J. E. Marsden
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The Hopf Bifurcation and Its Applications

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with contributions by

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To the courage of

G. Oyarzún

PREFACE

The goal of these notes is to give a reasonably complete, although not exhaustive, discussion of what is commonly referred to as the Hopf bifurcation with applications to specific problems, including stability calculations. Historically, the subject had its origins in the works of Poincaré [1] around 1892 and was extensively discussed by Andronov and Witt [1] and their co-workers starting around 1930. Hopf's basic paper [1] appeared in 1942. Although the term "Poincaré-Andronov-Hopf bifurcation" is more accurate (sometimes Friedrichs is also included), the name "Hopf Bifurcation" seems more common, so we have used it. Hopf's crucial contribution was the extension from two dimensions to higher dimensions.

The principal technique employed in the body of the text is that of invariant manifolds. The method of Ruelle-Takens [1] is followed, with details, examples and proofs added. Several parts of the exposition in the main text come from papers of P. Chernoff, J. Dorroh, O. Lanford and F. Weissler to whom we are grateful.

The general method of invariant manifolds is common in dynamical systems and in ordinary differential equations; see for example, Hale [1,2] and Hartman [1]. Of course, other methods are also available. In an attempt to keep the picture balanced, we have included samples of alternative approaches. Specifically, we have included a translation (by L. Howard and N. Kopell) of Hopf's original (and generally unavailable) paper. These original methods, using power series and scaling are used in fluid mechanics by, amongst many others, Joseph and Sattinger [1]; two sections on these ideas from papers of Iooss [1-6] and

Kirchgässner and Kielhoffer [1] (contributed by G. Childs and O. Ruiz) are given.

The contributions of S. Smale, J. Guckenheimer and G. Oster indicate applications to the biological sciences and that of D. Schmidt to Hamiltonian systems. For other applications and related topics, we refer to the monographs of Andronov and Chaiken [1], Minorsky [1] and Thom [1].

The Hopf bifurcation refers to the development of periodic orbits ("self-oscillations") from a stable fixed point, as a parameter crosses a critical value. In Hopf's original approach, the determination of the stability of the resulting periodic orbits is, in concrete problems, an unpleasant calculation. We have given explicit algorithms for this calculation which are easy to apply in examples. (See Section 4, and Section 5A for comparison with Hopf's formulae). The method of averaging, exposed here by S. Chow and J. Mallet-Paret in Section 4C gives another method of determining this stability, and seems to be especially useful for the next bifurcation to invariant tori where the only recourse may be to numerical methods, since the periodic orbit is not normally known explicitly.

In applications to partial differential equations, the key assumption is that the semi-flow defined by the equations be smooth in all variables for $t > 0$. This enables the invariant manifold machinery, and hence the bifurcation theorems to go through (Marsden [2]). To aid in determining smoothness in examples we have presented parts of the results of Dorroh-Marsden [1]. Similar ideas for utilizing smoothness have been introduced independently by other authors, such as D. Henry [1].

Some further directions of research and generalization are given in papers of Jost and Zehnder [1], Takens [1, 2], Crandall-Rabinowitz [1, 2], Arnold [2], and Kopell-Howard [1-6] to mention just a few that are noted but are not discussed in any detail here. We have selected results of Chafee [1] and Ruelle [3] (the latter is exposed here by S. Schecter) to indicate some generalizations that are possible.

The subject is by no means closed. Applications to instabilities in biology (see, e.g. Zeeman [2], Gurel [1-12] and Section 10, 11); engineering (for example, spontaneous "flutter" or oscillations in structural, electrical, nuclear or other engineering systems; cf. Aronson [1], Ziegler [1] and Knops and Wilkes [1]), and oscillations in the atmosphere and the earth's magnetic field (cf. Durand [1]) are appearing at a rapid rate. Also, the qualitative theory proposed by Ruelle-Takens [1] to describe turbulence is not yet well understood (see Section 9). In this direction, the papers of Newhouse and Peixoto [1] and Alexander and Yorke [1] seem to be important. Stable oscillations in nonlinear waves may be another fruitful area for application; cf. Whitham [1]. We hope these notes provide some guidance to the field and will be useful to those who wish to study or apply these fascinating methods.

After we completed our stability calculations we were happy to learn that others had found similar difficulty in applying Hopf's result as it had existed in the literature to concrete examples in dimension ≥ 3. They have developed similar formulae to deal with the problem; cf. Hsü and Kazarinoff [1, 2] and Poore [1].

The other main new result here is our proof of the validity of the Hopf bifurcation theory for nonlinear partial differential equations of parabolic type. The new proof, relying on invariant manifold theory, is considerably simpler than existing proofs and should be useful in a variety of situations involving bifurcation theory for evolution equations.

These notes originated in a seminar given at Berkeley in 1973-4. We wish to thank those who contributed to this volume and wish to apologize in advance for the many important contributions to the field which are not discussed here; those we are aware of are listed in the bibliography which is, admittedly, not exhaustive. Many other references are contained in the lengthy bibliography in Cesari [1]. We also thank those who have taken an interest in the notes and have contributed valuable comments. These include R. Abraham, D. Aronson, A. Chorin, M. Crandall, R. Cushman, C. Desoer, A. Fischer, L. Glass, J. M. Greenberg, O. Gurel, J. Hale, B. Hassard, S. Hastings, M. Hirsch, E. Hopf, N. D. Kazarinoff, J. P. LaSalle, A. Mees, C. Pugh, D. Ruelle, F. Takens, Y. Wan and A. Weinstein. Special thanks go to J. A. Yorke for informing us of the material in Section 3C and to both he and D. Ruelle for pointing out the example of the Lorentz equations (See Example 4B.8). Finally, we thank Barbara Komatsu and Jody Anderson for the beautiful job they did in typing the manuscript.

Jerrold Marsden
Marjorie McCracken

TABLE OF CONTENTS