problem is represented by the equations

$$\nabla^2 M = -q \qquad \nabla^2 w = -M/D$$

7.12 What external action will deflect a square plate so that

$$w = -\frac{M}{2D(1 + \nu)}(x^2 + y^2)$$

7.13 What external action will deflect a square plate so that

$$w = \frac{M_1 - \nu M_2}{2D(1 - \nu^2)} x^2 + \frac{M_2 - \nu M_1}{2D(1 - \nu^2)} y^2$$

7.14 What external action will deflect a square plate so that

$$w = \frac{M}{2D(1 - \nu)}(x^2 - y^2)$$

7.15 Beginning with Eqs. (7.15) derive the equation $\nabla^4 w = q/D$.

7.16 An elliptical plate has zero displacement along its edge and is deflected so that

$$w = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2$$

How is the plate loaded and how is it supported along its edge?

7.17 Three points equally spaced along a straight line are acted upon respectively by twisting moments M_1, M_2, M_3. Show that these moments are statically equivalent to three forces acting perpendicular to the line at the three points.

7.18 Show that as a consequence of Prob. 7.17, a uniform distribution of twisting moments $M_{xy} = M$ around the four edges of a square plate is statically equivalent to four concentrated forces acting at the corners of the plate. Show that this is equivalent to the plate loaded as shown in Fig. 6.9b.

7.19 A simply supported, square plate has a load

$$q = q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

Determine the displacement and the total reactions along the edges and at the corners. How do you think these reactions compare with those measured on an actual plate?

7.20 Show that the sum of all the edge reactions in Prob. 7.19 is equal to the applied load.

7.21 A simply supported rectangular plate carries a uniform load q. Show that the resulting displacement is

$$w = \frac{16q}{\pi^6 D} \sum_{m=1,2,3}^{\infty} \sum_{n=1,2,3}^{\infty} \frac{1}{mn(m^2/a^2 + n^2/b^2)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{7.26}$$

7.22 Show that the strain energy of bending and twisting in a plate is given by

$$V = \tfrac{1}{2} D \int\int \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1 - \nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2\right\}\right] dx\, dy \tag{7.27}$$

7.23 Show that the following expression represents the displacement of a simply supported rectangular plate with a concentrated force P applied at the point $x = x_1$, $y = y_1$

$$w = \frac{4P}{\pi^2 abD} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \dfrac{m\pi x_1}{a} \sin \dfrac{n\pi y_1}{b} \sin \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b}}{\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2} \tag{7.28}$$

7.24 Applying the method of Green's function, integrate Eq. (7.28) to deduce Eq. (7.16) for a simply supported plate with a uniform load q.

7.25 A circular plate of radius R has displacement

$$w = \frac{q}{64D}\left(\frac{5 + \nu}{1 + \nu} R^2 - r^2\right)(R^2 - r^2) \tag{7.29}$$

Show that the plate is simply supported and carries a uniform load q.

7.26 A circular plate of radius R has displacement

$$w = \frac{P}{16\pi D}\left[\frac{3 + \nu}{1 + \nu}(R^2 - r^2) + 2r^2 \ln \frac{r}{R}\right] \tag{7.30}$$

Show that the plate is simply supported and carries a concentrated force P at the center.

7.27 A circular plate of radius R has displacement

$$w = \frac{P}{16\pi D}\left(R^2 - r^2 + 2r^2 \ln \frac{r}{R}\right) \tag{7.31}$$

Show that the plate is built-in along the edge and carries a concentrated force P at the center.

7.28 Prove that at a corner of a polygonal, simply supported plate $M_{xy} = 0$ unless the corner is 90°. Note that from Eqs. (7.4) and (7.13) it can be concluded that χ_x, χ_y, χ_{xy} transform in the same manner as the strains ϵ_x, ϵ_y, ϵ_{xy}, that is, according to Mohr's circle.

7-7 THE LOAD CARRYING ACTION OF A SHELL

A shell is a thin curved plate that resists loads primarily by forces parallel to the middle surface. Examples of shells are a dome, the fuselage of an airplane, the exterior of a rocket, an eggshell, etc. All of these structures have thin walls that are curved. It is essential that a shell be built with curvature because it is this feature that enables it to act as a shell rather than as a plate.

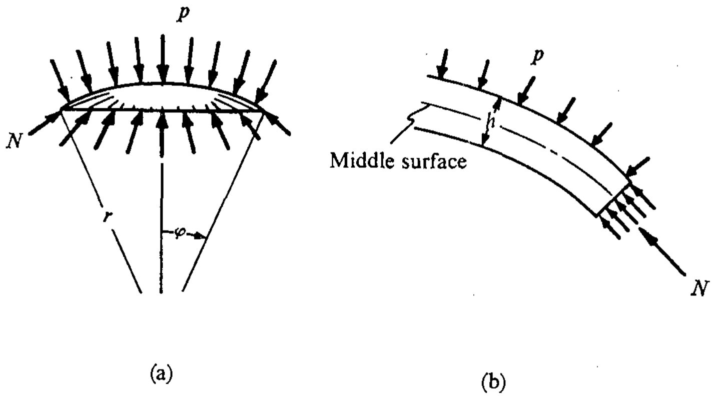


FIGURE 7.12

The normal loads applied to a flat plate are supported by bending action with negligible extension of the middle surface of the plate. On the other hand, when normal loads are applied to a shell the action is essentially the reverse of the plate behavior. In this case the extensions and contractions of the middle surface are not negligible and the stresses produced by these strains carry the applied load, whereas the bending stresses are small.

The manner in which the middle-surface forces carry the applied load is made clear by considering a thin spherical shell of radius r and thickness h, subjected to a uniform external pressure p, as shown in Fig. 7.12. A free-body diagram of a portion of the shell is shown in Fig. 7.12b which shows the pressure p being held in equilibrium by the middle-surface forces N per unit length. The total upward force exerted by the forces N must be equal to the total downward force exerted by p

$$(N \sin \varphi)(2\pi r \sin \varphi) = \pi(r \sin \varphi)^2 p$$

or

$$N = \frac{pr}{2}$$

It is seen that this result is independent of φ so this value of N will hold in equilibrium a very small (infinitesimal) piece of shell as well as a large piece.

The necessity for a true shell to have a thin wall is made clear by the following analysis. The in-plane force N of the shell shown in Fig. 7.12 corresponds to a uniform biaxial stress

$$\sigma = -\frac{N}{h} = -\frac{pr}{2h}$$

This stress produces a compressive strain*

$$\epsilon = \frac{\sigma}{E} - \nu\frac{\sigma}{E} = -\frac{pr}{2hE}(1 - \nu)$$

The strain ϵ shortens the radius of the shell to a length r_1, where

$$r_1 = r\left[1 - \frac{pr}{2hE}(1 - \nu)\right]$$

The corresponding change in curvature is

$$\Delta\chi = \left(\frac{1}{r_1} - \frac{1}{r}\right) = \frac{1}{r}\left[\frac{1}{1 - \dfrac{(1 - \nu)pr}{2hE}} - 1\right]$$

$$= \frac{(1 - \nu)p}{2hE}\left[1 + \frac{(1 - \nu)pr}{2hE} + \cdots\right]$$

For small strains we may write $\Delta\chi = [(1 - \nu)p]/2hE$ and, hence, using the expression for bending moment in terms of curvatures from plate theory, the bending moment is given by

$$M = -D(\Delta\chi_1 + \nu\Delta\chi_2) = -D(1 - \nu^2)\frac{p}{2hE} = -\frac{ph^2}{24}$$

The maximum bending stress produced by this moment is

$$\sigma_b = -\frac{M(h/2)}{I} = \frac{ph^2}{24}\cdot\frac{h}{2}\cdot\frac{12}{h^3} = \frac{p}{4}$$

The ratio of this stress to the in-plane stress is

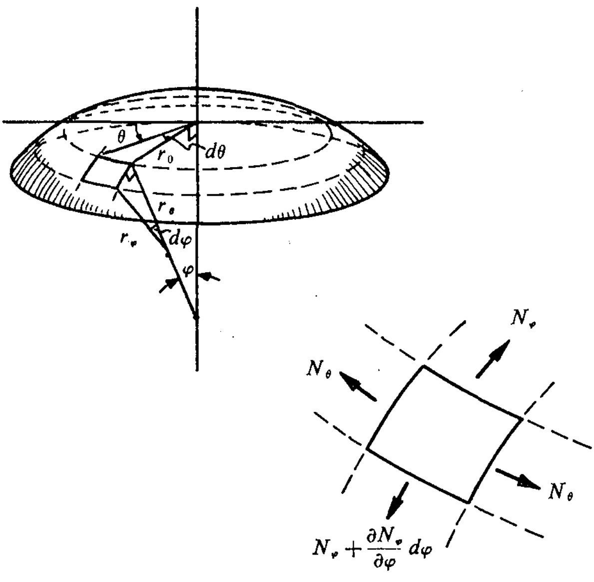
$$\frac{\sigma_b}{\sigma} = \frac{p}{4} \div \frac{pr}{2h} = \frac{h}{2r}$$

We thus see that the condition under which the bending stress is very small compared to the in-plane stress is that the thickness h is very small compared

* The stress normal to the middle surface is very small compared to σ so its effect on ϵ is neglected.

to the radius of curvature. The in-plane stresses are called *membrane stresses* as opposed to bending stresses. The membrane theory of shells deals with cases where the bending stresses are negligibly small.

The foregoing example was especially simple. If the external pressure p had not been uniform over the surface of the shell the inplane normal forces N would have varied throughout the shell and there would also have been in-plane shearing forces. It is also clear that if a concentrated force were applied to the shell it would cause a local bending around the point of application and in this region the bending moments could not be neglected in the analysis. Also, if the shell were restrained at certain points in such a way as to inhibit its natural deformation some concentrated reactions would be set up which would produce bending stresses that might not be small. We shall consider only shells and loadings for which the bending stresses are negligibly small. Such shells are called *elastic membranes.*



(a) (b)

FIGURE 7.13

Symmetrically Loaded Shells of Revolution. Shells of revolution that are loaded symmetrically about the axis of revolution are particularly simple to analyze. Such a shell of revolution is shown in Fig. 7.13 where it is seen that a point on the shell is located by the coordinates θ, φ, r_0. These rather special coordinates are used to describe the shape of the shell because they make *the equilibrium equations for the membrane forces particularly simple.* The curvatures of the shell are specified by r_θ which is the radius of curvature defined by two perpendicular lines passed through the two upper corners of the element, and r_φ which is the radius of curvature defined by two perpendicular lines passed through the two side corners of the element. The element is cut out by two meridional planes making an angle $d\theta$ with each other and two parallel planes that are perpendicular to the axis. The side edge of the element has a length $r_\varphi\, d\varphi$. In (b) are shown the in-plane forces N_φ and N_θ which are normal to the edge faces of the element. Because of symmetry there are no shearing forces acting on the element, and N_φ and N_θ do not vary with θ. The shell is sufficiently thin so that σ_φ and σ_θ can be taken to be constant across the thickness of the shell and hence

$$\begin{aligned} N_\varphi &= \sigma_\varphi h \\ N_\theta &= \sigma_\theta h \end{aligned} \tag{7.32}$$

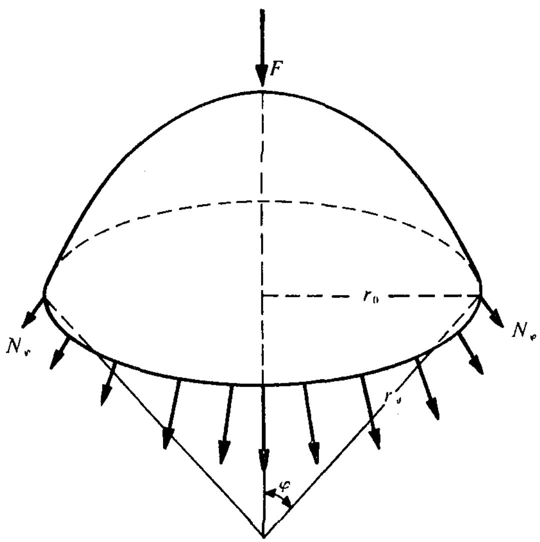


FIGURE 7.14

In the case of a shell of revolution with an axially-symmetric load the force N_φ can be determined directly from equilibrium of the shell. In Fig. 7.14 is shown the freebody of that part of the shell which is cut off by the angle φ. The force F represents the resultant of all the applied loads acting above the angle φ. The N_φ are shown as positive and from symmetry N_φ is a constant around the shell. Equilibrium requires that the total vertical component of N_φ be equal to F

$$2\pi r_0(N_\varphi \sin \varphi) + F = 0$$

or

$$N_\varphi = -\frac{F}{2\pi r_0 \sin \varphi} \tag{7.33}$$

The force N_θ must be determined from the equilibrium of an element of the shell. In Fig. 7.15 are shown three views of the element. The applied load is represented by its two perpendicular components p_r, p_φ. Equilibrium in a

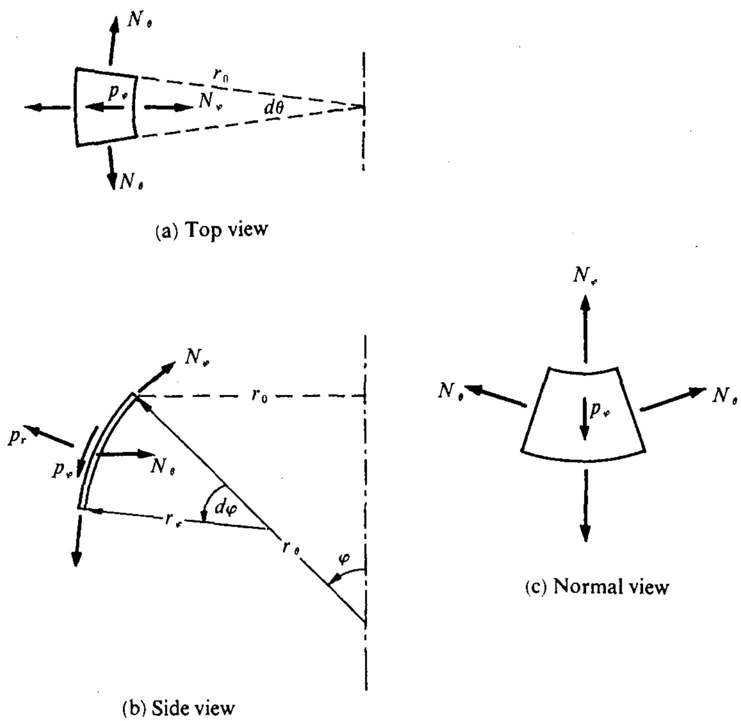


FIGURE 7.15

direction normal to the shell involves p_r, N_φ and N_θ, as the action of p_r is resisted by the components of N_φ and N_θ normal to the shell. The total normal force is the product of p_r by the area of the element on which it acts

$$p_r(r_0\, d\theta) r_\varphi\, d\varphi$$

The total force on the top edge of the element is $N_\varphi r_0\, d\theta$, and to within a higher order infinitesimal difference the force on the bottom edge is the same.

These two forces have a resultant normal to the shell equal to

$$(N_\varphi r_0\, d\theta)\, d\varphi$$

On each side of the element there is a force $N_\theta r_\varphi\, d\varphi$. Both of these lie in a horizontal plane and make an angle $d\theta$ with each other. Their resultant is a force $N_\theta r_\varphi\, d\varphi\, d\theta$ which lies in a horizontal plane and points toward the axis. It has a component normal to the shell given by

$$N_\theta r_\varphi\, d\varphi\, d\theta \sin \varphi$$

Equating the sum of all the normal forces to zero and canceling $d\theta\, d\varphi$ gives

$$-p_r r_0 r_\varphi + N_\varphi r_0 + N_\theta r_\varphi \sin \varphi = 0$$

Dividing through by $r_0 r_\varphi$ gives

$$\frac{N_\varphi}{r_\varphi} + \frac{N_\theta \sin \varphi}{r_0} = p_r$$

Recalling that $r_0 = r_\theta \sin \varphi$, we may write the conditions of equilibrium of the shell

$$\frac{N_\varphi}{r_\varphi} + \frac{N_\theta}{r_\theta} = p_r$$

$$N_\varphi = -\frac{F}{2\pi r_0 \sin \varphi} \tag{7.34}$$

Equations (7.34) determine the membrane stresses in a shell of revolution with axially symmetrical loading. Unlike the plane stress problem of Chapter 2, a compatibility equation is not required because the membrane is free to move normal to its surface and the strains associated with these motions will be such as to insure that they are consistent with the stresses.

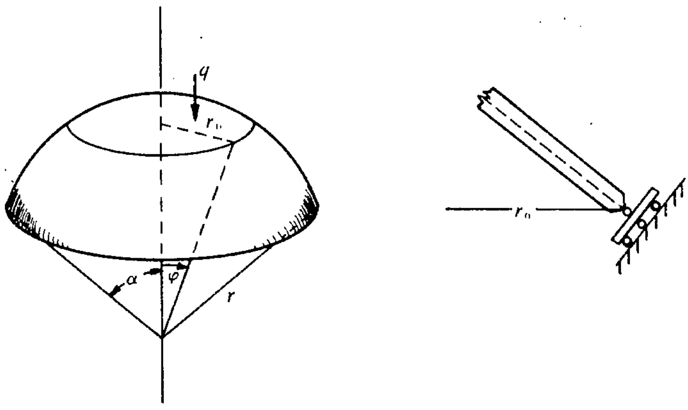
A Spherical Dome Under Its Own Weight. The spherical shell shown in Fig. 7.16 has a uniform weight q per unit area and hence the total weight above φ is

$$F = 2\pi r \int_0^\varphi (q \sin \varphi) r\, d\varphi = 2\pi r^2 (1 - \cos \varphi) q$$

Equation (7.34) then give

$$N_\varphi = -qr \frac{1}{1 + \cos \varphi}$$

$$N_\theta = qr \left(\frac{1}{1 + \cos \varphi} - \cos \varphi \right) \tag{7.35}$$



(a) (b)

FIGURE 7.16

It is seen that the circumferential force N_θ is compressive when φ is small and is tensile when φ is large, the cross-over point being when N_θ is equal to zero, that is, when

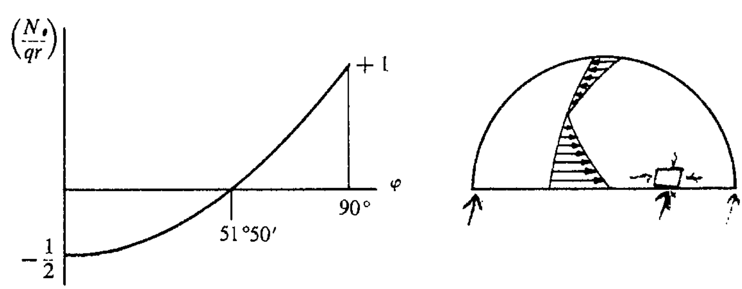
$$\frac{1}{1 + \cos\varphi} - \cos\varphi = 0$$

or

$$\varphi = 51°50'$$

This is shown in Fig. 7.17.

At the edge of the shell ($\varphi = \alpha$) the membrane solution (7.35) gives N_φ as the only force acting. Strictly, the only way to ensure this is to have the shell



(a) (b)

FIGURE 7.17

supported as shown in Fig. 7.16b so that the edge is free to move as the shell deforms under the stresses. For example, the circumferential strain in the shell is

$$\epsilon_\theta = \frac{\sigma_\theta}{E} - \nu \frac{\sigma_\varphi}{E} = \frac{N_\theta}{Eh} - \nu \frac{N_\varphi}{Eh}$$

and this produces a change in radius

$$\Delta r_0 = r_0 \epsilon_\theta \tag{7.36}$$

Customarily the edge of the shell is built into a support which inhibits any Δr_0 at this point. As a consequence the support exerts some forces on the shell which produce bending in the vicinity of the edge. A more elaborate analysis is required to evaluate the effect of these non-membrane forces.* For example, see Section 3-11 for such an analysis.

7-8 CYLINDRICAL SHELL

The simply supported cylindrical shell, shown in Fig. 7.18a, is a special type of shell in that it has only a single curvature. To apply the membrane theory of shells we assume that the end support is such that it can resist the

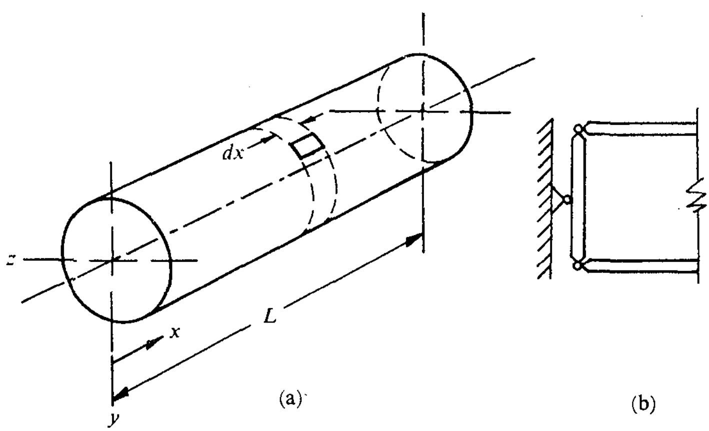


FIGURE 7.18

* Many special examples of shells are discussed in the book *Stresses in Shells*, by W. Flügge, Springer (1960).

membrane forces that are exerted by the shell but will not develop any non-membrane forces. As shown in (b) there are two types of end conditions; the structure as a whole is simply supported (no beam bending moment at the supports); and in addition the edge of the shell is simply supported by a rigid diaphragm that can develop tangential shearing forces acting on the shell but cannot develop any transverse shearing force or plate bending moment.

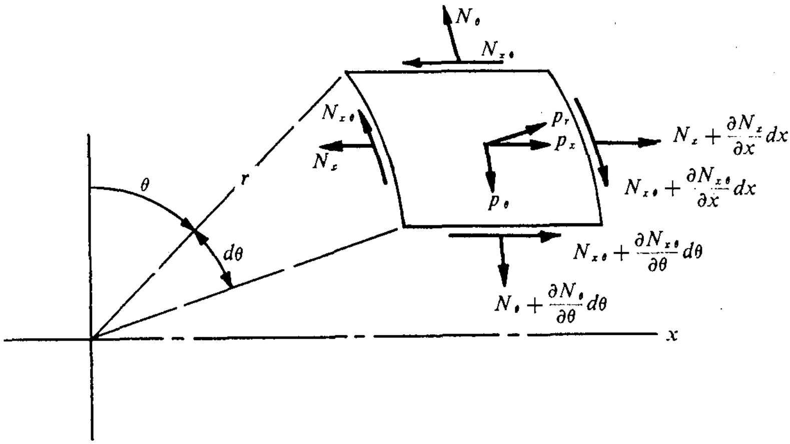


FIGURE 7.19

The freebody diagram of an element of the shell is shown in Fig. 7.19. The shearing force per unit length $N_{x\theta}$ is the integral of the shearing stress $\tau_{x\theta}$ over the thickness of the shell. Equilibrium of the shell in the x- and θ-directions requires

$$\frac{\partial N_x}{\partial x} dx(rd\theta) + \frac{\partial N_{x\theta}}{\partial \theta} d\theta\, dx + p_x\, dx\, rd\theta = 0$$

$$\frac{\partial N_\theta}{\partial \theta} d\theta\, dx + \frac{\partial N_{x\theta}}{\partial x} dx(rd\theta) + p_\theta\, dx\, rd\theta = 0$$

In the r-direction normal to the shell, equilibrium requires

$$N_\theta\, dx\, d\theta - p_r\, dx\, rd\theta = 0$$

Dividing out the infinitesimals these equations can be put in the following forms:

$$
\begin{aligned}
N_\theta &= rp_r \\
\frac{\partial N_\theta}{r\,\partial\theta} + \frac{\partial N_{x\theta}}{\partial x} &= -p_\theta \\
\frac{\partial N_x}{\partial x} + \frac{\partial N_{x\theta}}{r\,\partial\theta} &= -p_x
\end{aligned}
\tag{7.37}
$$

The first of these equations determines N_θ and, knowing this, the second equation can be used to determine $N_{x\theta}$. The third equation will then determine N_x. For example,

$$
\begin{aligned}
N_{x\theta} &= -\int\left(p_\theta + \frac{\partial N_\theta}{r\,\partial\theta}\right)dx + f_1(\theta) \\
N_x &= -\int\left(p_x + \frac{\partial N_{x\theta}}{r\,\partial\theta}\right)dx + f_2(\theta)
\end{aligned}
\tag{7.38}
$$

The $f_1(\theta)$ and $f_2(\theta)$ are arbitrary functions of integration which appear because partial derivatives were integrated. The significance of these functions is that they may be used to satisfy the boundary conditions, for example, $f_1(\theta)$ can be adjusted to give any desired distribution of $N_{x\theta}$ around the edge $x = 0$. This is made clear by expanding f_1 and f_2 in Fourier series

$$
\begin{aligned}
f_1(\theta) &= \sum_{n=1}^{\infty} A_n \sin n\theta + \sum_{n=0}^{\infty} B_n \cos n\theta \\
f_2(\theta) &= \sum_{n=1}^{\infty} C_n \sin n\theta + \sum_{n=0}^{\infty} D_n \cos n\theta
\end{aligned}
\tag{7.39}
$$

In this form the coefficients A_n, B_n, C_n, D_n are arbitrary constants of integration. The coefficient B_0, for example, represents a uniform shearing force $N_{x\theta} = B_0$ acting at each end of the shell, which means that a torque $2\pi r^2 B_0$ is applied at each end of the shell.

An interesting solution is that for the simply supported cylindrical shell of Fig. 7.18 full of water in which case $p_r = p_0 - \gamma r \cos\theta$, where p_0 is the fluid pressure at mid-height of the pipe and γ is the weight of the fluid per unit volume. In this case

$$
\begin{aligned}
N_\theta &= r(p_0 - \gamma r \cos\theta) \\
N_{x\theta} &= \gamma r\left(\frac{L}{2} - x\right)\sin\theta \\
N_x &= -\frac{\gamma}{2}\,x(L - x)\cos\theta
\end{aligned}
\tag{7.40}
$$

As may be verified, these values of N_x and $N_{x\theta}$ are the same as those given by applying the technical theory of bending ($\sigma_x = -My/I_z$). A pressure which does not vary with x can be expressed in a Fourier series

$$p_r = \sum_n p_n \cos n\theta + \sum_n f_n \sin n\theta$$

It is found that pressure terms with $n = 2$ or greater force the shell out of round. For example, if the shell is only partly full of water the fluid pressure tends to deform the shell out of round and M_θ bending moments are required to resist this. For a very thin shell the resulting stress distribution differs from that for the shell completely full of water as shown in Fig. 7.20.* Of course, if the wall of the shell is sufficiently thick so that it does not tend to go out-of-round the technical theory of bending can be used to determine N_x and $N_{x\theta}$ even if the shell is only partly full.

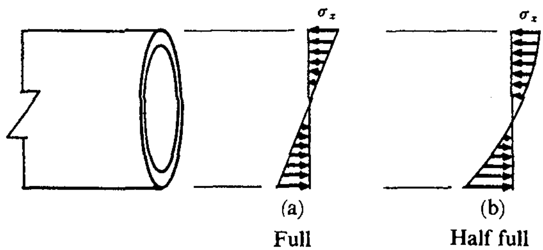


FIGURE 7.20

In most practical problems the loading and boundary conditions will be such that some shell bending will be involved and the membrane solution will not be entirely adequate. In these cases the general theory of cylindrical shells must be used.

It can be seen that the general theory of cylindrical shells will involve the in-plane forces N_x, N_θ, $N_{x\theta}$, and will also involve the plate-type moments, M_x, M_θ, $M_{x\theta}$ as well as Q_x and Q_θ. The large deflection of plates involves similar forces and moments. It should also be noted that for very thin shells the question of buckling may become important.

Problems

7.29 Prove that the shell shown in Fig. 7.12 has only normal in-plane forces N.

7.30 Show that the equation for r_1 in Section 7-7 is correct and that $\Delta\chi = [(1 - \nu)p]/2hE$ is a good approximation for $\Delta\chi$.

* Flügge, *loc. cit.*, p. 263.

7.31 A paraboloid dome is described by the equation $z = 2ar_0^2$ and is loaded by a uniform vertical load of q pounds per square foot of horizontal projection. Deduce that:

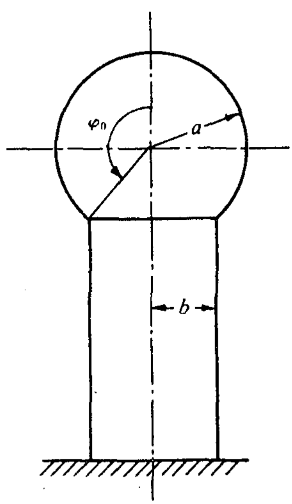
$$N_\varphi = -\frac{q}{8a\cos\varphi}$$
$$N_\theta = -\frac{q\cos\varphi}{8a} \tag{7.41}$$

7.32 If a spherical dome is subjected to wind pressure, p_r will not be axially symmetrical but will vary over the shell. In this case in-plane shearing forces $N_{\varphi\theta}$ will be developed and N_φ will vary with θ. Draw a freebody diagram of an element of the shell and derive the three equations of equilibrium.

7.33 Derive Eqs. (7.35) for the spherical dome loaded by its own weight.

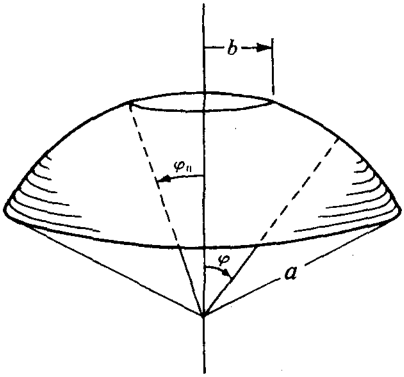
7.34 A water tank formed of a spherical shell supported on a cylindrical pipe is completely full of water. What conditions must be satisfied at the juncture of sphere and cylinder to make the spherical shell behave as a membrane? Show that the membrane stresses are given by

$$N_\varphi = \frac{\gamma a^2}{6}\left(1 - \frac{\cos^2\varphi}{1+\cos\varphi}\right)$$
$$N_\theta = \frac{\gamma a^2}{6}\left(5 - 6\cos\varphi + \frac{2\cos^2\varphi}{1+\cos\varphi}\right) \tag{7.42}$$



7.35 A spherical dome of uniform wall thickness has a hole of radius b as shown. Show that the stresses produced in the shell by its own weight q are:

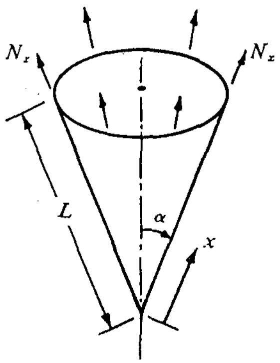
$$N_\varphi = -aq\left(\frac{\cos\varphi_0 - \cos\varphi}{\sin^2\varphi}\right)$$
$$N_\theta = aq\left(\frac{\cos\varphi_0 - \cos\varphi}{\sin^2\varphi} - \cos\varphi\right) \tag{7.43}$$



7.36 Derive Eqs. (7.40) for the cylindrical shell full of water.

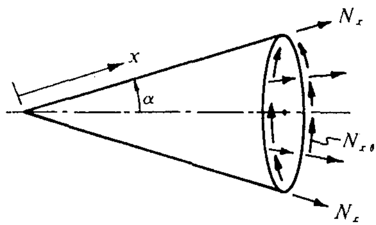
7.37 Show that Eqs. (7.40) agree with the solution of the technical theory of bending.

7.38 A vertical, conical shell (right circular cone) is full of water. Derive expressions for N_θ and N_x.



7.39 A right circular conical shell has its axis horizontal and is full of water. Derive the differential equations of equilibrium:

$$\frac{\partial}{\partial x}(xN_x) + \frac{1}{\sin\alpha}\frac{\partial N_{x\theta}}{\partial\theta} - N_\theta = 0$$
$$\frac{\partial N_\theta}{\partial\theta} + \sin\alpha\frac{\partial}{\partial x}(xN_{x\theta}) + N_{x\theta}\sin\alpha = 0 \tag{7.44}$$
$$\frac{N_\theta}{x\tan\alpha} = p_0 - (\gamma\sin\alpha)x\cos\theta$$

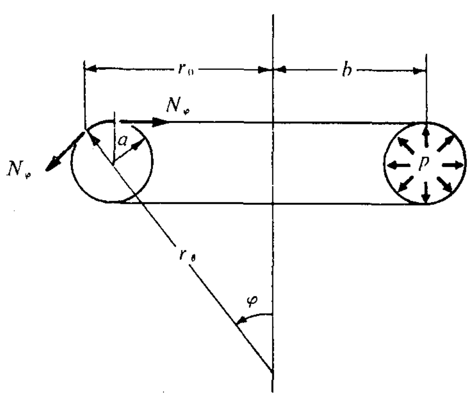


7.40 Compute the forces N_x, N_θ, and $N_{x\theta}$ in Prob. 7.39.

7.41 What is the physical significance of the A_1, C_1, and D_0 terms in Eq. (7.39)?

7.42 The toroidal shell shown in the diagram is subjected to an internal pressure p. By drawing an appropriate freebody diagram, derive the following expressions for N_φ and N_θ:

$$N_\varphi = \frac{pa(r_0 + b)}{2r_0} \qquad N_\theta = \frac{pa}{2} \tag{7.45}$$



7.43 For an axially symmetric shell with arbitrary loading, derive the following three equations of equilibrium of an element of the shell:

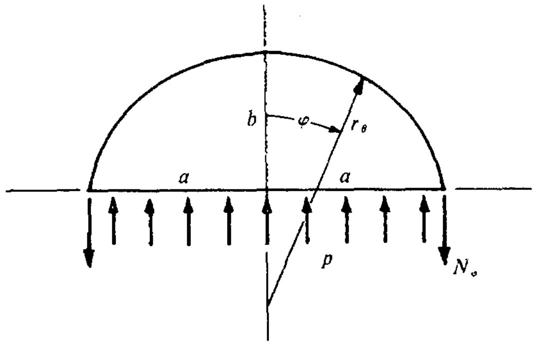
$$\begin{aligned} \frac{\partial(r_0 N_\varphi)}{\partial\varphi} + r_\varphi \frac{\partial N_{\theta\varphi}}{\partial\theta} - r_\varphi N_\theta \cos\varphi + p_\varphi r_\varphi r_0 &= 0 \\ r_\varphi \frac{\partial N_\theta}{\partial\theta} + \frac{\partial(r_0 N_{\theta\varphi})}{\partial\varphi} + r_\varphi N_{\theta\varphi} \cos\varphi + p_\theta r_0 r_\varphi &= 0 \\ \frac{N_\varphi}{r_\varphi} + \frac{N_\theta}{r_\theta} &= p_r \end{aligned} \tag{7.46}$$

7.44 Show that a shell in the form of an ellipsoid of revolution under internal pressure p, with semi-major axis b vertical, has stresses:

$$N_\varphi = \frac{pr_\theta}{2}, \quad N_\theta = pr_\theta\left(1 - \frac{r_\theta}{2r_\varphi}\right)$$

where

$$r_\varphi = \frac{a^2b^2}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} \qquad r_\theta = \frac{a^2}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/2}} \tag{7.47}$$



CHAPTER 8

APPLICATION TO VISCOUS AND PLASTIC BEHAVIOR OF MATERIALS

8-1 DEVIATIONS FROM LINEAR ELASTIC BEHAVIOR

Most solids are, for practical purposes, linearly elastic for sufficiently small stresses and strains but deviations from linearly elastic behavior will occur at sufficiently high stresses. Because of this, it is of practical importance to be able to make stress analyses which take into account inelastic behavior. The deviations from linearly elastic behavior are strongly dependent on the structure of the material; for example, the stress-strain behavior of rubber differs markedly from that of a crystalline solid, and the difference is largely due to the difference in atomic structure.

During elastic straining of a *crystalline* solid, the distances between atoms are slightly altered in a homogeneous manner, and this process is completely reversible. The elastic constants are a measure of the way the atomic forces vary with the distance between atoms. The relationship between atomic force and interatomic distance is fundamentally non-linear but the non-linearity is too small to measure within the range of elastic behavior of most crystals. In the case of single crystal metal "whiskers," which exhibit elastic behavior for strains of several per cent, small non-linearities can be observed in the tensile stress-strain curve.

The non-linear elastic deformation of rubber does not arise primarily from the non-linearity of the atomic force-distance relationship. The structure of rubber and other polymers consists of separate long chain molecules made up of atom-to-atom bonds with occasional cross links to other chains, the chains of molecules making up an irregular tangle. When the polymer is strained in tension, the chains tend to straighten and align themselves with the tensile force. This process gives rise to measurable non-linearity in the elastic stress-strain curve. The relatively strong atomic bonds along the molecular chains do not appreciably influence the stress-strain behavior until such high strains are reached that the chains are nearly straight.

The non-linear stress-strain behavior of materials at stresses which exceed the *elastic limit* is due to nonhomogeneous deformation on an atomic scale which involves rearrangement of atomic bonds or relative slippage of molecules. This rearrangement results in viscous or plastic behavior, or both. It may be reversible but time dependent, in which case it leads to viscoelastic behavior.

A linear polymer, such as latex, behaves as a viscous fluid. Since the long, convoluted molecules have no cross-links, a sustained shear stress will cause the molecules to slide over one another. When the molecular chains are very long, as in silicone putty, the material can show transient elastic behavior even with no cross links, due to straightening and crumpling of the long chains. The degree of elastic behavior is increased by cross links between the molecular chains. For example, vulcanization of rubber, which can be controlled to produce various densities of cross links, can produce a rubber which behaves as a viscous fluid or as a relatively hard, elastic material. The viscous behavior of polymers is very sensitive to temperature, as the sliding of molecular segments past one another is aided by thermal energy.

Plastic deformation in a crystalline material, such as a metal, is produced by the slipping of atoms past each other which results in the progressive growth of slipped areas on close-packed atomic planes. Such a slipped area is illustrated in Fig. 8.1a, where the material above the area bounded by *A-m-n* has slipped a distance b relative to the material below. Under the influence of a shear stress of appropriate magnitude, the boundary line *m-n* between the slipped and unslipped portions of *ABCD* will sweep across the plane resulting in a displacement, or slip b, over the entire plane as shown in Fig. 8.1b. The smallest incremental distance b is generally that between adjacent atoms on the slip plane *ABCD* in which case the regular pattern of atomic positions is not disrupted after the line *m-n* has passed over the slip plane. The line *m-n* is called a dislocation line, and it represents a linear defect in the atomic structure of the crystal. Near the dislocation line, the regular atomic arrangement is subjected to large distortions and, hence, a relatively large strain energy (approximately 5 ev/atom distance along the line) is associated with a dislocation. Most crystals when formed contain many dislocations (10^5 to 10^7 in. of dislocation line per cubic inch), and the dislocation density is observed to increase markedly with plastic strain. Approximately 5% of the work done in deforming a crystal by slip goes into the strain energy associated with newly formed dislocations, while the majority of the work goes into heating the material. The dislocation structure of crystalline solids may be changed by heat treatment and neutron irradiation and, therefore, such treatments can strongly influence the plastic behavior of the material. These treatments do not affect the forces between atoms so they do not affect the elastic constants. Dislocations can react strongly with impurity atoms and with other dislocations, and these reactions strongly influence both the onset of plastic flow and the stresses required to deform the material by various amounts.*

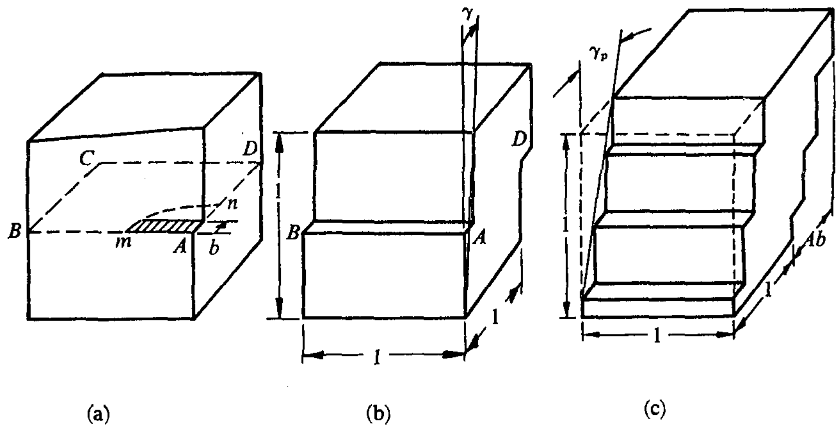


FIGURE 8.1

The plastic strain in a crystal is proportional to the number of dislocations which have moved and to the distance they moved, as may be deduced from Fig. 8.1. The plastic shear strain of a unit cube of crystal shown in Fig. 8.1b is equal to b when one dislocation has passed through the crystal. When a number of dislocations have moved in a crystal as illustrated in Fig. 8.1c, the plastic shear strain is

$$\gamma_p = Ab \tag{8.1}$$

where A is the slip plane area per unit volume swept out by the dislocations. The rate at which plastic shear strain takes place in the crystal is given by

$$\dot{\gamma}_p = b\,\frac{dA}{dt} = lbc_d \tag{8.2}$$

* For an analysis of dislocations and their influence on the stress-strain behavior of crystals, see Weertman, *Elementary Dislocation Theory*, MacMillan (1964)

where l is the length per unit volume of moving dislocations, and c_d is the dislocation velocity. The length per unit volume of moving dislocations increases with τ and γ_p, so we may write

$$l = F_1\left(\tau, \int d\gamma_p\right)$$

where the integral is taken over the entire strain history of the crystal. The dislocation velocity is a sensitive function of the shear stress, and is also a function of the strain history

$$c_d = F_2\left(\tau, \int d\gamma_p\right)$$

Equation (8.2) may therefore be written as

$$\dot{\gamma}_p = bF_1 F_2 \tag{8.3}$$

Equation (8.3) expresses the fundamental stress-plastic strain-time relationship of a crystal with one active set of slip planes. A fundamental description of the plastic stress-strain-time behavior therefore requires a knowledge of the number of dislocations which are moving at any time and the stresses required to move dislocations at different velocities. This information is available for only a very limited number of materials.

A detailed description of the inelastic stress-strain-time behavior of polymers and crystals is seen to be very complicated when the interaction of the atoms is specifically taken into account, and in most cases such descriptions are not available.* Because of this it is customary to approximate the stress-strain-time behavior of different materials by certain relationships which fit the observed behavior reasonably well. These approximations can be used only over the limited range of conditions for which the actual material behavior is known to agree with the approximate relations. The approximations discussed below consider several stress-strain-time relationships which are relatively simple to express mathematically.

8-2 SIMPLIFIED STRESS-STRAIN RELATIONS

In a discussion of material properties it is convenient to write the dilatational stress-strain relations and the distortional stress-strain relations separately since these two stress states often have markedly different properties. For

* For an analysis of the way the structure affects the behavior of materials see A. H. Cottrell, *Mechanical Properties of Matter*, Wiley (1964).

example, plastic and viscous properties are usually related to the distortional stresses and strains, whereas dilatation is usually elastic.

A general state of stress and strain can be separated into dilatational and distortional components, as shown by Eq. (1.37).

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} = \begin{vmatrix} \bar\sigma & 0 & 0 \\ 0 & \bar\sigma & 0 \\ 0 & 0 & \bar\sigma \end{vmatrix} + \begin{vmatrix} \sigma'_x & \tau'_{xy} & \tau'_{xz} \\ \tau'_{xy} & \sigma'_y & \tau'_{yz} \\ \tau'_{xz} & \tau'_{yz} & \sigma'_z \end{vmatrix}$$

where $\bar\sigma = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$ and $\sigma'_x = \sigma_x - \bar\sigma$; $\tau'_{xy} = \tau_{xy}$; etc.

$$\begin{vmatrix} \epsilon_x & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_y & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_z \end{vmatrix} = \begin{vmatrix} \bar\epsilon & 0 & 0 \\ 0 & \bar\epsilon & 0 \\ 0 & 0 & \bar\epsilon \end{vmatrix} + \begin{vmatrix} \epsilon'_x & \epsilon'_{xy} & \epsilon'_{xz} \\ \epsilon'_{xy} & \epsilon'_y & \epsilon'_{yz} \\ \epsilon'_{xz} & \epsilon'_{yz} & \epsilon'_z \end{vmatrix}$$

where $\bar\epsilon = \frac{1}{3}(\epsilon_x + \epsilon_y + \epsilon_z)$ and $\epsilon'_x = \epsilon_x - \bar\epsilon$; $\epsilon'_{xy} = \epsilon_{xy} = \frac{1}{2}\gamma_{xy}$, etc. The distortional stresses and strains are sometimes called deviator stresses and strains as they represent the deviation from a pure dilatation.

Linearly Elastic Material. The properties of an isotropic linearly elastic material are particularly simple. The stress-strain relations, Eq. (1.38), can be written:

$$\begin{gathered} \bar\sigma = C\bar\epsilon \\ \sigma'_x = 2G\epsilon'_x \qquad \sigma'_y = 2G\epsilon'_y \qquad \sigma'_z = 2G\epsilon'_z \\ \tau'_{xy} = 2G\epsilon'_{xy} \qquad \tau'_{yz} = 2G\epsilon'_{yz} \qquad \tau'_{zx} = 2G\epsilon'_{zx} \end{gathered} \tag{8.4}$$

where $\epsilon'_{xy} = \epsilon_{xy} = \frac{1}{2}\gamma_{xy}$, etc. All of the distortional stress-strain relations have the same form, so for convenience we shall write them as*

$$\tau'_{ij} = 2G\epsilon'_{ij} \tag{8.4a}$$

where it is understood that i may be x, y, or z and similarly for j, so that the normal stresses and strains are written τ'_{xx}, ϵ'_{xx}, etc., and the shearing stresses and strains are written τ'_{xy}, ϵ'_{xy}, etc.

It is found that for metals the compression modulus C is essentially a constant over the stress range of engineering interest and, therefore, the viscous and plastic deformations are associated only with the distortional stresses and strains.

* This system of notation is discussed in detail in Appendix I. Note that Eq. (8.4a) really represents nine stress-strain relations including $\tau'_{xy} = 2G\epsilon'_{xy}$ *and its counterpart* $\tau'_{yx} = 2G\epsilon'_{yx}$, etc.

In the following paragraphs we shall present some commonly used idealized stress-strain relations. Only the distortional stress-strain relations will be presented, it being understood that in each case the dilatational stress-strain relation is

$$\bar{\sigma} = \frac{E}{1 - 2\nu}\,\bar{\epsilon}$$

The Maxwell Solid. The properties of a linearly elastic material are analogous to those of a spring, as shown in Fig. 8.2a. A material that is analogous to a spring and dashpot in series, Fig. 8.2b, is called a Maxwell solid and its behavior is elasto-viscous since it is a viscous fluid with elastic

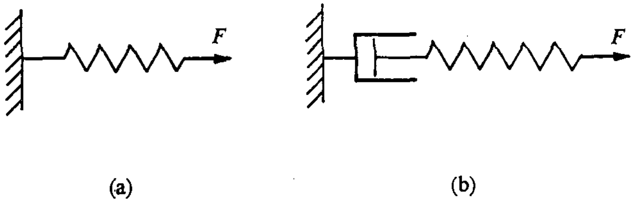


FIGURE 8.2

properties. The force in the dashpot is proportional to the relative velocity of its two parts. Since the dashpot has a velocity even for constant F it is the strain rate rather than the strain that is significant. The stress-strain relations* for this material are

$$\dot{\epsilon}'_{ij} = \frac{\tau'_{ij}}{2\mu} + \frac{\dot{\tau}'_{ij}}{2G} \tag{8.5}$$

where

$$\dot{\epsilon}'_{ij} = \frac{\partial \epsilon'_{ij}}{\partial t}, \quad \dot{\tau}'_{ij} = \frac{\partial \tau'_{ij}}{\partial t}$$

The strain rate $\dot{\epsilon}'_{ij}$ has a viscous component $\tau'_{ij}/2\mu$, and an elastic component $\dot{\tau}'_{ij}/2G$. A viscous fluid would have the relations

$$\dot{\epsilon}'_{ij} = \frac{\tau'_{ij}}{2\mu}$$

where μ is the coefficient of viscosity. A linearly elastic material would have the relations

$$\dot{\epsilon}'_{ij} = \frac{\dot{\tau}'_{ij}}{2G}$$

* The deformation is assumed to be sufficiently slow so that acceleration effects can be neglected.

If the stress-strain relations (8.5) are written in terms of the total stresses and strains and in terms of the ordinary notation they assume the form

$$\dot{\epsilon}_x = \frac{1}{E}(\dot{\sigma}_x - \nu\dot{\sigma}_y - \nu\dot{\sigma}_z) + \frac{1}{3\mu}(\sigma_x - \tfrac{1}{2}\sigma_y - \tfrac{1}{2}\sigma_z) \qquad \dot{\gamma}_{xy} = \frac{1}{G}\dot{\tau}_{xy} + \frac{1}{\mu}\tau_{xy}$$

$$\dot{\epsilon}_y = \frac{1}{E}(\dot{\sigma}_y - \nu\dot{\sigma}_z - \nu\dot{\sigma}_x) + \frac{1}{3\mu}(\sigma_y - \tfrac{1}{2}\sigma_z - \tfrac{1}{2}\sigma_x) \qquad \dot{\gamma}_{yz} = \frac{1}{G}\dot{\tau}_{yz} + \frac{1}{\mu}\tau_{yz}$$

$$\dot{\epsilon}_z = \frac{1}{E}(\dot{\sigma}_z - \nu\dot{\sigma}_x - \nu\dot{\sigma}_y) + \frac{1}{3\mu}(\sigma_z - \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_y) \qquad \dot{\gamma}_{xz} = \frac{1}{G}\dot{\tau}_{xz} + \frac{1}{\mu}\tau_{xz}$$

For a straight bar pulled in tension, the only nonzero stress is σ_x and, for this case, the equations reduce to

$$\dot{\epsilon}_x = \frac{\dot{\sigma}_x}{E} + \frac{\sigma_x}{3\mu}$$

$$\dot{\epsilon}_y = -\frac{\nu\dot{\sigma}_x}{E} - \frac{\sigma_x}{6\mu}$$

$$\dot{\epsilon}_z = -\frac{\nu\dot{\sigma}_x}{E} - \frac{\sigma_x}{6\mu}$$

The first of these equations can be written

$$\frac{\partial\sigma_x}{\partial t} + \frac{E}{3\mu}\sigma_x = E\dot{\epsilon}_x$$

Suppose that at time $t = 0$ the bar is given an initial strain $\epsilon_x = \epsilon_0$, and this strain is held constant, $\dot{\epsilon}_x = 0$ for $t > 0$. The differential equation for this problem is

$$\frac{\partial\sigma_x}{\partial t} + \frac{E}{3\mu}\sigma_x = 0$$

from which, writing t_e for $\frac{3\mu}{E}$, there is obtained

$$\sigma_x = ce^{-t/t_e}$$

At $t = 0$, $\sigma_x = E\epsilon_0$, so

$$\sigma_x = E\epsilon_0 e^{-t/t_e}$$

It is seen that the stress dies away exponentially (relaxes). The quantity $t_e = 3\mu/E$ has the dimensions of time and it is called the relaxation time for elongation. If a tube of this material is twisted in torsion there is a relaxation time for shear, $t_s = \mu/G$.

The Kelvin-Voight Solid. The stresses in a visco-elastic material may be functions of the strains, the strain rates, and the stress rates. A special case of practical importance is that of linear visco-elastic behavior in which the stresses are linear functions of the strains and the strain rates. One example of such behavior, which is analogous to that of a spring and dashpot in parallel as shown in Fig. 8.3a, is said to describe a Kelvin or Voight solid. The stress-strain relations of such a linear visco-elastic solid are

$$\tau'_{ij} = 2G\epsilon'_{ij} + 2\mu\dot{\epsilon}'_{ij} \tag{8.6}$$

where μ is the coefficient of viscosity and $\dot{\epsilon}'_{ij}$ is the strain rate. When careful measurements are made of stressed steel bars, for example, it is found that there is indeed a small amount of internal damping, a part of which can be described by the last term in Eq. (8.6).

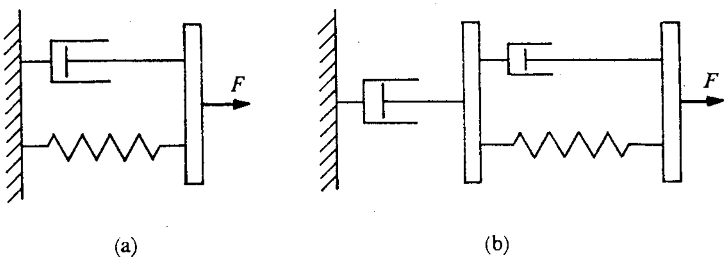


FIGURE 8.3

Most rocks have properties that can be represented by viscous terms in the stress-strain relations. Even at ordinary temperatures and pressures, rocks will deform permanently if the applied loads are maintained for a sufficiently long time. From the observed behavior of rocks it can be concluded that the model shown in Fig. 8.3b is a better analog than either a spring and dashpot in series, or a spring and dashpot in parallel. The models shown in Fig. 8.3 exhibit so-called elastic after-working; that is, if a long-time force F is suddenly removed, it will take some time for the system to return to a zero energy state.

Elasto-Plastic Material. The so-called elasto-plastic material is analogous to a spring in series with a friction slider, as shown in Fig. 8.4a. So long as the force F is smaller than the force F_0 that is required to overcome the frictional resistance of the slider the system behaves elastically. When the force F reaches the value F_0 any further pulling on the spring will cause the slider to slip with F remaining equal to F_0. The stress-strain relations for an elasto-plastic material are

$$\dot{\epsilon}'_{ij} = \frac{1}{2G}\dot{\tau}'_{ij} + \dot{f}\tau'_{ij}$$

The first term on the right hand side of this equation represents the elastic deformation and the second term represents the plastic deformation. The significance of the coefficient $\dot{f}$ can be explained as follows. An additional statement is required to specify the behavior of the stresses during yielding.

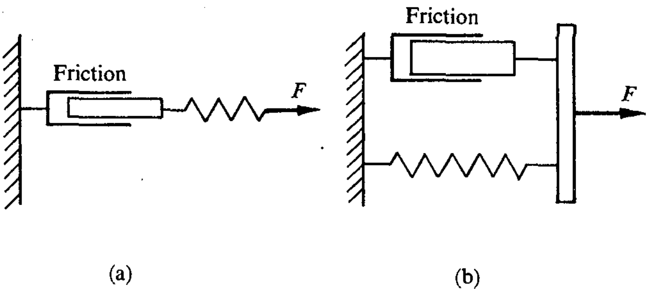


FIGURE 8.4

The most commonly used yield condition, proposed by Hencky and Mises, is that yielding occurs when the shear strain energy density reaches a certain value. When the strain energy density, Eq. (1.46), is expressed in terms of σ'_x, $\bar{\sigma}$, τ'_{xy}, etc., it is found that the cross product terms $\sigma'_x\bar{\sigma}$, $\sigma'_y\bar{\sigma}$, $\sigma'_z\bar{\sigma}$ drop out because of the relation $\sigma'_x + \sigma'_y + \sigma'_z = 0$. Therefore, the strain energy density can be written as the sum of the distortional and dilatational strain energy densities

$$V_0 = V'_0 + \bar{V}_0$$

where

$$V'_0 = \sum \frac{1}{2}\tau'_{ij}\,\epsilon'_{ij} = \frac{1}{2}\frac{\sum(\tau'_{ij})^2}{2G} = \frac{1}{2}\frac{I'_2}{2G} \tag{8.7}$$

and

$$I'_2 = \sum(\tau'_{ij})^2$$

The shear strain energy, in expanded form is written as

$$V'_0 = \frac{1}{2}\frac{1}{2G}[(\tau'_{xx})^2 + (\tau'_{yy})^2 + (\tau'_{zz})^2 + (\tau'_{xy})^2 + (\tau'_{yx})^2 + (\tau'_{yz})^2 + (\tau'_{zy})^2 + (\tau'_{xz})^2 + (\tau'_{zx})^2]$$

The condition for yielding* is thus $I_2' = k$. The value of the constant k is determined as follows. If a bar is pulled in uniaxial tension so that all stresses are zero except σ_x, the distortional state of stress is

$$\tau'_{xx} = \tfrac{2}{3}\sigma_x, \quad \tau'_{yy} = -\tfrac{1}{3}\sigma_x; \quad \tau'_{zz} = -\tfrac{1}{3}\sigma_x$$

$$\tau'_{xy} = \tau'_{yz} = \tau'_{xz} = 0$$

For this state of stress $I_2' = (\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9})\sigma_x^2 = \tfrac{2}{3}\sigma_x^2$ and, therefore, the constant k has the value $\tfrac{2}{3}\sigma_0^2$, where σ_0 is the yield stress in simple tension. The elasto-plastic material is, therefore, described by the equations

$$\dot{\epsilon}'_{ij} = \frac{1}{2G}\dot{\tau}'_{ij} + f\tau'_{ij}$$

$$I_2' = \tfrac{2}{3}\sigma_0^2 \tag{8.8}$$

The total strain rate is the sum of the elastic component plus the plastic component, $\dot{\epsilon}'_{ij} = (\dot{\epsilon}'_{ij})_e + (\dot{\epsilon}'_{ij})_p$, where

$$(\dot{\epsilon}'_{ij})_e = \frac{1}{2G}\dot{\tau}'_{ij}$$

$$(\dot{\epsilon}'_{ij})_p = f\tau'_{ij}$$

From the latter equation it is seen that, by squaring both sides and summing, the following relation must hold†

$$\sum (\dot{\epsilon}'_{ij})_p^2 = f^2 \sum (\tau'_{ij})^2$$

This equation is usually written

$$\dot{J}_2' = f^2 I_2'$$

where

$$\dot{J}_2' = \sum (\dot{\epsilon}'_{ij})_p^2; \quad I_2' = \sum (\tau'_{ij})^2 = \tfrac{2}{3}\sigma_0^2$$

It is thus seen that f is related to the plastic straining by

$$f = \sqrt{\dot{J}_2'/I_2'} = \sqrt{\tfrac{3}{2}\dot{J}_2'}/\sigma_0 \tag{8.9}$$

The quantity f is thus determined by the strain rates.

* I_2' is the second stress invariant; see Appendix I. $I_2' = k$ is the same yield condition as that given in Eq. (1.60).

† The summations are taken over the nine stresses and the nine strains.

Some materials, as the plastic strains increase, require increasingly larger stresses to produce the strains. Such materials are said to be strain hardening. Most metals exhibit strain hardening. A material with strain hardening is analogous to a spring and friction slider in series with the magnitude of the friction coefficient being a function of displacement.

Plasto-Elastic Material. The analog of a plasto-elastic material is a spring and a friction slider in parallel, as shown in Fig. 8.4b. For such a material there is an internal dissipation of energy for all straining. Most solids do have a small component of internal damping of this type.

Plastic-Rigid Material. A plastic-rigid material has an infinitely large modulus of compressibility C and infinitely large modulus of rigidity G. The elastic strains are zero, therefore, and only plastic strains are produced. The material properties (no strain hardening) are expressed by

$$\begin{aligned} \dot{\epsilon}_{ij} &= \dot{f}\tau'_{ij} \\ I'_2 &= \tfrac{2}{3}\sigma_0^2 \end{aligned} \tag{8.10}$$

For this material there will be no strains until the stresses reach such a value that the yield condition is satisfied. At this point the strains are indeterminant, for any attempt to increase the loads on the body will just produce straining without increasing the stresses, so there is no one-to-one relation between the stresses and the strains.

In the ordinary notation the expanded form of the equations of a plastic-rigid material are

$$\begin{aligned} \dot{\epsilon}_x &= \tfrac{2}{3}\dot{f}(\sigma_x - \tfrac{1}{2}\sigma_y - \tfrac{1}{2}\sigma_z) & \dot{\gamma}_{xy} &= 2\dot{f}\tau_{xy} \\ \dot{\epsilon}_y &= \tfrac{2}{3}\dot{f}(\sigma_y - \tfrac{1}{2}\sigma_z - \tfrac{1}{2}\sigma_x) & \dot{\gamma}_{yz} &= 2\dot{f}\tau_{yz} \\ \dot{\epsilon}_z &= \tfrac{2}{3}\dot{f}(\sigma_z - \tfrac{1}{2}\sigma_x - \tfrac{1}{2}\sigma_y) & \dot{\gamma}_{xz} &= 2\dot{f}\tau_{xz} \end{aligned} \tag{8.11}$$

$$I'_2 = \frac{(\sigma_x - \sigma_y)^2}{3} + \frac{(\sigma_y - \sigma_z)^2}{3} + \frac{(\sigma_z - \sigma_x)^2}{3} + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{xz}^2 = \tfrac{2}{3}\sigma_0^2$$

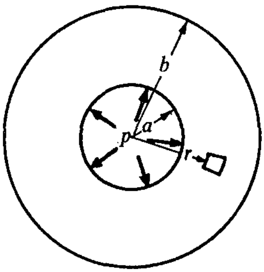
The plastic-rigid material is an idealization of a ductile steel that has plastic strains large in comparison to the elastic strains.

Yielding of a Plastic-Rigid, Thick-Walled Cylinder. An example of a plastic-rigid problem is the cylinder shown in Fig. 8.5a, subjected to an internal pressure p. Since the material is plastic-rigid there are no strains until p reaches a limiting value for which the material is in a plastic state. Any attempt to increase the pressure will merely cause the radius a to increase, it being assumed that the axial strain is zero ($\epsilon_z = 0$). The equation of equilibrium in the radial direction can be written

$$\frac{\partial \sigma_r}{\partial r} = -\frac{\sigma_r - \sigma_\theta}{r} \tag{8.12}$$

The stress-strain relations in polar coordinates are

$$\begin{aligned} \dot{\epsilon}_r &= \tfrac{2}{3}\dot{f}(\sigma_r - \tfrac{1}{2}\sigma_z - \tfrac{1}{2}\sigma_\theta) \\ \dot{\epsilon}_\theta &= \tfrac{2}{3}\dot{f}(\sigma_\theta - \tfrac{1}{2}\sigma_r - \tfrac{1}{2}\sigma_z) \\ \dot{\epsilon}_z &= \tfrac{2}{3}\dot{f}(\sigma_z - \tfrac{1}{2}\sigma_\theta - \tfrac{1}{2}\sigma_r) = 0 \end{aligned} \tag{8.13}$$



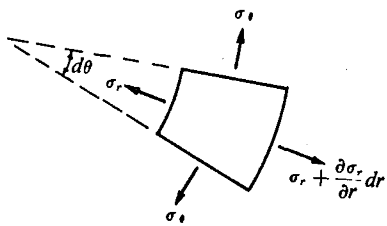


FIGURE 8.5

From the last of these equations, we have

$$\sigma_z = \frac{\sigma_r + \sigma_\theta}{2}$$

Using this, the stress-strain relations can be written

$$\begin{aligned} \dot{\epsilon}_r &= \tfrac{1}{2}\dot{f}(\sigma_r - \sigma_\theta) \\ \dot{\epsilon}_\theta &= \tfrac{1}{2}\dot{f}(\sigma_\theta - \sigma_r) \end{aligned} \tag{8.14}$$

and the Hencky-Mises yield condition can be written

$$I_2' = \tfrac{1}{2}(\sigma_r - \sigma_\theta)^2 = \tfrac{2}{3}\sigma_0^2$$

or

$$(\sigma_r - \sigma_\theta) = \pm \frac{2}{\sqrt{3}}\,\sigma_0 \tag{8.15}$$

If this is substituted in the equation of equilibrium, there is obtained

$$\frac{d\sigma_r}{dr} = \mp \frac{2}{\sqrt{3}} \frac{\sigma_0}{r}$$

The solution of this equation is

$$\sigma_r = \mp \frac{2}{\sqrt{3}} \sigma_0 \ln r + C$$

The boundary condition to be imposed is that on the exterior of the cylinder the stress is zero:

$$(\sigma_r)_{r=b} = 0$$

and this determines the constant C, giving

$$\sigma_r = \pm \frac{2\sigma_0}{\sqrt{3}} \ln \frac{b}{r}$$

The negative sign corresponds to an internal pressure. The value of σ_r at $r = a$ specifies the internal pressure that will just produce plastic yielding

$$p = \frac{2\sigma_0}{\sqrt{3}} \ln \frac{b}{a} \tag{8.16}$$

The stresses σ_θ and σ_z can be determined from the yield condition and from the stress-strain relations to give the complete plastic stress distribution

$$\begin{aligned} \sigma_r &= -\frac{2}{\sqrt{3}} \sigma_0 \ln \frac{b}{r} \\ \sigma_\theta &= +\frac{2}{\sqrt{3}} \sigma_0 \left(1 - \ln \frac{b}{r}\right) \\ \sigma_z &= +\frac{2}{\sqrt{3}} \sigma_0 \left(\tfrac{1}{2} - \ln \frac{b}{r}\right) \end{aligned} \tag{8.17}$$

The stress distribution is plotted in Fig. 8.6. This may be compared with the elastic stress distribution given by Eqs. (5.8), (5.9), and (5.11):

$$\sigma_r = -p \frac{1 - b^2/r^2}{1 - b^2/a^2}$$

$$\sigma_\theta = -p \frac{1 + b^2/r^2}{1 - b^2/a^2}$$

$$\sigma_z = -p \frac{2\nu}{1 - b^2/a^2}$$

A similar solution can be made for the elastic-plastic cylinder,* in this case yielding begins at the inner face and as the pressure increases the plastic front moves outward until a condition of full plasticity is reached; at this instant the stress distribution is the same as given by Eqs. (8.17).

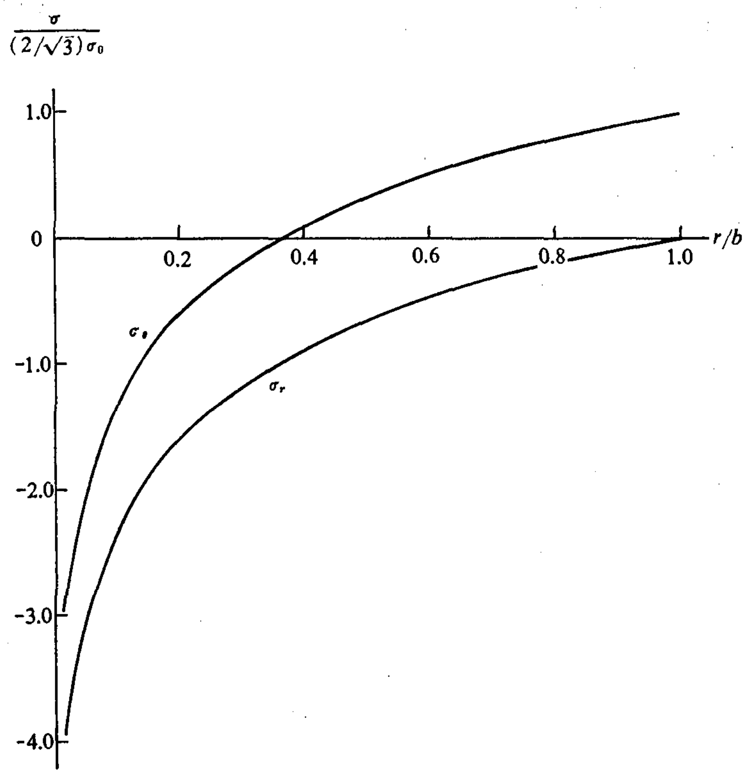


FIGURE 8.6

8-3 THE YIELD SURFACE IN STRESS SPACE

As an aid to visualization, the yield condition is sometimes discussed in geometrical terms with reference to the so-called stress space. The state of stress of an element is described by the three principal stresses σ_1, σ_2, σ_3, that is, the state of stress is the point located by the radius vector

$$\mathbf{s} = \sigma_1\mathbf{e}_1 + \sigma_2\mathbf{e}_2 + \sigma_3\mathbf{e}_3$$

where σ_1, σ_2, and σ_3 are measured along the three axes of stress-space shown in Fig. 8.7. The line *a-a* makes an equal angle with each of the coordinate axes, so any point on it corresponds to a state of pure dilatation ($\sigma_1 = \sigma_2 = \sigma_3$).

* See Hoffman and Sachs, *Theory of Plasticity*, McGraw-Hill (1953).

The component of s along *a-a* is given by the dot product of s with the unit vector lying along *a-a*

$$\mathbf{s}\cdot\frac{1}{\sqrt{3}}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)=\frac{1}{\sqrt{3}}(\sigma_1+\sigma_2+\sigma_3)=\bar{\sigma}\sqrt{3}$$

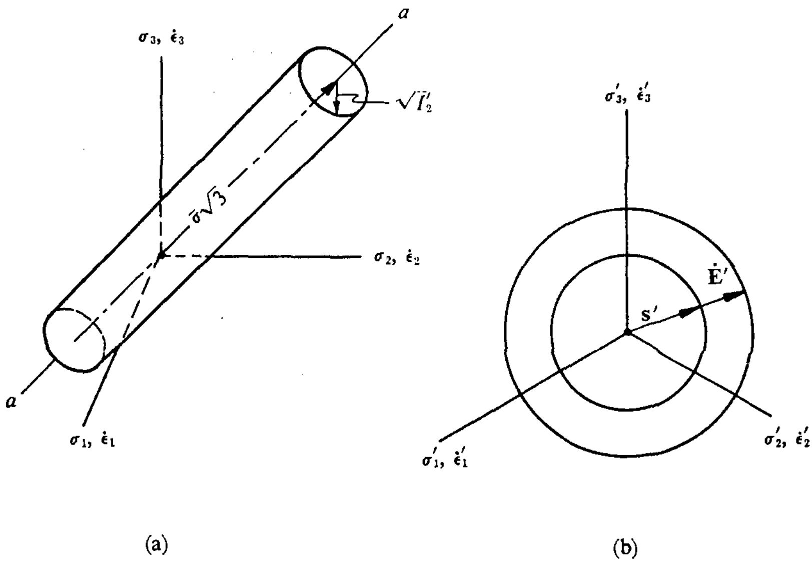


FIGURE 8.7

The component of s perpendicular to *a-a* has a magnitude equal to the absolute value of the cross product of s on the unit vector along *a-a*

$$\left|\mathbf{s}\times\frac{1}{\sqrt{3}}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)\right|=\left|\frac{(\sigma_2-\sigma_3)}{\sqrt{3}}\mathbf{e}_1+\frac{(\sigma_3-\sigma_1)}{\sqrt{3}}\mathbf{e}_2+\frac{(\sigma_1-\sigma_2)}{\sqrt{3}}\mathbf{e}_3\right|$$

$$=\{\tfrac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]\}^{1/2}$$

$$=\sqrt{I_2'}$$

It is thus seen that the Hencky-Mises yield condition, $I_2'=\frac{2}{3}\sigma_0^2$ describes a cylindrical surface of radius $\sqrt{I_2'}$ with its axis lying along line *a-a*, that is, the stress-point must lie on this surface if the yield condition is satisfied.

If the principal strain rates $\dot\epsilon_1$, $\dot\epsilon_2$, and $\dot\epsilon_3$ are plotted along the *1*, 2, 3 axes, the strain rate point is specified by the vector:

$$\dot{\mathbf{E}}=\dot\epsilon_1\mathbf{e}_1+\dot\epsilon_2\mathbf{e}_2+\dot\epsilon_3\mathbf{e}_3$$

It is seen that a point on line *a-a* represents a state of pure dilatational strain, and

$$\sqrt{J_2} \equiv \sqrt{\dot{\epsilon}_1^2 + \dot{\epsilon}_2^2 + \dot{\epsilon}_3^2}$$

is the radius of a cylindrical surface whose axis lies along *a-a*.

For a distortional state of stress the vectors $\mathbf{s}'$ and $\dot{\mathbf{E}}'$ lie in the plane passing through the origin and perpendicular to line *a-a*. Since a plastic-rigid material has $\dot{\epsilon}_{ij} = \dot{f}\tau'_{ij}$, it follows that

$$\frac{\dot{\epsilon}_1}{\sigma'_1} = \frac{\dot{\epsilon}_2}{\sigma'_2} = \frac{\dot{\epsilon}_3}{\sigma'_3} = \dot{f}$$

and, hence, the vectors $\mathbf{s}'$ and $\dot{\mathbf{E}}'$ have the same direction, as shown in Fig. 8.7. This fact has an interesting interpretation in terms of work done and energy dissipated as discussed in the following paragraphs.

Plastic Work. When a plastic-rigid material is strained, energy is dissipated. Since, in a plastic-rigid material the dilatational stresses do no work, the energy dissipation density (dissipation per unit volume) can be written

$$\begin{aligned}\dot{W}_0 &= \sigma'_x\dot{\epsilon}_x + \sigma'_y\dot{\epsilon}_y + \sigma'_z\dot{\epsilon}_z + \tau'_{xy}\dot{\gamma}_{xy} + \tau'_{yz}\dot{\gamma}_{yz} + \tau'_{xz}\dot{\gamma}_{xz}\\ &= \tau'_{xx}\dot{\epsilon}_{xx} + \tau'_{yy}\dot{\epsilon}_{yy} + \tau'_{zz}\dot{\epsilon}_{zz} + \tau'_{xy}\dot{\epsilon}_{xy} + \tau'_{yx}\dot{\epsilon}_{yx} + \tau'_{yz}\dot{\epsilon}_{yz} + \tau'_{zy}\dot{\epsilon}_{zy} \qquad (8.18)\\ &\quad + \tau'_{xz}\dot{\epsilon}_{xz} + \tau'_{zx}\dot{\epsilon}_{zx}\\ &= \sum \tau'_{ij}\dot{\epsilon}_{ij}\end{aligned}$$

The plastic-rigid material has stress-strain relations

$$\dot{\epsilon}_{ij} = \dot{f}\tau'_{ij}$$

$$I'_2 = \tfrac{2}{3}\sigma_0^2$$

and, hence, the energy dissipation density can be written in the following forms:

$$\dot{W}_0 = \sum \tau'_{ij}\dot{\epsilon}_{ij} = \sum \dot{f}(\tau'_{ij})^2 = \dot{f}I'_2$$

$$\dot{W}_0 = \sum \tau'_{ij}\dot{\epsilon}_{ij} = \sum \frac{1}{\dot{f}}(\dot{\epsilon}_{ij})^2 = \frac{1}{\dot{f}}\dot{J}_2$$

Multiplying these and taking the square root gives

$$\dot{W}_0 = \sqrt{I'_2\dot{J}_2} = \sigma_0\sqrt{\tfrac{2}{3}\dot{J}_2} \qquad (8.19)$$

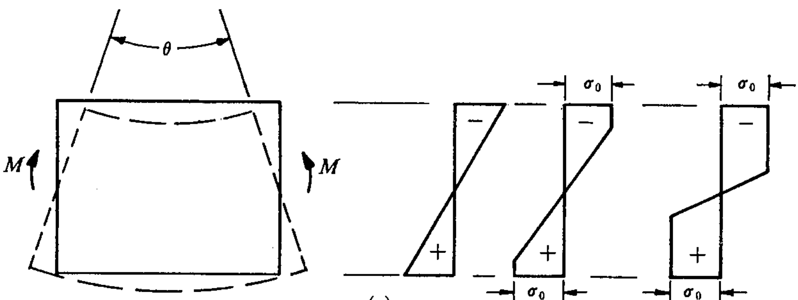
It is thus seen that with reference to stress space the energy dissipation density can be written:

$$\dot{W}_0 = \mathbf{s}' \cdot \dot{\mathbf{E}}' \tag{8.20}$$

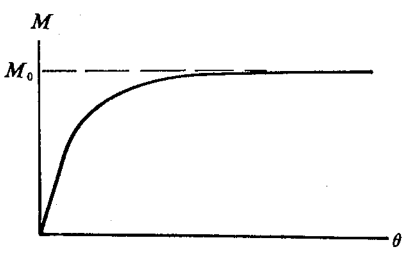
The total rate of energy dissipation in a body is

$$\dot{W} = \iiint \dot{W}_0 \, dx \, dy \, dz = \iiint (\mathbf{s}' \cdot \dot{\mathbf{E}}') \, dx \, dy \, dz$$

when $I_2' = \frac{2}{3}\sigma_0^2$. This is equal to the rate of doing work by the applied forces on the surface of the body.



(a)



(b)

FIGURE 8.8

From the foregoing expression, there can be deduced the principle of maximum plastic work: Of all states of stress that satisfy equilibrium in the interior of a body and satisfy the boundary conditions on the surface, and for which $I_2' = \frac{2}{3}\sigma_0^2$, the correct state of stress maximizes the rate of doing work by the surface forces. This is seen to be true since only the correct stress distribution has $\mathbf{s}'$ in the same direction as $\dot{\mathbf{E}}'$ and, therefore, the product $\mathbf{s}_1' \cdot \dot{\mathbf{E}}'$ of an incorrect stress $\mathbf{s}_1'$ cannot exceed the product $\mathbf{s}' \cdot \dot{\mathbf{E}}'$ for the correct stress ($|\mathbf{s}_1'| = |\mathbf{s}'| = \sqrt{I_2'}$).

8-4 YIELD HINGE IN A BEAM

When a beam that is made of an elastic-plastic material is subjected to an increasing bending moment, the bending stresses will at first be linearly distributed, as shown in Fig. 8.8a. For larger bending moments, the top and bottom portions of the beam will reach the yield point stress σ_0. For increasingly large bending moments the stress distribution approaches a rectangular shape. The relation between M and the angle θ is shown in Fig. 8.8b. At first there is a linear relation between M and θ, then as yielding

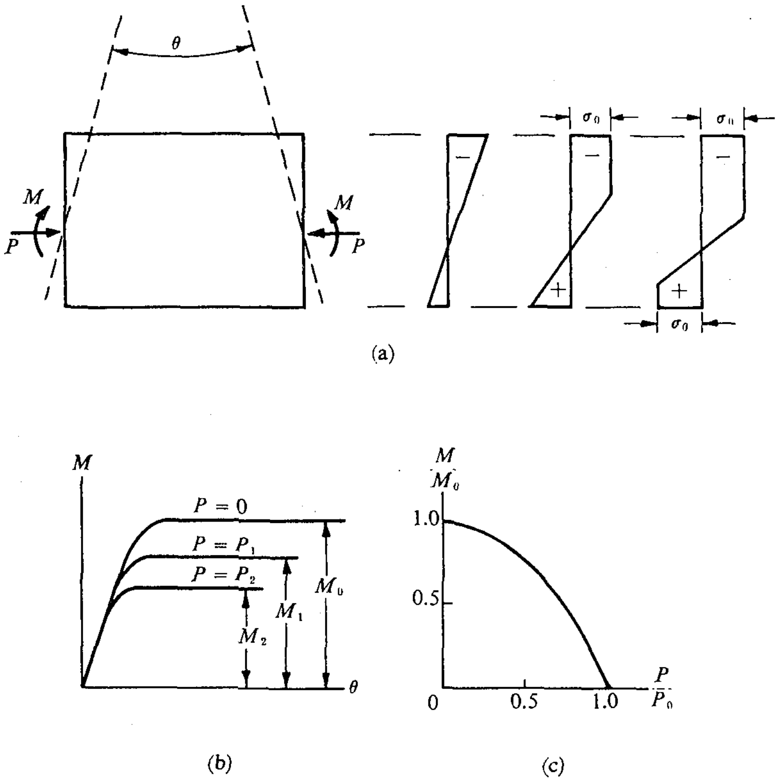


FIGURE 8.9

progresses through the beam the moment approaches the yield moment, or fully plastic moment M_0. For a *rectangular cross section the fully plastic* moment is 50% larger than the maximum, completely elastic moment. For an I-beam the fully plastic moment is approximately 15% larger than the maximum fully elastic moment.

If there is an axial force P acting on the beam in addition to the bending moment, the stress distribution will be as shown in Fig. 8.9a. It is seen in Fig. 8.9c that the effect of the axial force P is such as to reduce the value of M for which the beam becomes *fully plastic*. This effect is usually exhibited in the form of a so-called interaction diagram, shown in Fig. 8.9c. This diagram gives the values of M and P for which the beam section becomes fully plastic (see Prob. 8.14).

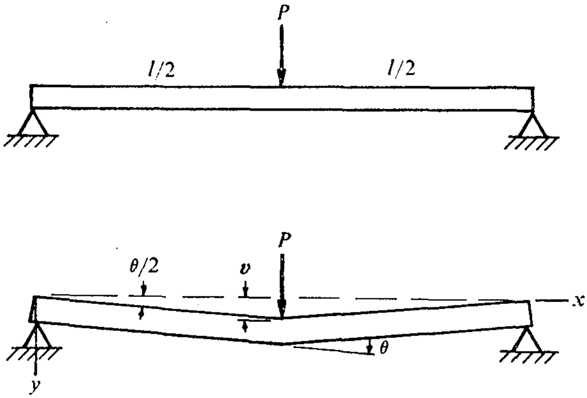


FIGURE 8.10

If the simply supported beam in Fig. 8.10 is subjected to a sufficiently large load P, it will develop a plastic region which will limit the maximum bending moment at this point to the plastic moment M_0*. During the plastic displacement the beam behaves as if there were a hinge under the load P and as if the hinge had so much frictional resistance that a moment M_0 was required to make it move. The energy dissipated by this yield hinge is provided by the work done by P. Assuming that the elastic deflection of P is so small compared to the plastic deflection that it may be neglected (plastic-rigid beam), a work-energy balance requires

$$Pv = M_0\theta$$

Since in this problem,

$$v = \frac{l}{2}\tan\frac{\theta}{2} \cong \frac{l\theta}{4}$$

* It is assumed that there is no interaction between the shearing force $P/2$, and the bending moment, and that the yield moment is the same as for a beam in pure bending. If the shear force is large the interaction must be taken into account.

the load that produces failure is

$$P_0 = \frac{4M_0}{l}$$

For a load P that is smaller than P_0 the factor of safety is given by

$$n = \frac{P_0}{P} = \frac{4M_0}{Pl}$$

The foregoing analysis assumes that a yield hinge with moment M_0 would be developed. It is, of course, not impossible that the beam could fail in some way without developing a hinge, so in applying this analysis to a beam or column it must be established that a hinge will indeed develop and then the correct value of the yield moment must be determined.

8-5 PLASTIC COLLAPSE OF BEAMS AND FRAMES

A beam that is built-in at each end and loaded as shown in Fig. 8.11a will develop yield hinges at the points of maximum bending moments.* Three hinges are required to form a collapse mechanism, as shown in Fig. 8.11b.

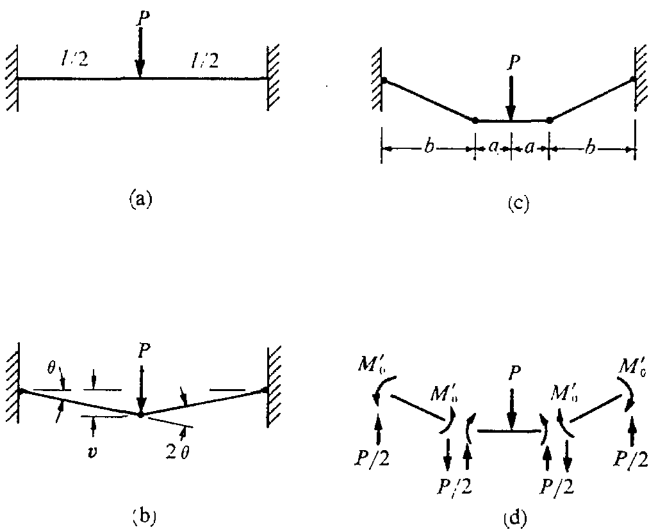


FIGURE 8.11

* Such problems are discussed in the book by B. G. Neal, *The Plastic Methods of Structural Analysis*, John Wiley and Sons (1963).

The principle of virtual work requires that during plastic straining

$$P\,\delta v = 4M_0\,\delta\theta$$

since for small displacements v is equal to $\frac{1}{2}l\theta$ it follows that $\delta v = \frac{1}{2}l\delta\theta$ and the collapse load is given by

$$P_0 = \frac{8M_0}{l}$$

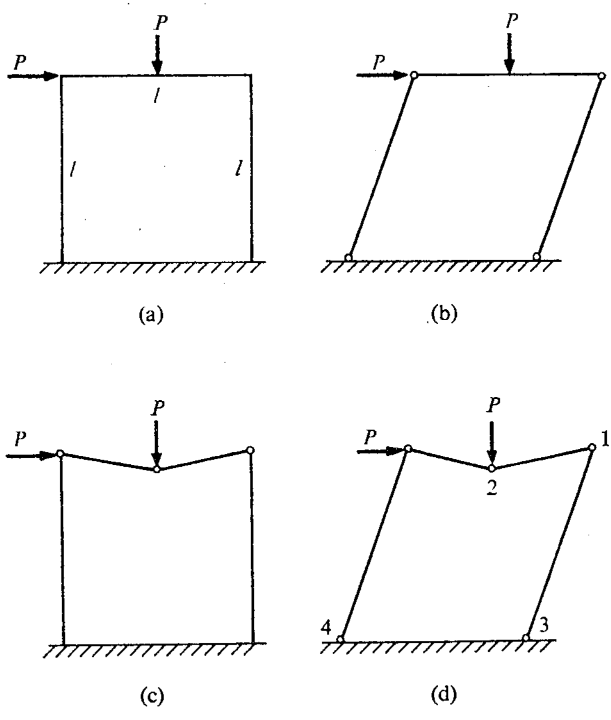


FIGURE 8.12

If we assume an incorrect yield mechanism, as shown in Fig. 8.11c, the corresponding collapse load is determined by

$$P_2\,\delta v = 4M_0\,\delta\theta = 4M_0\,\frac{\delta v}{b}$$

and hence,

$$P_2 = \frac{8M_0}{l - 2a}$$

This is larger than the collapse load calculated from the correct collapse mechanism.

The frame shown in Fig. 8.12 has the same yield moment M_0 everywhere, and some possible collapse mechanisms are shown. The correct collapse mechanism can, of course, always be found by the following procedure. First, make an elastic analysis of the bending moments in the frame. The first yield hinge will develop at the point of maximum bending moment. Suppose this is hinge No. 3 in Fig. 8.12d, then for greater loads another elastic analysis can be made subject to the condition that the bending moment at point 3 is equal to M_0. The second yield hinge will form at the point of greatest elastic moment, and so forth. For complicated frames the amount of analysis required becomes prohibitive, and approximate methods of analysis are used.*

8-6 FAILURES DUE TO PLASTIC STRAINING

It is possible for failures of actual structures to occur even when the loads are not sufficiently large to cause plastic collapse. This can be explained with reference to the simple structure shown in Fig. 8.13. Suppose the members

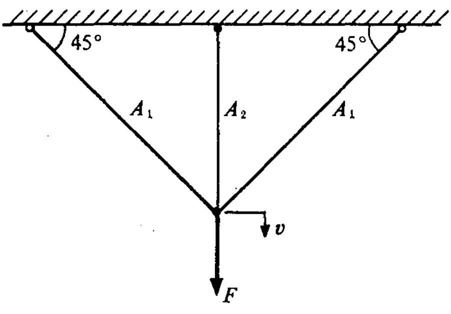


FIGURE 8.13

are made of ductile steel; then for small values of the applied force F the members will be strained elastically. When F is increased to a certain value, member 2 will reach its yield point. When the force is increased still more, one of the other members will reach its yield point, and the structure will "collapse." It is seen that there are two distinct ranges of noncollapse loading, one in which all strains are elastic, and one in which there is plastic straining. If the load F is applied and removed successively, this may or may not lead to failure as explained by the following examples. When the force F is applied, the joint will be given a vertical displacement v and we shall keep

* For a discussion of such problems see the book *Plastic Analysis of Structures*, by P. G. Hodge, McGraw-Hill (1963).

track of the forces F_1 and F_2 in the members as the displacement increases and decreases. In Fig. 8.14a it is seen that the forces F_1 and F_2 increase elastically until F_2 reaches the yield value. For increasing displacements v, member 2 is stretched plastically. When the displacement has reached a certain value v_2, the members 1 being still in the elastic range, the applied force is gradually decreased. During the unloading process the force in member 2 changes to compression and the member deforms plastically until displacement v_4 is reached, as shown in Fig. 8.14a. At this point the external force is equal to zero, and the forces in the three members meeting at the joint are in equilibrium, that is, $2(F_1/\sqrt{2}) = F_2$. There are, therefore, residual stresses in the unloaded system. If the force F is again applied the displacement will increase from v_4 to v_5 and will again reach v_2. If the system is again unloaded the displacement will again go from v_2 to v_3 and v_4. Repeated cycles of loading and unloading will thus trace out the hysteresis loop 2-3-4-5. The area within the loop is $2F_0(v_2 - v_5)$ and this is the energy that is dissipated in plastic working per cycle. If the loading is repeated for a sufficient number of cycles, member 2 will fail in fatigue.

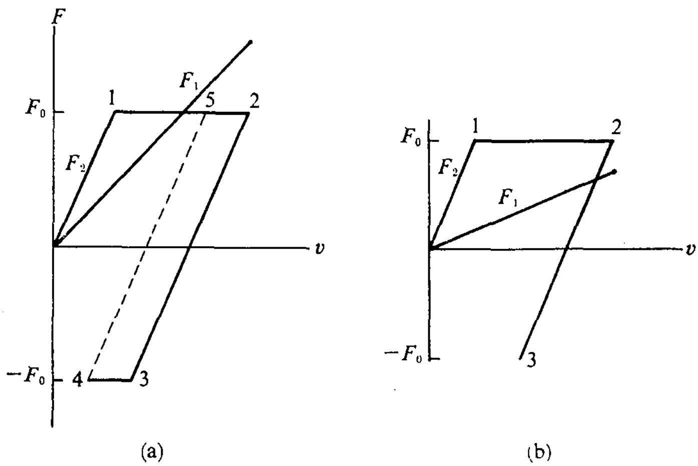


FIGURE 8.14

Another possibility for the behavior of the system under repeated loadings is illustrated in Fig. 8.14b. When the load is applied, member 2 is stretched plastically until the displacement is v_2. If the load is now gradually removed the displacement decreases to v_3 at which point $2(F_1/\sqrt{2}) = F_2$. This differs from the preceding case in that member 2 is not strained plastically in compression. If the load is reapplied, the displacement will increase from v_3 to v_2 without producing plastic strains. The loading-unloading cycle can now be repeated without any plastic straining, that is, with no dissipation of energy, and it is said that there has been a shakedown of the system. This terminology derives from the so-called "shakedown" cruise of a new naval vessel.

In general, it can be said that for a sequence of loading and unloading of one or more applied forces, either the structure will eventually shake down so that no energy is dissipated during the cycle, or it will not shake down but will fail in fatigue.

8-7 PLASTIC TORSION

If a circular steel bar is twisted by an increasing torque the shearing stresses will increase as shown in Fig. 8.15. In (a) there is shown the fully elastic state

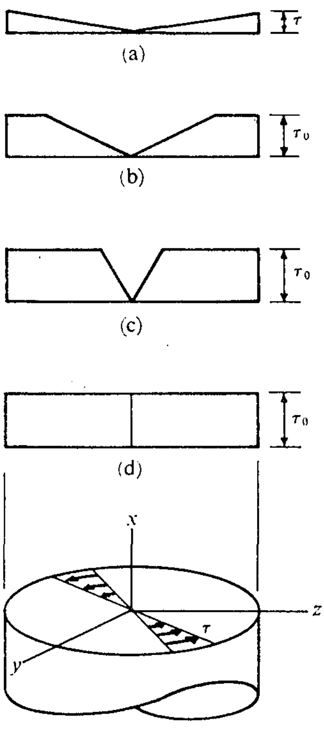


FIGURE 8.15

of stress with linear variation. If the torque is increased, the yield point stress τ_0, will be reached at the surface of the bar (b) and for a large angle of twist a fully plastic stress distribution (d) will be attained. The stress distribution in a prismatic bar of noncircular cross section is governed by the equation of equilibrium

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

Since there is only the torsional shear stress at any point in the bar, the Hencky-Mises yield condition is expressed by

$$\sqrt{\tau_{xy}^2 + \tau_{xz}^2} = \tau_0$$

where τ_0 is the yield stress in shear.

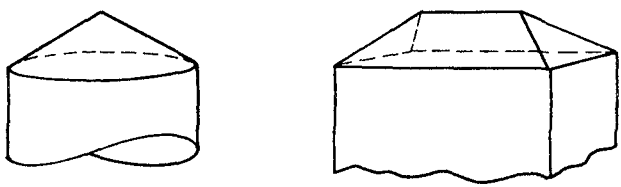
If the shear stresses are calculated from a stress function according to

$$\frac{\partial \Phi}{\partial z} = \tau_{xy}; \quad \frac{\partial \Phi}{\partial y} = -\tau_{xz}$$

the equation of equilibrium is automatically satisfied and the yield condition is expressed by:

$$\sqrt{\left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2} = \tau_0 \tag{8.21}$$

This equation states that the magnitude of the gradient (maximum slope) of Φ is everywhere equal to τ_0.



Round bar

(a)

Rectangular bar

(b)

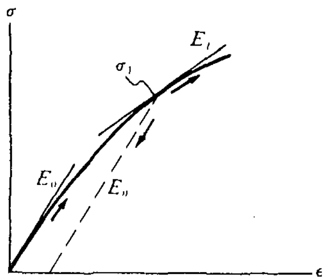
FIGURE 8.16

The yielding of the bar in torsion can be interpreted in terms of the so-called sand heap analogy. It is known that when sand is poured in a heap, the slope of the sides of the heap will build up until the angle of repose α_0 is reached, after which any additional sand grains merely roll down the slope. Therefore, if sand is poured onto the end of the bar it will represent the stress function Φ, with τ_0 corresponding to the angle of repose. Some sketches of sand heaps are shown in Fig. 8.16.

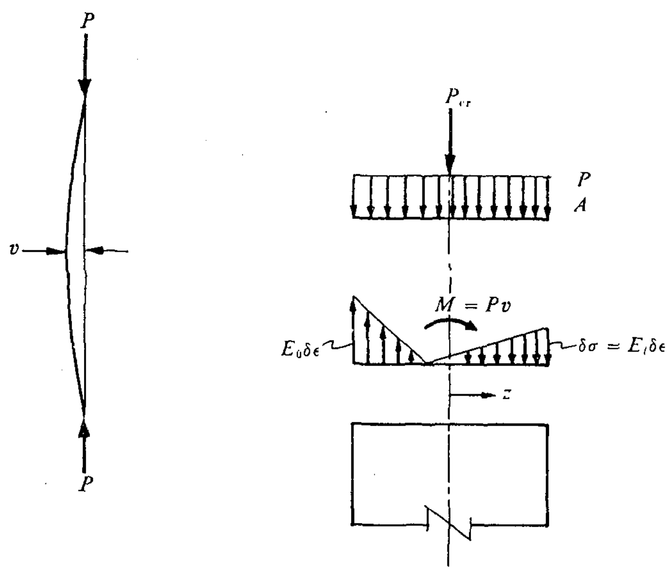
8-8 INELASTIC BUCKLING OF COLUMNS

If a column is made of nonlinear material whose stress-strain diagram is shown in Fig. 8.17a, two different modes of buckling can be forseen. This is because the unloading of the material follows a different path than the loading. For example, as seen in Fig. 8.17a, if the material is stressed to a value σ_1 and a positive (loading) increment of strain $\delta\epsilon$ is given to the material, the increment of stress is

$$\delta\sigma = E_t\,\delta\epsilon$$



(a)



(b) (c)

FIGURE 8.17

where E_t is the slope of the tangent to the stress-strain curve at point 1. For a negative (unloading) increment of strain the increment of stress is

$$\delta\sigma = E_0 \, \delta\epsilon$$

where E_0 is the slope of the unloading curve. One mode of buckling is when a perfectly straight column is subjected to the critical load. If the column is then given a small lateral displacement v part of the column will receive an increment of unloading strain and the other side will be given an increment of loading strain, as shown in Fig. 8.17c. Equilibrium requires that the moment M be equal to Pv. The conditions of equilibrium can be applied to the stress distribution

$$\int \sigma \, dA = P_{\text{cr}} \qquad (\delta\sigma = E_t \, \delta\epsilon \text{ for loading, } \delta\sigma = E_0 \, \delta\epsilon \text{ for unloading})$$

$$\int \sigma z \, dA = M \qquad \left(\epsilon = \frac{\partial^2 v}{\partial x^2} z\right)$$

and it will be found that the bending moment is related to the displacement by

$$M = E_r I \frac{\partial^2 v}{\partial x^2}$$

where E_r is the *reduced modulus*. In the case of a square section:

$$E_r = \left(\frac{2}{\sqrt{1/E_t} + \sqrt{1/E_0}}\right)^2 \tag{8.22}$$

It is seen, therefore, that the critical load of a pin-ended strut is just the Euler buckling load with E_r for the modulus of elasticity

$$P_{\text{cr}} = \frac{\pi^2 E_r I}{l^2} \tag{8.23}$$

A second possible mode of buckling is as follows. Suppose that initially the column is not quite straight. In this case, for each increment of axial load δP there is a corresponding increment of lateral displacement δv. This is indicated in Fig. 8.18. During this process there is an increment of strain due to the increase in axial stress $\delta P/A$ plus an increment due to the displacement δv, so that

$$\delta\epsilon = \delta\epsilon_p + \delta\epsilon_v$$

If the column is almost straight $\delta\epsilon_v$ will be small compared to $\delta\epsilon_p$ and the entire column behaves as if its modulus of elasticity were the tangent modulus

E_t. So in the initial buckling process the column behaves as if its buckling load were $P_{cr} = \pi^2 E_t I/l^2$. The tangent modulus E_t is somewhat smaller than the reduced modulus E_r. Experimental determination of buckling loads shows that the effective modulus E_e is closer to E_t than to E_r, and that

$$E_t < E_e < E_r$$

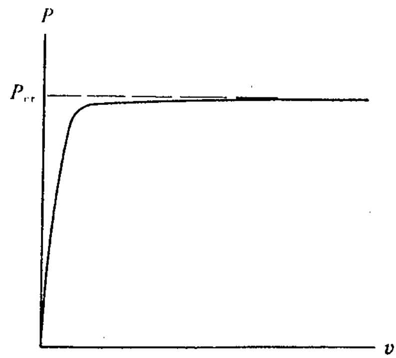


FIGURE 8.18

8-9 PLASTIC EXTENSION, DRAWING, AND ROLLING

There are a number of different forming processes used to reduce the thickness of a sheet of metal, or to reduce the diameter of a wire, etc. All of these processes involve putting the material into a plastic state and, while it is in this state, to achieve the desired change in proportions. A certain amount of work is expended during the plastic deformation. For example, consider the rectangular block h_0, l_0, which is deformed in plane strain ($\epsilon_z = 0$) so that it has the dimensions h, l, as shown in Fig. 8.19. The material is assumed to be incompressible (plastic-rigid) so that the volume remains constant

$$hl = h_0 l_0$$

or

$$\frac{d}{dt}(hl) = h\dot{l} + \dot{h}l = 0$$

This may also be written

$$\frac{\dot{l}}{l} = -\frac{\dot{h}}{h}$$

which states that $\dot{\epsilon}_x = -\dot{\epsilon}_y$. The energy dissipation per unit volume is

$$\begin{aligned} \dot{W}_0 &= \sqrt{I_2' \dot{J}_2'} = \sqrt{\tfrac{2}{3}\sigma_0^2}\,\sqrt{\dot{\epsilon}_x^2 + \dot{\epsilon}_y^2 + \dot{\epsilon}_z^2} \\ &= \sigma_0 \sqrt{\tfrac{2}{3}} \sqrt{\left(\frac{\dot{l}}{l}\right)^2 + \left(\frac{\dot{h}}{h}\right)^2} \\ &= \frac{2}{\sqrt{3}}\,\sigma_0\,\frac{\dot{h}}{h} \end{aligned}$$

This may also be written

$$dW_0 = \frac{2}{\sqrt{3}}\,\sigma_0\,\frac{dh}{h}$$

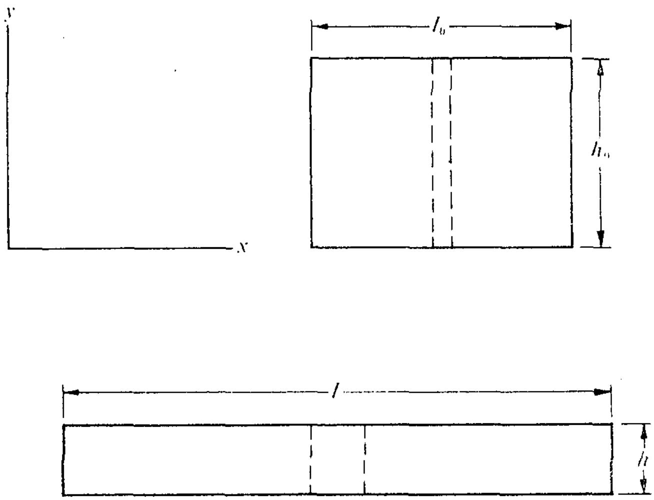


FIGURE 8.19

The total work done per unit volume is

$$W_0 = \int_h^{h_0} \frac{2}{\sqrt{3}}\,\sigma_0\,\frac{dh}{h} = \frac{2}{\sqrt{3}}\,\sigma_0 \ln \frac{h_0}{h} \tag{8.24}$$

This is the minimum amount of work required to reduce the thickness of the plastic-rigid material from h_0 to h. Usually in the process of reducing the thickness the cross-section shown dotted in Fig. 8.19 will be deformed out of plane and an additional amount of work is expended on this. Even more important than this, the process of reducing the thickness usually involves high external pressures with correspondingly high friction forces which dissipate energy. This situation is shown in Fig. 8.20a where a sheet of material is being drawn through a die. A freebody diagram of an element is shown in (b), where μ is the coefficient of friction. Since the die angle α is small, the following approximations are made. The element is assumed to undergo plastic straining the same as that shown in Fig. 8.19, that is, we assume that σ_x and σ_y are principal stresses. It is also assumed that $\sigma_y = -p$. The Hencky-Mises yield condition for plane strain can be written

$$\sigma_x - \sigma_y = \pm \frac{2}{\sqrt{3}} \sigma_0$$

or

$$\sigma_x + p = \pm \frac{2}{\sqrt{3}} \sigma_0 \tag{8.25}$$

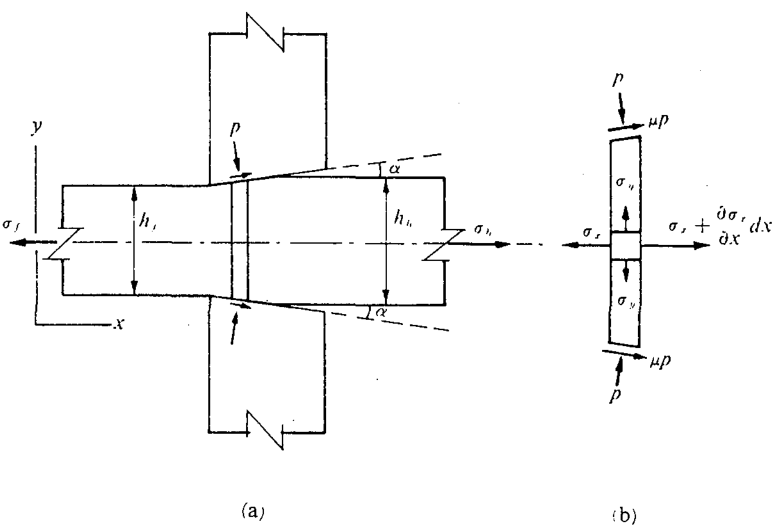


FIGURE 8.20

The equation of equilibrium of the element is

$$\frac{d}{dx}(\sigma_x 2x \tan \alpha) + 2p \tan \alpha + 2\mu p = 0$$

where x is measured from the point of intersection of the two dotted lines in Fig. 8.20a. Using the yield condition, this can be written

$$x \frac{d\sigma_x}{dx} - \beta\sigma_x = -\frac{2\sigma_0}{\sqrt{3}}(1 + \beta) \tag{8.26}$$

$$\beta = \frac{\mu}{\tan \alpha}$$

The integral of this equation is

$$\sigma_x = Cx^{\beta} + \frac{2\sigma_0}{\sqrt{3}}\left(1 + \frac{1}{\beta}\right)$$

If there is a backpull stress σ_b the constant C is determined from the condition $\sigma_x = \sigma_b$ at $x = x_b$

$$\sigma_x = \left[\sigma_b - \frac{2\sigma_0}{\sqrt{3}}\left(1 + \frac{1}{\beta}\right)\right]\left(\frac{x}{x_b}\right)^{\beta} + \frac{2\sigma_0}{\sqrt{3}}\left(1 + \frac{1}{\beta}\right) \tag{8.27}$$

The draw stress required to pull the sheet through the die can be written

$$\sigma_f = \frac{2\sigma_0}{\sqrt{3}}\left(1 + \frac{1}{\beta}\right)\left[1 - \left(\frac{h_f}{h_b}\right)^{\beta}\right] + \sigma_b\left(\frac{h_f}{h_b}\right)^{\beta} \tag{8.28}$$

It is seen that the larger the backpull stress σ_b the larger will be the draw stress σ_f and the smaller will be the die pressure p. It is also seen from the yield condition that σ_f cannot exceed $2\sigma_0/\sqrt{3}$ and this imposes a limit on the reduction in thickness h_f/h_b. Although a number of approximations were involved in the foregoing analysis, Eq. (8.28) does illustrate the essential features of the process. When applying Eq. (8.28) to an actual drawing process it would be advisable to treat σ_0 and β as unknown constants whose precise values are to be determined from test measurements made on the drawing process.

An analysis similar to the foregoing can be made for wire-drawing through a circular die, and a slightly more complicated analysis can be made of the reduction of thickness of a sheet passing between a pair of rolls.*

Problems

8.1 A rod of elasto-viscous material, Eq. (8.5), has a constant axial stress σ applied at $t = 0$. Describe the subsequent behavior of the system.

8.2 The rod of Prob. 8.1 has a constant strain rate $\dot{\epsilon}$ applied at time $t = 0$. Deduce that the resulting stress is described by

$$\sigma_x = 3\mu\dot{\epsilon}(1 - e^{-t/t_e}) \qquad t_e = \frac{3\mu}{E}$$

* Many examples of such analyses of metal forming are discussed in Hoffman and Sachs, *Theory of Plasticity*, McGraw-Hill (1953).

8.3 A thin-walled circular tube is made of elasto-viscous material. At time $t = 0$ the tube is twisted in torsion so that the shear strain rate is a constant, $\dot{\gamma}_{r\theta} = \dot{\gamma}$. Deduce that the shear stress is given by

$$\tau_{r\theta} = \mu\dot{\gamma}(1 - e^{-t/t_s}) \qquad t_s = \frac{\mu}{G}$$

8.4 A rod made of linear visco-elastic material, Eq. (8.6), has a constant stress $\sigma_x = \sigma$ applied at time $t = 0$. Deduce that the axial strain is given by

$$\epsilon_x = \frac{\sigma}{E}\left(1 - \frac{E}{3G}e^{-t/t_s}\right) \qquad t_s = \frac{\mu}{G}$$

Note that at $t = 0$ the solution shows $\epsilon_x = \bar{\epsilon} = \sigma(1 - 2\nu)/3E$.

8.5 Deduce the appropriate stress-strain relations for a plasto-elastic material.

8.6 Derive the stress-strain equations in expanded notation, Eq. (8.11), for a plastic-rigid material from Eqs. (8.10).

8.7 Derive the Hencky-Mises yield condition in the form given in Eq. (8.11) from the statement $I'_2 = \frac{2}{3}\sigma_0^2$.

8.8 Show that for plane strain the Hencky-Mises yield condition reduces to $(\sigma_1 - \sigma_2)/2 = \pm\sigma_0/\sqrt{3}$, where σ_1 and σ_2 are the in-plane principal stresses.

8.9 Show that for the thick-walled plastic-rigid cylinder the integral of σ_z over the cross section is just equal to $2\pi a^2 p$ and, hence, that a capped cylinder will have $\epsilon_z = 0$.

8.10 The maximum shear-stress yield condition states that during yielding the maximum shear stress is equal to the yield value τ_0. Show that in stress-space, Fig. 8.7, this condition specifies a hexagonal yield surface inscribed in the circular cylinder specified by the Hencky-Mises yield condition. What is the maximum discrepancy between the two conditions as regards magnitude of stress vector?

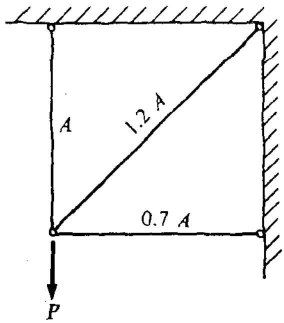
8.11 Draw a sketch of an element of volume showing the stresses and deduce that the rate of energy dissipation per unit volume is $\dot{W}_0 = \sum \tau'_{ij}\dot{\epsilon}_{ij}$ for a plastic-rigid material.

8.12 The rate of internal energy dissipation for a plastic-rigid material can be written

$$\dot{W} = \iiint (\sum \tau'_{ij}\dot{\epsilon}_{ij})\, dx\, dy\, dz$$

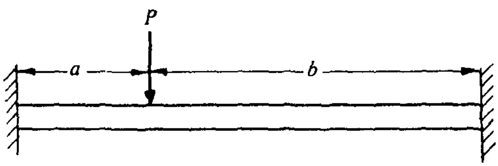
Set $\dot{\epsilon}_x = \partial\dot{u}/\partial x$, $\dot{\epsilon}_y = \partial\dot{v}/\partial y$, etc., and integrate by parts the terms under the integral signs and show that the resulting expression is just the rate of work done by the forces applied to the surface of the body.

8.13 The square framework shown in the diagram has areas of members as noted. What is the collapse load P_0?

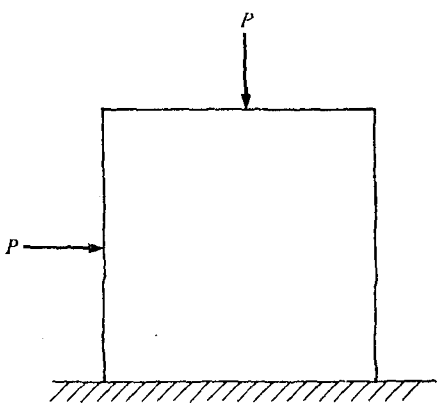


8.14 A rectangular, elasto-plastic beam with an axial load P is subjected to a bending moment M. Show by sketching the stress distribution that in the limit, for increasing M, the fully plastic stress distribution is made up of two rectangles. Show that the interaction diagrams of Fig. 8.9c is described by $M/M_0 = 1 - (P/P_0)^2$.

8.15 Compute the collapse load for the built-in beam shown in the diagram.



8.16 The square frame shown in the diagram is made of an elasto-plastic beam having the same cross section throughout. Try several collapse mechanisms and compute the collapse load P_0.



8.17 Try several collapse mechanisms for the square frame shown on page 322. All members have the same cross section.

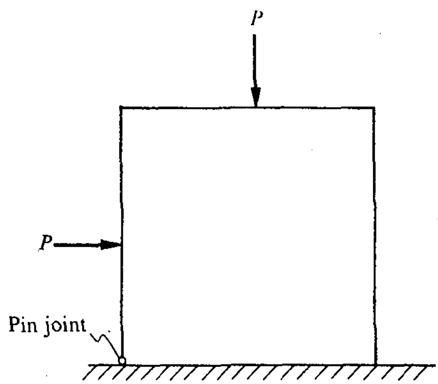
8.18 Derive Eq. (8.22) for the reduced modulus of elasticity.

8.19 Compute the completely plastic twisting moment for a circular bar in torsion and compare it with the maximum possible elastic moment.

8.20 The block shown in Fig. 8.19 is elongated by the applied stresses $\sigma_x = +\sigma$, $\sigma_y = -\sigma$. Compute the work done by these applied stresses and show that it agrees with Eq. (8.24).

8.21 A circular wire is drawn through a die to reduce its area from A_b to A_f. Making an approximate analysis along the lines of Fig. 8.20, show that the draw stress is given by

$$\sigma_f = \sigma_0\left(1 + \frac{1}{\beta}\right)\left[1 + \left(\frac{A_f}{A_b}\right)^\beta\right] + \sigma_b\left(\frac{A_f}{A_b}\right)^\beta$$



PROBLEM 8.17

CHAPTER 9

ELASTIC WAVE PROPAGATION

9-1 THE WAVE EQUATION

When a time-varying force acts upon an elastic body, the character of the resulting displacement depends strongly upon the rate at which the force varies. If the rate of variation is sufficiently slow the displacement at any instant is essentially the static displacement corresponding to the instantaneous value of the applied force; that is, the displacements of all points in the body are in phase with the force. If the rate of variation is somewhat greater, the displacements at different points in the body will usually be out of phase with the applied force and out of phase with each other. If the rate of variation is extremely high, as it would be in the case of a suddenly applied force, the initiation of displacement will travel outward from the point of application of the force and it is said that there is a propagating wave.

The nature of wave propagation is best explained in terms of the simplest elastic body, namely, a stretched elastic string. The string has a tension stress σ and an axial force $H = \sigma A$, as shown in Fig. 9.1a. The lateral displacement v in the x-y plane is assumed to be sufficiently small so that H can be taken to be a constant. The mass density is ρ, and the mass per unit length ρA, is constant over the length of the string. The freebody diagram of an element with positive slope and curvature, and having a mass ($\rho A\ dx$) is shown in Fig. 9.1b. The D'Alembert force (inertia force) is shown as a dotted arrow. The slope of the string ($\partial v/\partial x$) is assumed to be sufficiently small so that the length of the element can be taken to be dx rather than $ds = (dx^2 + dv^2)^{1/2}$. Equilibrium of the element is expressed by

$$H \frac{\partial^2 v}{\partial x^2} = \rho A \frac{\partial^2 v}{\partial t^2}$$

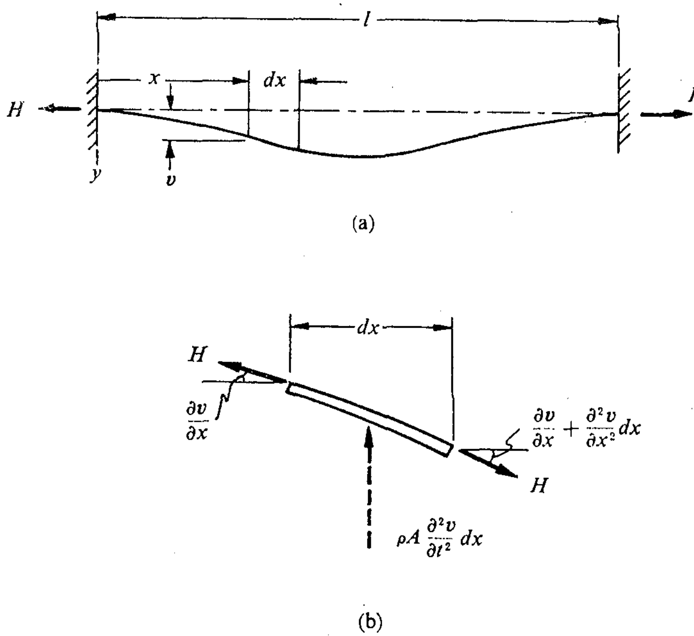


FIGURE 9.1

Using the notation $H/\rho A = \sigma/\rho = c^2$, the foregoing equation can be written

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} \tag{9.1}$$

This so-called *wave equation* was first derived by D'Alembert.* If a time-varying, distributed transverse load $p = f(x, t)$ acts on the string, the equation of motion is

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} - \frac{p}{H} \tag{9.1a}$$

* J. D'Alembert (1717–1783), "Researches on the curve formed by a stretched string set into motion," *Hist. Acad. Sci. Berlin* (1747). This paper also contained the solution of the equation in the form of wave propagation.

If $\partial p/\partial t$ is sufficiently small, the induced acceleration $\partial^2 v/\partial t^2$, will be negligibly small and the equation reduces to

$$\frac{d^2v}{dx^2} = -\frac{p}{H} \tag{9.1b}$$

which is the equation of equilibrium for a static load p. If the acceleration term cannot be neglected, Eqs. (9.1) and (9.1a) describe wave propagation, or vibration, depending upon the point of view we adopt.

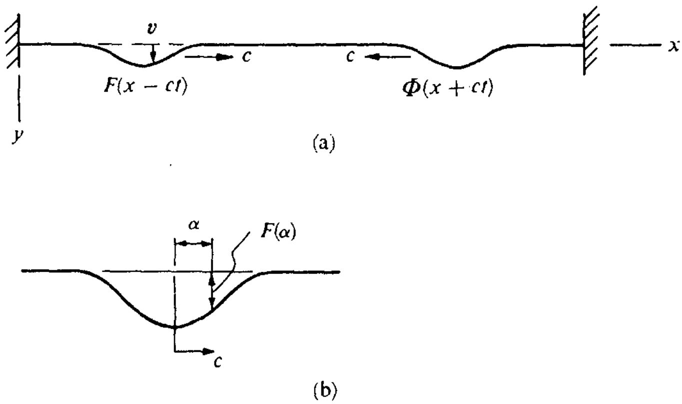


FIGURE 9.2

Wave Propagation. Equation (9.1) describes a physical system in which a displacement of specified shape can travel with constant velocity c. In other words, the solution of the equation can be written

$$v = F(\alpha) = F(x - ct) \tag{9.2}$$

where F is an arbitrary function* whose argument is $\alpha = (x - ct)$. As may be verified, this satisfies the wave equation. $F(x - ct)$ represents a disturbance of fixed shape traveling in the positive x-direction with velocity c. The function $\Phi(x + ct)$ also is a solution of the wave equation and it represents a disturbance traveling in the negative x-direction. These solutions are illustrated in Fig. 9.2a. As shown in (b), the variable α is measured in a

* $\partial F/\partial x = (dF/d\alpha)(\partial\alpha/\partial x) = F'$; $\partial F/\partial t = (dF/d\alpha)(\partial\alpha/\partial t) = -cF'$

coordinate system traveling with the wave. The pulse, $F(x - ct)$ is generated by moving the left support of the string in an appropriate manner. When the pulse $v = +F_1(x - ct)$ impinges on the fixed support it is reflected (reversed) as $v = -F_1(-x - ct)$. This reflection is illustrated in Fig. 9.3. The condition at the fixed support is $v = 0$ and this condition would be satisfied at a point in an infinite string if the two pulses shown in (b) reach the point at the same instant, for in this case the sum of the two is zero, as shown in (c). This process of two passing waves is, thus, the same as having the pulse $F(x - ct)$

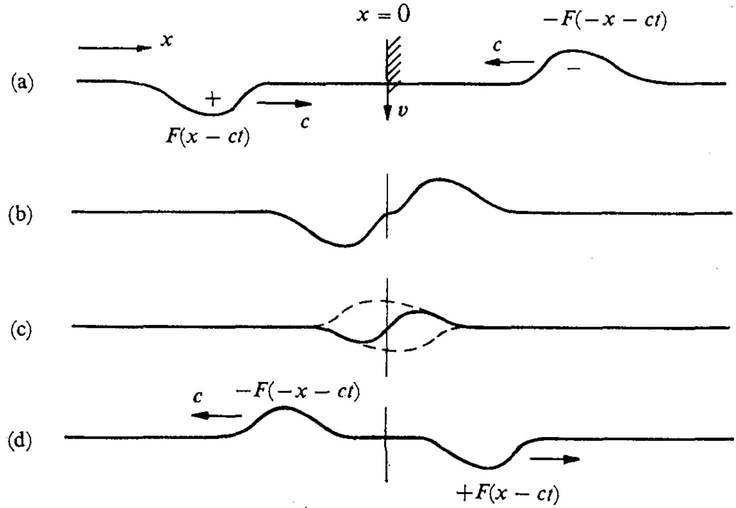


FIGURE 9.3

reflect from a fixed support. When the reflected pulse, $-F(-x - ct)$, reaches the left support it will reflect as $+F(x - ct)$ and will have made one complete cycle, that is, the system will have returned to its original state.

The stretched string can also have zero-slope as an end condition ($\partial v/\partial x = 0$), as shown in Fig. 9.4a, where the end of the string is attached to a frictionless slide. It is seen that in this case the pulse $F(x - ct)$ is reflected as $+F(-x - ct)$.

Vibrations. Instead of asking for the traveling wave solution of the wave equation we can take another point of view and seek a solution of the form

$$v = f(x)T(t)$$

Substituting this v into the wave equation and separating variables gives

$$c^2 \frac{d^2f(x)/dx^2}{f(x)} = \frac{d^2T(t)/dt^2}{T(t)}$$

The left-hand side of this equation is a function of x only while the right-hand side is a function of t only, so each must be equal to a constant which is taken to be $-\omega^2$. Thus,

$$\frac{d^2f(x)}{dx^2} + \frac{\omega^2}{c^2} f(x) = 0$$

and

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0$$

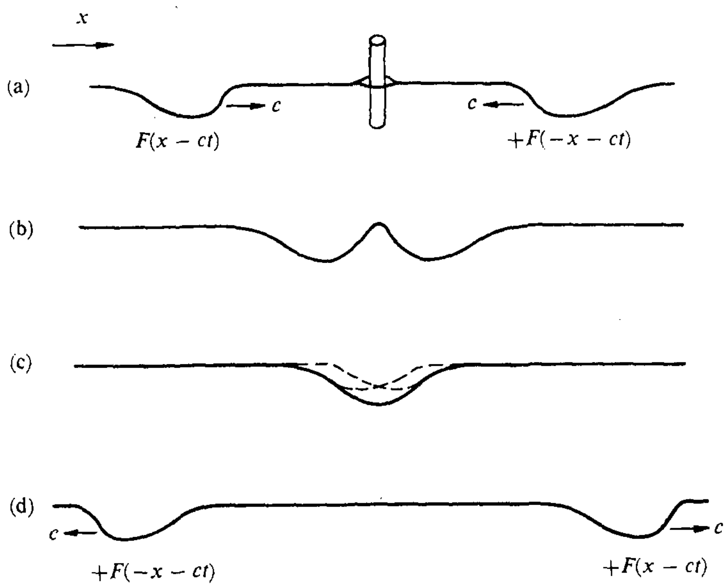


FIGURE 9.4

The solution of these equations may be written as

$$f(x) = A \sin \frac{\omega}{c} x + B \cos \frac{\omega}{c} x$$

and (9.3)

$$T(t) = A_1 \sin \omega t + B_1 \cos \omega t$$

The second of these expressions indicates a vibratory solution since $T(t)$ varies periodically with time. The first of these expressions contains three unknown constants, A, B, and ω, and in order to represent the stretched string with fixed supports these must have values that will satisfy the conditions $v = 0$ at $x = 0, l$. The first of these conditions gives $B = 0$ and the second gives

$$0 = A \sin \frac{\omega}{c} l$$

or

$$\frac{\omega}{c} l = n\pi$$

or

$$\omega = \omega_n = \frac{n\pi c}{l} \qquad (n = 1, 2, 3, \ldots) \tag{9.4}$$

There are thus an infinite number of terms of the form of Eq. (9.3) that will satisfy the wave equation and since superposition holds, the general solution for the string with fixed supports can be written

$$v = \sum_n C_n \sin \frac{n\pi}{l} x \left(\sin \frac{n\pi c}{l} t + D_n \cos \frac{n\pi c}{l} t \right) \tag{9.5}$$

The constants C_n and D_n are determined by the initial ($t = 0$) conditions. If $v = 0$ at $t = 0$, $D_n = 0$ and Eq. (9.5) becomes

$$v = \sum_n C_n \sin \frac{n\pi}{l} x \sin \frac{n\pi c}{l} t \tag{9.5a}$$

The constants C_n are determined by the velocity of the string at $t = 0$. The term $\sin n\pi x/l$ is the nth *normal mode* of vibration and it vibrates harmonically with a natural frequency $\omega_n/2\pi = nc/2l$ cycles per second. The first three modes are shown in Fig. 9.5. These are called standing waves as opposed to traveling waves. The vibratory solution of the wave equation was first given by Daniel Bernoulli.*

The relationship between Eq. (9.5a) and the traveling pulse $F(x - ct)$ can be seen by equating the two solutions at a particular time $t = t_0$:

$$\sum_n \left(C_n \sin \frac{n\pi c t_0}{l} \right) \sin \frac{n\pi}{l} x = F(x - ct_0)$$

* Daniel Bernoulli (1700–1782), "Reflections and enlightenments on the new vibrations of strings," *Hist. Acad. Sci. Berlin* (1953).

The left side of this equation is the Fourier expansion of $F(x - ct_0)$ and the coefficients of the series are given by

$$\left(C_n \sin \frac{n\pi c t_0}{l}\right) = \frac{\int_0^l F(x - ct_0) \sin (n\pi x/l)\, dx}{\int_0^l \sin^2 (n\pi x/l)\, dx}$$

The constants C_n are, therefore, related to the shape of the traveling pulse. It is thus seen that a problem can be analyzed either as a wave propagation or as a vibration (standing waves). An analysis of a problem as a wave propagation is usually simpler unless a number of reflections are involved, in which case a vibration analysis will usually be simpler. If the string is infinitely long, there can be no reflections and, hence, no standing waves and the problem cannot be analyzed as a vibration.

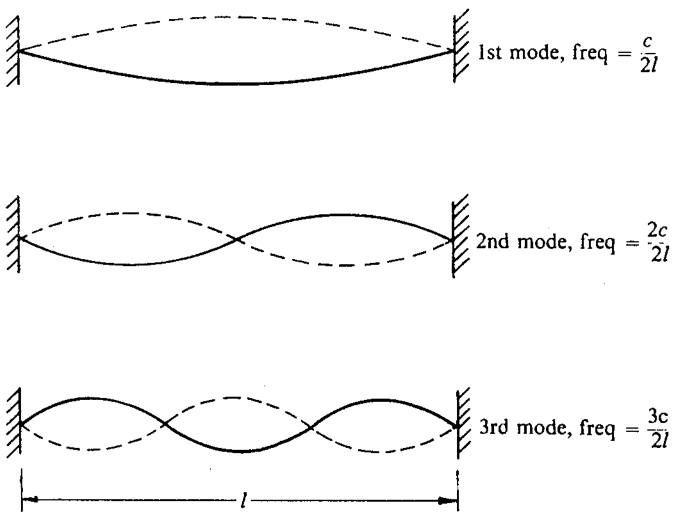


FIGURE 9.5

As an example of the explicit solution of a vibration problem we shall solve the case of a stretched string subject to the initial conditions $v = 0$, $dv/dt = \dot{v}_0$ at $t = 0$ where $\dot{v}_0$ is a uniform initial velocity. From Eq. (9.5)

$$\frac{dv}{dt} = \sum_n C_n \frac{n\pi c}{l} \sin \frac{n\pi}{l} x \cos \frac{n\pi c}{l} t$$

and at $t = 0$, $dv/dt = \dot{v}_0$ hence

$$\dot{v}_0 = \sum_n C_n \frac{n\pi c}{l} \sin \frac{n\pi}{l} x$$

The constants C_n must be determined so that the right-hand side of this equation is the Fourier series representation of $\dot{v}_0$. This requires

$$C_n \frac{n\pi c}{l} = \frac{\int_0^l \dot{v}_0 \sin \frac{n\pi}{l} x dy}{\int_0^l \sin^2 \frac{n\pi}{l} x dy} = \frac{\dot{v}_0 \frac{l}{n\pi}(1 - \cos n\pi)}{l/2}$$

$$C_n = \frac{4}{\pi^2} \frac{\dot{v}_0}{c} \frac{l}{n^2} \qquad (n = 1, 3, 5, \ldots)$$

Hence, the motion of the string for $t \geqq 0$ is

$$v = \frac{4\dot{v}_0 l}{\pi^2 c} \sum_n \frac{1}{n} \sin \frac{n\pi}{l} x \sin \frac{n\pi}{l} ct \tag{9.5b}$$

or

$$v = \frac{4}{\pi} \dot{v}_0 \sum_n \frac{1}{\omega_n} \sin \frac{n\pi}{l} x \sin \omega_n t \qquad (n = 1, 3, 5, \ldots)$$

Dispersive Waves. In the case of the stretched elastic string a pulse propagates without changing its shape. The reason for this is as follows. The pulse can be represented by a Fourier series:

$$v = F(x - ct) = \sum_n C_n \sin \frac{n\pi}{l} (x - c_n t)$$

Noting that $c_n = \lambda_n \omega_n / 2\pi$, this can be written

$$v = \sum_n C_n \sin \frac{2\pi}{\lambda_n} \left(x - \frac{\lambda_n \omega_n}{2\pi} t\right)$$

where λ_n is the wavelength and $\omega_n/2\pi$ is the frequency in cycles per second. If a typical term of the series, say,

$$\sin \frac{2\pi}{\lambda_n} \left(x - \frac{\lambda_n \omega_n}{2\pi} t\right)$$

is substituted in the differential equation of the string

$$\frac{\partial^2 v}{\partial x^2} = \frac{\rho}{\sigma} \frac{\partial^2 v}{\partial t^2}$$

there is obtained

$$\left(\frac{2\pi}{\lambda_n}\right)^2 \sin\frac{2\pi}{\lambda_n}\left(x - \frac{\lambda_n\omega_n}{2\pi}t\right) = \frac{\rho}{\sigma}\,\omega_n^2 \sin\frac{2\pi}{\lambda_n}\left(x - \frac{\lambda_n\omega_n}{2\pi}t\right)$$

which requires

$$\left(\frac{2\pi}{\lambda_n}\right)^2 = \frac{\rho}{\sigma}\,\omega_n^2$$

from which

$$\frac{\lambda_n\omega_n}{2\pi} = \sqrt{\frac{\sigma}{\rho}} = c_n = \text{constant}$$

The velocity of propagation $\sqrt{\sigma/\rho}$ is thus independent of the wavelength. Since all wavelengths travel with the same velocity, it follows that the pulse shape must be preserved.

A different result is obtained in the case of a beam. We may again suppose that the shape of a traveling wave is given by

$$v = \sum_n C_n \sin\frac{2\pi}{\lambda_n}\left(x - \frac{\lambda_n\omega_n}{2\pi}t\right)$$

However, when the *n*th term is substituted in the Bernoulli-Euler equation of a uniform beam*

$$EI\,\frac{\partial^4 v}{\partial x^4} = -\rho A\,\frac{\partial^2 v}{\partial t^2}$$

there is obtained

$$c_n = \frac{\lambda_n\omega_n}{2\pi} = \frac{2\pi}{\lambda_n}\sqrt{\frac{EI}{\rho A}}$$

This shows that the velocity of propagation c_n of a wave in a beam is inversely proportional to the wavelength λ_n. Because each sine wave travels with a different velocity, the shape of the pulse continually changes as indicated in Fig. 9.6, where it is seen that the wave spreads out (disperses) as it travels. A wave that behaves in this manner is said to be *dispersive*. The solutions of the wave equation (Eq. 9.1) are non-dispersive. A wave $\sin (2\pi/\lambda_n)(x - c_n t)$ is said to have a *phase velocity* c_n. A combination of such waves may form a dispersive pulse as shown in Fig. 9.6. If a pulse is composed of component sinusoidal waves whose wave lengths are close to λ_0, so that $\lambda_0 - \Delta\lambda \leq \lambda \leq \lambda_0 + \Delta\lambda$ the energy in the pulse may travel at a velocity different from

* See footnote in Chapter 3, page 111.

that of the phase velocity of the individual waves. In this case, the profile of the pulse travels with *group velocity*

$$c_g = \frac{d\omega}{d\left(\dfrac{2\pi}{\lambda}\right)} = c - \lambda \frac{dc}{d\lambda}$$

See Prob. 9.29 for the special case when the pulse is formed of two-component harmonic waves.

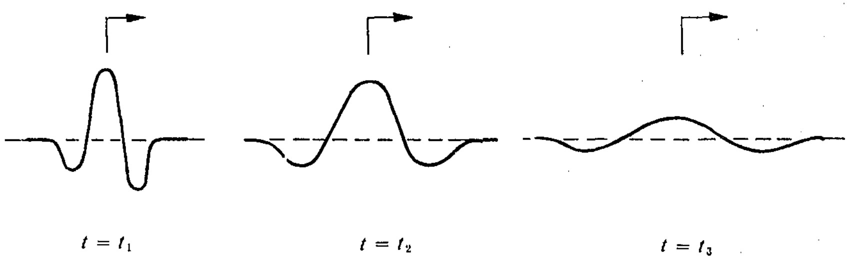


FIGURE 9.6

The equation for the velocity of propagation of a sine wave along a beam is sometimes called the frequency equation, as the same equation can arise when examining the problem from the vibration point of view. For example, a simply supported beam of length l can vibrate with normal modes

$$v = \sum_n C_n \sin \frac{n\pi}{l} x \sin \omega_n t$$

Each term of this solution satisfies the end conditions and if it is substituted in the differential equation there is obtained

$$EI\left(\frac{n\pi}{l}\right)^4 = \rho A \omega_n^2$$

This is the frequency equation which determines ω_n, and since $n\pi/l = 2\pi/\lambda_n$ it is seen that this equation is, indeed, the same as the equation that was derived from the wave propagation point of view. A dispersive system will always have a frequency equation of the form

$$\frac{\omega_n \lambda_n}{2\pi} = f(\omega_n) = f\left(\frac{2\pi c_n}{\lambda_n}\right)$$

A nondispersive system will have a frequency equation of the form

$$\frac{\omega_n \lambda_n}{2\pi} = c = \text{constant}$$

The Shear Beam. Certain physical systems are sometimes idealized as so-called *shear beams*. The beam shown in Fig. 9.7 is a shear beam and its distinguishing characteristic is that it undergoes only a shearing type of deformation with no bending deformation. Hooke's law for the beam relates the shearing force Q_x with the shearing deformation,* $\partial v/\partial x$

$$Q_x = k'AG\frac{\partial v}{\partial x} \tag{9.6}$$

where k' is a numerical constant depending upon the shape of the cross section. The freebody diagram of an element, shown in Fig. 9.7, indicates the D'Alembert force $\rho A\, \partial^2 v/\partial t^2$. Equilibrium requires

$$\frac{\partial Q_x}{\partial x} = \rho A \frac{\partial^2 v}{\partial t^2} \tag{9.7}$$

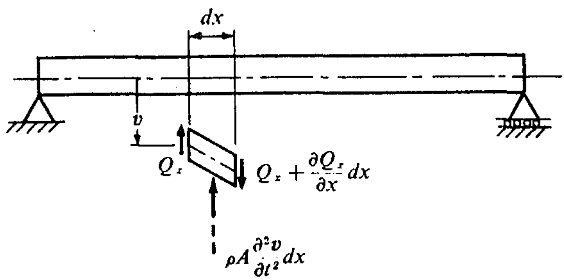


FIGURE 9.7

Substituting $Q_x = k'AG\, \partial v/\partial x$ into this equation, there is obtained

$$k'G\frac{\partial^2 v}{\partial x^2} = \rho\frac{\partial^2 v}{\partial t^2} \tag{9.8}$$

This is the wave equation and, hence, all of the remarks made about the stretched string also apply to the shear beam. The velocity of propagation along the shear beam is

$$c = \sqrt{\frac{k'G}{\rho}}$$

If a pulse $F(x - ct)$ travels along the beam, it can be shown that the kinetic energy of the pulse is equal to the strain energy in the pulse. The total energy transported by the pulse is the sum of the two.

* The displacement v is that of the neutral axis of the beam and $\partial v/\partial x$ is γ_{xy} at the neutral axis, as determined in the technical theory of bending.

Axial Stress Propagation. Another especially simple example of the wave equation is the idealized stretched, elastic string shown in Fig. 9.8, subjected to disturbances in the form of axial tension or compression; that is, the displacements are $u = f(x, t)$, $v = w = 0$. Hooke's law for the stretched string is

$$H = H_0 + EA \frac{\partial u}{\partial x} \tag{9.9}$$

where H_0 is the initial tension force in the string and A is the cross-sectional area. From the freebody diagram of an element shown in Fig. 9.8, it is seen that equilibrium requires

$$\frac{\partial H}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{9.10}$$

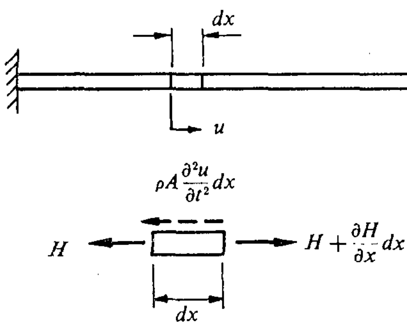


FIGURE 9.8

Substituting for H from Eq. (9.9) there is obtained

$$E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \tag{9.11}$$

This is again the wave equation with velocity of propagation:

$$c = \sqrt{\frac{E}{\rho}}$$

It should be noted that the foregoing analysis is somewhat idealized since nothing is stated about the displacements normal to the axis of the string. If Poisson's ratio is zero, there are no lateral displacements or strains, or if the diameter of the string approaches zero the lateral displacements approach zero. Wave propagation in a rod of finite diameter with nonzero Poisson's ratio is discussed in Section 9-7.

9-2 GENERAL EQUATIONS OF MOTION

When elastic bodies undergo time-varying stresses and strains the D'Alembert body force is

$$-\rho \frac{\partial^2 \mathbf{d}}{\partial t^2}$$

where ρ is the density of the material and $\mathbf{d}$ is the vector displacement

$$\mathbf{d} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

The components of the D'Alembert force are

$$X = -\rho \frac{\partial^2 u}{\partial t^2} \qquad Y = -\rho \frac{\partial^2 v}{\partial t^2} \qquad Z = -\rho \frac{\partial^2 w}{\partial t^2}$$

The general equations of equilibrium with D'Alembert body forces are:

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} &= \rho \frac{\partial^2 v}{\partial t^2} \\
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} &= \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{9.12}$$

When these equations are expressed in terms of the displacements (see Eqs. (2.16)), they assume the form:

$$\begin{aligned}
(\lambda + G) \frac{\partial e}{\partial x} + G\nabla^2 u &= \rho \frac{\partial^2 u}{\partial t^2} \\
(\lambda + G) \frac{\partial e}{\partial y} + G\nabla^2 v &= \rho \frac{\partial^2 v}{\partial t^2} \\
(\lambda + G) \frac{\partial e}{\partial z} + G\nabla^2 w &= \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{9.13}$$

In these equations e is the dilatation

$$e = \epsilon_x + \epsilon_y + \epsilon_z$$

and λ is Lamé's constant

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} = \frac{2\nu G}{1 - 2\nu}$$

Equations (9.13) tell us that there are two characteristic wave velocities for an infinite, homogeneous, isotropic, linearly elastic medium. This conclusion can be deduced as explained in the following paragraphs.

Dilatational Waves. When Eqs. (9.13) are differentiated respectively by x, y, z and added, there is obtained the single equation

$$\nabla^2 e = \frac{1}{c_1^2}\frac{\partial^2 e}{\partial t^2} \tag{9.14}$$

$$c_1 = \sqrt{\frac{\lambda + 2G}{\rho}} = \sqrt{\frac{G}{\rho}\frac{(2 - 2\nu)}{(1 - 2\nu)}}$$

This is the three-dimensional wave equation with velocity of propagation c_1. For example, the solution $e = F(x - c_1 t)$ satisfies Eq. (9.14) and represents a plane wave traveling in the x-direction (at any x, e is the same for all y and z). The solution $e = (1/r)F(r - ct)$, where $r = (x^2 + y^2 + z^2)^{1/2}$, will satisfy the wave equation (Eq. (9.14)), and represents a wave of spherical symmetry traveling away from the point of origin.* The amplitude of the wave decreases as $1/r$. This is, of course, required if energy is to be conserved. The total strain energy plus kinetic energy in a spherical shell of fixed thickness dr and increasing radius $r = c_1 t$ remains constant. Since the volume of this shell increases as r^2 the energy density must decrease as $1/r^2$ and, hence, the displacements and strains must decrease as $1/r$.

Dilatational waves are sometimes referred to as compressional waves. It is seen that as $\nu \to \frac{1}{2}$ (incompressible material), $c_1 \to \infty$.

Rotational Waves. When the first two of Eqs. (9.13) are differentiated respectively by y and x, and the first equation is subtracted from the second there is obtained

$$G\nabla^2\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

* Note that in spherical coordinates with polar symmetry the Laplacian is

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)$$

Recalling that the rotation is defined by

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

We see that the preceding equation can be written

$$G\nabla^2\omega_z = \rho\,\frac{\partial^2\omega_z}{\partial t^2}$$

A similar equation holds for ω_x and ω_y, so, in vector form the rotation satisfies the wave equation

$$\nabla^2\boldsymbol{\omega} = \frac{1}{c_2^2}\,\frac{\partial^2\boldsymbol{\omega}}{\partial t^2}$$
$$c_2 = \sqrt{\frac{G}{\rho}} \tag{9.15}$$

Rotational waves thus propagate with velocity c_2 which is smaller than c_1. For a Poisson's ratio equal to 0.25 the rotational velocity of propagation is given in terms of the dilatational velocity by $c_2 = c_1/\sqrt{3}$.

Shear Waves. If a wave is generated so that the displacement $\mathbf{d}$ has only shearing strains associated with it, the dilatation e, will be zero, and Eqs. (9.13) state that

$$\nabla^2\,\mathbf{d}_s = \frac{1}{c_2^2}\,\frac{\partial^2\,\mathbf{d}_s}{\partial t^2}$$
$$c_2 = \sqrt{\frac{G}{\rho}} \tag{9.16}$$

where $\mathbf{d}_s$ is any displacement for which the dilatation is zero. Shearing strains thus propagate with the same velocity as rotations, and $\mathbf{d}_s$ will in general involve both shear and rotation.

9-3 DISPLACEMENT POTENTIAL FUNCTIONS

The propagation of dilatational, rotational, and shear waves is treated in a unified manner by introducing the scalar potential φ and the vector potential $\boldsymbol{\psi} = \psi_1\mathbf{i} + \psi_2\mathbf{j} + \psi_3\mathbf{k}$. As in the plane stress problem and in the torsion problem, the introduction of potential functions simplifies the mathematical treatment. The equations of elasticity can be satisfied by taking the displacement to be given by

$$\mathbf{d} = \operatorname{grad} \varphi + \operatorname{curl} \boldsymbol{\psi} \tag{9.17}$$

This specifies the x, y, z components of displacement to be

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z} \\ v &= \frac{\partial \varphi}{\partial y} + \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x} \\ w &= \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} \end{aligned} \tag{9.18}$$

The dilatation is expressed by

$$e = \nabla^2 \varphi \tag{9.19}$$

When the first of the equations of equilibrium (9.13) is expressed in terms of φ and ψ, and the terms are collected, the following equation is obtained

$$\frac{\partial}{\partial x}\left[(\lambda + 2G)\nabla^2\varphi - \rho \frac{\partial^2 \varphi}{\partial t^2}\right] + \frac{\partial}{\partial y}\left[G\nabla^2\psi_3 - \rho \frac{\partial^2 \psi_3}{\partial t^2}\right] - \frac{\partial}{\partial z}\left[G\nabla^2\psi_2 - \frac{\partial^2 \psi_2}{\partial t^2}\right] = 0$$

For this equation to be satisfied it is sufficient that each of the expressions in the brackets be zero. Similar equations are obtained from the second and third equations of equilibrium (9.13). We conclude, therefore, that if φ and $\boldsymbol{\psi}$ satisfy the equations

$$\begin{aligned} \nabla^2 \varphi &= \frac{1}{c_1^2} \frac{\partial^2 \varphi}{\partial t^2} \\ \nabla^2 \boldsymbol{\psi} &= \frac{1}{c_2^2} \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} \end{aligned} \tag{9.20}$$

the equations of elasticity will be satisfied. The second of equations (9.20) states that displacements and strains that are equivoluminal (no volume change) will propagate with velocity c_2; such waves will involve only shearing strains and rotational displacements. The first of equations (9.20) states that a displacement that is the gradient of a scalar function φ will propagate with velocity c_1. In general,* since $e = \operatorname{div} \mathbf{d}$ and $\boldsymbol{\omega} = \frac{1}{2} \operatorname{curl} \mathbf{d}$ and since $\operatorname{curl}(\operatorname{div} \mathbf{d}) = 0$ and $\operatorname{div}(\operatorname{curl} \mathbf{d}) = 0$ it is seen that *rotational waves* travel with velocity c_2 and *irrotational waves* travel with velocity c_1. Some examples of irrotational and rotational waves are discussed in the following section.

9-4 PLANE WAVES IN AN INFINITE CONTINUUM

A planar wave (planar strain) is defined as a wave in which the displacement depends only upon one coordinate and time, i.e., displacement $= F(x, t)$. Plane waves in an infinite medium can be of two types. One has the displacement normal to the plane of the wave (irrotational), that is, the displacement is in the direction of propagation. The second type has the displacement parallel to the plane of the wave (rotational), that is, normal to the direction of propagation. The two types of displacement are represented by:

$$\text{TYPE 1} \quad u_1 = F(x, t); \quad v_1 = w_1 = 0$$

$$\text{TYPE 2} \quad u_2 = 0; \quad v_2 = F(x, t); \quad w_2 = 0$$

The Type 1 displacement represents a state of planar strain with $\epsilon_y = \epsilon_z = \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$, and for this state of stress Hooke's law (Eqs. (1.34)) reduces to

$$\begin{aligned} \sigma_x &= (\lambda + 2G)\frac{\partial u_1}{\partial x} \\ \sigma_y = \sigma_z &= \frac{\lambda}{\lambda + 2G}\sigma_x \end{aligned} \tag{9.21}$$

The equations of equilibrium (9.12) reduce to

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 u_1}{\partial t^2}$$

*Note that $\operatorname{div} \mathbf{d} = \nabla \cdot \mathbf{d} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \mathbf{d}$

$$\tfrac{1}{2} \operatorname{curl} \mathbf{d} = \tfrac{1}{2} \nabla \times \mathbf{d} = \boldsymbol{\omega}$$

Substituting for σ_x from Eq. (9.21), the equilibrium equation becomes

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{c_1^2}\frac{\partial^2 u_1}{\partial t^2}$$
$$c_1 = \sqrt{\frac{\lambda + 2G}{\rho}} \tag{9.22}$$

The general solution of Eq. (9.22) for propagation in the positive x-direction can be written

$$u_1 = F(x - c_1 t)$$

The particle velocity is given by

$$\frac{\partial u_1}{\partial t} = -c_1 F'(x - c_1 t)$$

where F' is defined by

$$F'(\alpha) = \frac{dF(\alpha)}{d\alpha}, \quad \alpha = x - c_1 t$$

The strain is given by

$$\epsilon_x = \frac{\partial u_1}{\partial x} = F'(x - c_1 t)$$

The corresponding stress is

$$\sigma_x = (\lambda + 2G)\epsilon_x = \rho c_1^2 F'(x - c_1 t)$$
$$\sigma_x = \rho c_1^2 \frac{\partial u_1}{\partial x} = -\rho c_1 \frac{\partial u_1}{\partial t} \tag{9.23}$$

Equation (9.23) relates the stress to the particle velocity in the wave.

This wave is called a plane tension or compression wave. Seismologists designate such a wave as a P wave (Primary wave) because having the largest propagation velocity c_1, it appears on the seismogram first. The displacements associated with such a wave may be derived from the scalar potential φ, with $\Psi = 0$. This wave has both dilatation and distortion.

For the Type 2 displacements all of the strains are zero except γ_{xy} and Hooke's law reduces to

$$\tau_{xy} = G\gamma_{xy} = G\frac{\partial v_2}{\partial x}$$

The equations of equilibrium reduce to

$$\frac{\partial \tau_{xy}}{\partial x} = \rho \frac{\partial^2 v_2}{\partial t^2}$$

Substituting for τ_{xy} there is obtained

$$\frac{\partial^2 v_2}{\partial x^2} = \frac{1}{c_2^2} \frac{\partial^2 v_2}{\partial t^2}$$

The solution of this equation is

$$v_2 = F(x - c_2 t) + G(x + c_2 t)$$

This wave is a plane shear wave. Seismologists distinguish between vertical and horizontal shear waves according to whether the displacement v_2 is vertical or horizontal. They are called SV and SH waves, respectively. The displacements associated with shear waves may be derived from the vector potential Ψ with $\varphi = 0$. These waves have both shear and rotation.

Plane waves satisfy the conditions for non-dispersive wave propagation as do also certain spherical waves and cylindrical waves. However, a transient disturbance in an infinite medium that does not satisfy these conditions will generate a wave whose front travels with velocity c_1 and whose rear travels with velocity c_2 but the distribution of stress and strain behind the front will be continually changing. The solution of wave problems, except for the simplest cases, leads to very complicated mathematical analysis.

9-5 WAVES IN NONHOMOGENEOUS MEDIA

When the elastic constants do not have the same values throughout the material the nature of the wave propagation becomes much more complex.* The simplest nonhomogeneous medium is one composed of two materials having elastic moduli E, G, for $x > 0$ and E_1, G_1 for $x < 0$; that is, there are two layers with boundary at $x = 0$. One extreme for this case is $E_1 = G_1 = 0$, which represents a half-space with a free boundary at $x = 0$. The other extreme is $E_1 = G_1 = \infty$, which represents a half-space with a rigid boundary at $x = 0$. When one or more boundaries are involved, the possibility of wave reflection and refraction at a boundary arises. When boundaries are present it also becomes possible to analyze the problem from the point of view of vibrations and normal modes.

* W. M. Ewing, W. S. Jardetsky, and F. Press, *Elastic Waves in Layered Media*, McGraw-Hill (1957).

The simplest case of reflection and refraction at a boundary is that of a plane wave approaching a boundary that is normal to the direction of propagation, and we shall examine this case in detail. Let the boundary be the yz-plane as shown in Fig. 9.9 and let the incident wave have displacements

$$u_1 = A_1 \sin \frac{2\pi}{\lambda_a}(x - c_a t)$$

$$v_1 = w_1 = 0$$

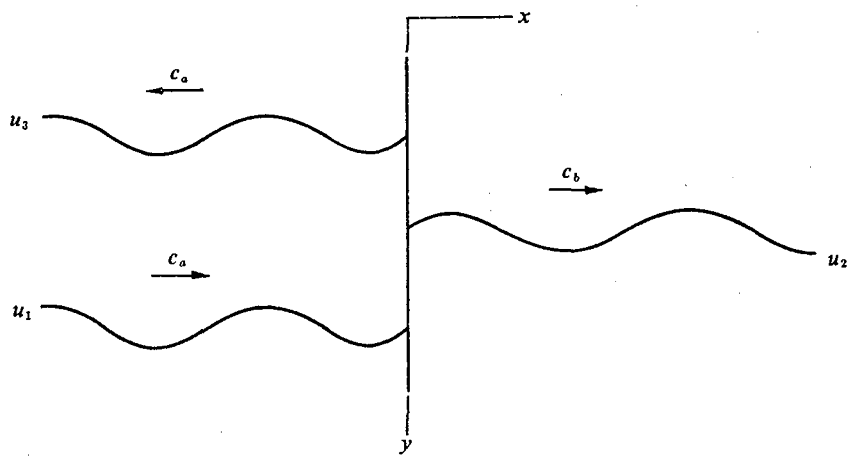


FIGURE 9.9

Let the refracted wave have displacements

$$u_2 = A_2 \sin \frac{2\pi}{\lambda_b}(x - c_b t)$$

$$v_2 = w_2 = 0$$

The reflected wave will have displacements

$$u_3 = A_3 \sin \frac{2\pi}{\lambda_a}(x + c_a t)$$

$$v_3 = w_3 = 0$$

At an interface the stresses must be continuous and the displacements must be continuous. In the present case the first of these conditions requires that the stress as computed in each medium must be the same at the interface, that is,

$$\sigma_1 + \sigma_3 = \sigma_2$$

or, in terms of the displacements, using Eq. (9.23)

$$\rho_a c_a^2 \frac{\partial(u_1 + u_3)}{\partial x} = \rho_b c_b^2 \frac{\partial u_2}{\partial x}$$

The second condition requires that the particle velocity computed in each medium must be the same at the interface

$$\frac{\partial(u_1 + u_3)}{\partial t} = \frac{\partial u_2}{\partial t}$$

Substituting for u_1, u_2, and u_3, setting $x = 0$ and canceling terms, these two conditions become

$$\frac{\rho_a c_a^2}{\lambda_a}(A_1 + A_3)\cos 2\pi \frac{c_a}{\lambda_a} t = \frac{\rho_b c_b^2}{\lambda_b} A_2 \cos 2\pi \frac{c_b}{\lambda_b} t$$

$$\frac{c_a}{\lambda_a}(-A_1 + A_3)\cos 2\pi \frac{c_a}{\lambda_a} t = -\frac{c_b}{\lambda_b} A_2 \cos 2\pi \frac{c_b}{\lambda_b} t$$

It is seen that in order for these equations to be satisfied it is necessary that $c_a/\lambda_a = c_b/\lambda_b$, that is, as viewed from fixed points, the frequencies of u_1 and u_2 as they pass must be the same. The preceding two equations, therefore, reduce to

$$A_1 + A_3 = \frac{\rho_b c_b}{\rho_a c_a} A_2$$

$$-A_1 + A_3 = -A_2$$

From these it is found that the amplitude ratio of reflected to incident wave is

$$\frac{A_3}{A_1} = \frac{1 - \rho_a c_a/\rho_b c_b}{1 + \rho_a c_a/\rho_b c_b} \tag{9.24a}$$

and the amplitude of refracted to incident wave is

$$\frac{A_2}{A_1} = \frac{2}{1 + \rho_b c_b/\rho_a c_a} \tag{9.24b}$$

It is seen that the reflection and refraction are governed by the ratio $\rho_a c_a/\rho_b c_b$, where the subscript b refers to the medium containing the refracted wave. The same result would be obtained for an incident wave of arbitrary shape since it can be expanded into a series of sine terms. The same result is also obtained for an incident plane shear wave. The product of mass density and wave velocity ρc is called the *acoustic impedance* of the medium.

The amplitude ratio for the reflected wave ranges from $+1$ for a rigid boundary $(\rho_a c_a/\rho_b c_b \to 0)$, to -1 for a free boundary $(\rho_a c_a/\rho_b c_b \to \infty)$. The refracted wave has an amplitude ratio that ranges from zero for $(\rho_b c_b/\rho_a c_a \to \infty)$, to $+2$ for $(\rho_b c_b/\rho_a c_a \to 0)$. It is thus seen that if the wave passes from one medium to another which has a much lower value of ρc, the amplitude of the transmitted wave is essentially doubled; that is, it has approximately the same amplitude as does the surface at a free boundary when the wave is completely reflected. The foregoing remarks refer to the amplitude of the displacement; the stress reflects and refracts in a manner just opposite to the displacement.

A more general case of reflection and refraction occurs when the incident wave approaches the boundary surface at an oblique angle. In this case it is found that an incident wave of either dilatation or shear will usually reflect both a dilatation and a shear wave with different directions of propagation, and will also refract a dilatation and a shear wave with different directions of propagation. An oblique plane wave reflecting from a free surface will usually reflect both a dilatational and a shear wave. However, for certain special angles there is only one reflected wave and this is of opposite kind; this process is called *mode conversion*. Below a certain critical angle of incidence for a shear wave the nature of the reflected waves changes and motion is obtained that decreases exponentially with distance below the surface. Such waves which propagate along interfaces or surfaces are treated in the following section.

9-6 SURFACE WAVES

It can be shown that a wave can be propagated along an interface between two media under certain conditions with a velocity different from the velocities c_1 and c_2 in either medium. The simplest example is when the boundary is a free surface lying in the xy-plane as shown in Fig. 9.10, where a plane wave is traveling in the x-direction. This so-called Rayleigh wave, named after its discoverer, has an amplitude of motion diminishing exponentially with z. As may be verified, the general equations of motion are satisfied by the solution

$$\varphi = Ae^{-\alpha_n z} \sin \frac{2\pi}{\lambda_n}(x - ct)$$

$$\psi_2 = Be^{-\beta_n z} \cos \frac{2\pi}{\lambda_n}(x - ct) \tag{9.25}$$

$$\psi_1 = \psi_3 = 0$$

where

$$\alpha_n = \frac{2\pi}{\lambda_n}\sqrt{1 - \frac{c^2}{c_1^2}}$$

$$\beta_n = \frac{2\pi}{\lambda_n}\sqrt{1 - \frac{c^2}{c_2^2}}$$

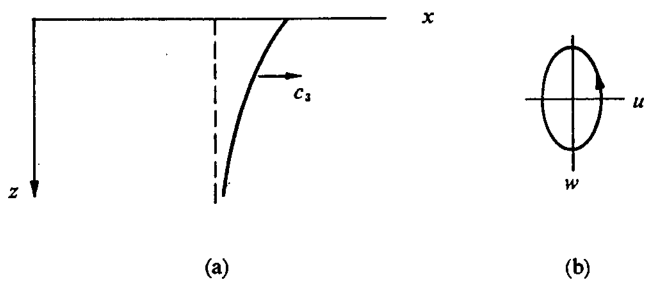


FIGURE 9.10

A particle traverses a path as shown in Fig. 9.10b. There are three unknown constants in the solution, A, B, and c and these are used to satisfy the boundary conditions $\sigma_z = \tau_{xz} = 0$ at $z = 0$. The displacements are given by

$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi_2}{\partial z}$$

$$v = 0$$

$$w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_2}{\partial x}$$

Substituting these in Hooke's law (Eqs. 1.34) there is obtained

$$\begin{aligned}\sigma_z &= (\lambda + 2G)\left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi_2}{\partial x\, \partial z}\right) + \lambda\left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \psi_2}{\partial x\, \partial z}\right) \\ \tau_{xz} &= G\left(2\,\frac{\partial^2 \varphi}{\partial x\, \partial z} - \frac{\partial^2 \psi_2}{\partial z^2} + \frac{\partial^2 \psi_2}{\partial x^2}\right)\end{aligned} \tag{9.26}$$

When the Rayleigh solution is substituted in these equations and σ_z and τ_{xz} are set equal to zero at $z = 0$ the following pair of equations is obtained

$$\left(2 - \frac{c^2}{c_2^2}\right)A + 2\sqrt{1 - \frac{c^2}{c_2^2}}\,B = 0$$
$$2\sqrt{1 - \frac{c^2}{c_1^2}}\,A + \left(2 - \frac{c^2}{c_2^2}\right)B = 0 \tag{9.27}$$

The amplitude ratio of φ to ψ_2 is

$$\frac{A}{B} = \frac{2\sqrt{1 - c^2/c_2^2}}{(2 - c^2/c_2^2)}$$

The determinant of the coefficients of Eqs. (9.27) must equal zero, which gives the following equation for determining c

$$\left(\frac{c^2}{2c_2^2} - 1\right)^4 = \left(1 - \frac{c^2}{c_1^2}\right)\left(1 - \frac{c^2}{c_2^2}\right) \tag{9.28}$$

It can be shown that this equation has a real root:

$$0 < c_3 < c_2$$

providing that Poisson's ratio lies between 0 and 0.5. The Rayleigh wave velocity c_3 varies from approximately 87% to 96% of c_2 as Poisson's ratio varies from 0 to 0.5. For $\nu = 0.25$ the surface wave velocity is $c_3 = 0.919c_2$.

A special feature of the Rayleigh wave is that its energy is concentrated near the free surface (stresses drop off exponentially with z). Therefore, if there is generated a Rayleigh wave symmetrical about the z-axis the energy density must diminish inversely as the distance r from the z-axis, whereas, in the case of body waves, the energy density of dilatational and shear waves diminishes inversely as the square of the distance from the point of origin, since they propagate spherically. It can, therefore, be inferred that at large distances from the point of origin of such a surface disturbance the amplitude of the Rayleigh wave at the surface will be larger than the amplitudes of the dilatational and shear waves. This is, indeed, borne out by the seismograph records of earthquake ground motions. At large distances from the epicenter of an earthquake the first wave to be recorded is the dilatational or P wave. Because of its slower velocity the shear wave (S-wave) arrives later but usually with a larger amplitude because of the shear-type mechanism which generates earthquake motions. The last to arrive is the Rayleigh wave with amplitude larger than the S-wave. It is therefore possible to compute the distance to the point of origin of the earthquake using the different arrival times of the waves when c_1, c_2, and c_3 are known. In sound granite c_1 is approximately 18,000 to 20,000 ft/sec; whereas in firm alluvium it may be 2000 to 4000 ft/sec, and in very soft soil it may be as low as 200 to 400 ft/sec.

A. E. H. Love showed that in the case of a uniform elastic layer overlying a rigid base, it is possible for shear waves to propagate in the layer with displacements parallel to the plane of the layer and perpendicular to the direction of propagation (Love waves). R. Stoneley showed that a wave associated with the interface between two different media will propagate parallel to the interface if the properties of the material satisfy certain conditions (Stoneley waves). Also, if a layer of one material is sandwiched between two wide layers of another material, a wave associated with the single layer and the two interfaces can, under certain conditions, propagate along the layer (channel waves).

9-7 LONGITUDINAL WAVES IN RODS

The axially propagated waves in a rod are dispersive, and hence the analysis of this problem is very complex. That this is so can be made physically plausible by considering the problem of a circular rod of radius R that is

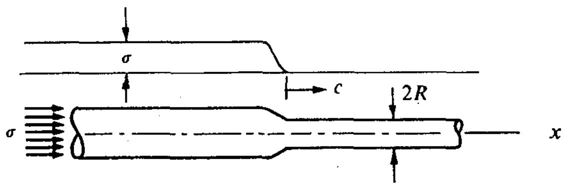


FIGURE 9.11

subjected to a step-function force at one end as shown in Fig. 9.11. If the waves were nondispersive, there would be a step function stress σ traveling down the bar with velocity c. Traveling with the wave front there would be a discontinuity in radius (shown exaggerated in the figure) because of the lateral strain $\epsilon_r = \nu\sigma_x/E$. Any abrupt change in radius would be associated with large shearing stresses which are generated by the wave front. The shearing stresses and strains must therefore be included in the analysis and, in fact they become particularly important when the wave front is sharp, i.e. when the front contains components with wavelengths of the order of the bar diameter. An approximate solution for wavelengths long compared to the radius of the bar gives the frequency equation*

$$c = c_0\left(1 - \nu^2\pi\frac{R}{\lambda^2}\right) \tag{9.29}$$

* H. Kolsky, *Stress Waves in Solids*, Oxford University Press (1953).

where $c_0 = \sqrt{E/\rho}$, and λ is the wave length. It is seen that for very long wavelengths ($R/\lambda \ll 1$) the velocity of propagation may be taken to be a constant $c \to c_0$, and these waves are nondispersive. This agrees with the result obtained for axial stress propagation along a stretched string Eq. (9.11) which is valid only if all components of the propagating wave have wavelengths that are large compared to the diameter of the string, that is if $R/\lambda \ll 1$.

Let us assume that the wave equation is applicable so that when the end of the rod, shown in Fig. 9.11, is given a sinusoidal displacement $u = u_0 \sin \omega t$ the displacement at any point x is given by

$$u = u_0 \sin \omega\left(t - \frac{x}{c_0}\right)$$

The velocity of propagation is $c_0 = \omega\lambda/2\pi$ and, hence, the wavelength can be written $\lambda = 2\pi c_0/\omega$ and the condition $R/\lambda \ll 1$ can be expressed as

$$\frac{R}{\lambda} = \frac{R\omega}{c_0 2\pi} = \frac{\tau}{T} \ll 1$$

where T is the period of the wave and τ is the time required for the wave to travel a distance R.

Experimental measurements of strain waves in rods show that so long as the rise time of the wave is at least six times greater than the time for a c_0 wave to travel one rod diameter, the wave equation is a good approximation for the problem, and no appreciable dispersion of the wave occurs.

9-8 VIBRATION OF BEAMS

It was seen in Section 9-1 that the flexural waves of a beam are highly dispersive. Because of this it is customary to analyze beam problems from the point of view of vibrations. For example, the Bernoulli-Euler equation for a uniform beam is

$$EI\frac{\partial^4 v}{\partial x^4} = -\rho A\frac{\partial^2 v}{\partial t^2}$$

A vibratory solution of this equation must be of the form

$$v = f(x) \sin \omega t$$

which, when substituted in the differential equation, gives

$$\frac{d^4 f(x)}{dx^4} = \frac{\rho A \omega^2}{EI} f(x)$$

The solution of this equation is

$$f(x) = A \sin \alpha x + B \cos \alpha x + C \sinh \alpha x + D \cosh \alpha x$$

where

$$\alpha = \sqrt[4]{\frac{\rho A \omega^2}{EI}}$$

There are five unknown constants in this equation, A, B, C, D, and ω. A total of four boundary conditions can be imposed and these will give four equations from which B/A, C/A, D/A, and ω can be determined. The equation which determines ω will be a transcendental equation (the frequency equation) whose roots $\omega_1, \omega_2, \ldots, \omega_n, \ldots$ are called the natural frequencies of the system. Each ω_n will determine the amplitude ratios so that the general solution can be written*

$$v = \sum_n A_n f_n(x, \omega_n) \sin \omega_n t$$

The function $f_n(x, \omega_n)$ satisfies the boundary conditions and it gives the shape of the nth normal mode of vibration. The natural frequency of the nth mode is $(\omega_n/2\pi)$ cycles per second. As the name implies, the normal modes have the property of orthogonality

$$\int_0^l f_n f_m \, dx = 0 \qquad (n \neq m)$$

The Bernoulli-Euler beam equation is not entirely satisfactory in describing the dynamic behavior of a beam. For example, the frequency $\omega_n/2\pi$ becomes infinite as the wavelength goes to zero ($n \to \infty$); or, in other words, the velocity of propagation becomes infinite. The solution of the Bernoulli-Euler equation is valid only if the higher modes are not excited.

The Bernoulli equation was derived by substituting $-\rho \, \partial^2 v/\partial t^2$ for the load in the static beam equation. However, for short wavelengths and high frequencies the angular acceleration of the beam cross section is large, and because of this each element of beam has a D'Alembert moment acting on it, as shown in Fig. 9.12. The resulting equation for a uniform beam is

$$EI \frac{\partial^4 v}{\partial x^4} = -\rho A \frac{\partial^2 v}{\partial t^2} + \rho I \frac{\partial^4 v}{\partial x^2 \, \partial t^2} \tag{9.30}$$

* For a simply supported beam of length l the mode shapes and frequencies are

$$f_n(x) = \sin \frac{n\pi}{l} x \qquad \omega_n = \left(\frac{n\pi}{l}\right)^2 \sqrt{\frac{EI}{\rho}}$$

This is called the Rayleigh beam equation. The last term represents the effect of rotary inertia of the element.

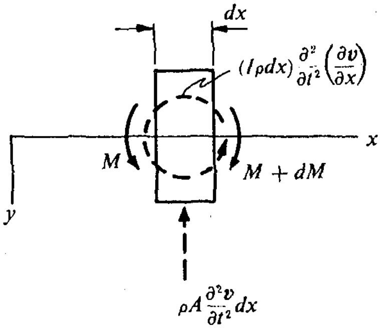


FIGURE 9.12

Since short wavelengths (higher modes) are associated with high shearing strains this effect should properly be included in the analysis. The shearing deformation of a beam is described by the equation

$$Q_x = k'AG\frac{\partial v_s}{\partial x}$$

where k' is a numerical factor depending upon the shape of the cross section, and v_s is the displacement produced by the shearing deformation ($v_s \to 0$ as $G \to \infty$). The bending deformation also contributes to the displacement, so that the total displacement is

$$v = v_b + v_s$$

The equation of vertical equilibrium of the element is

$$\frac{\partial Q_x}{\partial x} = \rho A\frac{\partial^2 v}{\partial t^2}$$

or

$$k'AG\frac{\partial^2 v_s}{\partial x^2} = \rho A\frac{\partial^2 v}{\partial t^2} \tag{9.31a}$$

The moment equation of equilibrium of the element reduces to

$$\frac{\partial M}{\partial x} - \rho I\frac{\partial^3 v_b}{\partial x\,\partial t^2} + Q_x = 0$$

or

$$EI\frac{\partial^4 v_b}{\partial x^4} - \rho I\frac{\partial^4 v_b}{\partial x^2\,\partial t^2} + k'AG\frac{\partial^2 v_s}{\partial x^2} = 0 \tag{9.31b}$$

The second term in this equation arises from the rotary inertia. By suitably combining Eqs. (9.31a) and (9.31b) there is obtained

$$EI\frac{\partial^4 v}{\partial x^4} - \rho I\left(1 + \frac{E}{k'G}\right)\frac{\partial^4 v}{\partial x^2\,\partial t^2} + \rho A\frac{\partial^2 v}{\partial t^2} + \frac{\rho^2 I}{k'G}\frac{\partial^4 v}{\partial t^4} = 0 \tag{9.32}$$

This is the so-called Timoshenko beam equation. When the ratio G/E is set equal to infinity the equation reduces to the Rayleigh equation and when G/E goes to zero it reduces to the equation of a shear beam. There are two modes of wave propagation, one in which the bending strains predominate and the other in which the shear strains predominate. The Timoshenko equation gives a good description of the dynamics of the beam so long as the shortest wavelength involved is at least five times greater than the depth of the beam.

Problems

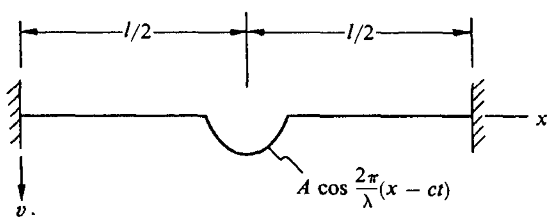
9.1 A pulse starts from one end of a stretched string that is attached to a rigid support at one end and to a sliding support at the other end. How many reflections does it make and how much time is required to complete one cycle? If there are sliding supports at each end, how long does one cycle take?

9.2 A stretched string has a rigid support at one end and a sliding support at the other. Derive the normal modes of vibration and their frequencies.

9.3 A stretched string has a sliding support at each end. Derive the normal modes of vibration and their frequencies.

9.4 Show that two oppositely directed sinusoidal wave trains of the same velocity, wavelength, and amplitude represent a standing wave.

9.5 A sine pulse is traveling along a stretched string; at time $t = 0$ it is as shown in the diagram. What is the normal mode solution?



9.6 A shear-force pulse travels along a shear beam that is built-in at one end and free at the other. Make a sketch showing how the pulse reflects at each end. How long a time is required to complete one cycle?

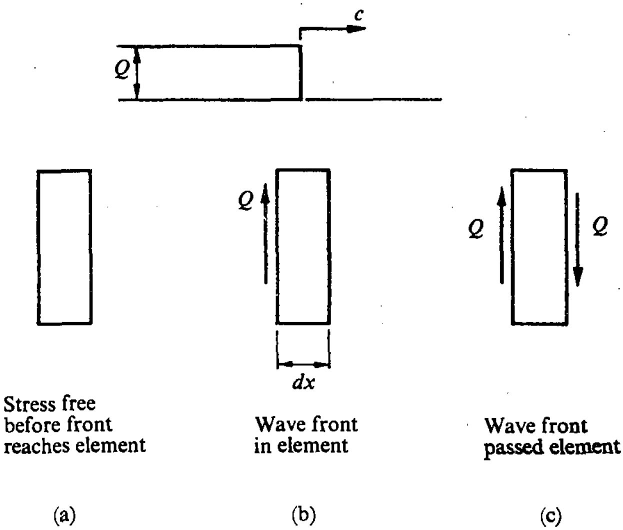
9.7 Using the trigonometric relation

$$\sin A \sin B = \tfrac{1}{2}[\cos(A + B) + \cos(A - B)]$$

put Eq. (9.5b) into the form of a traveling wave solution.

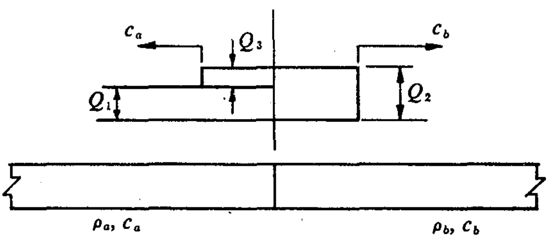
9.8 A step-function shear force Q propagates with unknown velocity c along a shear beam, as shown in the diagram. Write the equations of impulse-momentum and of work-energy for the element of length dx while the wave front passes by. From these equations deduce that

$$c = \sqrt{\frac{k'G}{\rho}}$$



9.9 A shear beam of uniform cross section is made of two materials as shown in the diagram. A step-function shear force Q propagates along section a and impinges on the interface between material a and material b. A wave Q_3 is reflected back from the interface and a wave Q_2 is transmitted through the interface. The conditions that must be satisfied at the interface are that the velocity and the shear force are continuous across the interface. By writing the equation of impulse-momentum for an element of infinitesimal length on each side of the interface, deduce that

$$\frac{Q_3}{Q_1} = \frac{1 - \rho_a c_a/\rho_b c_b}{1 + \rho_a c_a/\rho_b c_b}; \quad \frac{Q_2}{Q_1} = \frac{2}{1 + \rho_a c_a/\rho_b c_b}$$



9.10 A compression pulse propagates along a slender rod, as described by the wave equation. The rod is built-in at one end and free at the other. Make a sketch showing how the pulse $\sigma = F(x - ct)$ reflects at the two ends. Do the same for a displacement pulse, $u = f(x - ct)$.

9.11 Show that the analysis used to derive Eq. (9.11) for axial propagation along a rod is strictly correct for a material described by Eq. (9.21).

9.12 Derive Eqs. (9.18) expressing the displacements in terms of the potential functions.

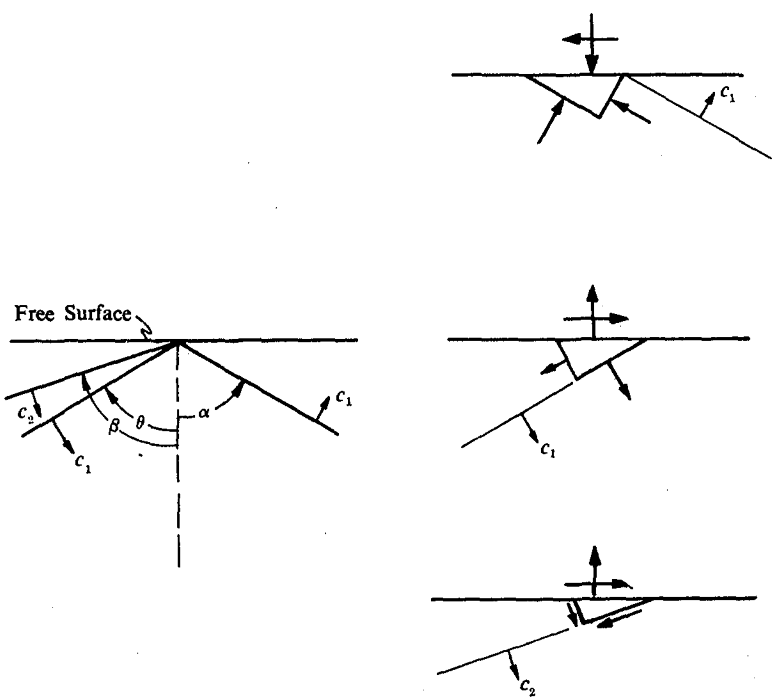
9.13 Verify that the dilatation is given by $\nabla^2\varphi$.

9.14 What are the wave velocities c_1 and c_2 in steel? ($E = 30 \times 10^6$ psi; $G = 12 \times 10^6$ psi; specific gravity is 7.86).

9.15 Describe the stresses and velocities in an infinite slab of elastic material with plane faces at $x = 0$ and $x = x_0$ if $u = v = w = 0$ for $t < 0$ and

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} = \dot{u}_0 \quad \text{in the time interval} \quad 0 < t < \frac{2x_0}{c_1}$$

9.16 Show that the boundary conditions on a free surface require both shear and dilatation wave reflections when a plane dilatation wave impinges at an angle of incidence $= \alpha$.



9.17 Write an expression for the angles of reflection of the dilatation and shear waves in Prob. 9.16. Deduce the smallest angle of incidence for which these relations are possible, for an incident plane shear wave. (Note that the speeds of the incident and reflected waves along the free surface must be equal in order to satisfy the boundary conditions.)

9.18 Obtain an equation for axial propagation of torsional waves in a circular bar.

9.19 Verify that the Rayleigh wave of Eqs. (9.25) satisfies the wave equations.

9.20 Obtain Eqs. (9.27) for the boundary conditions for a surface wave.

9.21 A long, steel, oil-well drill stem, 5.5 in. in diameter is lowered at a velocity of 3 ft/sec when its upper end is suddenly stopped. Describe the subsequent motion of the bar, assuming that it is free at the lower end.

9.22 What stress is induced in the drill stem of Prob. 9.21? What rise-time of stopping force can be tolerated if the wave equation is to be applicable with reasonable accuracy?

9.23 Two long steel bars of the same diameter and length are impacted together axially with relative velocity $\dot{u}_0$. Assuming that the wave equation is applicable, describe the consequent motion of the bars.

9.24 Two identical long rods are subjected to axial impact. What relative velocity will cause the rods to yield if they are made of steel with a yield point of 40,000 psi?

9.25 Assume a standing wave solution for a simply-supported Bernoulli-Euler beam and find the frequency equation.

9.26 Find the frequency of the first flexural mode of vibration of a freely vibrating beam (zero shear and moment at its ends), after deriving the frequency equation $\cos \alpha l = 1/\cosh \alpha l$.

9.27 Find the frequency of the first flexural mode of vibration of a Bernoulli beam built in at one end and free at the other end, after deriving the frequency equation $\cos \alpha l = -1/\cosh \alpha l$.

9.28 Show that the energy in a traveling wave is carried half in the form of strain energy and half in the form of kinetic energy.

9.29 A resultant wave is formed of two harmonic waves

$$f = A \cos \frac{2\pi}{\lambda_1}(x - c_1 t) + A \cos \frac{2\pi}{\lambda_2}(x - c_2 t).$$

Show that the resultant wave can be expressed

$$f = 2A \left[\cos \frac{2\pi}{\lambda_3}(x - c_3 t)\right] \cos \frac{2\pi}{\lambda_4}(x - c_4 t)$$

where

$$\lambda_3 = \frac{1}{2}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \qquad \lambda_4 = \frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)$$

$$c_3 = \frac{1}{2}\left(\frac{c_1}{\lambda_1} - \frac{c_2}{\lambda_2}\right) \qquad c_4 = \frac{1}{2}\left(\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right)$$

Show by means of a sketch that f represents an oscillatory pulse of wave length λ_4 whose individual loops are traveling with velocity c_4 but whose amplitude profile has a wave length λ_3 and travels with velocity c_3. Explain why the energy in the pulse travels with group velocity c_3 instead of wave velocity c_4. Show that if $(\lambda_1 - \lambda_2) = \Delta\lambda$, the velocity c_3 agrees with the general definition

$$c_g = \frac{d\omega}{d\left(\dfrac{2\pi}{\lambda}\right)}.$$

CHAPTER 10

NUMERICAL METHODS

10-1 SOLUTION OF PROBLEMS BY NUMERICAL METHODS

It frequently happens that one requires the solution of a differential equation but that it is not possible to find the solution in a form suitable for calculating numerical values. In this case, numerical methods of solution are used. Such calculations are commonly done by means of a digital computer, and matrix methods are used. In this chapter less formal methods of analysis will be discussed which will make clear how the physical behavior is related to the numerical analysis. The nature of such a problem and a method of solution is illustrated by the following example.

The bending deflection of a cantilever beam whose depth varies linearly, as shown in Fig. 10.1 is described by the following equations:

$$EI\frac{d^2v}{dx^2} = M \tag{10.1}$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = p \tag{10.2}$$

$$I = \frac{b}{12}\left[h_0 - (h_0 - h_1)\frac{x}{l}\right]^3$$

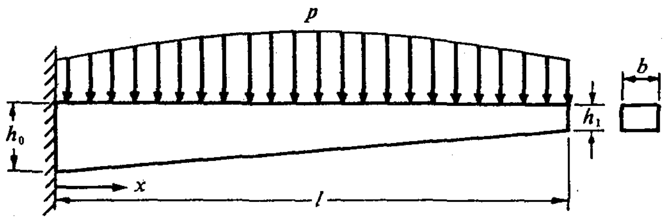


FIGURE 10.1

A suitable solution of Eq. (10.2) is not available and if it is desired to know the deflection v for the particular load p it is best determined by a numerical procedure. Let us divide the beam into N equal segments, $\Delta x = l/N$, as shown in Fig. 10.2 and let P_n be the load lying in the range

$$\left(x_n - \frac{\Delta x}{2}\right) < x < \left(x_n + \frac{\Delta x}{2}\right)$$

and let I_n be the uniform moment of inertia of the nth segment, taken equal to I at $x = x_n - \Delta x/2$. It is now easy to calculate the bending moment, M_n at each point. For sufficiently small Δx we may take the bending moment over the nth segment to be a constant M_n which is the moment at $x = x_n - \Delta x/2$. It is then easy to calculate the slope tan θ_n and displacement v_n at each x_n. Assuming that Δx and θ_n are small

$$\theta_n = \theta_{n-1} + \Delta\theta_n = \theta_{n-1} + \frac{M_n}{EI_n}\Delta x \tag{10.3}$$

$$v_n = v_{n-1} + \Delta v_n = v_{n-1} + \theta_{n-1}\Delta x \tag{10.4}$$

From these the θ_n and v_n can be calculated successively starting from $x = 0$.

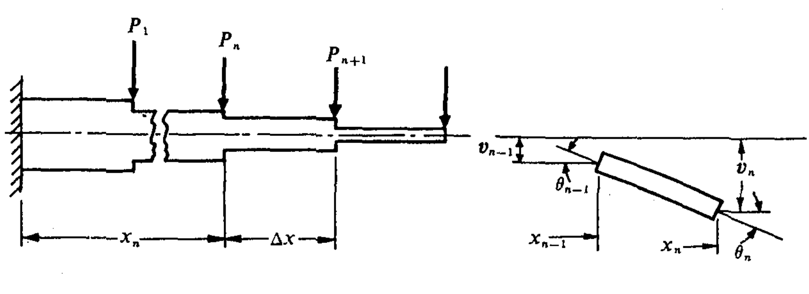


FIGURE 10.2

Equations (10.3) and (10.4) are recurrence relations from which it is easy to deduce that

$$\theta_n = \sum_{i=1}^{i=n} \frac{M_i\,\Delta x}{EI_i} \tag{10.5}$$

$$v_n = \sum_{i=1}^{i=n-1} \theta_i\,\Delta x \tag{10.6}$$

It is seen that these are just the first and second integrals of Eq. (10.1) expressed as summations.

It is clear that from Eqs. (10.5) and (10.6) the slope and displacement can be determined to any desired degree of accuracy by taking the increments Δx sufficiently small. However, if a high degree of accuracy is required, an

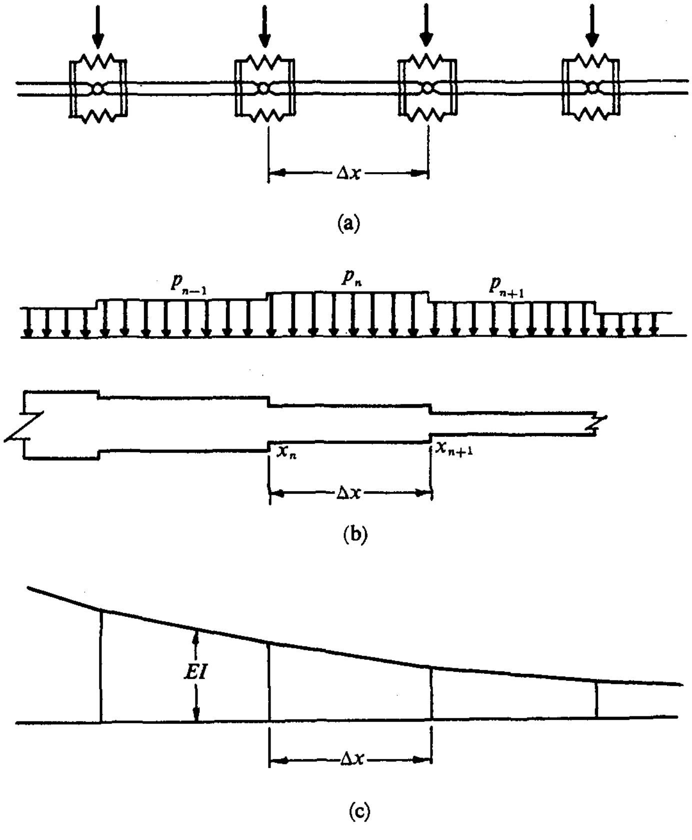


FIGURE 10.3

inordinately large number of increments may be required. In this case it may be advantageous to look for ways of decreasing the number of increments while maintaining the same accuracy. Also, if many problems are to be done, it will be desirable to look for ways to decrease the number of increments so as to reduce the amount of labor involved. The general procedure involved in doing this is illustrated by the special cases shown in Fig. 10.3. In (a) is shown the simplest approximation to the beam which consists of rigid,

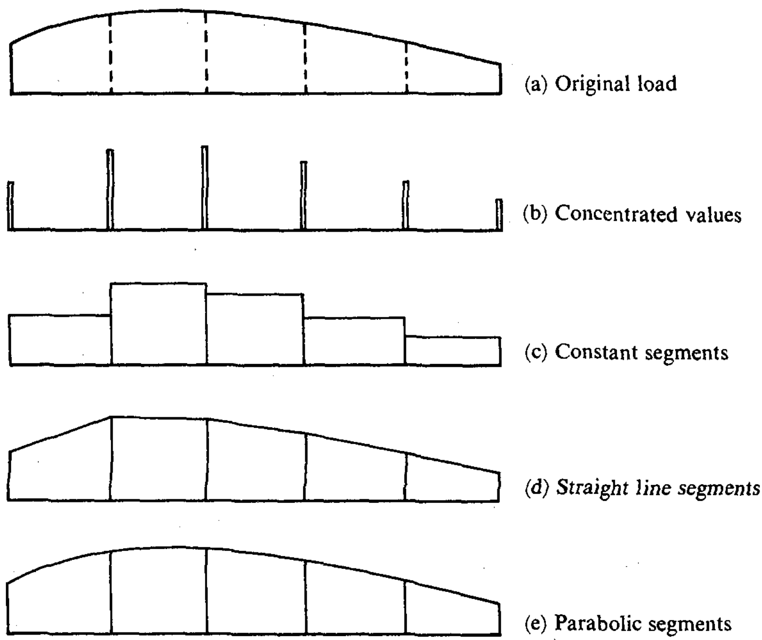


FIGURE 10.4

weightless pieces that are connected by hinges and springs. The springs are such as to give the same bending rigidity per Δx as the prototype beam. The applied forces are lumped at the hinge points. A better accuracy is given by the approximation shown in Fig. 10.3b where each segment has a uniform EI and the applied load is spread uniformly over the segment. The shear, moment and deformation of such a segment can be calculated and from these the behavior of the beam as a whole can be calculated. A further improvement in accuracy can be achieved by taking EI to vary linearly between points as shown in Fig. 10.3c; the variable load could also be represented by a load that varies linearly over the segment. The general nature of this method of improving the accuracy is illustrated in Fig. 10.4.

In (a) is shown a certain curve, say a distribution of load p, and in (b), (c), (d) and (e) are shown increasingly refined approximations to the given curve. Each additional refinement gives greater accuracy but at a cost of an increase in complexity of computations. The question then arises, how to balance the degree of refinement with the length of increment Δx to give the desired accuracy with the least amount of work. This aspect of the problem is discussed in books on numerical methods.*

10-2 NUMERICAL SOLUTION OF COLUMN BUCKLING

To illustrate the numerical solution of a problem, we shall compute the buckling load of the uniform, cantilever column shown in Fig. 10.5a. This problem is somewhat more difficult than computing merely the stresses and deflections of a beam. To solve the problem we shall use the following iterative procedure. We shall first guess at the displacements of the buckled column. From these, the bending moments produced by a vertical load P can be computed, and from the bending moments the corresponding displacements can be calculated. A repetition of this procedure will converge on the correct displacements. This procedure is an example of the relaxation method discussed in Section 10.4. It should be noted that if the assumed load P is below the critical load the computed displacements will be less than the assumed displacements and will converge to zero, whereas, if the assumed load is greater than the critical load the computed displacements will be larger than the assumed displacements and will diverge upon repetition, that is, the column will buckle. The critical load is the only load for which the displacements as computed from the bending moments are consistent with the bending moments as computed from the displacements.

The column whose length is 300 in., is divided into six equal segments $x = 50$ in. for this calculation, however, the expressions will be written for the more general case of unequal increments to illustrate the procedure. We shall use the simplest model (Fig. 10.3a) to represent the segments. As shown in Fig. 10.5c, the total angular deformation is lumped at the top of the element. We take $\theta_0 = 0$, and $\Delta\theta_n = M_n\,\Delta x_n/EI_n$, where M_n is the bending moment at x_n. The slope of the section above x_4 and the displacement at x_4, for example, are

$$\theta_4 = \Delta\theta_0 + \Delta\theta_1 + \Delta\theta_2 + \Delta\theta_3 = \sum_{i=0}^{3} \frac{M_i\,\Delta x_i}{EI_i}$$

$$v_4 = \theta_1\,\Delta x_1 + \theta_2\,\Delta x_2 + \theta_3\,\Delta x_3 = \sum_{i=0}^{3} \theta_i\,\Delta x_i$$

* See, for example, Salvadori and Baron, *Numerical Methods in Engineering*, Prentice-Hall (1959).

The computations (slide-rule accuracy) are shown in Table 10.1. The fourth column shows the assumed displacements, with 1.0 in. at the top. M_n is the bending moment at each point produced by an assumed load, $P_0 = 10{,}000$ lb.

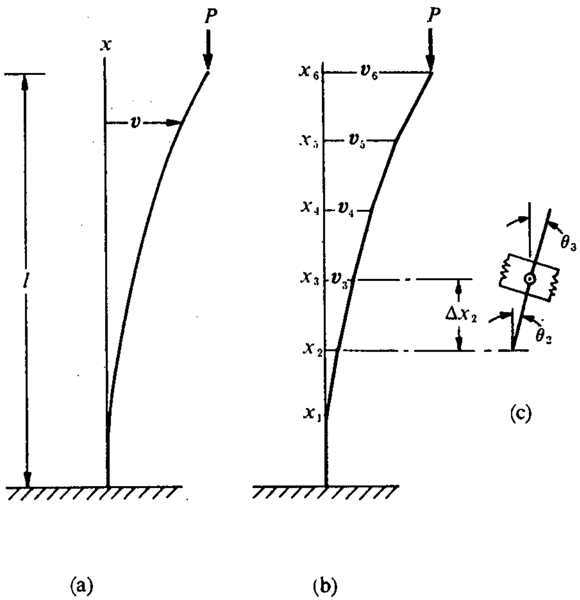


FIGURE 10.5

The next columns show the computed $\Delta\theta_n$, θ_n, $\theta_n\,\Delta x_n$, and v_n. It is seen that the assumed load of 10,000 lb produces a calculated displacement of only 0.213 in. at the top. The load of 10^4 lb is therefore less than the critical load so we must make a new estimate of P_{cr}. We designate this estimate P_{I} and choose it to give a deflection of 1.0 in. at the top of the column. A load of 46,800 lb will produce a displacement of 1.0 in. at the top since the computed v_n are linear in P, and the corresponding displacements with this load are shown in the column headed $(v_n)_{\mathrm{I}}$. The column headed $(v_n)_{\mathrm{II}}$ shows the results obtained from an iteration of the process. The corresponding buckling load is therefore $P_{\mathrm{II}} = 43{,}000$ lb. The Euler buckling load for the original column is

$$P_{cr} = \frac{\pi^2 EI}{4l^2} = 41{,}000\ \text{lb}$$

It is seen that the load P_{II} is approximately 5% in error. One more iteration would give a load P_{III} having a somewhat smaller error. A really significant improvement in accuracy would require either a larger number of segments or a better representation of the properties of the segments, as in Fig. 10.3b.

10-3 CALCULATIONS WITH FINITE DIFFERENCES

The formal methods of numerical calculations are frequently developed from the differential equations rather than from the physics of the problem and this is usually done in the language of finite differences. The differential equations of elasticity express the manner in which stresses, or displacements

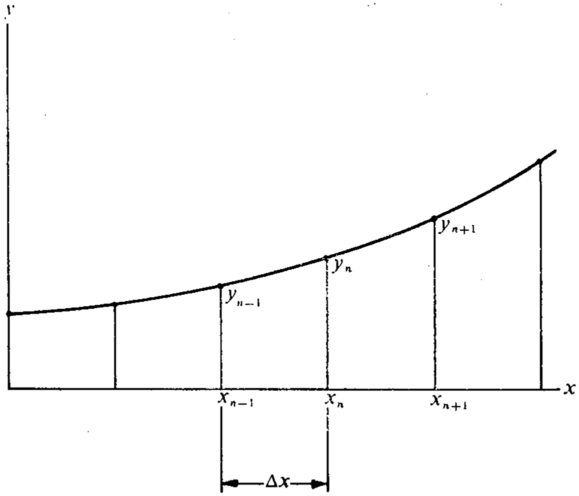


FIGURE 10.6

vary throughout a continuum. The finite difference representation of these equations is written in terms of the stresses or displacements at discrete points in the continuum. We may illustrate this representation by considering a continuous function y in one independent variable x. The curve $y = f(x)$ shown in Fig. 10.6 is continuous but we shall be concerned only with the values of y at discrete points x_n located by dividing the x-axis into equal increments Δx. By definition the first derivative of y is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(y_n - y_{n-1})}{\Delta x}$$

If Δx does not go to zero the expression in the parentheses is an approximation to the derivative. The numerator of this approximation is called the *first difference* of y

$$\Delta y = y_n - y_{n-1} \tag{10.7}$$

TABLE 10.1

n	EI_n (in.^{4})	Δx_n (in.)	$(v_n)_0$ (in.)	$M_n = P(v_6 - v_n)$ $P = 10^4$ lb	$\Delta\theta_n = \frac{M_n}{EI_n}\Delta x_n$	$\theta_n = \left(\sum_0^{n-1} \Delta\theta_i\right)$	$\theta_n \Delta x_n$	v_n (in.) $= \left(\sum_0^{n-1} \theta_i \Delta x_i\right)$ $P = 10^4$ lb	$(v_n)_{\mathrm{I}}$ (in.) $P_{\mathrm{I}} = 4.68(10^4)$ lb	$(v_n)_{\mathrm{II}}$ (in.) $P_{\mathrm{II}} = 4.3(10^4)$ lb
6	—	—	1.0	0	—	—	—	21.3×10^{-2}	1.0	1.0
5	15×10^8	50	0.75	0.25×10^4	0.833×10^{-4}	12.82×10^{-4}	6.41×10^{-2}	14.9	0.695	0.691
4	15	50	0.5	0.5	1.67	11.15	5.57	9.32	0.435	0.427
3	15	50	0.35	0.65	2.16	8.99	4.50	4.82	0.225	0.215
2	15	50	0.2	0.80	2.66	6.33	3.16	1.66	0.0775	0.0718
1	15	50	0.1	0.90	3.0	3.33	1.66	0	0	0
0	15	50	0	1.0	3.33	0	0	0	0	0

The *second difference* of y is

$$\Delta^2 y = \Delta(\Delta y) = y_{n+1} - 2y_n + y_{n-1} \tag{10.8}$$

Bending deflections of a beam can be expressed as a function of one independent variable x. In finite difference form the differential equation of a beam, $EI\, d^2v/dx^2 = M$ is written

$$\frac{EI_n}{(\Delta x)^2}(v_{n+1} - 2v_n + v_{n-1}) = M_n \tag{10.9}$$

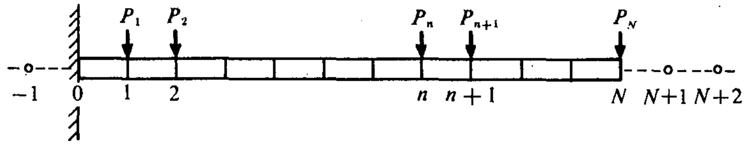


FIGURE 10.7

The equation $EI\, d^4v/dx^4 = p$, for a uniform beam, is written in finite difference form as

$$\frac{EI_n}{(\Delta x)^4}(v_{n+2} - 4v_{n+1} + 6v_n - 4v_{n-1} + v_{n-2}) = \frac{P_n}{\Delta x} \tag{10.10}$$

where $P_n = p(\Delta x)$. Such an equation can be written at each point x_n. If the beam is divided so that there are N different unknown v_n's there will be N equations of the foregoing type that can be written and therefore the problem has been reduced to the solution of N simultaneous algebraic equations for the N unknowns $v_1, \ldots, v_N$. If the deflections of the cantilever beam, shown in Fig. 10.7, are to be computed it will be necessary to impose the following boundary conditions:

$$v_0 = 0 \qquad \Delta^2 v_N = 0$$

$$\Delta v_0 = 0 \qquad \Delta^3 v_N = 0$$

The difference expressions for the points at $n = 0, N$ contain the terms v_{-1}, v_{N+1}, and v_{N+2}. If the fictitious points at $n = -1, N+1, N+2$ are introduced there will be a total of $N + 4$ unknown displacements. Equation (10.10) can be written for N points and these, together with the four boundary equations, give a total of $N + 4$ equations for determining the $N + 4$ displacements.

We next consider a problem in two independent variables. The soap-film shown in Fig. 10.8 has a uniform tension S per unit length in all directions and is subjected to a pressure p which produces small displacements w. For this situation equilibrium is described by the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p}{S} \tag{10.11}$$

If the rectangular area covered by the film is subdivided into a grid as shown in Fig. 10.8, the finite difference form of Eq. (10.11) is

$$\frac{\Delta_n^2 w_{n,m}}{(\Delta x)^2} + \frac{\Delta_m^2 w_{n,m}}{(\Delta y)^2} = \frac{-P_{n,m}}{S\,\Delta x\,\Delta y} \tag{10.12}$$

where

$$\Delta_n^2 w_{n,m} = w_{n+1,m} - 2w_{n,m} + w_{n-1,m}$$

$$\Delta_m^2 w_{n,m} = w_{n,m+1} - 2w_{n,m} + w_{n,m-1}$$

$$P_{n,m} = \begin{cases} \text{the applied force concentrated at the point} \\ x_n,\ y_n \text{ (it is equal to the intensity } p \text{ at the} \\ \text{point multiplied by the area } \Delta x\,\Delta y) \end{cases}$$

It is customary to make $\Delta y = \Delta x$ and in this case the difference equation would have the form

$$\Delta_n^2 w_{n,m} + \Delta_m^2 w_{n,m} = -\frac{P_{n,m}}{S} \tag{10.12a}$$

An equation of the form (10.12) can be written for each interior grid point so that there are just as many equations as there are unknown deflections and, hence, the solution of the partial differential Eq. (10.11) has been reduced to the solution of a set of algebraic equations. Equation (10.12a) may be interpreted as the equation of a stretched elastic net, as shown in Fig. 10.8b, having forces $S\,\Delta x$ and $S\,\Delta y$ in the strings and supporting a load $P_{n,m}$ at each node. We may write differential equations in three or more independent variables in finite difference form by following the procedure used for the two independent variables.

In the foregoing discussion the derivative was approximated by

$$\frac{dv}{dx} \approx \frac{v_n - v_{n-1}}{\Delta x}$$

More accurate approximations can be made by taking into account the values of v at adjacent points, for example,

$$\left(\frac{dv}{dx}\right)_n \approx \frac{(v_{n+1} - v_n) + (v_n - v_{n-1})}{2\Delta x} = \frac{v_{n+1} - v_{n-1}}{2\Delta x}$$

Note that v_n does not appear in this expression.

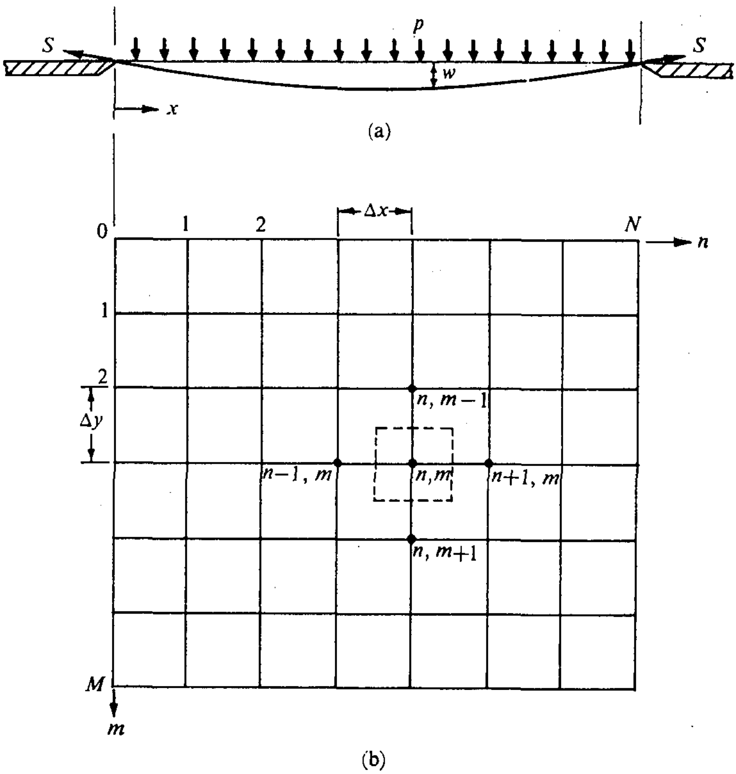


FIGURE 10.8

10-4 RELAXATION METHOD

There are many methods of obtaining numerical solutions to a set of simultaneous linear equations. The most appropriate method will depend upon the nature and number of equations, and the computing equipment that is available. A different method of solution would be indicated if an electronic digital computer were available than if only pencil and paper were available.

One of the simplest and easiest pencil and paper methods is the so-called "relaxation method." The nature of this method will be illustrated by the following example in one independent variable.

Let us consider a uniform, fixed-ended beam, uniformly loaded, whose finite difference equation is given by Eq. (10.10). As shown in Fig. 10.9 the beam is divided into four equal segments. We note that a fixed end merely means that the end of the beam does not rotate. One way to achieve this is to have an infinitely long beam continuous over an infinite number of equally spaced supports and loaded in such a way that the spans on the two sides of any support are symmetrical. This will insure that with respect to a point of support ($n = 0$), the displacements are symmetrical ($v_{+n} = v_{-n}$), which means zero rotation at the support. We *wish, then, to solve for the deflections* of the beam shown in Fig. 10.9, where this span is flanked by identical spans.

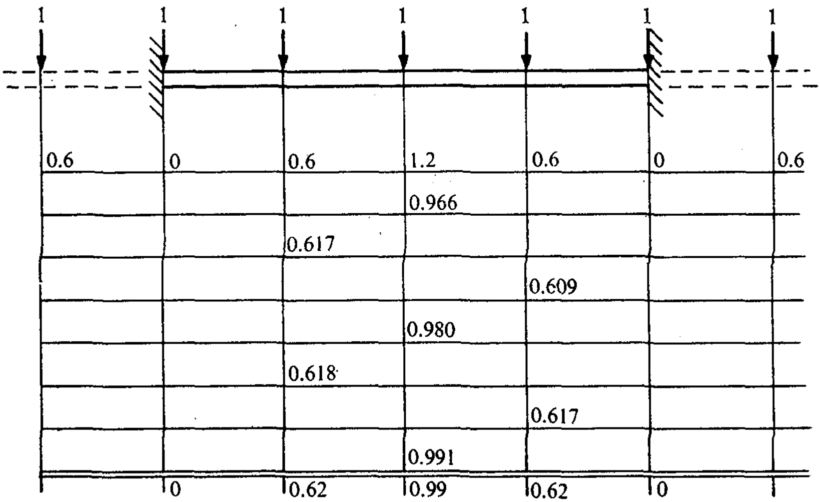


FIGURE 10.9

It will be convenient to solve the equations in the normalized form

$$v_{n+2} - 4v_{n+1} + 6v_n - 4v_{n-1} + v_{n-2} = 1 \tag{10.13}$$

The computed displacements must then be multiplied by $P_n(\Delta x^3)/EI$. The relaxation method *looks at the problem as follows. The foregoing equation* is put in the form

$$v_n = \tfrac{1}{6}[1 - (v_{n+2} - 4v_{n+1} - 4v_{n-1} + v_{n-2})] \tag{10.14}$$

For any numerical values of the displacements that may be substituted in the right side, this equation will give the correct value of v_n for unit value of $P_n(\Delta x)^3/EI$. The solution of the problem is thus automatically obtained by the following procedure:

1. Make a reasonable guess of the values of v_n at the grid points. Such a guess is shown in the top line of numbers in Fig. 10.9.
2. Apply Eq. (10.14) to each point, successively, remembering that at each side of the supports $v_{0+1} = v_{0-1}$, and $v_{4+1} = v_{4-1}$, etc.
3. Each time a new value of v_n is computed it will require a recomputation at adjacent points. The calculations are continued until a satisfactory degree of accuracy is obtained.

For example, on the basis of the assumed values of v shown in Fig. 10.9, the calculated value of v_2 is

$$v_2 = \tfrac{1}{6}\{1 - [0 - 4(0.6) - 4(0.6) + 0]\} = 0.966$$

With this value of v_2 there is obtained a new value of $v_1 = 0.617$, and then $v_3 = 0.609$, etc. After the number of operations shown in Fig. 10.9 there is obtained

$$v_0 = 0; \quad v_1 = 0.62; \quad v_2 = 0.99; \quad v_3 = 0.62; \quad v_4 = 0$$

If the calculations were carried to the limit the values would be $v_1 = 0.625$ and $v_2 = 0.99$. The computed displacement at midspan is

$$v_1 = \frac{0.99P_n(\Delta x)^3}{EI} = \frac{0.99p\left(\frac{L}{4}\right)^4}{EI} = \frac{0.99}{256}\frac{pL^4}{EI}$$

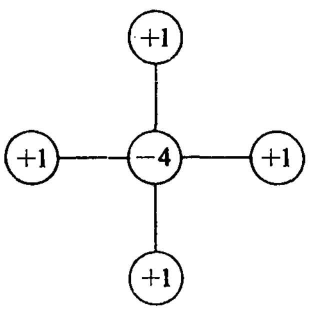
The correct value is approximately 30% smaller than this. The poor accuracy results from the fact that $\Delta x = L/4$ is too large to be a good approximation to dx.

It is obvious that the successive point-by-point method used in the preceding problem was not the most efficient. For example, it would have been better to have operated on points x_1 and x_3 simultaneously. In this regard, the essential thing to note is that the initial guess contains the correct solution plus a component of error and that the procedure should be such as to eliminate the error as quickly as possible.*

If Eq. (10.12a) is to be solved by the relaxation method the following procedure is convenient. A grid, such as shown in Fig. 10.8, is drawn to a suitably large scale and the values of $P_{n,m}$ and the initial values of $w_{n,m}$ are marked at the nodes. The equation can be written in the form

$$\left|\begin{array}{ccc} & w_{n,m-1} & \\ & + & \\ w_{n-1,m} & -\ 4w_{n,m}\ + & w_{n+1,m} \\ & + & \\ & w_{n,m+1} & \end{array}\right| = \frac{P_{n,m}}{S}$$

This equation is indicated schematically by the following form



$$= \frac{P_{n,m}}{S}$$

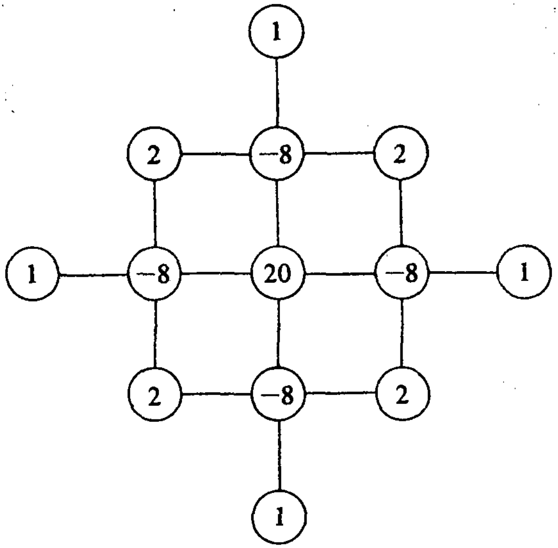
If this is drawn on a transparent overlay to the same scale as the grid it can be used as follows. The overlay is placed on the grid, centered at point n,m. The sum of the four adjacent values of w minus $4w_{n,m}$ should be equal to $P_{n,m}/S$. If it is not, change the value of $w_{n,m}$ so that the relation is satisfied. Repeat at the adjacent points, etc.

If a plate problem is to be solved numerically, the equation is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}$$

* F. S. Shaw, *Relaxation Methods*, Dover (1953).

which in finite difference form, with $\Delta x = \Delta y$, is represented by



$$= \frac{P_{n,m}(\Delta x)^2}{D}$$

Grid points outside of the edges of the plate must be used to represent built-in edges, simply supported edges, etc.

10-5 MOMENT DISTRIBUTION

The method of moment distribution, the original relaxation procedure,* was developed to solve statically indeterminate problems of beams and frames. The method lends itself to a physical interpretation so that with only

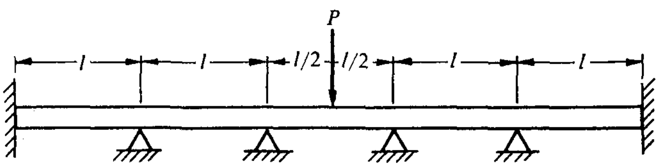


FIGURE 10.10

a few simple physical concepts a solution can be derived for a problem that in mathematical terms would be extremely complex. A simple example of the type of problem suited to moment distribution is the beam on six supports, shown in Fig. 10.10. This system is statically indeterminate to the sixth

* Developed by Professor Hardy Cross of the University of Illinois.

degree, *in that the bending moments at each of the six supports must be* evaluated before the bending moment at any point of a span can be determined with the equations of statics. The equation of three moments (Eq. (3.67)) could be used and its application would give six equations involving the six unknown bending moments $M_1, M_2, \ldots, M_6$. The problem can be solved by the method of moment distribution by repetitive solving of a simple problem. We first restrain the beam against rotation at all supports and then relax these restraints, one at a time so that the internal moments are in balance with the applied loads. Relaxation of the restraints allows rotation of the beam at the supports, and this rotation is easily related to the internal moments that are developed. If the left end of the uniform beam, shown in Fig. 10.11a, is rotated through an angle θ_L, it may be readily verified that the following end moments are developed

$$M_L = \frac{4EI\theta_L}{l}; \quad M_R = \frac{2EI\theta_L}{l} \tag{10.15}$$

It is seen that when a moment M_L is applied to the left-hand end, a moment is developed at the right-hand end equal to $M_R = \frac{1}{2}M_L$. We may use this fact to solve the simple problem shown in Fig. 10.11b where the external moment

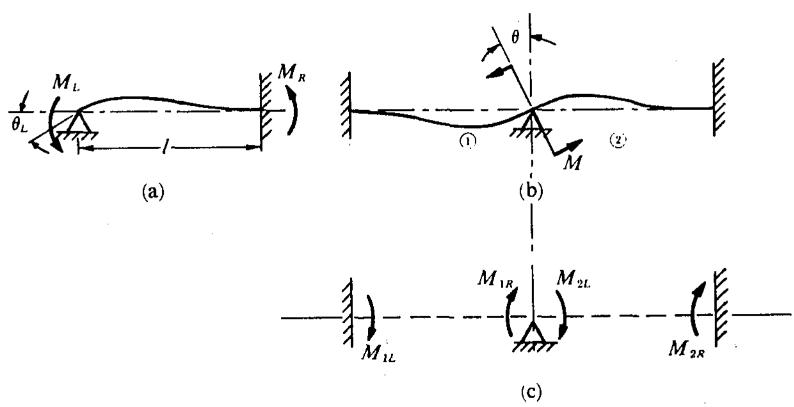


FIGURE 10.11

M is applied to a two-span beam at the point of the second support. The effect of M is to produce a rotation θ, as shown, and end moments will be produced in the two beams as shown in Fig. 10.11c. It should be noted that

M_{1L}, M_{1R} and M_{2L}, M_{2R} shown in the figure are the moments *exerted on the joints* by the beams. We know that

$$M_{1R} = \frac{4EI_1}{l_1}\,\theta, \qquad M_{2L} = \frac{4EI_2}{l_2}\,\theta$$

thus

$$\frac{M_{1R}}{I_1/l_1} = \frac{M_{2L}}{I_2/l_2}$$

Equilibrium requires

$$M_{1R} + M_{2L} = M$$

so we can solve for M_{1R} and M_{2L}

$$\begin{aligned} M_{1R} &= k_1 M \\ M_{2L} &= k_2 M \end{aligned} \tag{10.16}$$

where

$$k_1 = \frac{I_1/l_1}{I_1/l_1 + I_2/l_2}; \quad k_2 = \frac{I_2/l_2}{I_1/l_1 + I_2/l_2}$$

The quantity I/l is a measure of the stiffness of the beam against rotation so k_1 and k_2 are called the stiffness factors. The values of the moments M_{1L} and M_{2R} are given by:

$$M_{1L} = c_1 M_{1R}$$

$$M_{2R} = c_2 M_{2L}$$

The sign convention for these moments is that a positive moment tends to rotate the joint in a clockwise direction. The coefficients have values:

$$c_1 = 0.5; \quad c_2 = 0.5$$

The coefficients c_1 and c_2 are called carry-over factors, for a reason which will become apparent. If the beams were not uniform but had varying moments of inertia the k and c factors would have different values.

The basic problem in moment distribution is that shown in Fig. 10.11c, that is, when a moment M is applied to the center joint it is resisted by the moments

$$M_{1R} = k_1 M$$

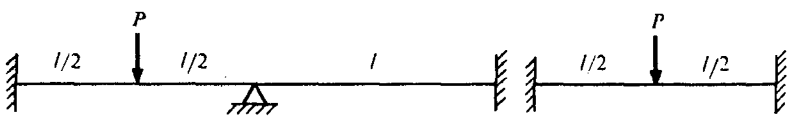
$$M_{2L} = k_2 M$$

and we may say that the moment M is distributed according to the stiffness factors k_1 and k_2. We may also say that a fraction c_1 of the moment M_{1R} is carried over to the left-hand end of beam (1), and that a fraction c_2 of M_{2L} is carried over to the right-hand end of beam (2).

Example 1

We shall now solve the problem shown in Fig. 10.12a. If the center support is fixed against rotation, the left-hand span will be in the condition shown in (b). It is easy to calculate that the fixed-end moments at each end of (b) are

$$M_{FE} = \tfrac{1}{8}Pl = 0.125Pl$$



(a) (b)

	(0.5)	(0.5)	
+0.125 Pl	−0.125 Pl	0	0
0	+0.0625 Pl	+0.0625 Pl	0
+0.0318 Pl	0	0	+0.0318 Pl
+0.1568 Pl	−0.0625 Pl	+0.0625 Pl	+0.0318 Pl

(c)

FIGURE 10.12

These moments are shown in the first row in (c). To hold the center joint against rotation thus requires an external fixing moment equal to $+0.125Pl$. This is apparent from the diagram where it is seen that the center joint is unbalanced by $-0.125Pl$. If the joint is released, that is, if it is allowed to rotate until it is balanced, there will be generated $M_{1R} = +0.0625Pl$ and $M_{2L} = +0.0625Pl$, since the stiffness factors are 0.5 for each. These moments are shown in the second line in the diagram. Corresponding to these moments there are carry-over moments one half as large, and these are shown in the third line. The fourth line shows the sum at each joint

$$M_{1L} = +0.1568Pl \qquad M_{2L} = +0.0625Pl$$

$$M_{1R} = -0.0625Pl \qquad M_{2R} = +0.0318Pl$$

In making the calculations it would be sufficient to know that the fixed-end moments are the same at each end of beam (1) and to express this fact as shown in the first row in Fig. 10.13. Note that the stiffness factors have been marked on the diagram for reference. The center joint is unbalanced by -1, so to balance it a $+0.5$ is added in the second row. A carry-over of one-half is made in the third row. At this stage, the joints are balanced and the problem is solved

$$M_{1L} = 1.25(Pl/8)$$

$$M_{1R} = 0.5(Pl/8)$$

$$M_{2R} = 0.25(Pl/8)$$

	(0.5)	(0.5)		
+1	−1	0	0	—FEM
0	+0.5	+0.5	0	—Balance
+0.25	0	0	+0.25	—Carryover
+1.25	−0.5	+0.5	+0.25	

FIGURE 10.13

Example 2

We shall now solve the problem shown in Fig. 10.10. In physical terms, the procedure that will be used is to hold all joints fixed and then release one until it is balanced and then fix it, then repeat the process with another joint,

	(0.5)	(0.5)		(0.5) (0.5)	(0.5)	(0.5)		(0.5) (0.5)	
0	0	0	0	+1	−1	0	0	0	0
0	0	0	−0.5	−0.5	+0.5	+0.5	0	0	0
0	0	−0.25	0	+0.25	−0.25	0	+0.25	0	0
0	+0.125	+0.125	−0.125	−0.125	+0.125	+0.125	−0.125	−0.125	0
0.0625	0	−0.0625	+0.0625	+0.0625	−0.0625	−0.0625	+0.0625	0	−0.0625
0	+0.0312	+0.0312	0	0	0	0	−0.0312	−0.0312	0
0.016	0	0	+0.016	0	0	−0.016	0	0	−0.016
0	0	0	−0.008	−0.008	+0.008	+0.008	0	0	0
0.0785	+0.156	−0.156	−0.555	+0.555	−0.555	+0.555	+0.156	−0.156	−0.0785

FIGURE 10.14

etc. Actually, in doing the problem we shall balance all joints at the same time and make all carry-overs at the same time. The solution is shown in Fig. 10.14 where we see, for example, that the moments at the two center supports are

$$M = 0.555(Pl/8)$$

Example 3

A frame of five identical members rigidly connected at the joints is loaded as shown in Fig. 10.15. The solution has been carried out for two balances and two carry-overs. Note that the frame is prevented from swaying sideways.

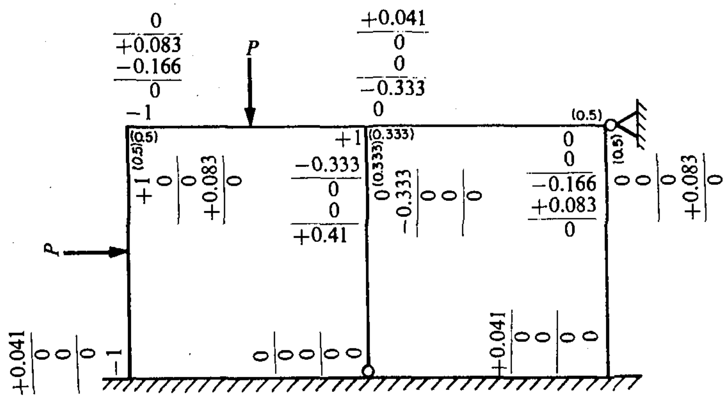


FIGURE 10.15

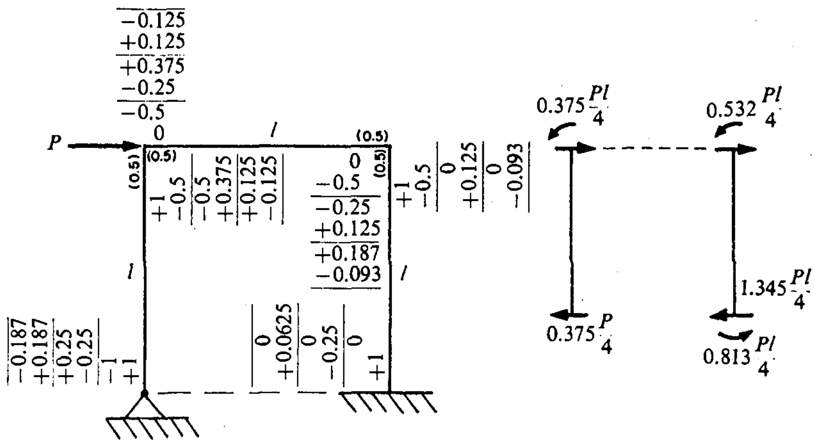
Example 4

A simple frame made of three identical members is loaded as shown in Fig. 10.16. With the joints fixed against rotation, the action of P is to displace the frame sideways until a moment of $Pl/4$ is developed at the top and bottom of each vertical member. Fixed end moments of $+1$ are shown and three balances have been made. In (b) there are shown the resulting moments and shears at the ends of the columns. The total shear in the frame is

$$(0.375 + 0.813 + 0.532)P/4 = 0.43P$$

so the moments in (a) must be multiplied by (1/0.43) to make the shear equal to the required P. The reason the shear diminished is that during the balance and carry-over the frame was restrained against side sway in the same manner as shown in Fig. 10.15 and the restraint developed a reaction force equal to $0.57P$. In physical terms the foregoing solution consisted of first displacing the top of the frame sideways, with all joints fixed against rotation, until the total shear was equal to P. The frame was then fixed against lateral movement and the joints were permitted to rotate. When solving problems of frames that have side sway, such as a multi-story building subjected to wind forces, it is necessary to make two kinds of balances. First, the moments at the joints must be balanced (and carried over) and second, the total shear in each story must be balanced against the actual shear force in the story (and moments carried over).*

One of the essential features of moment distribution is that a single complicated problem, Fig. 10.10, is replaced by repetitive solving of a simple problem, Fig. 10.11c. This was done by fixing all joints and then relaxing only one at a time. The second essential feature is that the simple problem was solved by imposing a displacement and computing the internal moments, Eq. (10.15). The process of moment distribution can thus be described as fixing all joints (or nodes) and then one at a time displacing a joint (rotation



(a) (b)

FIGURE 10.16

or side sway) so that the internal forces or moments are in balance with the external forces or moments. This same procedure could be applied to any problem. For example, the stretched elastic net shown in Fig. 10.8b could be solved by fixing all of the nodes against displacement and then displacing (balancing) one at a time. The basic problem is shown in Fig. 10.17 where it is seen that the stiffness factor k for each of the four strings at a joint is 0.25, and the carry-over factor is -1. As may be verified, this is what the finite difference equation for the net states. It is, of course, immaterial whether the problem is looked at from the physical point of view, Fig. 10.17, or from the mathematical point of view, Eq. (10.12a). The formal relaxation method does save some work by making an initial guess at the total displacement of all of the nodes. This initial guess could also be made when applying moment

* See the book *Moment Distribution*, by J. M. Gere, Van Nostrand (1963).

distribution, but in this case it is more difficult to make a reasonable estimate of the displacements. A similar procedure could be applied to problems of trusses, etc.

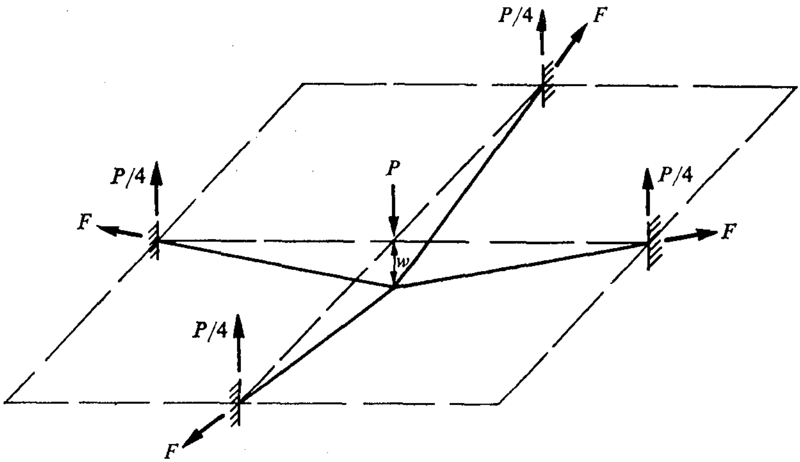


FIGURE 10.17

10-6 USE OF DIGITAL COMPUTERS

The development of large digital computers has provided a convenient tool for the numerical solution of problems. If the finite difference equations are written for a set of points as, for example Eq. (10.12), the problem will be described by a set of simultaneous algebraic equations. If the number of equations is not excessively large, the digital computer can quickly solve the equations. If the number of equations is too large for direct simultaneous solution, special methods of solution are used that exploit the properties of the computer rather than the physical behavior as in the relaxation and moment distribution methods.

A drawback in the use of the computer is that there is a tendency to lose some of the feel for the physics of the problem, but this is usually more than offset by the speed of solution and by the advantage of being able to solve very complex problems.

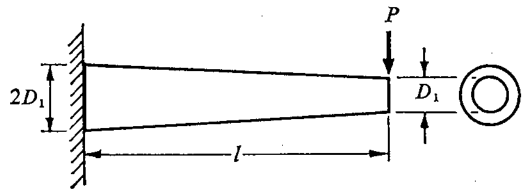
Problems

10.1 Show by means of a sketch that Eqs. (10.3) and (10.4) do give the slope and displacement.

10.2 Verify Eqs. (10.5) and (10.6).

10.3 What form do Eqs. (10.5) and (10.6) have if the bending moment is assumed to have a linear variation as in Fig. 10.4d?

10.4 Compute the displacement of the beam shown in the diagram. Divide the beam into five segments.



10.5 Compute the buckling load of the beam in Prob. 10.4.

10.6 Compute the deflections of the uniform cantilever beam shown in Fig. 10.7, using four equal segments and taking into account the actual deformation of the segment rather than lumping the flexibility as in Fig. 10.3a.

10.7 Do Prob. 10.6 using the relaxation procedure with Eq. (10.10).

10.8 Solve for the displacements of a uniformly loaded square membrane by a relaxation procedure using Eq. (10.12a). Make use of the twofold symmetry of the problem.

10.9 Solve the problem of torsion of a square shaft, using relaxation methods. First determine values of the stress function, then determine the magnitude of the maximum shear stress.

10.10 Derive Eq. (10.11) for a stretched, elastic membrane subjected to a uniform air pressure on one side.

10.11 Derive the equation that expresses equilibrium of a node of a stretched elastic net, as shown in Fig. 10.8b, and compare with the finite difference equation for a membrane.

10.12 Put into finite difference form the differential equation of buckling of a pin-ended strut, $EI(d^2v/dx^2) + Pv = 0$, and show that it is satisfied by the solution

$$v_n = c_1 \cos kn\,\Delta x + c_2 \sin kn\,\Delta x$$

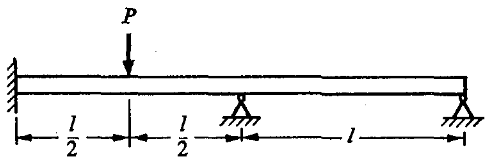
where

$$k^2 = \frac{P}{EI}$$

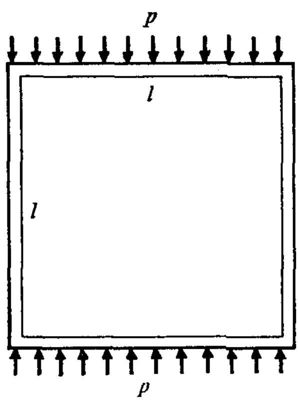
10.13 Solve for the deflection of the beam shown in Fig. 10.9 using $\Delta x = L/8$. Take advantage of the symmetry of the problem.

10.14 Verify Eq. (10.15), that $M_L = 4EI\theta_L/l$ and $M_R = 2EI\theta_L/l$ and deduce that $M = 4EI(\theta_L + \frac{1}{2}\theta_R)/l$.

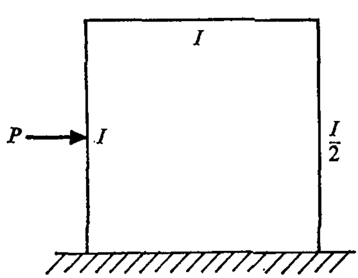
10.15 Solve for the joint moments of the beam shown in the diagram.



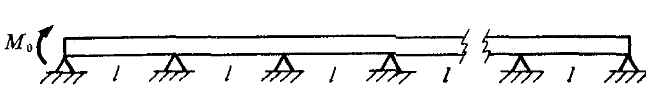
10.16 Solve for the joint moments of the uniformly loaded frame shown in the diagram. All members have the same E, I, and l.



10.17 Solve for the joint moments of the frame shown in the diagram. All members have the same length.



10.18 Solve for the joint moments of the 100-span beam shown in the diagram. Slide rule accuracy is sufficient.



10.19 Show that the finite difference equation

$$\Delta_n^2 w_{n,m} + \Delta_m^2 w_{n,m} = \frac{Pl}{F}$$

is a mathematical formulation of the equilibrium condition depicted in Fig. 10.17, when a force P is applied to one node with the adjacent nodes fixed against displacement.

APPENDIX I

STRESSES AND STRAINS IN TENSOR NOTATION

I-1 TENSOR NOTATION

If we wish to discuss the stress-strain properties of a material which is other than isotropic and linearly elastic, or if we wish to make general transformations of coordinates, we find our ordinary system of notation to be cumbersome. To overcome this we introduce a more compact system of notation which has the advantage of making it easier to grasp the concepts involved. In studying the theory of elasticity we are concerned with 9 stress components and 9 strain components in 3 dimensions, and the relations between them. In ordinary notation we are led to multiple equations with many terms and the concepts involved are obscured by the very size of the expressions. The new notation to be introduced is based on the principle of minimizing the number of symbols used and as far as possible representing each concept or operation by a single symbol. To illustrate this, consider our present notation for the stresses at a point

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

We note that two subscripts on a stress symbol are sufficient to specify the face of the element upon which the stress is acting and the direction of the stress, so that the subscripts alone are sufficient to tell whether it is a normal stress or a shearing stress. It is, therefore, sufficient to use the single letter τ to indicate that the symbol refers to a stress and to rely on the subscripts to tell which stress it is. With this convention we write the stresses at a point

$$\begin{vmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{vmatrix}$$

Let us now examine, from the same viewpoint, our notation for the three coordinates x, y, z. Here again, we have 3 different symbols, and we shall find it convenient to use *one* symbol only to indicate that it refers to a coordinate and to use a subscript to indicate which coordinate. In the new notation, the letter x will be used to indicate a coordinate and numerical subscripts will indicate the direction of the coordinate (x_1, x_2, x_3). The subscripts 1, 2, 3 thus indicate the x, y, z directions. The coordinate axes are also labeled 1, 2, 3 in the new notation. With this convention the stresses at a point are written

$$\begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}$$

These stresses are shown in Fig. I.1.

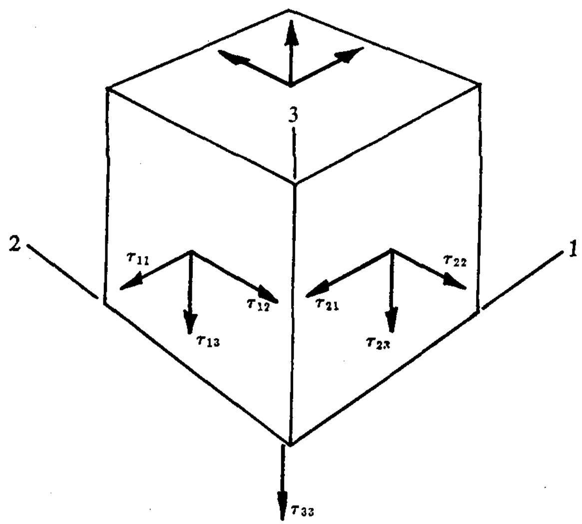


FIGURE I.1

On each face of an element there are 3 stress components, one normal and two shear, and the resultant stress on the face is a vector having these three for components. Let the unit vectors be $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$,* then the resultant stress on the 1-face is the vector

$$\overset{1}{\mathbf{T}} = \tau_{11}\mathbf{e}_1 + \tau_{12}\mathbf{e}_2 + \tau_{13}\mathbf{e}_3$$

The resultant stresses on the 2- and 3-faces are

$$\overset{2}{\mathbf{T}} = \tau_{21}\mathbf{e}_1 + \tau_{22}\mathbf{e}_2 + \tau_{23}\mathbf{e}_3$$

$$\overset{3}{\mathbf{T}} = \tau_{31}\mathbf{e}_1 + \tau_{32}\mathbf{e}_2 + \tau_{33}\mathbf{e}_3$$

Or, in general, we may say that the stress vector on the i-face ($i = 1, 2, 3$) is

$$\overset{i}{\mathbf{T}} = \tau_{i1}\mathbf{e}_1 + \tau_{i2}\mathbf{e}_2 + \tau_{i3}\mathbf{e}_3$$

or

$$\overset{i}{\mathbf{T}} = \sum_{j=1}^{3} \tau_{ij}\mathbf{e}_j$$

We shall use the following *summation* convention: whenever the summation sign

$$\sum_{j=1}^{3}$$

appears with a repeated subscript or superscript in the same term ($\tau_{ij}\mathbf{e}_j$) we shall *omit* the summation sign and understand that a repeated subscript means sum from 1 to 3 with respect to that index.† With this convention we write

$$\overset{1}{\mathbf{T}} = \tau_{1j}\mathbf{e}_j$$

$$\overset{2}{\mathbf{T}} = \tau_{2j}\mathbf{e}_j$$

$$\overset{3}{\mathbf{T}} = \tau_{3j}\mathbf{e}_j$$

* On each face of the element the directions of the unit vectors are taken to coincide with the directions of the *positive* stress components so that the resultant stress vector is $\overset{i}{\mathbf{T}} = \tau_{i1}\mathbf{e}_1 + \tau_{i2}\mathbf{e}_2 + \tau_{i3}\mathbf{e}_3$.

† If for some reason it is necessary to write a term with repeated subscript, and summation is not intended, this fact must be explicitly noted.

or, in general, the resultant stress on the i-face is

$$\overset{i}{\mathbf{T}} = \tau_{ij}\mathbf{e}_j \qquad (i = 1, 2, 3) \tag{I.1}$$

The Equations of Equilibrium. Consider an element with sides dx_1, dx_2, dx_3. The resultant *force* on the 1-face is $\overset{1}{\mathbf{T}}\, dx_2\, dx_3$ and on the 2- and 3-faces the resultant forces are $\overset{2}{\mathbf{T}}\, dx_3\, dx_1$ and $\overset{3}{\mathbf{T}}\, dx_1\, dx_2$. Note that these are the forces acting on the three faces which intersect at the corner, x_1, x_2, x_3, that is, they are composed of the stresses evaluated at the point x_1, x_2, x_3. The force on the other 1-face at $x_1 + dx_1$ is

$$\overset{1}{\mathbf{T}}\, dx_2\, dx_3 + \frac{\partial \overset{1}{\mathbf{T}}}{\partial x_1}\, dx_1\, dx_2\, dx_3$$

with similar expressions for the stresses on the other 2- and 3-faces. These forces are shown in Fig. I.2. Equilibrium of the element (no body forces) requires

$$\frac{\partial \overset{1}{\mathbf{T}}}{\partial x_1}\, dx_1\, dx_2\, dx_3 + \frac{\partial \overset{2}{\mathbf{T}}}{\partial x_2}\, dx_1\, dx_2\, dx_3 + \frac{\partial \overset{3}{\mathbf{T}}}{\partial x_3}\, dx_1\, dx_2\, dx_3 = 0$$

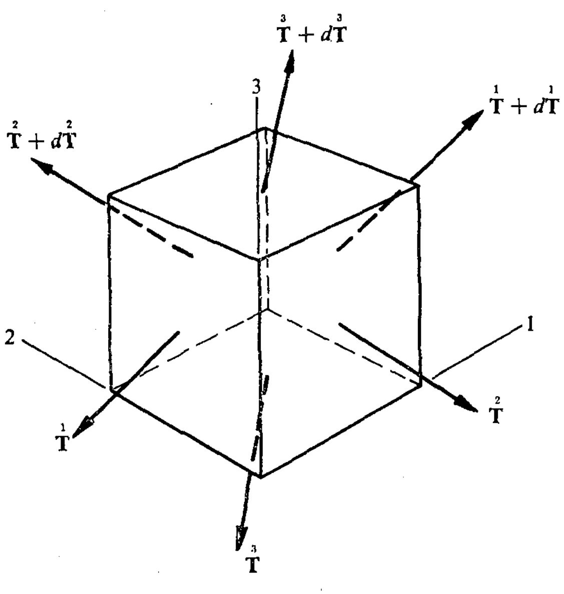


FIGURE I.2

or

$$\frac{\partial \overset{1}{\mathbf{T}}}{\partial x_1} + \frac{\partial \overset{2}{\mathbf{T}}}{\partial x_2} + \frac{\partial \overset{3}{\mathbf{T}}}{\partial x_3} = 0$$

or

$$\frac{\partial \overset{i}{\mathbf{T}}}{\partial x_i} = 0 \qquad \text{(summation convention)}$$

or

$$\frac{\partial(\tau_{ij}\mathbf{e}_j)}{\partial x_i} = 0 \qquad \text{(double summation convention)}$$

or

$$\frac{\partial \tau_{ij}}{\partial x_i}\mathbf{e}_j = 0$$

This is the vector equation of equilibrium. The three scalar equations are

$$\frac{\partial \tau_{ij}}{\partial x_i} = 0 \qquad (j = 1, 2, 3;\ i \text{ summed } 1, 2, 3)$$

We now introduce an additional convention in connection with the symbol for differentiation. The symbol $\partial/\partial x_i$ is interpreted as: $\partial/\partial x$ means "differentiate with respect to a coordinate," and the subscript i specifies which coordinate. For convenience, instead of the symbol $\partial/\partial x$ we shall write subscript comma (,). So the equilibrium equation is

$$\tau_{ij,i} = 0 \tag{I.2}$$

For purposes of illustration let us expand this equation and translate it into ordinary notation

1. The equations of equilibrium are

$$\tau_{ij,i} = 0$$

2. Sum with respect to i

$$\tau_{1j,1} + \tau_{2j,2} + \tau_{3j,3} = 0$$

3. Write $\partial/\partial x$ in place of (,)

$$\frac{\partial \tau_{1j}}{\partial x_1} + \frac{\partial \tau_{2j}}{\partial x_2} + \frac{\partial \tau_{3j}}{\partial x_3} = 0$$

4. The subscript j has values 1, 2, and 3

$$\frac{\partial\tau_{11}}{\partial x_1} + \frac{\partial\tau_{21}}{\partial x_2} + \frac{\partial\tau_{31}}{\partial x_3} = 0$$

$$\frac{\partial\tau_{12}}{\partial x_1} + \frac{\partial\tau_{22}}{\partial x_2} + \frac{\partial\tau_{32}}{\partial x_3} = 0$$

$$\frac{\partial\tau_{13}}{\partial x_1} + \frac{\partial\tau_{23}}{\partial x_2} + \frac{\partial_{33}\tau}{\partial x_3} = 0$$

5. Write stresses in σ_x, τ_{xy}, etc., notation

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} = 0$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} = 0$$

In the new notation, equilibrium of the element is expressed using a total of 7 written symbols ($\tau_{ij,i} = 0$) whereas the ordinary notation requires 72 different symbols. The physical interpretation of the equation of equilibrium is as follows

$$\tau_{ij,i} = 0$$

1. The equation refers to stresses, τ.
2. There are 3 equations since j may be either 1, 2, or 3.
3. There are 3 different stresses in each equation since i is summed from 1 to 3.
4. The 3 different stresses are on different faces ($i = 1, 2, 3$) but are all in the same direction (j).
5. Each stress is differentiated with respect to the coordinate corresponding to the face.
6. The sum of the 3 stress derivatives is zero.

I-2 TRANSFORMATION OF COORDINATES

Suppose the stresses τ_{ij} at a point are known for $i, j = 1, 2, 3$ and it is desired to find the stresses on a different face, say the 1′-face. Consider the corner of the element shown in Fig. I.3 where the resultant *stress* on the 1′-face is the vector:

$$\overset{1'}{\mathbf{T}} = \tau_{1'\beta'}\mathbf{e}_{\beta'}$$

where the β'-directions are those of the 1′, 2′, 3′ axes which are three orthogonal axes rotated with respect to the 1, 2, 3 axes. The stress components on the 1′-face are $\tau_{1'1'}$, $\tau_{1'2'}$, $\tau_{1'3'}$. Let $A_{1'}$ be the area of the 1′-face of the tetrahedron and A_1, A_2, A_3 be the respective areas of the other 3 faces. Then, as may be verified,

$$A_i = A_{1'}l_{i1'}$$

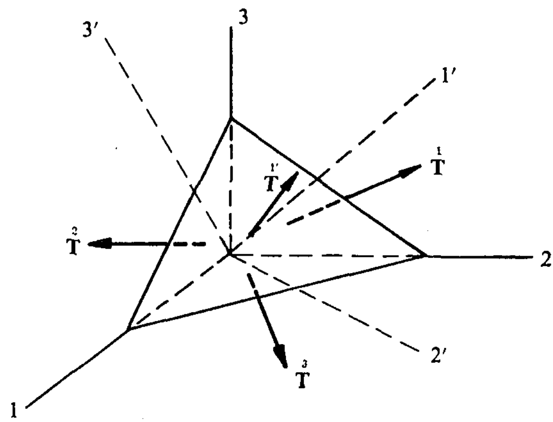


FIGURE I.3

where $l_{i1'}$ is the cosine of the angle between the i and 1′-directions. The resultant forces on each of the four faces of the tetrahedron are

$$\overset{1'}{\mathbf{T}}A_{1'},\quad \overset{1}{\mathbf{T}}A_1,\quad \overset{2}{\mathbf{T}}A_2,\quad \overset{3}{\mathbf{T}}A_3$$

Equilibrium of the tetrahedron requires the force on the 1′-face to be equal and opposite to the sum of the forces on the 1, 2 and 3-faces:

$$\overset{1'}{\mathbf{T}}A_{1'} = \overset{i}{\mathbf{T}}A_i \qquad \text{(summation convention)}$$

or

$$\overset{1'}{\mathbf{T}}A_{1'} = \overset{i}{\mathbf{T}}A_{1'}l_{i1'}$$

or

$$\overset{1'}{\mathbf{T}} = \overset{i}{\mathbf{T}}l_{i1'}$$

or

$$\tau_{1'\beta'}\mathbf{e}_{\beta'} = \tau_{ij}\mathbf{e}_j l_{i1'}$$

This is a vector equation, one side of which is expressed in terms of β' components and the other side of which is expressed in terms of j components. To obtain scalar equations from it we form the vector dot product with $\mathbf{e}_{\gamma'}$. This gives the component equation of equilibrium in the γ'-direction:

$$\tau_{1'\beta'}\mathbf{e}_{\beta'}\cdot\mathbf{e}_{\gamma'} = \tau_{ij}l_{i1'}\mathbf{e}_j\cdot\mathbf{e}_{\gamma'}$$

Since $\mathbf{e}_{\beta'}\cdot\mathbf{e}_{\gamma'} = \delta_{\beta'\gamma'}$* and $\mathbf{e}_j\cdot\mathbf{e}_{\gamma'} = l_{j\gamma'}$,

$$\tau_{1'\beta'}\delta_{\beta'\gamma'} = \tau_{ij}l_{i1'}l_{j\gamma'}$$

For $\beta' = \gamma'$ we obtain

$$\tau_{1'\beta'} = \tau_{ij}l_{i1'}l_{j\beta'}$$

or, more generally:

$$\tau_{\alpha'\beta'} = \tau_{ij}l_{i\alpha'}l_{j\beta'} \tag{I.3}$$

This is the transformation equation for stresses. There is a double summation (i, j) on the right-hand side and in expanded form it is

$$\begin{aligned}\tau_{\alpha'\beta'} &= \tau_{11}l_{1\alpha'}l_{1\beta'} + \tau_{12}l_{1\alpha'}l_{2\beta'} + \tau_{13}l_{1\alpha'}l_{3\beta'}\\ &+ \tau_{21}l_{2\alpha'}l_{1\beta'} + \tau_{22}l_{2\alpha'}l_{2\beta'} + \tau_{23}l_{2\alpha'}l_{3\beta'}\\ &+ \tau_{31}l_{3\alpha'}l_{1\beta'} + \tau_{32}l_{3\alpha'}l_{2\beta'} + \tau_{33}l_{3\alpha'}l_{3\beta'}\end{aligned}$$

The foregoing transformation could, of course, have been derived in standard notation rather than tensor notation but as may be verified in this case the procedure is much more laborious.

I-3 PRINCIPAL STRESSES

If the stresses, τ_{ij}, at a point are known, then the stresses $\tau_{\alpha'\beta'}$ with respect to a rotated coordinate system can be derived, as was shown in the preceding

* In an orthogonal coordinate system the unit vectors have the following properties: $\mathbf{e}_i\cdot\mathbf{e}_j = 0$ for $i \neq j$, and $= 1$ for $i = j$. These properties are expressed by the frequently used symbol δ_{ij}. This is called the Kronecker delta and has the property that

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

An example of its use is

$$\mathbf{A}\cdot\mathbf{B} = A_iB_j\,\delta_{ij} = A_1B_1 + A_2B_2 + A_3B_3$$

section, by writing the equation of equilibrium of the tetrahedron shown in Fig. I.3:

$$\overset{i}{\mathbf{T}}A_i = \overset{1'}{\mathbf{T}}A_{1'}$$

The stress vector $\overset{1'}{\mathbf{T}}$ can be expressed in terms of its 1′, 2′, 3′ components or in terms of 1, 2, 3 components:

$$\overset{1'}{\mathbf{T}} = \tau_{1'\beta'}\,\mathbf{e}_{\beta'} = \tau_{1'j}\mathbf{e}_j$$

When the 1′-plane is a principal plane, the stress vector $\overset{1'}{\mathbf{T}}$ lies along the 1′-axis and it follows that its 1, 2, 3 components are given by

$$\tau_{1'j} = Sl_{1'j} = S\,\delta_{ij}l_{1'i}$$

where S is the magnitude of $\overset{1'}{\mathbf{T}}$. The vector equation of equilibrium can therefore be written, when 1′ is a principal plane

$$\tau_{ij}A_i\mathbf{e}_j = Sl_{1'i}\,\delta_{ij}A_{1'}\mathbf{e}_j$$

or

$$(S\,\delta_{ij} - \tau_{ij})l_{1'i}\mathbf{e}_j = 0$$

or, in scalar form, writing α' for 1′,

$$(S\,\delta_{ij} - \tau_{ij})l_{i\alpha'} = 0$$

This set of three scalar equations determines the values of the three direction cosines $l_{1\alpha'}$, $l_{2\alpha'}$, $l_{3\alpha'}$, which specify the principal plane. In expanded form the three equations are

$$\begin{aligned}
(S - \tau_{11})l_{1\alpha'} - \tau_{12}l_{2\alpha'} - \tau_{13}l_{3\alpha'} &= 0 \\
-\tau_{12}l_{1\alpha'} + (S - \tau_{22})l_{2\alpha'} - \tau_{23}l_{3\alpha'} &= 0 \\
-\tau_{13}l_{1\alpha'} - \tau_{23}l_{2\alpha'} + (S - \tau_{33})l_{3\alpha'} &= 0
\end{aligned}$$

This set of equations has a solution *only if the determinant of the coefficient* of the $l_{i\alpha'}$ is zero. The determinantal equation is

$$S^3 - I_1S^2 + I_2S - I_3 = 0$$

where

$$\begin{aligned}
I_1 &= (\tau_{11} + \tau_{22} + \tau_{33}) = \tau_{ii} \\
I_2 &= (\tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2) \\
I_3 &= (\tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{23}\tau_{31} - \tau_{11}\tau_{23}^2 - \tau_{22}\tau_{13}^2 - \tau_{33}\tau_{12}^2)
\end{aligned}$$

The three values of S which are the roots of this equation are the values of the three principal stresses. Since the magnitudes of the principal stresses do not depend upon the orientation of the axes (1, 2, 3) with respect to which the state of stress is expressed, it follows that the coefficients I_1, I_2, I_3 of the foregoing equation must have the same values independent of the orientation of the coordinate axes. I_1, I_2, I_3 are said to be invariant. The stress invariants assume particularly simple forms when expressed in terms of principal stresses, σ_1, σ_2, σ_3

$$I_1 = (\sigma_1 + \sigma_2 + \sigma_3)$$

$$I_2 = (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

It is seen that when the values of the three stress invariants are known, it is possible to calculate the values of the three principal stresses. The three invariants thus describe the stress at a point, but they do not specify the orientation of the principal axes. The complete state of stress at a point is specified by I_1, I_2, I_3 plus three angles which orient the principal axes.

In terms of the distortional stresses, $\tau'_{ij} = \tau_{ij} - (I_1/3)\,\delta_{ij}$, the stress invariants, after a bit of algebraic manipulation, can be written

$$I_1 = \tau_{ii}$$

$$I_2 = \frac{I_1^2}{3} - \tfrac{1}{2}I'_2$$

$$I_3 = \left(\frac{I_1}{3}\right)^3 + \frac{I_1 I'_2}{3} + I'_3$$

where

$$I'_2 = \tau'_{ij}\tau'_{ij} = \tau'^2_{11} + \tau'^2_{22} + \tau'^2_{33} + 2\tau'^2_{12} + 2\tau'^2_{23} + 2\tau'^2_{13}$$

and I'_3 is the determinant of the array τ'_{ij}.

$$I'_3 = \begin{vmatrix} \tau'_{11} & \tau'_{12} & \tau'_{13} \\ \tau'_{12} & \tau'_{22} & \tau'_{23} \\ \tau'_{13} & \tau'_{23} & \tau'_{33} \end{vmatrix} = \tau'_{11}\tau'_{22}\tau'_{33} + 2\tau'_{12}\tau'_{23}\tau'_{31} - \tau'_{11}\tau'^2_{23} - \tau'_{22}\tau'^2_{13} - \tau'_{33}\tau'^2_{12}$$

It is seen that if I_1, I_2, I_3 are invariants then I'_2 and I'_3 must also be invariants. Since the τ'_{ij} stresses represent how much the actual state of stress deviates from a state of pure dilatation, I'_2 and I'_3 are sometimes called the deviatoric stress invariants. Whereas I_1 describes the dilatational state of stress, the values of I'_2 and I'_3 describe the distortional (or deviatoric) stress, except for the orientation of the element. It may be verified that $I_2'/2G$ is the distortional energy per unit volume (Eq. 1.58) and hence the Hencky–Mises yield condition (Eq. 1.59) can also be written $I_2' = \frac{2}{3}\sigma_0^2$, where σ_0 is the yield stress in uniaxial tension.

I-4 TRANSFORMATION OF STRAINS

In line with the principle of simplifying the notation, instead of u, v, w, we shall write, u_1, u_2, u_3 for the displacements of a point in the 1, 2, 3 directions respectively. To examine the strains at a point in a body consider an unstrained element dx_1, dx_2, dx_3, as shown in Fig. I.4a. We call the diagonal of the element the vector $\mathbf{R}$; it has, then, the three components dx_1, dx_2, dx_3.

$$\begin{aligned}\mathbf{R} &= dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3 \\ &= dx_i\mathbf{e}_i \\ &= R_i\mathbf{e}_i\end{aligned}$$

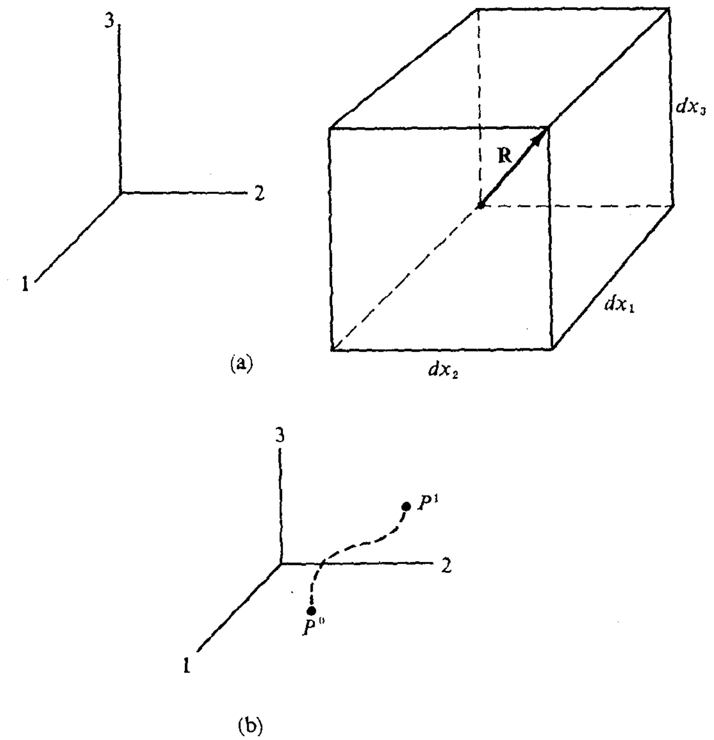


FIGURE I.4

After the element has undergone a strain, the vector $\mathbf{R}$ will be changed, and the diagonal of the strained element will be a vector $\mathbf{R}'$ where

$$\mathbf{R}' = R_i'\mathbf{e}_i$$

With the usual assumptions about small strains we shall examine the change in $\mathbf{R}$. To do this we must calculate the change

$$\mathbf{R}' - \mathbf{R} = \delta\mathbf{R}$$

or, examining the components of the vectors, we must calculate

$$R_i' - R_i = \delta R_i$$

Because of the straining, the R_1' component is larger than the R_1 component by

$$\delta R_1 = \frac{\partial u_1}{\partial x_1}\,dx_1 + \frac{\partial u_1}{\partial x_2}\,dx_2 + \frac{\partial u_1}{\partial x_3}\,dx_3$$

$$= u_{1,j}\,dx_j$$

or, in general,

$$\delta R_i = u_{i,j}\,dx_j$$

We shall see that it is convenient to write this

$$\delta R_i = \tfrac{1}{2}(u_{i,j} - u_{j,i})\,dx_j + \tfrac{1}{2}(u_{i,j} + u_{j,i})\,dx_j$$

Then, introducing the notation

$$\tfrac{1}{2}(u_{i,j} - u_{j,i}) = \omega_{ij}$$

$$\tfrac{1}{2}(u_{i,j} + u_{j,i}) = \epsilon_{ij}$$

we have

$$\delta R_i = \omega_{ij}\,dx_j + \epsilon_{ij}\,dx_j$$

The ω_{ij} are rigid body rotations of the element, and the ϵ_{ij} represent the strains, for example,

$$\epsilon_{11} = \tfrac{1}{2}(u_{1,1} + u_{1,1}) = u_{1,1} = \frac{\partial u_1}{\partial x_1}$$

This is the normal strain which in the standard notation was written $\partial u/\partial x = \epsilon_x$. Another example is

$$\epsilon_{12} = \tfrac{1}{2}(u_{1,2} + u_{2,1}) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

This is the shearing strain which in the standard notation was written ϵ_{xy}.

It can be shown (see Prob. I.3) that the strains transform in the same way as the stresses transform:

$$\epsilon_{\alpha'\beta'} = l_{i\alpha'}l_{j\beta'}\epsilon_{ij} \tag{I.4}$$

I-5 DERIVATION OF THE COMPATIBILITY EQUATIONS

The tensor notation permits us to discuss the compatibility equations by means of concise expressions which would not be possible using the ordinary scalar notation. Compatible strains may be defined as follows. Consider two points in an unstrained body, P^0 at the point x_1^0, x_2^0, x_3^0, and P^1 at the point x_1^1, x_2^1, x_3^1, as shown in Figure I.4b. The superscripts refer to the two particular fixed points. When the body is strained the displacement of P^0 has components u_j^0, that is, u_1^0, u_2^0, u_3^0, and the displacement of P^1 has components u_j^1. Knowing u_j^0 the displacement of u_j^1 can be computed by

$$u_j^1 = u_j^0 + \int_{P^0}^{P^1} du_j \qquad (j = 1, 2, 3) \tag{I.5}$$

The strains are said to be compatible if the same value of u_j^1 is obtained for all possible paths of integration between P^0 and P^1.

The du_j appearing in the integral in Eq. (I.5) can be expressed in terms of the derivatives of the strains, and the condition that the integral be independent of the path between the two points leads to certain equations involving the derivatives of the strains (compatibility equations). The procedure will only be sketched here.*

As may be verified,

$$du_j = u_{j,k}\,dx_k = (\epsilon_{jk} + \omega_{jk})\,dx_k$$

* For details see I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill (1956).

where the strains and rotations are given by

$$\epsilon_{jk} = \tfrac{1}{2}(u_{j,k} + u_{k,j})$$

$$\omega_{jk} = \tfrac{1}{2}(u_{j,k} - u_{k,j})$$

and hence,

$$u_j^1 = u_j^0 + \int_{P^0}^{P^1} (\epsilon_{jk} + \omega_{jk})\, dx_k$$

We note that an integration by parts gives

$$\int_{x_k^0}^{x_k^1} \omega_{jk}\, dx_k = x_k \omega_{jk} \Big|_{x_k^0}^{x_k^1} - \int_{P^0}^{P^1} x_k\, d\omega_{jk}$$

$$= x_k \omega_{jk} \Big|_{x_k^0}^{x_k^1} - \int_{P^0}^{P^1} x_k \omega_{jk,l}\, dx_l$$

Making use of the identity,

$$\int_{P^0}^{P^1} x_k^1\, \omega_{jk,l}\, dx_l = x_k^1 \omega_{jk} \Big|_{P^0}^{P^1}$$

we may write

$$\int_{P^0}^{P^1} \omega_{jk}\, dx_k = (x_k^1 - x_k^0)\omega_{jk}^0 + \int_{P^0}^{P^1} (x_k^1 - x_k)\omega_{jk,l}\, dx_l$$

Substituting this in the equation for u_j^1 and noting that

$$\omega_{jk,l} = \epsilon_{jl,k} - \epsilon_{kl,j}$$

the displacement u_j^1 can be written

$$u_j^1 = u_j^0 + (x_k^1 - x_k^0)\omega_{jk}^0 + \int_{P^0}^{P^1} U_{jl}\, dx_l$$

where

$$U_{jl} = \epsilon_{jl} + (x_k^1 - x_k)(\epsilon_{jl,k} - \epsilon_{kl,j})$$

For the integral to be independent of the path of integration it is necessary that the expression under the integral sign be an exact differential. The condition for this is

$$U_{jl,i} = U_{ji,l}$$

or

$$U_{jl,i} - U_{ji,l} = 0$$

When this equation is written in expanded form it is found that it reduces to

$$\epsilon_{jl,ki} + \epsilon_{ki,jl} - \epsilon_{kl,ji} - \epsilon_{ji,kl} = 0 \tag{I.6}$$

Equation (I.6) holds for $i = 1, 2, 3$; $j = 1, 2, 3$; $k = 1, 2, 3$; $l = 1, 2, 3$, so that it represents 81 equations; however, all but six of them are either identically zero or are duplicates. The six unique compatibility equations are specified by the following six sets of subscripts:

$$\begin{array}{llllll} j = 1 & 2 & 3 & 1 & 2 & 3 \\ l = 1 & 2 & 3 & 1 & 2 & 3 \\ k = 2 & 3 & 1 & 2 & 3 & 1 \\ i = 3 & 1 & 2 & 2 & 3 & 1 \end{array}$$

In expanded notation the six compatibility equations are

$$\begin{aligned}
&\frac{\partial^2 \epsilon_{xx}}{\partial y\,\partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x^2} - \frac{\partial^2 \epsilon_{xy}}{\partial x\,\partial z} - \frac{\partial^2 \epsilon_{xz}}{\partial y\,\partial x} = 0 \\
&\frac{\partial^2 \epsilon_{yy}}{\partial z\,\partial x} + \frac{\partial^2 \epsilon_{zx}}{\partial y^2} - \frac{\partial^2 \epsilon_{zy}}{\partial y\,\partial x} - \frac{\partial^2 \epsilon_{yx}}{\partial z\,\partial y} = 0 \\
&\frac{\partial^2 \epsilon_{zz}}{\partial x\,\partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial z^2} - \frac{\partial^2 \epsilon_{xz}}{\partial z\,\partial y} - \frac{\partial^2 \epsilon_{zy}}{\partial x\,\partial z} = 0 \\
&\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - \frac{\partial^2 \epsilon_{yx}}{\partial x\,\partial y} - \frac{\partial^2 \epsilon_{xy}}{\partial y\,\partial x} = 0 \\
&\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} - \frac{\partial^2 \epsilon_{zy}}{\partial y\,\partial z} - \frac{\partial^2 \epsilon_{yz}}{\partial z\,\partial y} = 0 \\
&\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} - \frac{\partial^2 \epsilon_{xz}}{\partial z\,\partial x} - \frac{\partial^2 \epsilon_{zx}}{\partial x\,\partial z} = 0
\end{aligned} \tag{I.7}$$

It will be seen that the fourth of Eq. (I.7) is the same as the compatibility equation derived for plane stress in Chapter 2.

By means of Hooke's law the compatibility equations can be written in terms of stresses. When this is done, the equations can be put in the following standard forms when the body forces are zero:

$$\begin{aligned}
&\left(\frac{1+\nu}{3}\right)\nabla^2\sigma_x + \frac{\partial^2\bar{\sigma}}{\partial x^2} = 0 \qquad &&\left(\frac{1+\nu}{3}\right)\nabla^2\tau_{xy} + \frac{\partial^2\bar{\sigma}}{\partial x\,\partial y} = 0 \\
&\left(\frac{1+\nu}{3}\right)\nabla^2\sigma_y + \frac{\partial^2\bar{\sigma}}{\partial y^2} = 0 \qquad &&\left(\frac{1+\nu}{3}\right)\nabla^2\tau_{yz} + \frac{\partial^2\bar{\sigma}}{\partial y\,\partial z} = 0 \\
&\left(\frac{1+\nu}{3}\right)\nabla^2\sigma_z + \frac{\partial^2\bar{\sigma}}{\partial z^2} = 0 \qquad &&\left(\frac{1+\nu}{3}\right)\nabla^2\tau_{zx} + \frac{\partial^2\bar{\sigma}}{\partial z\,\partial x} = 0
\end{aligned} \tag{I.8}$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\bar{\sigma} = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

I-6 DEFINITION OF A TENSOR

Suppose we have a scalar function $f(x_1, x_2, x_3)$ where the 1, 2, 3 coördinates pertain to an orthogonal set of axes. If we were to transform to another linear set of axes x_1', x_2', x_3' the transformation of f would be

$$f'(x_1', x_2', x_3') = f(x_1, x_2, x_3) \tag{I.9}$$

i.e., f' and f are identical in value at every point in space. All scalar functions transform according to Eq. (I.9) and we use this property to define a tensor of rank zero, that is, a scalar is a tensor of rank zero. The reason for this nomenclature will become clear in the course of the following definitions.

Let us consider again the scalar function $f(x_1, x_2, x_3)$ and form the components of the gradient. These are the projections of the vector gradient *upon* the axes:

$$\frac{\partial f}{\partial x_1}\mathbf{e}_1; \quad \frac{\partial f}{\partial x_2}\mathbf{e}_2; \quad \frac{\partial f}{\partial x_3}\mathbf{e}_3$$

The scalar components are $\partial f/\partial x_i$. The $\partial f/\partial x_i$ are rates of change along x_i. If we were to transform to another set of coordinates (linear) we would require the relation between the new and old coordinates

$$x_i' = \varphi_i'(x_1, x_2, x_3)$$

$$x_i = \varphi_i(x_1', x_2', x_3')$$

Now the rates of change along x_1', x_2', x_3' are $\partial f'/\partial x_i'$ but, according to the rules of differentiation,

$$\frac{\partial f'}{\partial x_i'} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x_i'}$$

This then is the transformation equation for the components of the gradient. In general, any vector $\mathbf{R}$ whose projections *on* the axes are R_i transforms according to

$$R_i' = R_j\frac{\partial x_j}{\partial x_i'} \tag{I.10}$$

An entity defined by a set of such components, that is, whose components transform according to Eq. (I.10) is called a *covariant tensor* of the first rank.

In a cartesian coordinate system the derivative in this expression is the direction cosine l_{ij}

$$\frac{\partial x_j}{\partial x'_i} = l_{ij}$$

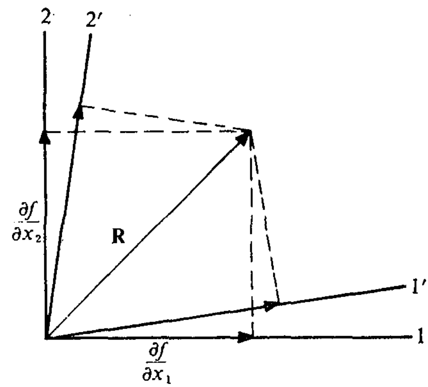


FIGURE I.5

The preceding *covariant* tensor was defined by components which were projections on the axes as shown in Fig. I.5. Let us now consider a vector **R**, Fig. I.6, where

$$\mathbf{R} = R_i\mathbf{e}_i = R'_j\mathbf{e}'_j$$

that is, the R_i and R'_j are now components *along* the axes. An example is the element of length

$$d\mathbf{s} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3 = dx_i\mathbf{e}_i = dx'_j\mathbf{e}'_j$$

Now in the x'-coordinate system

$$dx'_j = \frac{\partial x'_j}{\partial x_1}\,dx_1 + \frac{\partial x'_j}{\partial x_2}\,dx_2 + \frac{\partial x'_j}{\partial x_3}\,dx_3 = \frac{\partial x'_j}{\partial x_i}\,dx_i$$

That is,

$$R'_j = R_i\,\frac{\partial x'_j}{\partial x_i} \tag{I.11}$$

This transformation differs from Eq. (I.10) and it defines a *contravariant* tensor of the first rank. If we are dealing only with cartesian coordinate systems then

$$\frac{\partial x'_j}{\partial x_i} = l_{ij}$$

and there is no difference between covariant and contravariant tensors.

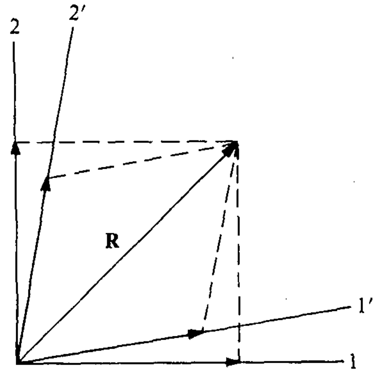


FIGURE I.6

The equation for the transformation of stresses which was derived in a preceding paragraph:

$$\tau'_{\alpha\beta} = l_{i\alpha} l_{j\beta} \tau_{ij} = \frac{\partial x'_\alpha}{\partial x_i} \frac{\partial x'_\beta}{\partial x_j} \tau_{ij}$$

is the transformation of a tensor of the *second* rank.

Many other quantities, in addition to vectors, stresses and strains are tensors. For example, the bending and twisting moments M_x, M_y, M_{xy} in a flat plate are components of a tensor of the second rank since they transform in the same way as do the stresses σ_x, σ_y, τ_{xy}. Also the mass moments and products of inertia are the components of a tensor of the second rank since they transform according to

$$I_{\alpha'\beta'} = I_{ij} l_{i\alpha'} l_{j\beta'}$$

where the I_{ij}'s are defined by the following integral over the mass of the body*

$$I_{ij} = \iiint (x_k x_k \, \delta_{ij} - x_i x_j) \, dm$$

* Note that $\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$.

The elastic properties of a material are also a tensor. The most general form of Hooke's law in tensor notation is the linear relation between the stresses and the strains

$$\tau_{ij} = C_{ijkl}\epsilon_{kl}$$

In a rotated cartesian coordinate system the stresses and strains are given by

$$\tau_{\alpha'\beta'} = l_{i\alpha'}l_{j\beta'}\tau_{ij}$$

$$\epsilon_{\gamma'\delta'} = l_{k\gamma'}l_{l\delta'}\epsilon_{kl}$$

The inverse of the last expression is

$$\epsilon_{kl} = l_{k\gamma'}l_{l\delta'}\epsilon_{\gamma'\delta'}$$

Therefore, Hooke's law in the rotated coordinate system is

$$\tau_{\alpha'\beta'} = l_{i\alpha'}l_{j\beta'}\tau_{ij} = l_{i\alpha'}l_{j\beta'}C_{ijkl}\epsilon_{kl} = l_{i\alpha'}l_{j\beta'}l_{k\gamma'}l_{l\delta'}C_{ijkl}\epsilon_{\gamma'\delta'}$$

However, in the rotated coordinate system

$$\tau_{\alpha'\beta'} = C_{\alpha'\beta'\gamma'\delta'}\epsilon_{\gamma'\delta'}$$

From the last two equations we see that

$$C_{\alpha'\beta'\gamma'\delta'} = l_{i\alpha'}l_{j\beta'}l_{k\gamma'}l_{l\delta'}C_{ijkl} \tag{I.12}$$

This is how the elastic constants transform from one coordinate system to another, and it shows that the elastic coefficients are components of a tensor of the fourth rank.

It is seen that all of the foregoing tensors are in a certain sense physical entities which are represented equally well by the components in one coordinate system as by those in another. The properties of tensors and analysis by means of tensors are discussed in many books.*

I-7 NONISOTROPIC STRESS-STRAIN RELATIONS

The values of elastic coefficients of materials are usually given for the non-tensor form of the coefficients, that is, instead of the tensor form of the equations

$$\tau_{ij} = C_{ijkl}\epsilon_{kl}$$

$$\epsilon_{ij} = B_{ijkl}\tau_{kl}$$

* See, for example, *Vector and Tensor Analysis*, by G. E. Hay, Dover (1953), and *Mathematical Theory of Elasticity*, by I. S. Sokolnikoff, McGraw-Hill (2nd Ed. 1956).

The non-tensor forms are used

$$\begin{aligned} \tau_i &= C_{ij}\epsilon_j \\ & \qquad\qquad (i, j = 1, 2, \ldots, 6) \\ \epsilon_i &= B_{ij}\tau_j \end{aligned} \tag{I.13}$$

where $\tau_1 = \sigma_x$, $\tau_2 = \sigma_y$, $\tau_3 = \sigma_z$, $\tau_4 = \tau_{xz}$, $\tau_5 = \tau_{yz}$, $\tau_6 = \tau_{xy}$, and $\epsilon_1 = \epsilon_x$, $\epsilon_2 = \epsilon_y$, $\epsilon_3 = \epsilon_z$, $\epsilon_4 = \gamma_{xz} = \epsilon_{xz} + \epsilon_{zx}$, $\epsilon_5 = \gamma_{yz} = \epsilon_{yz} + \epsilon_{zy}$, $\epsilon_6 = \gamma_{xy} = \epsilon_{xy} + \epsilon_{yx}$.

The properties of a material are specified by the 36 coefficients C_{ij} or the 36 coefficients B_{ij}. These are not all independent as can be shown by the following energy considerations. The strain energy density is

$$V_0 = \int_0^{\epsilon_i} \tau_i \, d\epsilon_i = \tfrac{1}{2}\tau_i\epsilon_i = \tfrac{1}{2}C_{ij}\epsilon_i\epsilon_j$$

It is seen from the first two terms that

$$\frac{\partial V_0}{\partial \epsilon_k} = \tau_k$$

and taking the derivative, with respect to ϵ_k, of the last term:

$$\begin{aligned} \tau_k &= \frac{\partial}{\partial \epsilon_k}\left(\tfrac{1}{2}C_{ij}\epsilon_i\epsilon_j\right) \\ &= \tfrac{1}{2}C_{ij}\epsilon_j \frac{\partial \epsilon_i}{\partial \epsilon_k} + \tfrac{1}{2}C_{ij}\epsilon_i \frac{\partial \epsilon_j}{\partial \epsilon_k} \\ &= \tfrac{1}{2}C_{kj}\epsilon_j + \tfrac{1}{2}C_{ij}\epsilon_i \end{aligned}$$

This can be written, since $\tau_k = C_{kj}\epsilon_j$:

$$C_{kj}\epsilon_j = \tfrac{1}{2}(C_{kj} + C_{jk})\epsilon_j$$

or

$$\tfrac{1}{2}(C_{kj} - C_{jk})\epsilon_j = 0$$

from which

$$C_{jk} = C_{kj} \tag{I.14}$$

Because of this symmetry of elastic coefficients there can be at most

$$\frac{36-6}{2}+6=21$$

twenty-one independent elastic coefficients:

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{vmatrix}$$

If the material is isotropic the array of elastic coefficients must be identical when transformed from one coordinate system to another. As may be verified, this condition is satisfied only when the array is as follows

$$\begin{vmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{vmatrix} \quad \text{with } C_{44}=\tfrac{1}{2}(C_{11}-C_{12})$$

The corresponding array of coefficients for the equation $\epsilon_i = B_{ij}\tau_j$, is

$$\begin{vmatrix} B_{11} & B_{12} & B_{12} & 0 & 0 & 0 \\ B_{12} & B_{11} & B_{12} & 0 & 0 & 0 \\ B_{12} & B_{12} & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{44} \end{vmatrix} \quad \text{with } B_{44}=2(B_{11}-B_{12})$$

In terms of our old notation the elastic coefficients for isotropic materials are

$$C_{44}=G \qquad B_{44}=\frac{1}{G}$$

$$C_{12}=\frac{\nu E}{(1+\nu)(1-2\nu)}=\lambda \qquad B_{12}=-\frac{\nu}{E}$$

$$C_{11}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \qquad B_{11}=\frac{1}{E}$$

A nonisotropic material that has three planes of symmetry, such as a cubic crystal, has an array

$$\begin{vmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{vmatrix}$$

This material is described by three independent elastic moduli, C_{11}, C_{12}, C_{44}. This array is given with respect to the axes of symmetry of the material. For axes rotated from these axes, the array would, in general, not have so many zeros in it. The array for rotated axes could be calculated by means of the tensor equation

$$C_{\alpha'\beta'\gamma'\delta'} = l_{i\alpha'} l_{j\beta'} l_{k\gamma'} l_{l\delta'} C_{ijkl}$$

It should be noted that the four subscript C's have the same values as the corresponding two subscript C's, but because of the tensor summations, the four subscript B's are not the same as the two subscript B's. They are smaller by the ratios shown in the following table.

	11 22 33	12 21 23 32 31 13
11 22 33	1	$\frac{1}{2}$
12 21 23 32 31 13	$\frac{1}{2}$	$\frac{1}{4}$

Problems

I.1 With reference to Fig. I.3, prove that $A_i = A_{1'} l_{i1'}$.

I.2 Show that Eq. (I.3) reduces for the plane stress case to the equations derived in Chapter 1.

I.3 Derive Eq. (I.4) by considering how u_i transforms and how $u_{i,j}$ transforms.

I.4 The strains ϵ_{ij} at a point are given by the array shown below. What is the strain $\epsilon_{\alpha'\beta'}$, $\alpha' = 1'$, $\beta' = 2'$ in the rotated coordinate system if the direction cosines $l_{i\alpha'}$ are as given below:

$$\epsilon_{ij} = \begin{vmatrix} 0.002 & 0 & 0.001 \\ 0 & -0.001 & -0.004 \\ 0.001 & -0.004 & 0 \end{vmatrix} \qquad \begin{array}{c|ccc} & 1' & 2' & 3' \\ \hline 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 2 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 3 & 0 & 0 & -1 \end{array}$$

I.5 What forms do the stress invariants I_1, I_2', I_3' assume for plane stress? Verify that for plane stress I_1, I_2', I_3' are indeed invariants.

I.6 Show how the principal distortional stresses can be determined when I_2' and I_3' are known.

I.7 Since the strains ϵ_{ij} and the stresses τ_{ij} are both tensors of the second rank they must both have the same transformational properties. What are the three strain invariants J_1, J_2, J_3? What are J_2' and J_3'?

I.8 Show that $I_2 = I_1^2/3 - \frac{1}{2}I_2'$. Note that $\tau_{ii}' = 0$ and, hence,

$$(\tau_{ii}')^2 = 0 = \tau_{11}'^2 + \tau_{22}'^2 + \tau_{33}'^2 + 2\tau_{11}'\tau_{22}' + 2\tau_{22}'\tau_{33}' + 2\tau_{33}'\tau_{11}'$$

Therefore, $\tau_{11}'^2 + \tau_{22}'^2 + \tau_{33}'^2 = -2(\tau_{11}'\tau_{22}' + \tau_{22}'\tau_{33}' + \tau_{33}'\tau_{11}')$.

I.9 The mass moments and products of inertia of a solid body can be expressed in tensor notation by

$$I_{ij} = \int (r^2\,\delta_{ij} - x_i x_j)dm$$

where m is the mass, $r^2 = (x_1^2 + x_2^2 + x_3^2)$, and $\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. Write the array of three moments of inertia and six products of inertia according to this definition of I_{ij}. Show that I_{ij} transforms as a tensor of the second rank.

APPENDIX **II**

THE MEASUREMENT OF STRAIN

II-1 STRAIN MEASUREMENT

The average strain over a long gage length l is easily determined by measuring δl; for example, a steel bar 10 ft long that is stressed to 30,000 psi in tension will increase in length by $\delta l = 0.12$ in. and there is no difficulty in measuring one-tenth of an inch. However, when the gage length is only a fraction of an inch, δl becomes too small to measure without employing special techniques. A variety of different mechanical and optical systems have been developed with which the average strain over a relatively short gage length can be measured.* The most widely used method for measuring strains utilizes the bonded electrical resistance strain gage. This gage has become a very important experimental tool since its development by E. Simmons at the California Institute of Technology in 1940.†

II-2 BONDED STRAIN GAGE

The operation of the bonded strain gage is based on the fact that in certain materials a change in strain is accompanied by a corresponding change in

* See, for example, *Handbook of Experimental Stress Analysis*, M. Hetenyi, Ed., John Wiley and Sons, New York (1957).

† See, for example, the publications of the Society for Experimental Stress Analysis.

electrical resistance. This behavior is particularly pronounced in a wire made of 60% copper-40% nickel alloy called *constantan*. A small zig-zag grid filament of 0.001-in. diameter wire is cemented between two pieces of paper or thin bakelite. A cementing agent is applied to the surface where the strain is to be measured and the gage is bonded to the surface as shown in Fig. II.1. A number of different types of these gages are used for various special purposes or service conditions, and gage lengths between 0.03 and 1.0 in. are available. These gages and other gages made of an alloy called "iso-elastic" are used for measuring rapidly varying dynamic strains.

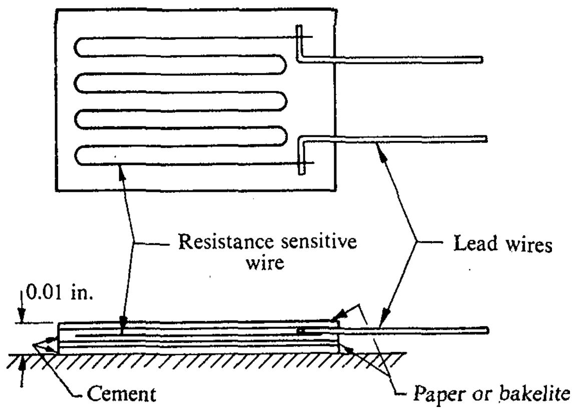


FIGURE II.1

A more recent development of this type of strain gage is the foil gage. The resistance element in this gage is formed of very thin metal foil instead of wire. For high temperature applications a foil gage with strippable paper backing is used. A thin coating of a ceramic type of cement is first baked onto the test material, and then the foil gage is placed on this with the exposed surface of the foil against the ceramic coating which has been covered with fresh ceramic cement. After the cement has dried the paper backing is stripped from the foil and a coat of ceramic cement is applied over it. This assembly is then baked so that the end product is a metal resistance element embedded within a thin layer of non-conducting ceramic material that is bonded to the surface of the test material. A number of other types of foil gages are also available.

Functioning of the Gage. The fraction change of resistance of the strain gage is directly proportional to the increment of strain to which it is subjected, if the strain does not exceed a few per cent. This relation is expressed by the equation

$$\frac{\delta R}{R} = K\,\delta\epsilon_x$$

where R = resistance of the gage

δR = change of resistance

$\delta\epsilon_x$ = change in strain

K = proportionality constant

The magnitude of the *gage factor* K, depends primarily on the metal used to make the resistance element of the gage. Most gages are made of constantan wire or foil and these have a gage factor of approximately 2. Semi-conductor material is used in some commercially available gages and these have a gage factor of approximately 140.

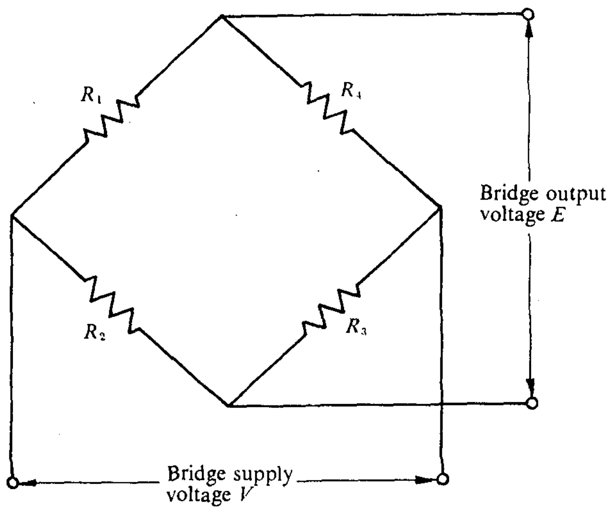


FIGURE II.2

The usual method of measuring with the gage is to let the change of resistance induce a voltage change in a Wheatstone bridge, shown in Fig. II.2. If the bridge is initially balanced ($E_0 = 0$) with $R_1 = R_2 = R_3 = R_4$, and input voltage V, small changes δR in the bridge resistors produce a voltage output

$$E = \frac{V}{4}\left(\frac{\delta R_1}{R_1} + \frac{\delta R_3}{R_3} - \frac{\delta R_2}{R_2} - \frac{\delta R_4}{R_4}\right)$$

If R_1 is a strain gage, and the other resistors are fixed, the voltage output is

$$E = \frac{V}{4}\frac{\delta R_1}{R} = \frac{V}{4} K\, \delta\epsilon_x$$

Measurement of E thus determines the change in strain when V and K are known. The Wheatstone bridge may be used to measure the sum or difference of two or more strains when two or more of the bridge resistors are strain gages.

It is advantageous to use as large a value of input voltage V as possible to increase the bridge output voltage E for a given strain. The practical limit on the value of V is set by the permissible gage heating due to the i^2R loss in the gage (i = current through the gage). The gage current is usually limited to approximately 20 milliamperes which produces a tolerable amount of heat in the gage. If the gage is not reasonably well protected from drafts or other effects which can influence the heat removed at the surface of the gage, spurious output voltages E will result as discussed below.

Commercially available instruments may be purchased which contain a voltage source V and a detection system which measures E. These instruments are provided with a gage factor setting and are calibrated directly in terms of strain. A typical system of this type will be able to measure strains as small as approximately 10^{-6} in./in.

Special gages are available for the determination of the state of strain on a surface, and for determination of the normal *stress* along a particular direction at the surface. The gages to measure the state of strain are called strain gage *rosettes*. A strain gage rosette is an assembly of three differently oriented gages, which measure the normal strain in three different directions (usually 45° or 60° apart). Mohr's circle for strain may be used together with the three strain gage readings to find the principal strains ϵ_1 and ϵ_2 and their orientation.

A so-called *stress gage* consists of two gages which measure strain along directions separated by an angle $\theta = 2\tan^{-1}\sqrt{\nu}$. As may be verified, the $\Delta R/R$ of the two gages in series is proportional to the normal stress acting along the bisector of the angle θ. Different stress gages must be employed for use on materials with different Poisson's ratio.

Temperature Compensation. The necessity for temperature compensation may be seen by considering the "apparent" strain which would be indicated by an active gage when the temperature of the gage and the part to which it is attached is changed. There are two effects to consider. First, a change in temperature produces a change in resistivity of the metal employed in the gage. Second, the coefficients of thermal expansion of the gage metal and the material in which the strain is being measured are in general different. Therefore, a temperature change produces a thermal strain in the gage since it is forced to follow the thermal expansion of the material to which it is bonded.

The "apparent" strain or error, $\delta\epsilon_T$ due to a temperature variation may be derived from the above considerations and is

$$\delta\epsilon_T = \left(\alpha_s - \alpha_g + \frac{\beta}{K}\right)\delta T$$

where α_s = the coefficient of thermal expansion of the material to which the gage is bonded,

α_g = the coefficient of thermal expansion of the gage metal, and

β = the temperature coefficient of resistivity of the gage metal.

It is possible to adjust β by the addition of small quantities of alloying elements to the gage metal without an appreciable effect on α_g. Alloys have been developed to make $\beta/k - \alpha_g = -\alpha_s$, for gages bonded to different materials such as steel, stainless steel, aluminum, titanium, and quartz. Gages made from the appropriate alloy are called temperature compensated gages, and $\delta\epsilon_T = 0$ for a limited range of temperatures. The range of temperature compensation is limited because resistivity and thermal expansion are not truly linear functions of temperature.

Apparent strains due to temperature effects can be eliminated by the proper use of dummy gages in a Wheatstone bridge circuit. If the measuring gage is R_1 in Fig. II.2, and R_2 is an identical dummy gage, bonded to a separate piece of the same material as that to which R_1 is bonded, the output due to a temperature change is

$$E = \frac{V}{4}\left(\frac{\delta R_1}{R_1} - \frac{\delta R_2}{R_2}\right) = \frac{V}{4}\,[K(\alpha_s - \alpha_g) + \beta](\delta T_1 - \delta T_2)$$

The bridge output is not affected by temperature changes that are the same in the active gage and the dummy gage.

II-3 BRITTLE COATING METHOD

There are two experimental methods that are commonly used to obtain an overall survey of the stress distribution of a stressed body as well as a quantitative measure of the stresses. They are the photoelastic method which is described in Appendix III, and the brittle coating method which is described in the following paragraphs. When there is a large stress gradient near the point of application of a concentrated load or near an abrupt change in shape of a body, a strain gage which averages over even a short gage length may give a reading which is far from the actual value of strain. In this case the photoelastic or brittle coating method must be used in order to measure the concentrated stress. The brittle coating method is particularly useful when the shape of the body to be analyzed is so complicated that an overall survey of the stress distribution is required to locate the points of high stress. Such a survey can often be followed by a strain gage analysis with the gages placed at the critical points indicated by the brittle coating tests.

The brittle coating method employs a special material applied in a thin coating to the surface of the body to be analyzed. The coating has the property that when properly cured on the surface of the body, a precalibrated threshold level of tensile strain in the body will cause the coating to crack. An actual structure or part, or a model of it, that has a brittle coating is observed while it is being stressed, and the magnitudes of the loading required to produce cracks at various points are determined. This procedure gives the values of loading required to produce a known principal strain at various points. A commonly encountered brittle coating is the oxide scale on the surface of hot rolled steel bars which cracks and flakes off when the base metal is stressed to the yield point. Two brittle coatings are commercially available which crack at strains below the elastic limit of structural materials. These coatings, supplied by the Magnaflux Corporation, are useful in nondestructive experimental stress analysis. The two coating materials are a lacquer with the trade name "Stresscoat" and a porcelain enamel with the trade name "All-Temp."

The ceramic coating may be used over a temperature range from $-60°F$ to 700°F. It is sprayed onto the part to be analyzed and is cured at a temperature of approximately 1000°F. The successful use of a ceramic coating depends on a controlled difference between the coefficients of expansion of the ceramic and the base metal. The coefficient of expansion of the ceramic coating is chosen to be 1.1 to 1.3 times that of the metal part to be analyzed to ensure that there will be residual tensile stresses in the coating. If the coefficient of expansion of the coating is greater than this it will crack or flake off when the part is cooled from the curing temperature, and if the coefficient of expansion of the ceramic coating is too low, it may not crack until the yield point of the base metal is reached. Coatings are available for use on carbon steel, chronium, austenitic stainless steel, cast iron and titanium.

The crack patterns developed in the ceramic are generally too fine to be seen with the unaided eye. The pattern is revealed when the coating is sprayed

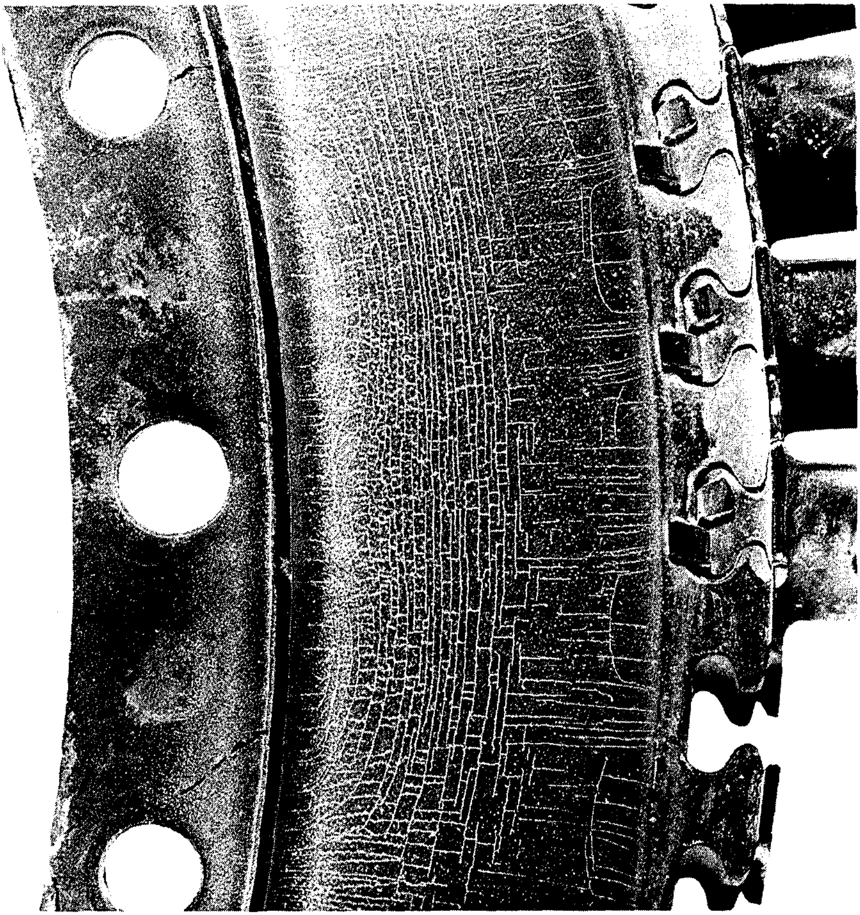


FIGURE II.3a Strain patterns developed in a ceramic coating on a compressor disk under operating conditions, courtesy of the Magnaflux Corporation.

with electrostatically charged particles which adhere to the cracks. Crack patterns which were developed on a compressor disk and on the compressor blades, under operating conditions, are shown in Fig. II.3. Since the closest crack spacing occurs where the tensile strain is greatest, the crack patterns revealed by the charged particles on the blades show clearly the regions of greatest and least strain. The cracks run along the trajectories of minimum strain and are orthogonal to the trajectories of *maximum principal* strain. The crack pattern on the compressor disk shows that many of the cracks due to hoop strain terminate on the cracks due to radial strain. This means that they developed *after* the cracks due to radial strain. Therefore the radial strain at those points was higher than the circumferential strain for the same

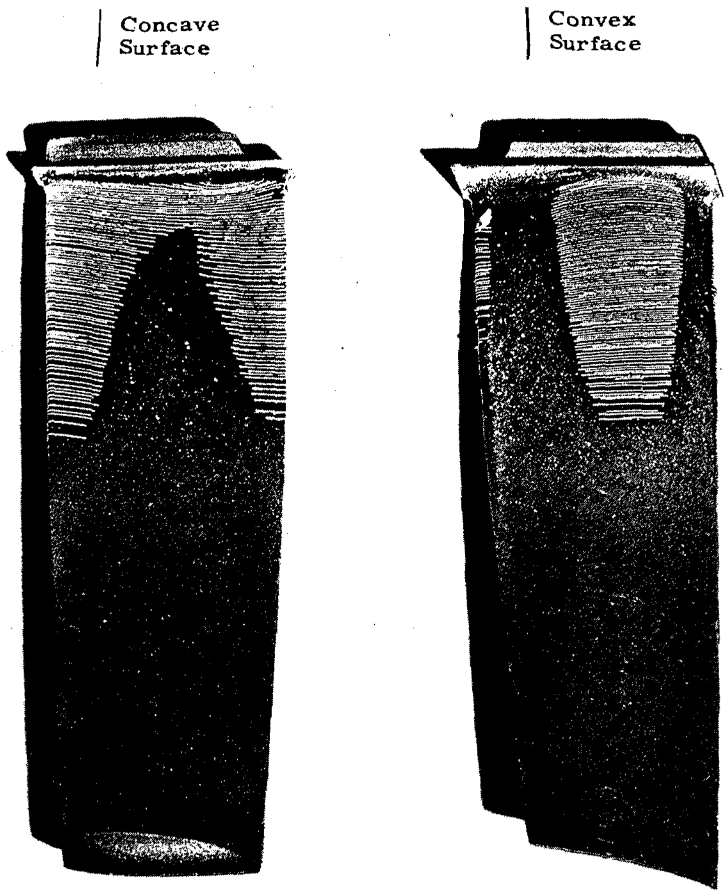


FIGURE II.3b Strain patterns developed in a ceramic coating on compressor blades under operating conditions, courtesy of the Magnaflux Corporation.

compressor speed since the cracks due to radial strain occurred first, at a lower rpm.

Brittle lacquer coatings are somewhat easier to apply and cure than ceramic coatings. The curing is done at or slightly above room temperature. Compressive strains may be studied with brittle lacquer by coating a stressed part or model and then removing the loads when the coating has cured. Brittle lacquer coatings are selected on the basis of the desired strain threshold for cracking independent of the material to be coated, and cracks in the lacquer can generally be detected with the unaided eye.

The chief difficulty encountered in the use of brittle lacquer is the tendency of the strained lacquer to creep with a consequent relaxation of stress. When the loading time is more than several minutes, the strain threshold for cracking increases appreciably. Another difficulty is that the strain threshold for cracking the lacquer is very sensitive to temperature and humidity. However, under laboratory conditions the method will give quantitative results with an accuracy of approximately 10%.

APPENDIX **III**

PHOTOELASTIC STRAIN MEASUREMENT

Certain optically isotropic materials, such as clear glass, lucite, etc., become optically nonisotropic when they are stressed. The discovery of this photoelastic effect is ascribed to Sir David Brewster who, in 1816, published an account of his observations on the color patterns exhibited when polarized white light was passed through a plate of glass in plane stress. A plate of CR-39 plastic has a photoelastic effect approximately fifteen times stronger than glass of the same thickness, and since it is also easy to machine models of plastic, this material is commonly used for photoelastic studies. Since the equations of equilibrium and the equation of compatibility for plane stress with no body forces are independent of the elastic constants, E and ν, the stress distribution is also independent of them. Therefore the stresses in a CR-39 beam are the same as in a steel beam of identical shape and loading.*

The essential features of the photoelastic method are exhibited by the apparatus shown in Fig. III.1. This consists of two parallel sheets of polaroid, P_1 and P_2. When a beam of monochromatic light passes through P_1 it is polarized in a vertical plane. The plane of polarization of P_2 is horizontal so that no light will pass through the two crossed polaroids. Also when a transparent plate of CR-39 is interposed between P_1 and P_2 no light will pass

* The photoelastic method of stress analysis was developed by J. C. Maxwell (1831–1879) in his paper "On the equilibrium of elastic solids," *Trans. Cambridge Phil. Soc.* (1850).

through P_2. However, if the plate is stressed, with principal stresses σ_1 and σ_2, light does pass through P_2. The relation between the intensity of light passing through P_2 and the magnitudes of σ_1 and σ_2 is shown in Fig. III.2 where it is seen that when $\sigma_1 = \sigma_2$ no light passes through but there is a

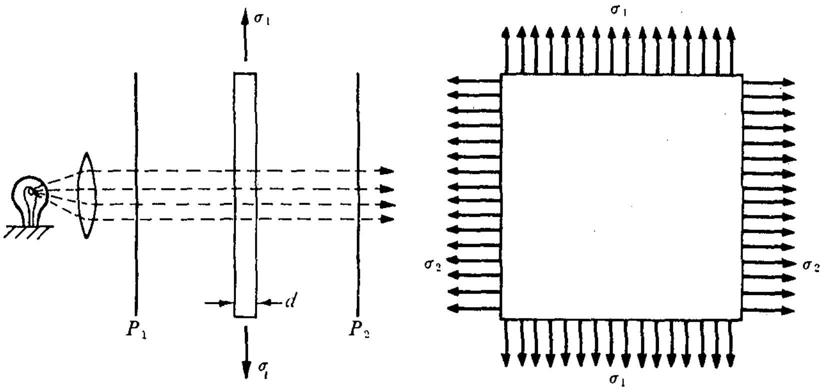


FIGURE III.1

cyclical variation of light intensity with increase of $(\sigma_1 - \sigma_2)$. Since $(\sigma_1 - \sigma_2)/2$ is the principal shear stress, a count of the number of times the model goes from dark through light to dark, etc., will determine the magnitude of the principal shear stress (this is the radius of Mohr's circle).

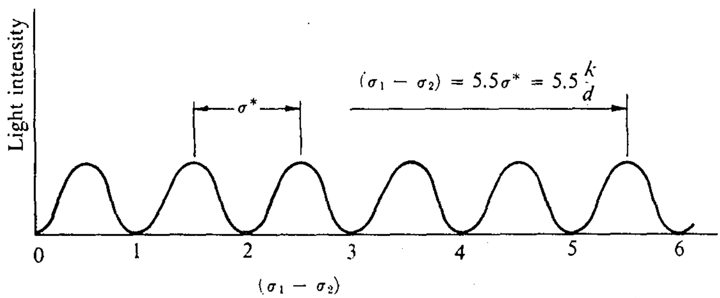


FIGURE III.2

The way in which the stressed specimen influences the polarized light is indicated in Fig. III.3a. After passing through the polarizer P_1 the plane polarized light can be represented by the "displacement" vector

$$A = a \cos \frac{2\pi}{\lambda}(x - ct)$$

where λ is the wavelength of the light and c is the propagation velocity of the light. The velocity of the light within the specimen is different from the velocity in air, and the velocity in the specimen is a function of the stresses. Therefore, the light A_1' issuing from the back face will be out of phase with the light entering the front face by an amount that depends upon the optical properties of the specimen and upon the stresses:

$$A' = a \cos \frac{2\pi}{\lambda} (x - ct - c\,\Delta t)$$

This expression can be written

$$A' \doteq a \cos \frac{2\pi}{\lambda} [x - ct - kd(\sigma_1 - \sigma_2)]$$

where d is the thickness of the specimen and k is an experimentally determined coefficient.

In Fig. III.3b there is shown a specimen with principal stresses σ_1, σ_2 making an angle α with respect to the planes of polarization, P_1 and P_2. After passing through the first polaroid the planar polarized light is represented by **A**. The components of **A** parallel to the principal stresses are

$$A_1 = A \cos \alpha = a \cos \alpha \cos \frac{2\pi}{\lambda} (x - ct)$$

$$A_2 = A \sin \alpha = a \sin \alpha \cos \frac{2\pi}{\lambda} (x - ct)$$

After passing through the specimen

$$A_1' = a \cos \alpha \cos \frac{2\pi}{\lambda} (x - ct - c\,\Delta t_1)$$

$$A_2' = a \sin \alpha \cos \frac{2\pi}{\lambda} (x - ct - c\,\Delta t_2)$$

Only the light that is polarized parallel to the plane of polarization of P_2 will emerge through it. The components of A_1' and A_2' in the plane of polarization of P_2 are

$$\begin{aligned} B &= A_1' \sin \alpha + A_2' \cos \alpha \\ &= a \sin \alpha \cos \alpha \left[\cos \frac{2\pi}{\lambda} (x - ct - c\,\Delta t_2) - \cos \frac{2\pi}{\lambda} (x - ct - c\,\Delta t_1)\right] \\ &= \left[a \sin 2\alpha \sin \frac{2\pi}{\lambda} \frac{c\,\Delta(t_2 - t_1)}{2}\right] \sin \frac{2\pi}{\lambda} (x - ct - c\,\Delta t_3) \end{aligned}$$

or

$$B = \left[a \sin 2\alpha \sin \frac{2\pi}{\lambda} kd \frac{(\sigma_1 - \sigma_2)}{2}\right] \sin \frac{2\pi}{\lambda} (x - ct - c\,\Delta t_3)$$

where $\Delta t_3 = \Delta t_1 + \Delta t_2$.

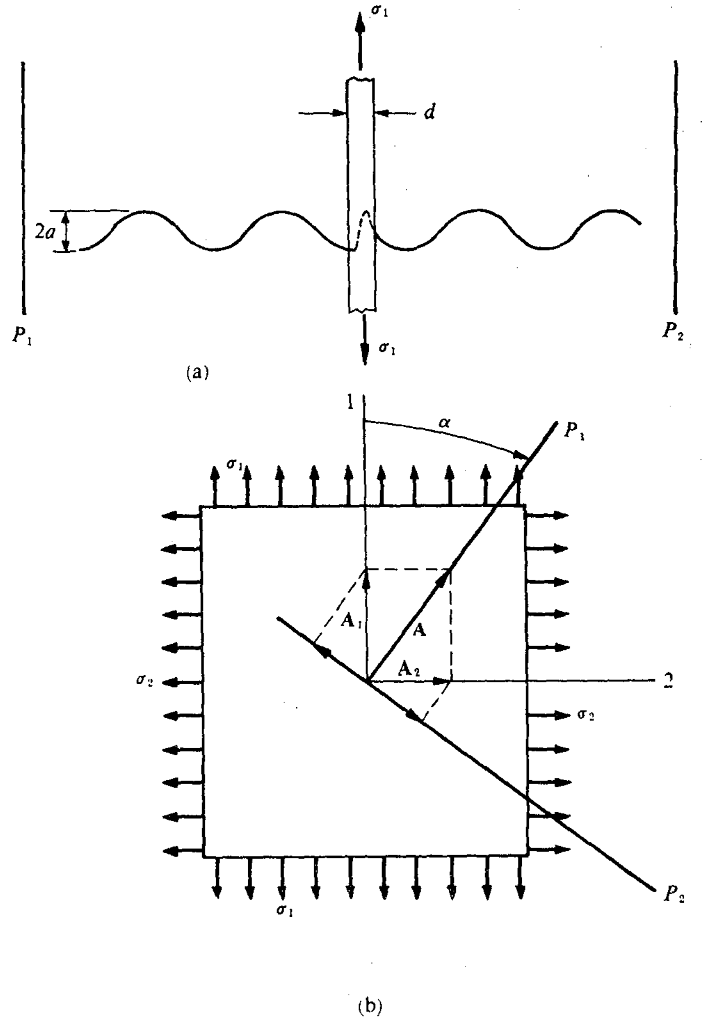


FIGURE III.3

The amplitude of the light vector **B** depends upon the term in brackets which shows that the intensity will vary with $(\sigma_1 - \sigma_2)$ as shown in Fig. III.2. In addition the amplitude of B depends upon the term $\sin 2\alpha$ which indicates that when the planes of polarization coincide with the directions of principal stresses ($\alpha = 0, \pi/2$) the intensity of emerging light will be zero. This provides a method of determining experimentally the orientation of the principal planes at a point in a stressed specimen.

If the CR-39 plate is a model of a stressed beam, then each element dx, dy of the beam will have the properties indicated by Figs. III.1, 2, 3. By observing this element, the magnitude of the principal shear stress and the orientation of the planes of principal stress at that point can be determined. In actual practice a more elaborate apparatus is used than that shown in Fig. III.1, the purpose being to improve the accuracy and the ease of operation.* Special optical components are introduced so that the effect of the angle α can be eliminated if it is desired to do so. When this is done, a photograph of a stressed beam as viewed through P_2 appears as shown in Fig. III.4, where the model is covered by a series of light and dark bands, called interference fringes.† Along each fringe the value of $(\sigma_1 - \sigma_2)$ is constant and, as shown in Fig. III.2, from one dark fringe to the adjacent dark fringe the value of $(\sigma_1 - \sigma_2)$ changes by K/d where K is the photoelastic constant of the material and d is the thickness of the model. Thus, the value of $(\sigma_1 - \sigma_2)$ along a given fringe is

$$(\sigma_1 - \sigma_2) = n\frac{K}{d}$$

where n is the fringe order, that is, the number of times that the point in the model has passed from light to light, etc. The fringe order can be determined by identifying the first fringe ($n = 1$) to appear as the model is loaded. Since the maximum stresses are on the boundaries of the model, the fringes shown in Fig. III.4 first appeared on a boundary and then moved to the interior as the loading was increased. The fringe order can also be determined if it is known that $\sigma_1 - \sigma_2 = 0$ at some point, for the fringe through that point has $n = 0$. For example, in Fig. III.5 the neutral axis of the beam has $(\sigma_1 - \sigma_2) = 0$ since along it $\sigma_1 = \sigma_2 = 0$.

A free boundary of a model is a principal plane on which one principal stress is zero ($\sigma_1 = 0$), hence, the fringe order at such a point determines the value of σ_2. Thus the state of stress is completely determined along the free boundary. Also, if a prescribed stress intensity is applied to the boundary of a model the complete state of stress at these points can be determined when the fringe order and the orientation of the principal plane are known. The

* See *Handbook of Experimental Stress Analysis*. M. Hetenyi, Ed., John Wiley and Sons, New York (1957). This Handbook also describes a technique for the photoelastic determination of three-dimensional stress distributions.

† This and most of the following photographs were taken to provide a light background that would show the loading apparatus.

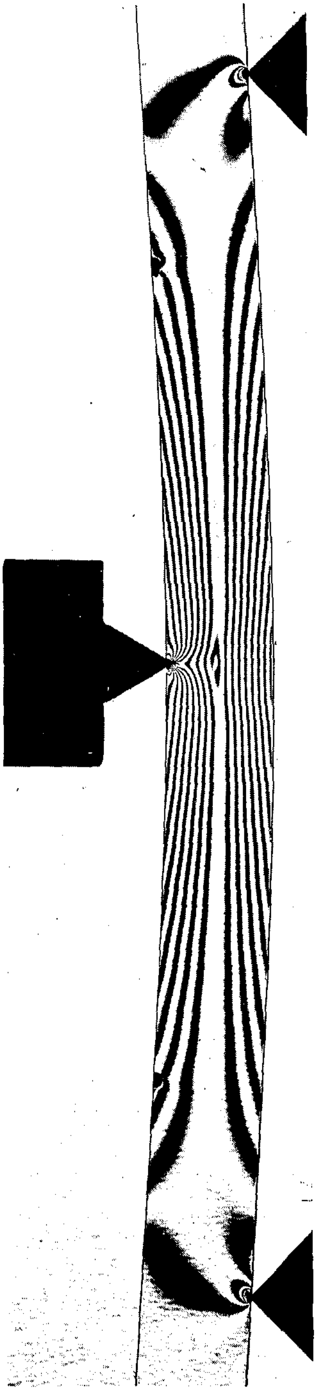


FIGURE III.4 Concentrated load on a simply supported beam.

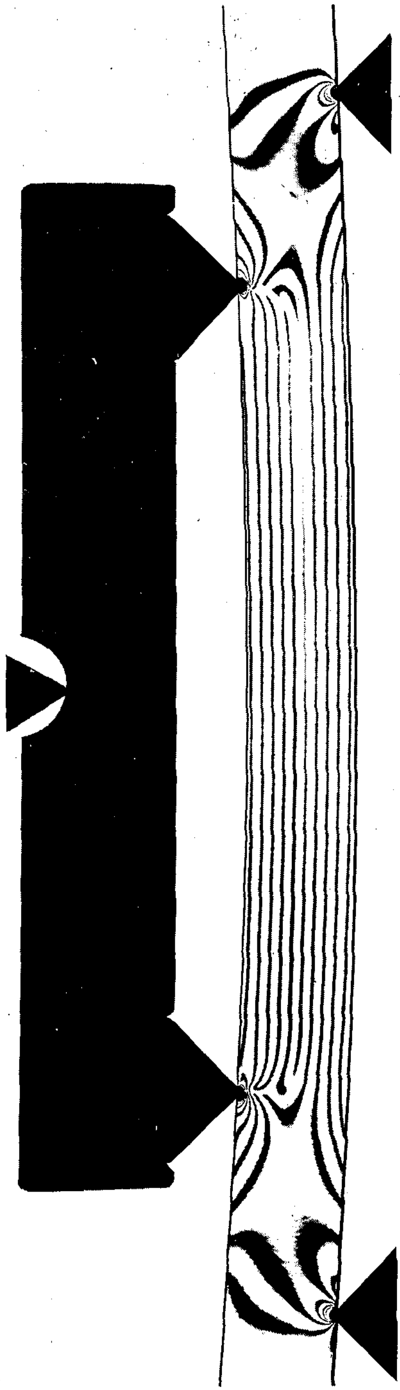


FIGURE III.5 Four point loading producing pure bending in the central portion of the beam.

photoelastic method determines the difference $(\sigma_1 - \sigma_2)$ and the orientation of the principal axes. If the sum $(\sigma_1 + \sigma_2)$ can be determined, then σ_1 and σ_2 will be known and the complete state of stress in the interior of the specimen will be known as indicated in Fig. III.6. The sum $(\sigma_1 + \sigma_2)$ can be determined by solving analytically or experimentally the compatibility equation

$$\frac{\partial^2(\sigma_1 + \sigma_2)}{\partial x^2} + \frac{\partial^2(\sigma_1 + \sigma_2)}{\partial y^2} = 0$$

with boundary conditions being the known values of $(\sigma_1 + \sigma_2)$ on the boundaries.

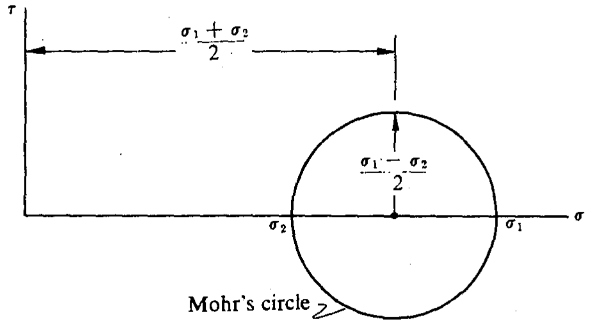


FIGURE III.6

The fringe pattern of a beam bent by couples at each end, Fig. III.5, shows that the stress distribution does not change over the central portion of the beam. In this region the stress distribution is that of pure bending. The fringes in this region are parallel and uniformly spaced, as they should be for $\sigma_x = My/I$, $\sigma_y = \tau_{xy} = 0$. The fringes in the beam shown in Fig. III.4 are not parallel since σ_x varies over the span and τ_{xy} varies over the depth.

Figure III.7 shows the fringes developed in a U-shaped beam by forces tending to spread the ends apart. It is seen that the stress on the inner curved surface is greater than that on the outer curved surface; the stress gradient is steepest at the inner surface; and the neutral axis is closer to the inner surface than to the outer surface. These facts are all in agreement with the analytic solution for the stresses in a curved beam in bending.

Figure III.8a shows a short compression member with a small axial force approximately uniformly distributed over each end. The first half fringe to appear as the load was applied is shown covering the entire member.* The

* A fringe in the light field model corresponds to a change from light through dark to light.

same member, with the same axial force, is shown in Fig. III.8b, but there the force is distributed over approximately one half of the area of the end. In Fig. III.8c the force is distributed over one end but concentrated at the

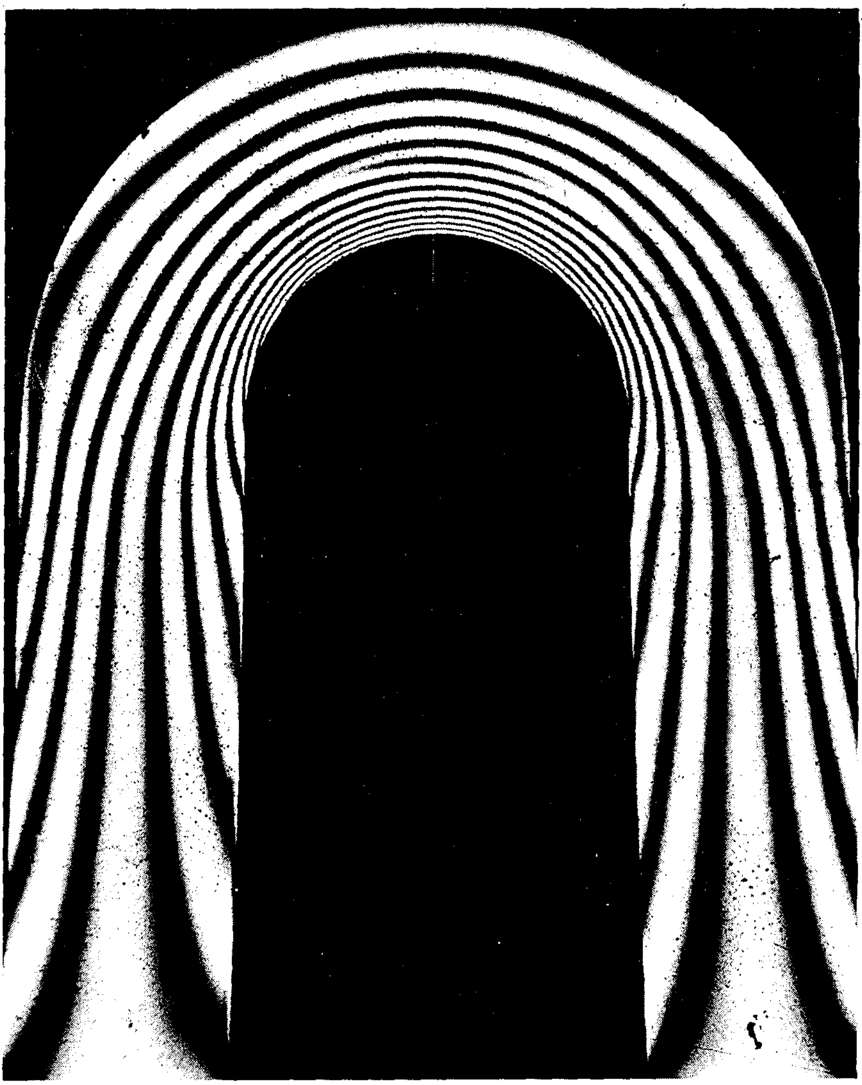
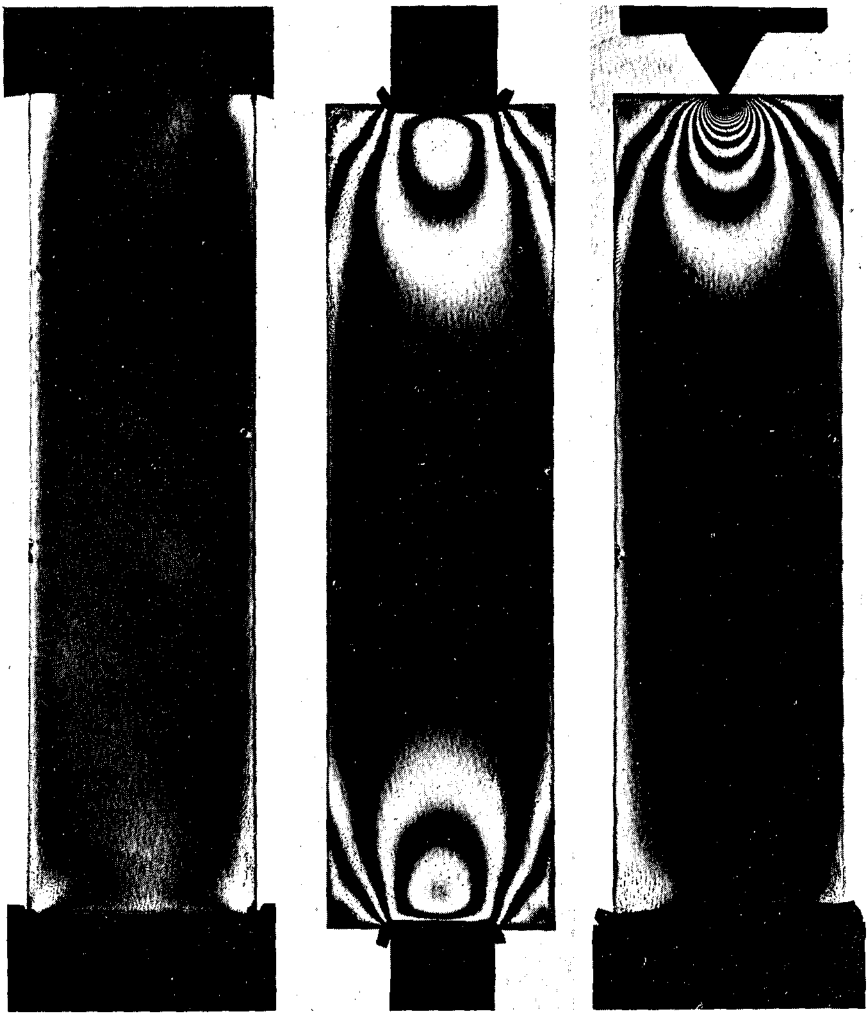


FIGURE III.7 Curved beam in bending.

center of the other end. These photographs illustrate St. Venant's principle, as do the patterns in Figs. III.4 and III.5.

In a photoelastic investigation considerable finesse is required in preparing the model and applying the loads. Also care must be taken to assure elastic behavior in the model, since plastic strains will change the distribution of stress and strain in the model from what they are in linearly elastic deformation. The test model should be an accurate reproduction of the prototype but any convenient reduction or increase in scale may be used. The stress



(a) (b) (c)

FIGURE III.8 A compression member subjected to a load over different portions of the ends.

pattern in the model per inch of thickness is identical to that in the prototype if the loads are in proportion to the scale. Differences in elastic constants of the model and prototype do influence the plane stress distribution when body forces are involved, as in a rotating disk.

A recent development in photoelastic stress analysis employs a photoelastic plastic that can be molded directly on the surface of a body. The plastic is bonded to the prototype which has been given a reflecting coating. Strains in the prototype are transmitted to the photoelastic plastic, and since the modulus of the plastic is much smaller than that of most structural materials, the thin layer of bonded plastic does not appreciably influence the strain distribution in the prototype. Light used in the photoelastic analysis is reflected off of the coating, thus passing through the plastic twice. Fringes resulting from strains in the plastic give a quantitative measure of the stresses on the surface of the prototype.

APPENDIX **IV**

VARIATIONAL METHODS

IV-1 VARIATION OF A FUNCTION

The concepts of the calculus of variations form the basis of a powerful method of analysis. These concepts are best explained with reference to an example. Consider the simply-supported beam, shown in Fig. IV.1, carrying a distributed load $p(x)$. The resulting displacement of the beam is:

$$v = f(x)$$

Suppose now an increment of load, δp, is added. The beam will be given an increment of deflection, δv. It is supposed that δp and δv are infinitesimals, that is,

$$\begin{aligned} \delta p &= \epsilon F(x) \\ \delta v &= \epsilon \varphi(x) \end{aligned} \tag{IV.1}$$

where ϵ is an infinitesimal quantity and $\varphi(x)$ is a function that satisfies the boundary conditions of the problem. We say that δp is the variation of the load p, and δv is the variation of the displacement v. A variation in displacement implies, of course, a variation in slope, curvature, etc. For example, given that the slope is $v^{\mathrm{I}} = \partial v/\partial x$, the slope of the varied curve is

$$\frac{\partial}{\partial x}(v + \delta v) = \frac{\partial v}{\partial x} + \frac{\partial(\delta v)}{\partial x} = v^{\mathrm{I}} + \delta v^{\mathrm{I}}$$

The variation of slope is thus given by

$$\delta v^{\mathrm{I}} = \frac{\partial}{\partial x}(\delta v)$$

that is,

$$\delta\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}(\delta v) \tag{IV.2}$$

Similarly, the variation of the second derivative is

$$\delta\left(\frac{\partial^2 v}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}(\delta v)$$

It should be noted that the variation of the derivative is just the derivative of the variation for all orders of the derivative.

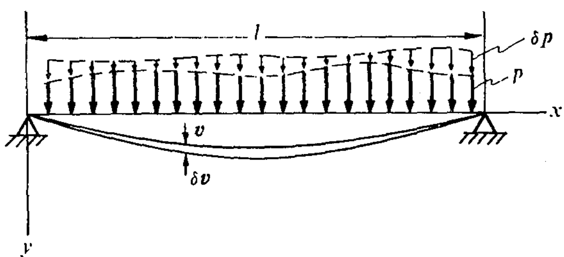


FIGURE IV.1

The second concept from the calculus of variations is the notion of the variation of an integral. For example, the strain energy of bending of the beam in Fig. IV.1 is

$$V = \int_0^l \tfrac{1}{2}EI(v^{\mathrm{II}})^2\,dx \tag{IV.3}$$

If the displacement is varied, the strain energy will be varied:

$$V + \delta V = \int_0^l \tfrac{1}{2}EI(v^{\mathrm{II}} + \delta v^{\mathrm{II}})^2\,dx$$

Subtracting V from $V + \delta V$, there is obtained

$$\begin{aligned}\delta V &= \int_0^l \tfrac{1}{2}EI[(v^{\mathrm{II}} + \delta v^{\mathrm{II}})^2 - (v^{\mathrm{II}})^2]\,dx \\ &= \int_0^l \tfrac{1}{2}EI[(v^{\mathrm{II}})^2 + 2v^{\mathrm{II}}\,\delta v^{\mathrm{II}} + (\delta v^{\mathrm{II}})^2 - (v^{\mathrm{II}})^2]\,dx \\ &= \int_0^l [EIv^{\mathrm{II}}\,\delta v^{\mathrm{II}} + \tfrac{1}{2}EI(\delta v^{\mathrm{II}})^2]\,dx\end{aligned}$$

The second term in this expression is a second-order infinitesimal, $(\delta v^{\mathrm{II}})^2 = \epsilon^2[\varphi^{\mathrm{II}}(x)]^2$, and may be dropped, so that

$$\delta V = \int_0^l (EIv^{\mathrm{II}}\,\delta v^{\mathrm{II}})\,dx \tag{IV.4}$$

This is the variation of the integral, δV, corresponding to the variation in displacement, δv. It may be noted that the procedure for calculating the variation of the integral is similar to the procedure for calculating the derivative of a function, and the same result is obtained by writing:

$$\begin{aligned}\delta V &= \delta \int_0^l \tfrac{1}{2}EI(v^{\mathrm{II}})^2\,dx \\ &= \int_0^l \tfrac{1}{2}EI\,\delta(v^{\mathrm{II}})^2\,dx \\ &= \int_0^l \tfrac{1}{2}EI\frac{\partial}{\partial v^{\mathrm{II}}}(v^{\mathrm{II}})^2\,\delta v^{\mathrm{II}}\,dx \\ &= \int_0^l EIv^{\mathrm{II}}\,\delta v^{\mathrm{II}}\,dx\end{aligned}$$

The expression for δV given in Eq. (IV.4) can be modified as follows:

$$\delta V = \int_0^l EIv^{\mathrm{II}}\,\delta v^{\mathrm{II}}\,dx = \int_0^l EIv^{\mathrm{II}}\frac{\partial^2(\delta v)}{\partial x^2}\,dx$$

This may be integrated by parts,

$$\delta V = \int_0^l EIv^{\mathrm{II}}\frac{\partial^2(\delta v)}{\partial x^2}\,dx = EIv^{\mathrm{II}}\frac{\partial(\delta v)}{\partial x}\bigg|_0^l - \int_0^l EI\frac{\partial v^{\mathrm{II}}}{\partial x}\frac{\partial(\delta v)}{\partial x}\,dx$$

The integral may again be integrated by parts to give

$$\delta V = EIv^{\mathrm{II}}\,\delta v^{\mathrm{I}}\Big|_0^l - EIv^{\mathrm{III}}\,\delta v\Big|_0^l + \int_0^l EIv^{\mathrm{IV}}\,\delta v\,dx \tag{IV.5}$$

In this form it is the variation δv that appears under the integral sign, rather than δv^{II}.

The variation in work done by the applied load p, is expressed as follows:

$$\begin{aligned}\delta W &= \int_0^l \int_v^{v+\delta v} p\,dv\,dx \\ &= \int_0^l [p(v+\delta v) - pv]\,dx\end{aligned}$$

or

$$\delta W = \int_0^l p \, \delta v \, dx \tag{IV.6}$$

where the second order infinitesimal has been dropped. This is the increment of work done by p as it moves through δv.

It should be noted that in the special case of a linear system

$$W = \int_0^l \tfrac{1}{2} pv \, dx$$

The work done by the load $(p + \delta p)$ is

$$W + \delta W = \int_0^l \tfrac{1}{2}(p + \delta p)(v + \delta v) \, dx$$

The increment of work is given by the difference

$$\begin{aligned}
\delta W &= \int_0^l [\tfrac{1}{2}(p + \delta p)(v + \delta v) - \tfrac{1}{2}pv] \, dx \\
&= \int_0^l [\tfrac{1}{2}pv + \tfrac{1}{2}p \, \delta v + \tfrac{1}{2}v \, \delta p + \tfrac{1}{2}\delta p \, \delta v - \tfrac{1}{2}pv] \, dx \\
&= \int_0^l [\tfrac{1}{2}p \, \delta v + \tfrac{1}{2}v \, \delta p] \, dx
\end{aligned}$$

As seen in Fig. IV.2, if the relation between p and v is linear, say $p = kv$, it follows that

$$v \, \delta p = v \, \delta(kv) = kv \, \delta v = p \, \delta v$$

The variation in work is thus

$$\delta W = \int_0^l p \, \delta v \, dx$$

which agrees with Eq. (IV.6).

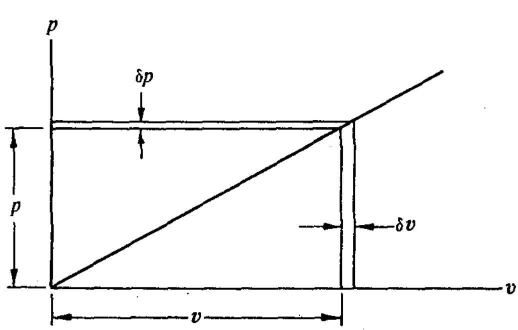


FIGURE IV.2

IV-2 DERIVATION OF THE DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS

Consider the beam shown in Fig. IV.3, with unspecified boundary conditions and an arbitrary load. The physical properties of the beam are specified by the statement that the total strain energy is

$$V = \int_0^l \tfrac{1}{2}EI(v^{\mathrm{II}})^2\,dx \tag{IV.7}$$

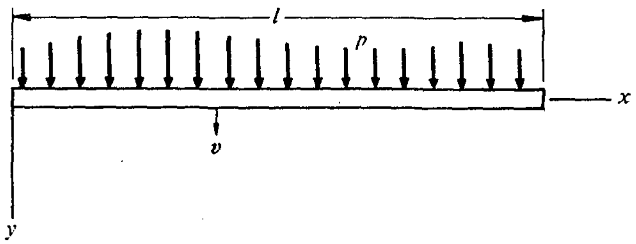


FIGURE IV.3

The principle of virtual work states that:

$$\delta V = \delta W$$

which, for the beam, is:

$$\delta \int_0^l \tfrac{1}{2}EI(v^{\mathrm{II}})^2\,dx = \int_0^l p\,\delta v\,dx$$

or

$$\int_0^l EIv^{\mathrm{II}}\,\delta v^{\mathrm{II}}\,dx = \int_0^l p\,\delta v\,dx$$

For our purpose it is necessary to obtain the variation δv under the integral sign, rather than δv^{II}, and this is achieved by writing

$$\int_0^l EIv^{\mathrm{II}}\frac{\partial^2}{\partial x^2}(\delta v)\,dx = \int_0^l p\,\delta v\,dx$$

and after twice integrating by parts, see Eq. (IV.5), we obtain

$$EIv^{\mathrm{II}}\,\delta v^{\mathrm{I}}\Big|_0^l - EIv^{\mathrm{III}}\,\delta v\Big|_0^l + \int_0^l EIv^{\mathrm{IV}}\,\delta v\,dx = \int_0^l p\,\delta v\,dx$$

Collecting terms according to the variations and using the relations $EIv^{\mathrm{II}} = M$, $-EIv^{\mathrm{III}} = Q$, the foregoing equation can be written

$$M\,\delta v^{\mathrm{I}}\Big|_0^l + Q\,\delta v\Big|_0^l + \int_0^l \left(EI\frac{\partial^4 v}{\partial x^4} - p\right)\delta v\,dx = 0$$

This equation must hold for arbitrary δv, and since the first two terms depend upon the values of δv and δv^{I} at $x = 0, l$ and the integral depends upon δv for $0 < x < l$, it is required that each term be equal to zero separately

$$\int_0^l \left(EI\frac{\partial^4 v}{\partial x^4} - p\right)\delta v\,dx = 0$$

$$M\,\delta v^{\mathrm{I}} = 0 \qquad \text{at } x = 0, l$$

$$Q\,\delta v = 0 \qquad \text{at } x = 0, l$$

Since δv is arbitrary, the expression within the brackets must be zero to ensure that the integral is zero. We therefore conclude that the statement $\delta V = \delta W$ is equivalent to the statements

$$EI\frac{\partial^4 v}{\partial x^4} - p = 0 \tag{IV.8}$$

$$M\,\delta v^{\mathrm{I}} = 0, \quad Q\,\delta v = 0 \qquad \text{at } x = 0, l$$

The first of these is the differential equation of the beam and the others are statements of the boundary conditions appropriate to the differential equation. For example,

$$M\,\delta v^{\mathrm{I}} = 0\begin{cases} M = 0: & \text{unrestrained} \\ \text{or} & \\ \delta v^{\mathrm{I}} = 0: & \text{built-in} \end{cases}$$

$$Q\,\delta v = 0\begin{cases} Q = 0: & \text{unsupported} \\ \text{or} & \\ \delta v = 0: & \text{immovable support} \end{cases}$$

The foregoing example is too simple a problem to justify the use of the variational method in deriving the differential equations. A more suitable example is the following.

The Trans-Arabian pipeline is an above-ground pipe, with 30 in. diameter and multiple 65 ft. spans, through which oil of mass per unit length ρ is flowing with velocity s as illustrated in Fig. IV.4. The differential equation of free vibrations may be derived by means of Hamilton's principle, which states that

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt = 0$$

where T is the total kinetic energy of the system. For this problem we prescribe that

$$T = \int_0^l \{\tfrac{1}{2}(m - \rho)\dot{v}^2 + \tfrac{1}{2}\rho[s^2 + (\dot{v} + sv^{\mathrm{I}})^2]\}\, dx$$

$$V = \int_0^l \tfrac{1}{2}EI(v^{\mathrm{II}})^2\, dx$$

$$W = 0$$

where $\dot{v} = \partial v/\partial t$, $v^{\mathrm{I}} = \partial v/\partial x$, etc., and m is the total mass per unit length of pipe plus oil. When the variations of T and V are calculated and the integrations by parts are performed, there is obtained the differential equation

$$EI\frac{\partial^4 v}{\partial x^4} + \rho s^2 \frac{\partial^2 v}{\partial x^2} + 2\rho s \frac{\partial^2 v}{\partial x\, \partial t} + m\frac{\partial^2 v}{\partial t^2} = 0 \qquad \text{(IV.9)}$$

The second term represents the centrifugal force resulting from the fact that the oil flows along a curved path, the third term represents the force resulting from the Coriolis acceleration of the oil and the last term is the D'Alembert inertia force.

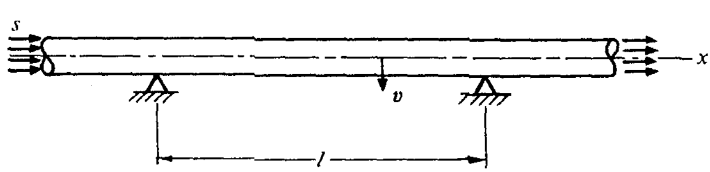


FIGURE IV.4

IV-3 APPROXIMATE SOLUTIONS

The variational method is particularly useful in deriving approximate solutions. As an example we shall analyze the problem of two aluminum sheets fastened together by an elastic glue and transmitting a tension force of F pounds per unit length, as shown in Fig. IV.5. The exact determination of the stress by means of the general equations of elasticity would be a very difficult problem. We simplify the problem by prescribing that the strain energy in the joint per unit width is:

$$V = \int_{-l/2}^{+l/2} \left(\tfrac{1}{2} F_1 \frac{du_1}{dx} + \tfrac{1}{2} F_2 \frac{du_2}{dx} + \tfrac{1}{2} \tau \gamma h_3 \right) dx$$

where

$$F_1 = Eh \frac{du_1}{dx}$$

$$F_2 = Eh \frac{du_2}{dx}$$

$$\tau = G_3 \gamma = \frac{G_3}{h_3} (u_2 - u_1)$$

Implicit in this statement are the assumptions that σ_x is uniform across the thickness of the sheets, that the shear stress τ_{xy} in the sheets does not produce appreciable strains, and that σ_x in the glue is relatively small. These assumptions can be made plausible on the basis that the elastic moduli of the glue, G_3, E_3, are small compared to the moduli of the plates, and that the thickness, h_3, of the glue is small, and that $2h + h_3 \ll l$.

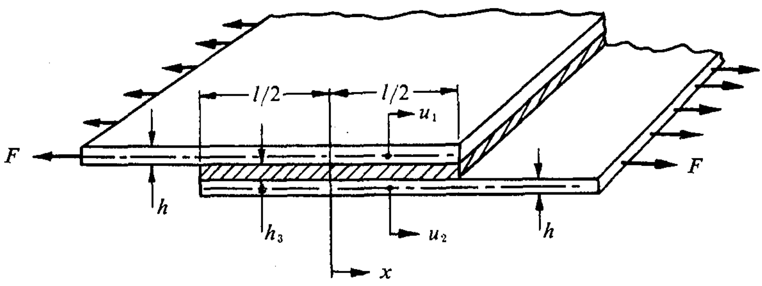


FIGURE IV.5

The principle of virtual work for this problem requires

$$\delta V = \delta W$$

where

$$\delta W = F\, \delta u_1 \big|_{x=-l/2} + F \delta u_2 \big|_{x=+l/2}$$

$$\delta V = \delta \int_{-l/2}^{+l/2} \left[\tfrac{1}{2} hE(u_1^{\text{I}})^2 + \tfrac{1}{2} hE(u_2^{\text{I}})^2 + \frac{G_3}{h_3} (u_1 - u_2)^2 \right] dx$$

or

$$\delta V = \int_{-l/2}^{+l/2} \left[hEu_1^{\mathrm{I}}\, \delta u_1^{\mathrm{I}} + hEu_2^{\mathrm{I}}\, \delta u_2^{\mathrm{I}} + \frac{G_3}{h_3}(u_1 - u_2)(\delta u_1 - \delta u_2) \right] dx$$

Writing $\delta V = \delta W$, integrating by parts and collecting terms, gives*

$$\int_{-l/2}^{+l/2} \left\{ \left[-hEu_1^{\mathrm{II}} + \frac{G_3}{h_3}(u_1 - u_2) \right] \delta u_1 + \left[-hEu_2^{\mathrm{II}} + \frac{G_3}{h_3}(u_2 - u_1) \right] \delta u_2 \right\} dx = 0$$

Since δu_1 and δu_2 are independent, each of the expressions within the parentheses must be equal to zero:

$$hEu_1^{\mathrm{II}} - \frac{G_3}{h_3}(u_1 - u_2) = 0$$

$$hEu_2^{\mathrm{II}} - \frac{G_3}{h_3}(u_2 - u_1) = 0$$

These are the differential equations that describe the problem. If the first of these is subtracted from the second, there is obtained

$$(u_2^{\mathrm{II}} - u_1^{\mathrm{II}}) - 2\frac{G_3}{Ehh_3}(u_2 - u_1) = 0$$

This can also be written

$$\frac{d^2\tau}{dx^2} - 4\beta^2\tau = 0$$

where

$$\beta^2 = \frac{1}{2}\frac{G_3}{Ehh_3}$$

This equation describes the variation of shearing stress along the joint. The solution of the equation is

$$\tau = C_1 \cosh 2\beta x + C_2 \sinh 2\beta x$$

From symmetry we see that $C_2 = 0$. The constant C_1 is determined from the equilibrium condition

$$\int_{-l/2}^{+l/2} \tau\, dx = F$$

* The terms evaluated on the boundaries disappear because the boundary conditions are satisfied.

This gives

$$\tau = \frac{\beta \cosh 2\beta x}{\sinh \beta l} F$$

A dimensionless plot of the stress is shown in Fig. IV.6. It is seen that when βl is small the shear stress is distributed quite uniformly across the joint but when βl is equal to 3 the stress is very high at the ends of the joint and very small in the middle region. The maximum shear stress is at $x = l/2$:

$$\tau_{\max} = \frac{\beta}{\tanh \beta l} F$$

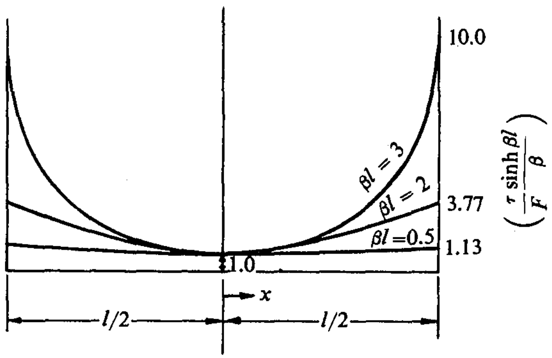


FIGURE IV.6

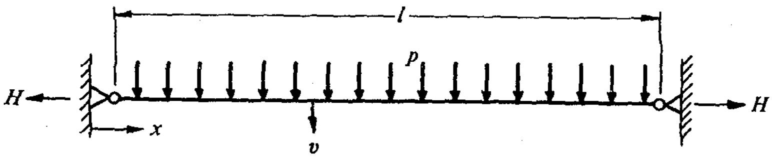
It is seen from the table that the maximum stress is decreased very little by increasing the length of the joint if βl is equal to 2.0 or more.

	βl				
	0.5	1.0	2.0	3.0	4.0
$\tanh \beta l =$	0.462	0.762	0.964	0.995	0.999

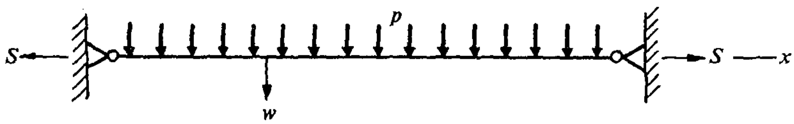
A more precise analysis could be made by taking account of the bending rigidity of the plates, the shearing strains across the thickness of the plates, etc. For any specified strain energy of the system, the variational method will produce the corresponding differential equations and boundary conditions.

Problems

IV.1 An elastic cord is stretched between two fixed points with an axial tension H as shown in the diagram. Derive the differential equation, assuming that the displacement v is small and that H can be taken to be a constant.



IV.2 A stretched elastic membrane with membrane tension S has displacement w produced by pressure p. From the equation $\delta V = \delta W$ deduce the differential equation of equilibrium, and the boundary conditions.



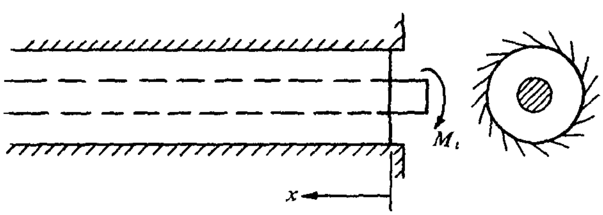
IV.3 Derive Eq. (IV.9) for the vibrations of a pipe containing flowing fluid.

IV.4 Deduce the equation for the vibrations of a beam that has shearing deformations as well as bending deformations, Eq. (9.32). The properties of the beam are described by

$$M = EIv_b^{II} \qquad Q = kAGv_s^{I}$$

$$v = v_b + v_s \qquad T = \int_0^l (\tfrac{1}{2}\rho A(\dot{v})^2 + \tfrac{1}{2}\rho AI(\dot{v}_b^{I})^2)\,dx$$

IV.5 A circular steel rod is embedded in a rubber matrix that is fixed at its outer radius. Derive a differential equation that describes the rotation of the rod $\varphi(s)$ when a moment M_t is applied at one end. Assume Eq. (6.12) is applicable and that the rubber has a resisting torque per unit length $M_t = k\varphi$.



IV.6 Show that for a rectangular flat plate in bending with load q on the surfaces of the plate, the variation of strain energy and work can be written

$$\delta V = -\int_0^a \int_0^b [M_x(\delta\chi_x) + M_y(\delta\chi_y) + 2M_{xy}(\delta\chi_{xy})]\,dx\,dy$$

$$\delta W = \int_0^a \int_0^b q(\delta w)\,dx\,dy$$

According to the principle of virtual work, $\delta V = \delta W$, so expressing the curvatures in terms of derivatives of the displacements deduce that the differential equation of equilibrium is

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q$$

and deduce the correct statement of the boundary conditions.

IV.7 Show that for the torsion of a prismatic bar of length l twisted by a moment M the strain energy and work can be expressed

$$V = l\int\int\frac{1}{2G}\left\{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2\right\} dx\,dy$$

$$W = \tfrac{1}{2}M\theta l$$

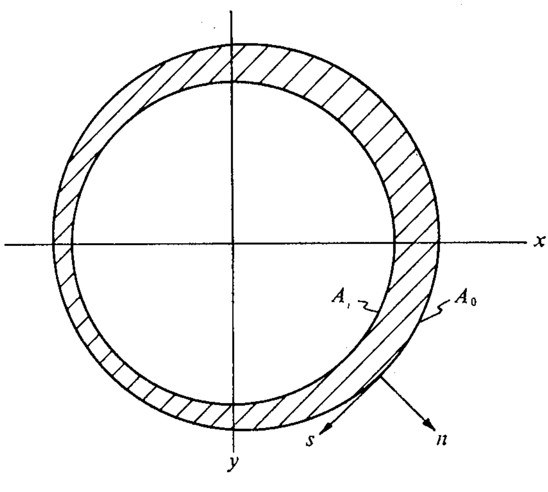
where Φ is the stress function and θ is the twist per unit length. Writing the equation $\delta V = \delta W$, deduce the differential equation and the boundary conditions.

IV.8 Do Prob. IV.7 for a tubular shaft. Show that the derived boundary conditions can be stated in terms of the membrane anology as

$$\oint_{A_0} S\frac{\partial w}{\partial n}\,ds = pA_0$$

$$\oint_{A_i} S\frac{\partial w}{\partial n}\,ds = pA_i$$

Note that these are the boundary conditions discussed in Section 6.6.



IV.9 By using $\delta V = \delta W$ derive the correct differential equation and the correct statement of the boundary conditions for the problem discussed in Section IV.3.

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