Applications of Global Analysis in Mathematical Physics

Jerry Marsden

University of California, Berkeley

Mathematics Lecture Series

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Introduction.

These notes are based on a series of ten lectures given at Carleton University, Ottawa, from June 21 through July 6, 1973. The notes follow the lectures fairly closely except for a few minor amplifications.

The purpose of the lectures was to introduce some methods of global analysis which I have found useful in various problems of mathematical physics. Many of the results are based on work done with P. Chernoff, D. Ebin, A. Fischer and A. Weinstein. A more complete exposition of some of the points contained here may be found in Chernoff-Marsden [1] and Marsden-Ebin-Fischer [1] as well as in references cited later.

"Global Analysis" is a vague term. It has, by and large, two more or less distinct subdivisions. On the one hand there are those who deal with dynamical systems emphasizing topological problems such as structural stability (see Smale [2]). On the other hand there are those who deal with problems of nonlinear functional analysis and partial differential equations using techniques combining geometry

and analysis. It is to the second group that we belong.

One of the first big successes of global analysis (in the second sense above) was Morse theory as developed by Palais [7] and Smale [3] and preceeded by the ideas of Leray-Schauder, Lusternik-Schnirelman and Morse. The result is a beautiful geometrization and powerful extension of the classical calculus of variations. (See Graff [1] for more up-to-date work.)

It is in a similar spirit that we proceed here. Namely we want to make use of ideas from geometry to shed light on problems in analysis which arise in mathematical physics. Actually it comes as a pleasant surprize that this point of view is useful, rather than being a mere language convenience and an outlet for generalizations. As we hope to demonstrate in the lectures, methods of global analysis can be useful in attacking specific problems.

The first three lectures contain background material. This is basic and more or less standard. Each of the next seven lectures discusses an application with only minor dependencies, except that lectures 4 and 5, and 9 and 10 form units. Lectures 4 and 5 deal with hydrodynamics and 9 and 10 with general relativity. Lecture 6 deals with miscellaneous applications, both mathematical and physical, of the concepts of symmetry groups and conserved quantities. Lecture 7 studies quantum mechanics as a hamiltonian system and discusses, e.g. the Bargmann-Wigner theorem. Finally lecture 8 studies a general method for obtaining global (in time) solutions to certain evolution equations.

It is a pleasure to thank Professors V. Dlab, D. Dawson and M. Grmela for their kind hospitality at Carleton.

1. Infinite Dimensional Manifolds.

Basic Calculus.

We shall let E, F, G, ... denote Banach spaces. Let $U \subset E$ be open and let $f : U \to F$ be a given mapping. We say f is Fréchet differentiable at $x_0 \in U$ if there is a continuous (= bounded) linear map $Df(x_0) : E \to F$ such that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\|h\| < \delta$ implies

$$\|f(x_0 + h) - f(x_0) - Df(x_0)\cdot h\| \leq \varepsilon\|h\| .$$

The map $Df(x_0)$ is necessarily unique.

Let $L(E, F)$ denote the space of all continuous linear maps from E to F together with the operator norm

$$\|T\| = \sup_{\|x\|\leq 1} \|T\cdot x\|$$

so that $L(E, F)$ is a Banach space. Let $L_s(E, F)$ denote the same space with the strong operator topology; i.e. the topology of pointwise convergence.

If f is Fréchet differentiable at each $x \in U$ and if $x \mapsto Df(x) \in L(E, F)$ (resp. $L_s(E, F)$) is continuous, we say f is