SECTION 9

BIFURCATIONS IN FLUID DYNAMICS AND THE PROBLEM OF TURBULENCE

This section shows how the results of Sections 8 and 8A can be used to establish the bifurcation theorem for the Navier-Stokes equations. Alternatively, although conceptually harder (in our opinion) methods are described in Sections 9A, 9B and Sattinger [5] and Joseph and Sattinger [1].

The new proof of the bifurcation theorems using the center manifold theory as given in Section 8 allow one to deduce the results very simply for the Navier-Stokes equations with a minimum of technical difficulties. This includes all types of bifurcations, including the bifurcation to invariant tori as in Section 6 or directly as in Jost and Zehnder [1]. All we need to do is verify that the semiflow of the Navier-Stokes equations is smooth (in the sense of Section 8A); the rest is then automatic since the center manifold theorem immediately reduces us to the finite dimensional case (see

Section 8 for details). We note that already in Ruelle-Takens [1] there is a simple proof of the now classical results of Velte [1] on stationary bifurcations in the flow between rotating cylinders from Couette flow to Taylor cells.

The first part of this section therefore is devoted to proving that the semiflow of the Navier-Stokes equations is smooth. We use the technique of Dorroh-Marsden [1] (see Section 8A) to do this.

This guarantees then, that the same results as given in the finite dimensional case in earlier sections, including the stability calculations hold without change.

The second part of the section briefly describes the Ruelle-Takens picture of turbulence. This picture is still conjectural, but seems to be gaining increased acceptance as time goes on, at least for describing certain types of turbulence. The relationship with the global regularity (or "all time") problem in fluid mechanics is briefly discussed.

Statement of the Smoothness Theorem.

Before writing down the smoothness theorem, let us recall the equations we are dealing with. For homogeneous incompressible viscous fluids, the classical Navier-Stokes equations are, as in Section 1,

$$(\mathrm{NS})\quad \begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - \nu\Delta v = -\mathrm{grad}\, p + b_t, \quad b_t = \text{external force} \\ \mathrm{div}\, v = 0 \\ v = 0 \quad \text{on} \quad \partial M \quad \begin{array}{l}\text{(or prescribed on } M\text{, possibly depending} \\ \text{on a parameter } \mu)\end{array} \end{cases}$$

Here M is a compact Riemannian manifold with smooth boundary

∂M, usually an open set in $\mathbb{R}^3$.

The <u>Euler equations</u> are obtained by supposing $\nu = 0$ and changing the boundary condition to $v || \partial M$:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\text{grad } p + b_t \\ \text{div } v = 0 \\ v || \partial M \end{cases}$$

The pressure $p(t,x)$ is to be determined from the incompressibility condition in these equations.

The Euler equations are a <u>singular limit</u> of the Navier-Stokes equations. Taking the limit $\nu \to 0$ is very subtle and is the subject of much recent work. The sudden disappearance of the highest order term and the associated sudden change in boundary conditions is the source of the difficulties and is the reason why boundary layer theory and turbulence theory are so difficult. This point will be remarked on later.

Note that Euler equations are reversible in the sense that if we can solve them for all sets of initial data and for $t \geq 0$, then we can also solve them for $t < 0$. This is because if v_t, $t \geq 0$ is a solution, then so is $w_t = -v_{-t}$, $t < 0$.

For $s \geq 0$ an integer and $1 < p < \infty$, let $W^{s,p}$ denote the Sobolev space of functions (or vector functions) on M whose derivatives up to order s are in L_p; another way of describing $W^{s,p}$ is to complete the C^∞ functions f in the norm

$$||f||_{s,p} = \sum_{0\leq\alpha\leq s} ||D^\alpha f||_{L_p}$$

where $D^{\alpha}f$ is the α^{th} total derivative of f. Details on Sobolev spaces can be found in Friedman [1].

We point out that in the non-compact case one must deal seriously with the asymptotic conditions and many of the results we discuss are not known in that case (see Cantor [1], and McCracken [2] however).

The following result is a special case of a general result proved in Morrey [1]. For a direct proof in this case, see Bourguignon and Brezis [1].

(9.1) Lemma. Hodge Decomposition): *Let M be as above. Let X be a $W^{s,p}$ vector field on M, $s \geq 0$, $p > 1$. Then X has a unique decomposition as follows:*

$$X = Y + \text{grad } p$$

where $\text{div } Y = 0$ *and* $Y\|_{\partial M}$. *It is also true that* $Y \in W^{s,p}$ *and* $p \in W^{s+1,p}$.

Let $\tilde{W}^{s,p} = \{$vector fields X on $M | X \in W^{s,p}$, $\text{div } X = 0$ and $X\|_{\partial M}\}$. By the Hodge theorem, there is a map $P: W^{s,p} \to \tilde{W}^{s,p}$ via $X \mapsto Y$. The problem of solving the Euler equations now becomes that of finding $v_t \in \tilde{W}^{s+1,p}$ such that

$$\frac{dv_t}{dt} + P((v_t \cdot \nabla)v_t) = 0 \tag{E}$$

(plus initial data). If $s > \frac{n}{p}$, the product of two $W^{s,p}$ functions is $W^{s,p}$ so $(v \cdot \nabla)v$ is in $W^{s,p}$ if $v \in W^{s+1,p}$. This kind of equation is thus an *evolution equation* on $\tilde{W}^{s,p}$ as in Section 8A.

Let

$$\tilde{W}_0^{s,p} = \{\text{vector fields } v \text{ on } M | v \text{ is of class } \tilde{W}^{s,p},\ \operatorname{div} v = 0 \text{ and } v = 0 \text{ on } \partial M\}.$$

If $s = 0$ this actually makes sense and the space is written J_p (see Ladyzhenskaya [1]).

The Navier-Stokes equations can be written: find $v_t \in \tilde{W}_0^{s,p}$ such that

$$\text{(NS)} \qquad \frac{dv_t}{dt} - \nu P \Delta v_t + P(v_t \cdot \nabla) v_t = 0$$

again an evolution equation in $\tilde{W}_0^{s,p}$. In the terminology of Section 8A, the Banach space X here is $\tilde{W}_0^{0,p} = J_p$ and $Y = \tilde{W}_0^{2,p}$. The bifurcation parameter is often $\mu = 1/\nu$, the Reynolds number.

The case $p \neq 2$ is quite difficult and won't be dealt with here, although it is very important. If $p = 2$ one generally writes

$$\tilde{H}^s = \tilde{W}^{s,2}, \quad \tilde{H}_0^s = \tilde{W}_0^{s,2} \quad \text{etc.}$$

(9.2) <u>Theorem</u>. <u>The Navier-Stokes equations in dimension</u> 2 <u>or</u> 3 <u>define a smooth local semiflow on</u> $\tilde{H}_0^2 \subset \tilde{H}^0 \equiv J$.

This semiflow satisfies conditions 8.1, 8.2, and the smoothness in 8.3 of Section 8, so the Hopf theorems apply. (The rest of the hypotheses in 8.3 and 8.4 depend on the particular problem at hand and must be verified by calculation.)

In other words, the technical difficulties related to the fact that we have partial rather than ordinary differential equations are automatically taken care of.

(9.3) <u>Remarks</u>. 1. If the boundaries are moving and speed is part of the bifurcation parameter μ, the same result holds by a similar proof. This occurs in, for instance the Taylor problem.

2. The above theorem is implicit in the works of many authors. For example, D. Henry has informed us that he has obtained it in the context of Kato-Fujita. It has been proved by many authors in dimension 2 (e.g. Prodi). The first explicit demonstration we have seen is that of Iooss [3,5].

3. For the case $p \neq 2$ see McCracken [2].

4. This smoothness for the Navier-Stokes semiflow is probably false for the Euler equations on $\tilde{H}^s$ (see Kato [6], Ebin-Marsden [1]). Thus it depends crucially on the dissipative term. However, miraculously, the flow of the Euler equations <u>is smooth</u> if one uses Lagrangian coordinates (Ebin-Marsden [1]).

We could use Lagrangian coordinates to prove our results for the Navier-Stokes equations as well, but it is simpler to use the present method.

Before proving smoothness we need a local existence theorem. Since this is readily available in the literature (Ladyzhenskaya [1]), we shall just sketch the method from a different point of view. (Cf. Soboleoskii [1].)

Local Existence Theory.

The basic method one can use derives from the use of integral equations as in the Picard method for ordinary differential equations. For partial differential equations this has been exploited by Segal [1], Kato-Jujita [1], Iooss [3,6], Sattinger [2], etc. (See Carroll [1] for general background.)

The following result is a formulation of Weissler [1]. (See Section 9A for a discussion of Iooss' setup.)

First some notation: $\mathbb{E}_0, \mathbb{E}_1, \mathbb{E}_2$ will be three Banach spaces with norms $||\cdot||_0, ||\cdot||_1, ||\cdot||_2$, with $\mathbb{E}_2 \subset \mathbb{E}_1 \subset \mathbb{E}_0$, with the inclusions dense and continuous. (Some of the spaces may be equal.) Let e^{tA} be a C^0 linear semigroup on $\mathbb{E}_0$ which restricts to a contraction semigroup on $\mathbb{E}_2$. Assume, for $t > 0$, $e^{tA}: \mathbb{E}_1 \to \mathbb{E}_2$ is a bounded linear map and let its norm be denoted $\mu(t)$. Our first assumption is:

A1) For $T > 0$ assume $\int_0^T \mu(\tau)d\tau < \infty$.

For the Navier-Stokes equations, $A = \nu P\Delta$, and we can choose either

(i) $\mathbb{E}_0 = J_2$, $\mathbb{E}_1 = \tilde{H}^1$ and $\mathbb{E}_2 = \tilde{H}_0^2$

or (ii) $\mathbb{E}_0 = \mathbb{E}_1 = \tilde{H}^{-1/2}$ = completion of J with the norm $||(-\nu P\Delta)^{-1/4}u||_{L_2}$, $\mathbb{E}_2 = \tilde{H}^1$ = domain of $(-\nu P\Delta)^{1/2}$.

The case (ii) is that of Kato-Fujita [1].

The fact that A1) holds is due to the fact that A is a negative self adjoint operator on J with domain $\tilde{H}_0^2$ (Ladyzhenskaya [1]); generates an analytic semigroup, so the norm of e^{tA} from $\tilde{H}_0^2$ to J_2 is $\leq C/t$. (See Yosida [1].) However we have the Sobolev inequality (derived most easily from the general Sobolev-Nirenberg-Gagliardo inequality from Nirenberg [1]:

$$||D^j f||_{L_p} \leq C||D^m f||^a_{L^r}||f||^{1-a}_{L^q}$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\,\frac{1}{q}, \quad \frac{j}{m} \le a \le 1$$

(if $m - j - \frac{n}{r}$ is an integer ≥ 1, $a = 1$ is not allowed)}, that

$$||f||_{H^1} \le ||f||_{H^2}^{1/2}\,||f||_{L^2}^{1/2}$$

so we can choose $\mu(t) \le C/\sqrt{t}$, so A1) holds. Similarly in case (ii) one finds $\mu(t) = C/t^{3/4}$ (Kato-Fujita [1]).

As for the nonlinear terms, assume

A2) $J_t\colon \mathbb{E}_2 \to \mathbb{E}_1$ is a semi-Lipschitz map (i.e., Lipschitz on bounded sets), locally uniformly in t with $J_t(\phi)$ continuous in (t,ϕ). We can suppose $J_t(0) = 0$ for simplicity.

Consider the "formal" differential equation

$$\frac{d\phi}{dt} = A\phi + J_t(\phi) \tag{9.1}$$

in integral equation form (see e.g., Segal [1])

$$W(t,t_0)\phi = e^{(t-t_0)A}\phi + \int_{t_0}^{t} e^{(t-\tau)A} J_\tau(W(\tau,t_0)\phi)\,d\tau \tag{9.2}$$

where $t > t_0$. (Adding an inhomogeneous term causes no real difficulties.)

(9.4) Theorem. *Under assumptions* A1) *and* A2), *the equation* (9.2) *defines a unique local semiflow* (*i.e.*, *evolution system*) *on* $\mathbb{E}_2$ (*in the sense of Section* 8A) *with* $W(t,t_0)\colon \mathbb{E}_2 \to \mathbb{E}_2$ *locally uniformly Lipschitz*, *varying continuously in* t.

Proof. The proof proceeds by the usual contraction mapping argument as for ordinary differential equations (see Lang [1]): Pick $0 < \alpha_0 < \alpha$, let $K_\alpha(t)$ be the Lipschitz constant of J_t from $\mathbb{E}_2$ to $\mathbb{E}_1$ on B_α the α ball about and pick T such that

$$\left(\int_0^{T-t_0} \mu(\tau)d\tau\right)\left(\sup_{\tau\in[t_0,T]} K_\alpha(\tau)\right) \le 1 - \frac{\alpha_0}{\alpha} . \qquad (9.3)$$

Now choose $\phi \in B_{\alpha_0}$ and let M be the complete metric space of C^0 maps Φ of $[0,T]$ to $\mathbb{E}_2$ with $\Phi(t_0) = \phi$, $\Phi(t) \in B_\alpha$ and metric

$$d(\Phi,\Psi) = \sup_{t\in[t_0,T]} ||\Phi(t)-\Psi(t)||_2$$

Define $\mathscr{F}$: $M \to M$ by

$$\mathscr{F}\Phi(t) = e^{(t-t_0)A}\phi + \int_{t_0}^{t} e^{(t-\tau)A}J_\tau(\Phi(\tau))d\tau .$$

From the definitions and (9.3) we have two key estimates: first

$$\begin{aligned} ||\mathscr{F}\Phi(t)||_2 &\le \alpha_0 + \int_{t_0}^{T} \mu(t-\tau)K_\alpha(\tau)\cdot\alpha\, d\tau \\ &\le \alpha_0 + \alpha\left(1 - \frac{\alpha_0}{\alpha}\right) = \alpha \end{aligned} \qquad (9.4)$$

(remember $J_\tau(0) = 0$ here), which shows $\mathscr{F}$ maps M to M and, in the same way

$$d(\mathscr{F}\Phi, \mathscr{F}\Psi) \le \left(1 - \frac{\alpha_0}{\alpha}\right)d(\Phi,\Psi) \qquad (9.5)$$

which shows $\mathscr{F}$ is a contraction.

The result now follows easily. $\square$

(9.5) Exercises. 1. Show that $W(t,t_0)$ has $\mathbb{E}_2$ Lipschitz constant given by α/α_0. Verify that $W(t,s)W(s,t_0) = W(t,t_0)$.

2. If ϕ_t is a maximal solution of (9.2) on $[0,T)$ and $T < \infty$, show $\limsup_{t \to T} ||\phi_t||_2 = \infty$; i.e., verify the continuation assumption 8.2.

3. Use the Sobolev inequalities to verify that $J_t(u) = P((u\cdot\nabla)u)$ satisfies the hypotheses in case (i) above. For case (ii), see Kato-Fujita [1].

Next we want to see that we actually have a solution of the differential equation. Make

A3) Assume that the domain of A as an operator in $\mathbb{E}_0$ is exactly $\mathbb{E}_2$.

(9.6) Theorem. If A1), A2) and A3) hold, then any solution of (9.2) solves (9.1) as an evolution system in $\mathbb{E}_0$ with domain $D = \mathbb{E}_2$; (in the terminology of Section 8A, $W(t,t_0)$ is the (time dependent) local flow of the operator $X(\phi) = A(\phi) + J_t(\phi)$, mapping $\mathbb{E}_2$ to $\mathbb{E}_0$). Solutions of (9.2) in $\mathbb{E}_2$ are unique.

Proof. Let $\phi \in \mathbb{E}_2$ and $\phi(t) = W(t,t_0)\phi \in \mathbb{E}_2$ be the solution of (9.2), so taking $t_0 = 0$ for simplicity,

$$\phi(t) = e^{tA}\phi + \int_0^t e^{(t-\tau)A} J_\tau(\phi(\tau))d\tau.$$

It is easy to verify that

$$\begin{aligned}\frac{1}{h}\{\phi(t+h)-\phi(t)\} &= \frac{1}{h}\{e^{hA}\phi(t)-\phi(t)\} - J_t(\phi(t)) \\ &+ \frac{1}{h}\int_t^{t+h}\{e^{(t+h-\tau)A}J_\tau(\phi(\tau))-J_t(\phi(t))\}d\tau\end{aligned} \tag{9.6}$$

writing

$$e^{(t+h-\tau)A}J_\tau(\phi(\tau)) - J_t(\phi(t))$$

$$= e^{(t+h-\tau)A}[J_\tau(\phi(\tau))-J_t(\phi(t))] + e^{t+h-\tau}J_t(\phi(t))-J_t(\phi(t))$$

one sees that the last term of (9.6) $\to 0$ as $h \to 0$ in $\mathbb{E}_1$ and hence in $\mathbb{E}_0$. The first term of (9.6) tend to $A(\phi(t)) - J_t(\phi(t))$ in $\mathbb{E}_0$ as $h \to 0$ since $\phi(t) \in \mathbb{E}_2$, the domain of A. □

Thus we can conclude that the Navier-Stokes equations define a local semiflow on $\tilde{H}_0^2$ and that this semiflow extends to a local semiflow on $\tilde{H}^1$ (via the integral equation).

Smoothness.

(9.7) Theorem. Let A1), A2) and A3) hold and assume

A4) $J_t\colon \mathbb{E}_2 \to \mathbb{E}_1$ is C^∞ with derivatives depending continuously on t.

Then the semiflow defined by equations (9.1) on $\mathbb{E}_2$ is a C^∞ semiflow on $\mathbb{E}_2$; i.e., each $W(t,t_0)$ is C^∞ with derivatives varying continuously in t in the strong topology (see Section 8).

Proof. We verify the hypotheses of (8A.31). Here we take $X = \mathbb{E}_0$ and $Y = \mathbb{E}_2$, with $D = Y$. Certainly a) holds by hypothesis. Since $Z(\phi_1,\phi_2)$ is the same type of operator as considered above, 9.4 shows b) holds. c') holds by the $\mathbb{E}_2$ Lipschitzness of $W(t,t_0)$ proven in (9.4) and d) is clear.

Hence $W(t,t_0)\colon \mathbb{E}_2 \to \mathbb{E}_2$ is Gateaux differentiable.

The procedure can be iterated. The same argument applies to the semiflow

$$\tilde{W}(t,t_0)(\phi,\psi) = (W(t,t_0)\phi,\ DW(t,t_0)\phi\cdot\psi)$$

on $\mathbb{E}_2 \to \mathbb{E}_2$.

Hence W is C^1 (see (8A.32)), and by induction is C^∞. $\square$

In the context of equation (9.1) the full power of the machinery in Section 8A; in particular the delicate results on time dependent evolution equations are not needed. One can in fact directly prove (9.7) by the same method as (8A.31). However it seems desirable to derive these types of results from a unified point of view.

(9.8) Problem. Assume, in addition that

A5) A generates an analytic semigroup.

Show that for $t > 0$ and $\phi \in \mathbf{E}_1$, $\phi(t)$ lies in the domain of every power of A and that $\phi(t)$ is a C^∞ function of t for $t > 0$. (Hint. Use (8A.33)). Also establish smoothness in ν if A is replaced by νA (see remarks following (8A.33)).

A more careful analysis actually shows that $W(t,t_0)$ are C^∞ maps on $\tilde{H}^1$ in the context of the Navier-Stokes equation (i.e., without assuming A3)). See Weissler [1].

Thus all the requisite smoothness is established for the Navier-Stokes equations, so the proof of (9.2) and hence the bifurcation theorems for those equations is established.

The Problem of Turbulence.

We have already seen how bifurcations can lead from stable fixed points to stable periodic orbits and then to stable 2-tori. Similarly we can go on to higher dimensional tori. Ruelle and Takens [1] have argued that in this or other situations, complicated ("strange") attractors can be expected and that this lies at the roots of the explanation of turbulence.

The particular case where tori of increasing dimension form, the model is a technical improvement over the idea of E. Hopf [4] wherein turbulence results from a loss of stability through successive branching. It seems however that strange attractors may form in other cases too, such as in the Lorenz equations (see Section 4B) [Strictly speaking, it has only a "strange" invariant set]. This is perfectly consistent with the general Ruelle-Takens picture, as are the closely related "snap through" ideas of Joseph and Sattinger [1].

In the branching process, stable solutions become unstable, as the Reynolds number is increased. Hence turbulence is supposed to be a necessary consequence of the equations and in fact of the "generic case" and just represents a complicated solution. For example in Couette flow as one increases the angular velocity Ω_1 of the inner cylinder one finds a shift from laminar flow to Taylor cells or related patterns at some bifurcation value of Ω_1. Eventually turbulence sets in. In this scheme, as has been realized for a long time, one first looks for a stability theorem and for when stability fails (Hopf [2], Chandresekar [1], Lin [1] etc.).

For example, if one stayed closed enough to laminar flow, one would expect the flow to remain approximately laminar. Serrin [2] has a theorem of this sort which we present as an illustration:

(9.9) <u>Stability Theorem. Let</u> $D \subset \mathbb{R}^3$ <u>be a bounded domain and suppose the flow</u> v_t^ν <u>is prescribed on</u> ∂D <u>(this corresponds to having a moving boundary, as in Couette flow). Let</u> $V = \max_{\substack{x \in D \\ t \geq 0}} \|v_t^\nu(x)\|$, $d =$ <u>diameter of</u> D <u>and</u> ν <u>equal the viscosity. Then if the Reynolds number</u> $R = (Vd/\nu) \leq 5.71$, v_t^ν <u>is universally</u> L^2 <u>stable among solutions of the Navier-Stokes equations.</u>

Universally L^2 stable means that if $\bar{v}_t^\nu$ is <u>any</u> other solution to the equations and with the same boundary conditions, then the L^2 norm (or energy) of $\bar{v}_t^\nu - v_t^\nu$ goes to zero as $t \to 0$.

The proof is really very simple and we recommend reading Serrin [1,2] for the argument. In fact one has local stability in stronger topologies using Theorem 1.4 of Section 2A and the ideas of Section 8.

Chandresekar [1], Serrin [2], and Velte [3] have analyzed criteria of this sort in some detail for Couette flow.

As a special case, we recover something that we expect. Namely if $v_t^\nu = 0$ on ∂M is any solution for $\nu \to 0$ then $v_t^\nu \to 0$ as $t \to \infty$ in L^2 norm, since the zero solution is universally stable.

A traditional definition (as in Hopf [2], Landau-

Lifschitz [1]) says that turbulence develops when the vector field v_t can be described as $v_t(w_1,\ldots,w_n) = f(tw_1,\ldots,tw_n)$ where f is a quasi-periodic function, i.e., f is periodic in each coordinate, but the periods are not rationally related. For example, if the orbits of the v_t on the tori given by the Hopf theorem can be described by spirals with irrationally related angles, then v_t would such a flow.

Considering the above example a bit further, it should be clear there are many orbits that the v_t could follow which are qualitatively like the quasi-periodic ones but which fail themselves to be quasi-periodic. In fact a small neighborhood of a quasi-periodic function may fail to contain many other such functions. One might desire the functions describing turbulence to contain most functions and not only a sparse subset. More precisely, say a subset U of a topological space S is <u>generic</u> if it is a Baire set (i.e., the countable intersection of open dense subsets). It seems reasonable to expect that the functions describing turbulence should be generic, since turbulence is a common phenomena and the equations of flow are never exact. Thus we would want a theory of turbulence that would not be destroyed by adding on small perturbations to the equations of motion.

The above sort of reasoning lead Ruelle-Takens [1] to point out that since quasi-periodic functions are not generic, it is unlikely they "really" describe turbulence. In its place, they propose the use of "strange attractors." These exhibit much of the qualitative behavior one would expect from "turbulent" solutions to the Navier-Stokes equations and they are stable under perturbations of the equations; i.e., are "structurally stable".

For an example of a strange attractor, see Smale [1]. Usually strange attractors look like Cantor sets $\times$ manifolds, at least locally.

Ruelle-Takens [1] have shown that if we define a strange attractor A to be an attractor which is neither a closed orbit or a point, and disregarding non-generic possibilities such as a figure 8 then there are strange attractors on T^4 in the sense that a whole open neighborhood of vector fields has a strange attractor as well.

If the attracting set of the flow, in the space of vector fields which is generated by Navier-Stokes equations is strange, then a solution attracted to this set will clearly behave in a complicated, turbulent manner. While the whole set is stable, individual points in it are not. Thus (see Figure 9.1) an attracted orbit is constantly near unstable (nearly periodic) solutions and gets shifted about the attractor in an aimless manner. Thus we have the following reasonable definition of turbulence as proposed by Ruelle-Takens:

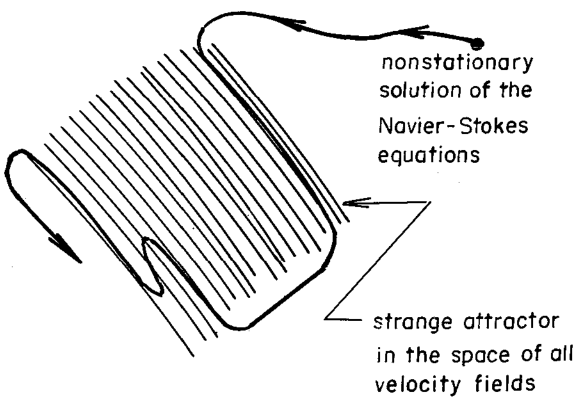

Figure 9.1

"...the motion of a fluid system is turbulent when this motion is described by an integral curve of a vector field X_μ which tends to a set A, and A is neither empty nor a fixed point nor a closed orbit."

One way that turbulent motion can occur on one of the tori T^k that occurs in the Hopf bifurcation. This takes place after a finite number of successive bifurcations have occurred. However as S. Smale and C. Simon pointed out to us, there may be an infinite number of other qualitative changes which occur during this onset of turbulence (such as stable and unstable manifolds intersecting in various ways etc.). However, it seems that turbulence can occur in other ways too. For example, in Example 4B.9 (the Lorenz equations), the Hopf bifurcation is subcritical and the strange attractor may suddenly appear as μ crosses the critical value without an oscillation developing first. See Section 12 for a description of the attractor which appears. See also McLaughlin and Martin [1,2], Guckenheimer and Yorke [1] and Lanford [2].

In summary then, this view of turbulence may be phrased as follows. Our solutions for small μ (= Reynolds number in many fluid problems) are stable and as μ increases, these solutions become unstable at certain critical values of μ and the solution falls to a more complicated stable solution; eventually, after a certain (finite) number of such bifurcations, the solution falls to a strange attractor (in the space of all time dependent solutions to the problem). Such a solution, which is wandering close to a strange attractor, is called turbulent.

The fall to a strange attractor may occur after a Hopf bifurcation to an oscillatory solution and then to invariant tori, or may appear by some other mechanism, such as in the Lorenz equations as explained above ("snap through turbulence").

Leray [3] has argued that the Navier-Stokes equations might break down and the solutions fail to be smooth when turbulence ensues. This idea was amplified when Hopf [3] in 1950 proved global existence (in time) of weak solutions to the equations, but not uniqueness. It was speculated that turbulence occurs when strong changes to weak and uniqueness is lost. However it is still unknown whether or not this really can happen (cf. Ladyzhenskaya [1,2].)

The Ruelle-Takens and Leray pictures are in conflict. Indeed, if strange attractors are the explanation, their attractiveness implies that solutions remain smooth for all t. Indeed, we know from our work on the Hopf bifucation that near the stable closed orbit solutions are defined and remain smooth and unique for all $t \geq 0$ (see Section 8 and also Sattinger [2]). This is already in the range of interesting Reynolds numbers where global smoothness is not implied by the classical estimates.

It is known that in two dimensions the solutions of the Euler and Navier-Stokes equations are global in t and remain smooth. In three dimensions it is unknown and is called the "global regularity" or "all time" problem.

Recent numberical evidence (see Temam et. al. [1]) suggests that the answer is negative for the Euler equations.

Theoretical investigations, including analysis of the spectra have been inconclusive for the Navier-Stokes equations (see Marsden-Ebin-Fischer [1] and articles by Frisch and others in Temam et.al. [1]).

We wish to make two points in the way of conjectures:

1. In the Ruelle-Takens picture, global regularity for all initial data is not an a priori necessity; the basins of the attractors will determine which solutions are regular and will guarantee regularity for turbulent solutions (which is what most people now believe is the case).

2. Global regularity, if true in general, will probably never be proved by making estimates on the equations. One needs to examine in much more depth the attracting sets in the infinite dimensional dynamical system of the Navier-Stokes equations and to obtain the a priori estimates this way.

<u>Two Major Open Problems</u>:

(i) identify a strange attractor in a specific flow of the Navier-Stokes equation (e.g, pipe flow, flow behind a cylinder, etc.).

(ii) link up the ergodic theory on the strange attractor, (Bowen-Ruelle [1]) with the statistical theory of turbulence (the usual reference is Batchellor [1]; however, the theory is far from understood; some of Hopf's ideas [5] have been recently developed in work of Chorin, Foias and others).

SECTION 9A

ON A PAPER OF G. IOOSS

BY G. CHILDS

This paper [3] proves the existence of the Hopf bifurcation to a periodic solution from a stationary solution in certain problems of fluid dynamics. The results are similar to those already described. For instance, in the subcritical case, the periodic solution is shown to be unstable in the sense of Lyapunov when the real bifurcation parameter (Reynold's number) is less than the critical value where the bifurcation takes place; it is shown to be (exponentially) stable if this value is greater than the critical value in the supercritical case.

Iooss, in contrast to the main body of these notes, makes use of a linear space approach for almost all of what he does. Specifically, his periodic solution is a continuous function to elements of a Sobolev space on a fundamental domain Ω in $\mathbb{R}^3$. However, the implicit function theorem is extensively used. The three main theorems of the paper will be stated and the proofs will be briefly outlined to illustrate this method.

First, we formulate a statement of the problem. Let I be a closed interval of the real line. Let $V(I)$ be a neighborhood in $\mathbb{C}$ of this interval. For each $\lambda \in V(I)$, L_λ is a closed, linear operator on a Hilbert space H. The family $\{L_\lambda\}$ is holomorphic of type (A) in $V(I)$ (see Kato [3], p. 375). Also, each L_λ is m-sectorial with vertex-γ_λ. Finally, L_λ has a compact resolvent in H. Let $\mathscr{D}$ be the common domain of the L_λ. Assume K is a Hilbert space such that $\mathscr{D} \subseteq K \subseteq H$ with continuous injections and such that $\forall U \in K$, $||I(t)U||_{\mathscr{D}} \leq ke^{\gamma_\lambda t}(1+t^{-\alpha})||U||_K$, $0 \leq \alpha < 1$ where $I_\lambda(t)$ is the holomorphic semi-group generated by $-L_\lambda$. Let M be a continuous bilinear form: $\mathscr{D} \times \mathscr{D} \to K$. Now we can state the problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} + L_\lambda U - M(U,U) = 0 \\ U \in C^0(0,\infty;\mathscr{D}) \cap C^1(0,\infty;H) \\ U(0) = U_0 \in \mathscr{D},\ U(0) = U(T) = U(2T) = \dots \text{ for some } T > 0 \end{cases} \tag{9A.1}$$

Iooss shows that the equation of perturbation (from a stationary solution) for some Navier-Stokes configurations is of the above form (see also Iooss [5] for more details). In order to find a solution of (9A.1), it is necessary to make some additional hypotheses. Let $\xi_0(\lambda) = \sup\limits_{\xi \in \sigma(-L_\lambda)} \{\mathrm{Re}\ \zeta\}$. Then:

(H.1) $\exists\ \lambda_c \in \mathbb{R}$, <u>a left hand neighborhood</u> $V^-(\lambda_c)$ <u>and a right hand neighborhood</u> $V^+(\lambda_c)$ <u>such that</u> $\xi_0(\lambda_c) = 0$, $\lambda \in V^-(\lambda_c) - \{\lambda_c\} \Rightarrow \xi_0(\lambda) < 0$, $\lambda \in V^+(\lambda_c) - \{\lambda_c\} \Rightarrow \xi_0(\lambda) > 0$.

(H.2) The operator L_{λ_c} admits as proper values pure imaginary numbers $\zeta_0 = i\eta_0$ and $\overline{\zeta}_0$. Also, these proper values are simple. For $\lambda \in V(\lambda_c)$ there exist two analytic functions ζ_1 and $\overline{\zeta}_1 \in \sigma(-L_\lambda)$ such that $\zeta_1(\lambda_c) = \zeta_0$. The spectrum $\sigma(L_\lambda)$ separates into $\{-\zeta_1\} \cup \{-\overline{\zeta}_1\} \cup \tilde{\sigma}(L_\lambda)$. This separation gives a decomposition of H into invariant subspaces:

$$\forall U \in H, \quad U = X + Y, \quad X = E_\lambda U, \quad Y = P_\lambda U;$$

$E_\lambda = E(-\zeta_1) + E(-\overline{\zeta}_1)$, $(L_\lambda+\zeta_1)U_1(\lambda) = 0$, $(L_\lambda^*+\overline{\zeta}_1)W_1(\lambda) = 0$, $(U_1(\lambda),W_1(\lambda))_H = 1$, $(U_1(\lambda),W(0))_H = 1$. The eigenvectors $U_1(\lambda)$, $\overline{U_1(\lambda)}$ are a basis for $E_\lambda H$. For $\lambda \in V(\lambda_c)$, $L_\lambda U = L_{\lambda_c}U + \sum_{n=1}^{\infty}(\lambda-\lambda_c)^n L^{(n)}U$; $U_1(\lambda) = U^{(0)} + (\lambda-\lambda_c)U^{(1)} + \ldots$; $W_1(\lambda) = W^{(0)} + (\lambda-\lambda_c)W^{(1)} + \ldots$; $\zeta_1(\lambda) = \zeta_0 + (\lambda-\lambda_c)\zeta^{(1)} + \ldots$. In particular, we have $\zeta^{(1)} = -(L^{(1)}U^{(0)},W^{(0)})_H$, $L_{\lambda_c} + \zeta_0 U^{(0)} = 0$, $L_{\lambda_c}^* + \overline{\zeta}_0 W^{(0)} = 0$, $(U^{(0)},W^{(0)})_H = 1$. Now, we can make the hypothesis:

(H.3) $\mathrm{Re}(L^{(1)}U^{(0)},W^{(0)})_H \neq 0$. By (H.1) this implies $\mathrm{Re}\,\zeta^{(1)} > 0$. These hypotheses are just the standard ones for the existence of the Hopf bifurcation. For the statement of the theorem we also need to know that $\gamma_0 = \gamma_{0_r} + i\gamma_{0_i} =$

$$-(M^{(0)}[U^{(0)},L_{\lambda_c}^{-1}M^{(0)}(U^{(0)},\overline{U^{(0)}})] +$$

$$M^{(0)}[\overline{U^{(0)}},(L_{\lambda_c}+2i\eta_0 1)^{-1}M(U^{(0)},U^{(0)})],\ W^{(0)})_H \quad \text{where}$$

$$M^{(0)}(U,V) = M(U,V) + M(V,U).$$

We now state Iooss'

Theorem 2. If the hypotheses (H.1), (H.2), (H.3), and

$\gamma_{0_r} \neq 0$ <u>are satisfied</u>, <u>there is a bifurcation to a non-trivial</u> T-<u>periodic solution of</u> (9A.1) <u>starting from</u> λ_c. <u>If</u> $\gamma_{0_r} > 0$, <u>the bifurcation takes place for</u> $\lambda \in V^+(\lambda_c)$, <u>whereas if</u> $\gamma_{0_r} < 0$, <u>it takes place for</u> $\lambda \in V^-(\lambda_c)$. <u>The solution</u> $\mathscr{U} \in C^0(-\infty,\infty;\mathscr{D})$ <u>is unique with the exception of</u> Arg a <u>corresponding to a translation in</u> t. <u>Finally</u>, $\mathscr{U}(t)$ <u>is analytic with respect to</u> $\varepsilon = \sqrt{|\lambda-\lambda_c|}$, <u>the period analytic with respect to</u> $\lambda - \lambda_c$ <u>and one can write</u> $\mathscr{U}(t,\varepsilon) = \varepsilon\,\mathscr{U}^{(1)}(t) + \varepsilon^2\,\mathscr{U}^{(2)}(t) + \ldots$ <u>where</u> $\mathscr{U}^{(i)}(t)$ <u>is</u> T-<u>periodic</u>. <u>Here</u> Arg a <u>is the phase of the</u> X(t) <u>oscillations</u>.

An outline of the proof will be given. Denote

$$\zeta_1(\lambda) = \xi(\lambda) + i\eta(\lambda)$$

$$N_\lambda = iE(-\zeta_1) - iE(-\overline{\zeta}_1).$$

Then the equations for the X and Y parts of $\mathscr{U}$ coming from (9A.1) are

$$\begin{cases} Y(t) = \tilde{B}_t(X+Y,X+Y;\lambda), \\ \dfrac{dX}{dt} - \eta N_\lambda X = \xi X + E_\lambda M(X+Y,X+Y) \equiv F(X,Y,;\lambda), \\ X(0) = X(T), \quad X \text{ and } Y \in C^0(-\infty,+\infty;\mathscr{D}), \end{cases} \tag{9A.2}$$

where $\tilde{B}_t(U,V;\lambda) = \int_{-\infty}^{t} I_\lambda(t-\tau)P_\lambda M[U(\tau),V(\tau)]d\tau$. By substituting $X(t) = A(t)U_1(\lambda) + \overline{A(t)U_1(\lambda)}$ in the right hand side of the equation for Y we obtain a right hand side which is analytic with respect to (λ,X,Y) in a neighborhood of $(\lambda_c,0,0)$ in $\mathbb{C} \times \{C^0(-\infty,+\infty;\mathscr{D})\}^2$. And the derivative with respect to Y is zero at $(\lambda_c,0,0)$. Then, denoting $X^{(0)}(t) =$

$A(t)U^{(0)} + \overline{A(t)U^{(0)}}$, and using the implicit function theorem, one has $Y(t) = \eta_t(X;\lambda) = \sum_{i,j\geq 2}^{\infty} (\lambda-\lambda_c)^i \eta_t^{(i,j)}(X^{(0)},\ldots,X^{(0)})$ where $\eta_t^{(i,j)}(\cdot,\ldots,\cdot)$ is a continuous functional, homogeneous of degree j. We must now solve $\frac{dX}{dt} = \eta N_\lambda X + F(X,\eta_t(X;\lambda);\lambda)$ with $X(0) = X(T)$, $X \in C^0(-\infty,\infty;\mathscr{D})$. The following form is assumed for X:

$$\begin{cases} X(t) = e^{\frac{2\pi t}{T}N_\lambda}\chi + \tilde{X}(t) \equiv \chi(t) + \tilde{X}(t) \\ \chi = \frac{1}{T}\int_0^T e^{-\frac{2\pi t}{T}N_\lambda} X(t)dt = aU_1(\lambda) + \overline{aU_1(\lambda)} \end{cases} \tag{9A.3}$$

Decomposing the equation for $X(t)$ according to (9A.3), one can solve for $\tilde{X}(t)$ using the implicit function theorem

$$\tilde{X}(t) = \mathscr{X}_t(\chi,\lambda,T) = \sum_{i,j,k\geq 2}^{\infty} (\lambda-\lambda_c)^i (T-T_0)^j \mathscr{X}^{(i,j,b)}(\chi;t),$$

$$t \in [0,T],$$

where $\mathscr{X}^{(i,j,k)}(\chi;t)$ is homogeneous of degree k with respect to χ. $\tilde{X}(t)$ is now replaced with $\mathscr{X}_t(\chi,\lambda,T)$ in the other equation (the one for χ). Splitting the result into real and imaginary parts one obtains

$$\xi + f(|a|^2,\lambda,T) = 0$$

$$\eta = \frac{2\pi}{T} + g(|a|^2,\lambda,T) = 0$$

with $f(0,\lambda,T) = g(0,\lambda,T) = 0$. The development in Taylor series about the point $(0,\lambda_c,T_0)$ has first term $-\gamma_0|a|^2$. It is in this way that a non-zero value of γ_{0_r} allows a solution for $|a|^2$ and T by the implicit function theorem. This completes the determination of $X(t)$, $Y(t)$ and therefore $\mathscr{U}(t,\varepsilon)$.

Now that we have the periodic solution it is desired to exhibit its stability properties. We consider a nearby solution $U(t)$ and set $U(t) = \mathscr{U}(t+\delta,\varepsilon) + U'(t)$. Then $U'(t)$ satisfies

$$\frac{\partial U'}{\partial t} = A_\varepsilon(t+\delta)U' + M(U',U')$$

$$U'(0) = U_0 = \mathscr{U}(\delta,\varepsilon) \in \mathscr{D}$$

$$U' \in C^0(0,\infty;\mathscr{D}) \cap C'(0,\infty;H)$$

where

$$A_\varepsilon(t+\delta) = -L_\lambda + M^{(0)}[\mathscr{U}(t+\delta,\varepsilon),\cdot], \ \lambda = \lambda_c + \varepsilon^2 \operatorname{sgn}(\lambda-\lambda_c).$$

Therefore, in order to study stability one examines the properties of the solution of the linearized equation:

$$\frac{\partial V}{\partial t} = A_\varepsilon(t+\delta)V, \quad V \in C^0(0,T_1;\mathscr{D}) \cap C^1(0,T_1;H)$$

$$V(0) = V_0 \in \mathscr{D}, \ T_1 < \infty$$

The solution of this equation is:

$$V(t) = I_\lambda(t)V_0 + \int_0^t I_\lambda(t-\tau)M^{(0)}[\mathscr{U}(t+\delta,\varepsilon),V(\tau)]d\tau.$$

We denote this solution as

$$V(t) = G_\varepsilon(t,\delta)V_0.$$

The stability properties will come from the properties of the spectrum of $G_\varepsilon(T,\delta)$, which plays the role of the Poincaré map.

We can now state Iooss'

<u>Theorem 3</u>. <u>The hypotheses of Theorem 2 being satisfied and the operator</u> $G_\varepsilon(T,\delta)$ <u>being defined by the equation above</u>, <u>the spectrum of</u> $G_\varepsilon(T,\delta)$ <u>is independent of</u> $\delta \in \mathbb{R}$. <u>It is constituted on the one hand</u>, <u>by two real simple proper</u>

values in a neighborhood of 1, which are 1 and $1 - 8\pi\xi^{(1)}(\lambda-\lambda_c) + o(\lambda-\lambda_c)$. On the other hand, the remainder of the spectrum is formed from a denumerable infinity of proper values of finite multiplicities, the only point of accumulation being 0, there remaining in the interior of a disk of radius $\zeta < 1$ independent of $\varepsilon \in \mathscr{V}(0)$.

The following is then a direct result of Lemma 5 of Judovich [10].

Corollary. The hypotheses of Theorem 2 being satisfied, if further $\gamma_{0_r} < 0$, the bifurcation takes place for $\lambda \in \mathscr{V}^-(\lambda_c)$ and the secondary solution is unstable in the sense of Lyapunov.

Now the proof of Theorem 3 is outlined. The operator $G_\varepsilon(T,\delta)$ is compact in $\mathscr{D}$. Hence, its spectrum is discrete. Let $\sigma \in$ spectrum of $G_\varepsilon(T,\delta)$. Then there exists $V \neq 0$ such that $\sigma V = G_\varepsilon(T,\delta)V$. Then, if $W = G_\varepsilon(nT-\delta,\delta)V$ with $n \in \mathbb{N}$ such that $\delta \leq nT$, $\sigma W = G_\varepsilon(T,0)W$. Hence, spectrum $(G_\varepsilon(T,\delta)) \subseteq$ spectrum $(G_\varepsilon(T,0))$. Similarly, the reverse containment holds. To establish $1 \in$ spectrum of $G_\varepsilon(T,\delta)$, it suffices to show $G_\varepsilon(T,0)\frac{\partial\mathscr{U}}{\partial t}(0,\varepsilon) = \frac{\partial\mathscr{U}}{\partial t}(0,\varepsilon)$. Note that $I_{\lambda_c}(T_0) = \lim_{\varepsilon\to 0} G_\varepsilon(T,0)$.

It can be shown that 1 is a semi-simple (multiplicity 2) proper value of $I_{\lambda_c}(T_0)$. With all proper values ζ_n of $I_{\lambda_c}(T_0)$ correspond a finite number of proper values of L_{λ_c} of the form $-T_0^{-1}(\mathrm{Log}\,|\zeta_n| + 2k\pi i)$. If one removes $i\eta_0$ from the spectrum of L_{λ_c}, the proper values ζ_i are remaining such that $\mathrm{Re}\,\zeta_i > \tilde{\xi} > 0 \Longrightarrow \mathrm{Log}\,|\zeta_n| < -\tilde{\xi}T_0 < 0$. Thus, the remainder of the spectrum of $I_{\lambda_c}(T_0)$ other than 1 is contained in a disk of radius strictly less than 1. Because of the continuity of the discrete spectrum the same is true of $G_\varepsilon(T,0)$ for $\varepsilon \in \mathscr{V}(0)$. The other proper value of $G_\varepsilon(T,0)$ or $G_\varepsilon(T,\delta)$ is found by studying the degenerate operator

$$\varepsilon^{-1}[E(\varepsilon)G_\varepsilon E(\varepsilon) - E(\varepsilon)] = \tilde{G}(\varepsilon)$$

where $E(\varepsilon)$ is $\frac{1}{2\pi i}\int_\Gamma [\zeta 1 - G_\varepsilon]^{-1} d\zeta$ with Γ being a circle of sufficiently small radius about 1 and $G_\varepsilon \equiv G_\varepsilon(T(\varepsilon),0)$. One uses the expansion $G_\varepsilon = I_{\lambda_c}(T_0) + \varepsilon G^{(1)} + \varepsilon\hat{G}(\varepsilon)$ where $\hat{G}(\varepsilon) = o(1)$ and methods of the theory of perturbations to obtain $\hat{G}(\varepsilon) = \varepsilon\tilde{G}^{(1)} + o(\varepsilon)$,

$$\tilde{G}^{(1)} = -T_0\begin{pmatrix} |a^{(1)}|^2\gamma_0 & \left(a^{(1)}\right)^2\gamma_0 \\ \left(\bar{a}^{(1)}\right)^2\gamma_0 & |a^{(1)}|^2\bar{\gamma}_0 \end{pmatrix}$$

(in the basis $\{U^{(0)},\overline{U^{(0)}}\}$). The bijection $\sigma \to \varepsilon^{-1}(\sigma^{-1})$ makes a correspondence between the proper values of $G(\varepsilon)$ and $\tilde{G}(\varepsilon)$. The proper values of $\tilde{G}^{(1)}$ are 0 and $-4T_0\gamma_{0_r}|a^{(1)}|^2 = -8\pi\xi^{(1)}\operatorname{sgn}\gamma_{0_r}$. This gives us 1 and $1 - 8\pi\xi^{(1)}(\lambda-\lambda_c) + o(\varepsilon^2)$ as proper values of $G_\varepsilon(T,\delta)$.

We now state

<u>Theorem 4. If the hypotheses of Theorem 2 are satisfied and</u> $\gamma_{0_r} > 0$, <u>the bifurcation takes place for</u> $\lambda \in \mathscr{V}^+(\lambda_c)$. <u>There exist</u> $\mu > 0$ <u>and a right hand neighborhood</u> $\mathscr{V}^+(\lambda_c)$ <u>such that if</u> $\lambda \in \mathscr{V}^+(\lambda_c)$, <u>and if one can find</u> $\delta_0 \in [0,T]$ <u>such that the initial condition</u> U_0 <u>satisfies</u>

$$||U_0 - \mathscr{U}(\delta_0,\varepsilon)|| \le \mu\varepsilon^2 \quad (\varepsilon = \sqrt{\lambda-\lambda_c}),$$

<u>then there exists</u> $\delta_\ell \in [0,T]$ <u>such that</u> $||U(t)-\mathcal{U}(t+\delta_\ell,\varepsilon)||_{\mathcal{D}} \to 0$ <u>exponentially when</u> $t \to \infty$, $U(t)$ <u>being the solution of (9A.1) satisfying</u> $U(0) = U_0$.

This case is not so simple since there is the proper value, 1. The theorem results from the following lemma:

<u>Lemma 9</u>. <u>Given</u> $V_0 \in V(\delta)$, <u>that is to say</u>, <u>satisfying</u> $E_\delta V_0 = \mathcal{G}(P_\delta V_0,\varepsilon,\delta)$ <u>and</u> $||P_\delta V_0||_{\mathcal{D}} \le \mu_1 \varepsilon^2$, <u>then the equation</u>,

$$\frac{\partial V}{\partial t} = A_\varepsilon(t+\delta)V + M(V,V), \quad V(0) = V_0,$$

<u>admits a unique solution in</u> $C^0(0,\infty;\mathcal{D}) \cap C^1(0,\infty;H)$ <u>such that</u> $||V(t)||_{\mathcal{D}} \le \mu_2 \varepsilon^2 e^{-\sigma/2\ t}$, $\forall t \ge 0$.

We must identify some of the terminology. First, E_δ is the projection onto a vector collinear to $\frac{\partial \mathcal{U}}{\partial t}(\delta,\varepsilon)$, and $P_\delta = 1 - E_\delta$. We have $\mathcal{G}(P_\delta V_0,\varepsilon,\delta) = E_\delta \check{\mathcal{B}}_0\{\eta_\tau[W_0(P_\delta V_0,\varepsilon,\delta),\varepsilon,\delta], \eta_t[\cdots];\varepsilon,\delta\}$. The notation on the right hand side is associated with the following problem:

$$V(t) = G_\varepsilon(t,\delta)W_0 + \hat{\mathcal{B}}_t(V,V;\varepsilon,\delta) + \check{\mathcal{B}}_t(V,V;\varepsilon,\delta)$$

with

$$\hat{\mathcal{B}}_t(U,V;\varepsilon,\delta) = \int_0^t G_\varepsilon(t-\tau,\tau+\delta)P_{\delta+\tau}M[U(\tau),V(\tau)]d\tau,$$

$$\check{\mathcal{B}}_t(U,V;\varepsilon,\delta) = -\int_t^\infty G_\varepsilon(t-\tau,\tau+\delta)E_{\delta+\tau}M[U(\tau),V(\tau)]d\tau,$$

where W_0 is such that $E_\delta W_0 = 0$, and where one searches for V in the Banach space $\mathcal{B}_\beta = \{V\colon t \to e^{\beta t}V(t) \in C^0(0,\infty;\mathcal{D})\}$, provided with the norm $|V|_\beta = \sup_{t\in(0,\infty)} ||e^{\beta t}V(t)||_{\mathcal{D}}$, with $\beta = \sigma/2$.

These estimates can be shown:

$$|G_\varepsilon(t,\delta)W_0|_\beta \leq M_1 ||W_0||_{\mathscr{D}},$$

$$|\hat{\mathscr{B}}_t(U,V;\varepsilon,\delta)|_\beta \leq M_2\gamma\sigma^{-1}|U|_\beta|V|_\beta,$$

$$|\check{\mathscr{B}}_t(U,V;\varepsilon,\delta)|_\beta \leq M_2\gamma\sigma^{-1}|U|_\beta|V|_\beta$$

where γ is a bound for the bilinear form M. It results that there exists a μ_0 independent of ε such that for $||W_0|| \leq \mu_0\varepsilon^2$, there exists a unique V in $\mathscr{B}$ satisfying the above problem. The solution is denoted $V(t) = \eta_t(W_0,\varepsilon,\delta)$. Now, $V(0) = \eta_0(W_0,\varepsilon,\delta) = W_0 + \check{\mathscr{B}}_0[\eta_\tau(W_0,\varepsilon,\delta),\ \eta_\tau(W_0,\varepsilon,\delta);\varepsilon,\delta]$ which gives after decomposition:

$$E_\delta V_0 = E_\delta \check{\mathscr{B}}_0[\eta_\tau(W_0,\varepsilon,\delta),\eta_\tau(W_0,\varepsilon,\delta);\varepsilon,\delta],$$

$$P_\delta V_0 = W_0 + P_\delta \check{\mathscr{B}}_0[\eta_\tau(W_0,\varepsilon,\delta),\eta_\tau(W_0,\varepsilon,\delta);\varepsilon,\delta].$$

After having remarked that $\frac{\partial}{\partial W_0}[\eta_\tau(W_0,\varepsilon,\delta)]_{W_0 = 0} = G_\varepsilon(t,\delta)$, it is easy to find μ_1 independent of ε such that if $||P_\delta V_0|| \leq \mu_1\varepsilon^2$, then the second of the above equations is solvable with respect to W_0 by the implicit function theorem, and $W_0 = W_0(P_\delta V_0,\varepsilon,\delta)$ satisfies $||W_0||_{\mathscr{D}} \leq \mu_0\varepsilon^2$. The notation is completely explained. The proof from here on is not too difficult. The uniqueness comes from the uniqueness of the solutions to (9A.1) on a bounded interval for ε sufficiently small. The lemma will be demonstrated as soon as it is shown that the solution of our above problem is the same as the solution of the equation in the lemma. But this follows immediately utilizing $\frac{\partial}{\partial t} G_\varepsilon(t-\tau,\tau+\delta) = A_\varepsilon(\tau+\delta)G_\varepsilon(t-\tau,\tau+\delta)$ in evaluating $\frac{\partial}{\partial t}\hat{\mathscr{B}}_t(V,V;\varepsilon,\delta)$ and $\frac{\partial}{\partial t}\check{\mathscr{B}}_t(V,V;\varepsilon,\delta)$. (It is seen that the manifold $V(\delta)$ is nothing more than the set of V_0 such that W_0 can be found with $||W_0|| \leq \mu_0 \varepsilon^2$ and satisfying the equations for $P_\delta V_0$ and $E_\delta V_0$.)

The techniques of Iooss are very similar to those of Judovich [1-12] (see also Bruslinskaya [2,3]). These methods are somewhat different from those of Hopf which were generalized to the context of nonlinear partial differential equations by Joseph and Sattinger [1].

Either of these methods is, nevertheless, basically functional-analytic in spirit. The approach used in these notes attempts to be more geometrical; each step is guided by some geometrical intuition such as invariant manifolds, Poincaré maps etc. The approach of Iooss, on the other hand, has the advantage of presenting results in more "concrete" form, as, for example, $\mathcal{U}(t,\varepsilon)$ in a Taylor series in ε. This is also true of Hopf's method. However, stability calculations (see Sections 4A, 5A) are no easier using this method.

Finally, it should be remarked that Iooss [6] presents analogous results to this paper for the case of the invariant torus (see Section 6).

SECTION 9B

ON A PAPER OF KIRCHGÄSSNER AND KIELHÖFFER

BY O. RUIZ

The purpose of this section is to present the general idea that Kirchgässner and Kielhöffer [1] follow in resolving some problems of stability and bifurcation.

<u>The Taylor Model</u>.

This model consists of two coaxial cylinders of infinite length with radii r_1' and r_2' $(r_1' < r_2')$ rotating with constant angular velocities ω_1 and ω_2. Due to the viscosity an incompressible fluid rotates in the gap between the cylinders. If λ is the Reynolds number $\lambda = \frac{r_1'\omega_1(r_2'-r_1')}{\nu}$ (ν the viscosity), we have for small values of λ a solution independent of λ, called the Couette flow. As λ increases, several types of fluid motions are observed, the simplest of which is independent of ϕ and periodic in z, when we consider cylindrical coordinates. If we restrict our considerations to these kinds of flows and require that the solution v be invariant under the groups of translations T_1 generated by $z \to z + 2\pi/\sigma$ and $\phi \to \phi + 2\pi$, $\sigma > 0$ and we consider the "basic" flow $V = (V_r, V_\phi, V_z), P$ in cylindrical coordinates, (we assume V, P are given) we may write the N-S equation in the form

$$\begin{aligned}
&\text{(a)} \quad D_t u - \tilde{\Delta} u + \lambda L(V)u + \lambda \Delta q = -\lambda N(u), \\
&\text{(b)} \quad \nabla \cdot u = 0, \quad u|_{r=r_1,r_2} = 0, \; u(Tx,t) = u(x,t), \\
&\qquad\qquad T \in T_1, \qquad\qquad (9B.1) \\
&\text{(c)} \quad u|_{t=0} = u^0
\end{aligned}$$

where

$$v = V + u, \quad p = P + q, \quad D_t = \frac{\partial}{\partial t},$$

$$\nabla = \left(\frac{\partial}{\partial r}, 0, \frac{\partial}{\partial z}\right), \quad \text{(gradient)}$$

$$\Delta = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \text{ Laplacian, } \tilde{\Delta}_{ik} = (\Delta - (1-\delta_{i3})/r^2)\delta_{ik}$$

$$L^0_{ik}(V) = -2V_\phi \delta_{i1}\delta_{2k}/r + (V_r \delta_{2k} + V_\phi \delta_{1k})\delta_{i2}/r,$$

$$L(V)u = L^0(V)u + (V\cdot\nabla)u + (u\cdot\nabla)V,$$

$$Q(u)_i = -u_\phi^2 \delta_{i1}/r + u_\phi u_r \delta_{i2}/r,$$

$$N(u) = (u\cdot\nabla)u + Q(u).$$

The Bénard model consists of a viscous fluid in the strip between 2 horizontal planes which moves under the influence of viscosity and the buoyance force, where the latter is caused by heating the lower plane. If the temperature of the upper plane is T_1, and the temperature of the lower plane is T_0 $(T_0 > T_1)$, the gravity force generates a pressure distribution which for small values of $T_1 - T_0$ is balanced by the viscous stress resulting in a linear temperature distribution. If however $T_1 - T_0$ is above a critical point of value, a convection motion is observed. Let α, h, g, ν, ρ, k denote the coefficient of volume expansion, the thickness of the layer, the gravity, the kinematic viscosity, the density and the coefficient of thermodynamic conductivity respectively. We use Cartesian coordinates where the x_3-axis points opposite to the force of gravity, $\tilde{\theta}$ denotes the temperature and p the pressure. By the N-S equations we have the following initial-value problem for an arbitrary reference flow V, T, P. If $\omega = (u,\theta)$, $v = V + u$, $p = P + q$, $\tilde{\theta} = T + \theta$, we have

$$\begin{aligned}
&\text{(a)} \quad D_t\omega - \tilde{\Delta}\omega + \lambda L(V)\omega + \nabla q = -N(\omega),\\
&\text{(b)} \quad \nabla\cdot\omega = 0, \quad \omega|_{x_3=0,1} = 0, \qquad\qquad (9B.2)\\
&\text{(c)} \quad \omega|_{t=0} = \omega^0
\end{aligned}$$

where $\lambda = \alpha g(T_0-T_1)h^3/\nu^2$ (Reynolds number or Grashoff number)

$$\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, 0),$$

$$\tilde{\Delta}_{ik} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\left(\delta_{ik} + \frac{1}{Pr}\delta_{i4}\right), \quad Pr = k/\nu$$

$$L^0_{ik} = -\delta_{i3}\delta_{k4} - \frac{1}{Pr}\delta_{i4}\delta_{k3},$$

$$L(V)\omega = L^0\omega + (V\cdot\nabla)\omega + (u\cdot\nabla)V,$$

$$N(\omega) = (u\cdot\nabla)\omega.$$

Since experimental evidence shows that the convection takes place in a regular pattern of closed cells having the form of rolls, we are going to consider the class of solutions such that

$$\omega(Tx,t) = \omega(x,t), \quad q(Tx,t) = q(x,t)$$
$$\text{where } T \in T_1, \text{ and } T_1$$

is the group generated by the translations

$$x_1 \to x_1 + 2\pi/\alpha, \quad x_2 \to x_2 + 2\pi/\beta, \quad \alpha^2 + \beta^2 \neq 0.$$

$$\left.\begin{aligned} u(Tx,t) &= Tu(x,t) \\ q(Tx,t) &= q(x,t) \\ \theta(Tx,t) &= \theta(x,t) \end{aligned}\right. \qquad T \in T_2$$

where T_2 is the group of rotations generated by

$$T_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note. It is possible to show that all differential operators in the differential equation preserve invariance under T_1 and T_2. An interesting fact is that a necessary condition for the existence of nontrivial solutions is $\alpha = 2\pi/n$, $n \in \{1,2,3,4,6\}$, and that there are only 7 possible combinations of n,α,β, which give different cell patterns (no cell structure, rolls, rectangles, hexagons, squares, triangles).

Functional-Analytic Approach.

The analogy between (9B.1) and (9B.2) in the Taylor's and

Bénard's models suggests to the authors an abstract formulation of the bifurcation and stability problem. In this part I am going to sketch the idea that the authors follow to convert the differential equations (9B.1) and (9B.2) in a suitable evolution equation in some Hilbert space.

We may consider D an open subset of R^3 with boundary ∂D which is supposed to be a two dimensional C^2-manifold. T_1 denotes a group of translations and Ω its fundamental region of periodicity which we may suppose is bounded. Assume

$$D = \bigcup_{T \in T_1} T\Omega \ .$$

Now we consider the following sets ($c\ell$ = closure)

$$C^{T,\infty}(\bar{D}) = \{\omega | \omega : c\ell(D) \to R^n, \text{ infinitely often differentiable in } c\ell(D),\ \omega(Tx) = \omega(x),\ T \in T_1\},$$

$$C_0^{T,\infty}(D) = \{\omega | \omega \in C^{T,\infty}(\bar{D}),\ \text{supp } \omega \subset D\},$$

$$C_{0,\infty}^{T,\infty}(D) = \{\omega | \omega \in C_0^{T,\infty}(D),\ \nabla \cdot u(x) = 0,\ \omega = (u,v),\ u \in R^3\}.$$

Defining

$$(v,\omega)_m = \sum_{|\gamma| \le m} (D^\gamma v, D^\gamma \omega)\ , \quad |v|_m = \{(v,v)_m\}^{1/2},$$

where

$$(D^\gamma v,\ D^\gamma \omega)_2 = \int_\omega (D^\gamma v(x) \cdot D^\gamma \omega(x))\,dx$$

and γ is a multiindex of length 3; one obtains the following Hilbert spaces

$$L_2^T = c\ell_{||_0} C_0^{T,\infty}(D)\ , \quad \mathcal{J}^T = c\ell_{||_0} C_{0,\sigma}^{T,\infty}(D),$$

$$\overset{\circ}{H}{}_{1,\sigma}^T = c\ell_{||_1} C_{0,\sigma}^{T,\infty}(D)\ , \quad H_m^T = c\ell_{||_m} C^{T,\infty}(\bar{D}),$$

For the Taylor problem we have

$$n = 3,\ D = (r_1,r_2) \times [0,2\pi)$$

$$\Omega = (r_1,r_2) \times [0,2\pi) \times [0,2\pi/\sigma)$$

and for the Bénard problem

$$n = 4,\quad D = \mathbb{R}^2 \times (0,1),\quad \mathbb{R} = \text{the real numbers}$$

$$\Omega = [0,2\pi/\alpha) \times [0,2\pi/\beta) \times (0,1)\ .$$

We may consider that the differential equations of the form (9B.1) or (9B.2) are written with operators in L_2^T.

From H. Weyl's lemma it is possible to consider $L_2^T = \overset{\circ}{J}{}^T \oplus G^T$, where G^T contains the set of ∇_q such that $q \in H_1^T$. We may use the orthogonal projection $P\colon L_2^T \to \overset{\circ}{J}{}^T$ for removing the differential equation $D_t\omega - \tilde{\Delta}\omega + \lambda L(V)\omega + \lambda\nabla q = -\lambda N(\omega)$ with the additional conditions of boundary and periodicity, to a differential equation in $\overset{\circ}{J}{}^T$. If we consider $q \in H_1^T$, $P\nabla q \equiv 0$ and since we look for solutions on $\overset{\circ}{J}{}^T$, we have $Pu = u$, and we may write the new equation in $\overset{\circ}{J}{}^T$ as

$$\frac{d\omega}{dt} + P\tilde{\Delta}u + \lambda PL(V) = -\lambda PN(\omega) \tag{9B.3}$$

with initial condition $\omega_{t=0} = \omega^0$.

The authors write (9B.3) in the form

$$\frac{d\omega}{dt} + \tilde{A}(\lambda)\omega + h(\lambda)R(\omega) = 0,\quad \omega|_{t=0} = \omega^0,$$

where $\tilde{A}(\lambda) = P\tilde{\Delta} + \lambda PL(V)$, $R(\omega) = PN(\omega)$ and where $h(\lambda) = \lambda$ for Taylor's model and $h(\lambda) = 1$ for Benard's model.

Also, they show using a result of Kato-Fujita, that it is possible to define fractional powers of $\tilde{A}(\lambda)$ by

$$\tilde{A}(\lambda)^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty \exp(-\tilde{A}(\lambda)t)t^{\beta-1}dt, \quad \beta > 0,$$

and that this operator is invertible.

The above fact is useful in resolving some bifurcation problems in the stationary case. If $A = P\tilde{\Delta}$, $M(V) = PL(V)$ the stationary equation form (9B.3) is

$$\text{(SP)} \qquad A\omega + \lambda M(V)\omega + h(\lambda)R(\omega) = 0$$

where V is any known stationary solution with $V \in$ Domain of A.

If we consider the substitution $A^{3/4}\omega = v$

$$K(V) = A^{-1/4}M(V)A^{-3/4} \qquad T(v) = A^{-1/4}R(A^{-3/4}v)$$

we may write (SP) in the form

$$v + \lambda K(V) + h(\lambda)T(v) = 0.$$

If one uses a theorem of Krasnoselskii, it is possible to show the following theorem.

Theorem 4.1. Let be $\lambda_j \in R$, $\lambda_j \neq 0$ and $(-\lambda_j)^{-1}$ be an eigenvalue of K of odd multiplicity, then

i) in every neighborhood of $(\lambda_j, 0)$ in $R \times \overset{\circ}{J}{}^{T}$ there exists (λ,ω), $0 \neq \omega \in D(A)$ such that ω solves the stationary equation (SP).

ii) if $(-\lambda_j)^{-1}$ is a simple eigenvalue, then there exists a unique curve $(\lambda(\alpha),\omega(\alpha))$ such that $\omega(\alpha) \neq 0$ for $\alpha \neq 0$ which solves (SP), moreover $(\lambda(0),\omega(0)) = (\lambda_j, 0)$.

Now if we assume that $V \in C(\overline{\Omega})$ and ∂D is a C^∞-manifold which is satisfied for the Taylor and Bénard Problem, and we consider solutions of (SP) on the space $H_{m+2} \cap H_{1,\sigma}$, $m > 3/2$, we may write the (SP) equation in the form

$$\text{(SP)}' \qquad A_m\omega + \lambda M\omega + h(\lambda)R(\omega) = 0$$

where A_m is an operator whose domain is $D(A_m) = H_{m+2} \cap H_{1,\sigma} \subset PH_m$, $A_m(\omega) = A\omega$, $\omega \in D(A)$. If besides we consider that in $R(\omega) = PN(\omega)$, $N(\omega)$ is an arbitrary polynomial operator including differentiation operators up to the order 2, and $K_m = MA_m^{-1}$ we may obtain the following theorem.

Theorem 4.2. *Let $V \in C(\overline{D})$, ∂D be a C^∞-manifold; let M such that there exists constants C_1 and C_2 such that $|M\omega|_{m+1} \leq C_1|\omega|_{m+2}$ and $|MA_m^{-1}\omega|_{m+1} \leq C_2|\omega|_m$. Then for every eigenvalue $(-\lambda_j)^{-1}$ of K_m, $\lambda_j \neq 0$, of odd multiplicity, $(\lambda_j,0)$ is a bifurcation point of (SP)'. The solutions ω are in $C(\overline{D})$ and fulfill the boundary condition $\omega|_{\partial D} = 0$.*

We may note that this theorem shows that we may obtain a strong regularity for the branching solutions. Besides, we note that in theorems 4.1 and 4.2 the existence of nontrivial solutions of (SP) is reduced to the investigation of the spectrum of K or K_m.

Now, we are going to apply these theorems to the Taylor's and Bénard models.

Taylor Model. For this model we take $K = A^{-1/4}M(v^0)A^{-3/4}$ where v^0 is the Couette flow. In cylindrical coordinates the solution is given by $v^0 = (0,v_\phi^0,0)$ where $v_\phi^0 = ar + b/r$

$$a = \frac{1}{r_1\omega_1} \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2} \qquad b = \frac{1}{r_1\omega_1} \frac{(\omega_1-\omega_2) r_1^2 r_2^2}{r_2^2 - r_1^2}$$

When $a \geq 0$, $v_\phi^0 \geq 0$, Synge shows that the Couette flow is locally stable.

For $a < 0$, $v_\phi^0(r) > 0$, Velte and Judovich proved the following theorem for the K operator.

Theorem 4.3. *Let be* $a < 0$, $v_\phi^0(r) > 0$ *for* $r \in (r_1, r_2)$, T_1 *the group of translations generated by* $z \to z + 2\pi/\sigma$; $\phi \to \phi + 2\pi$, $\sigma > 0$. *Then for all* $\sigma > 0$, *except at most a countable number of positive numbers, there exists a countably many sets of real simple eigenvalues* $(-\lambda_i)^{-1}$ *of* K. *Every point* $(\lambda_i, 0) \in R \times D(A)$ *is a bifurcation point of the stationary problem where exactly one nontrivial solution branch* $(\lambda(\alpha), \omega(\alpha))$ *emanates. These solutions are Taylor vortices.*

Strong experimental evidence suggest that all solutions branching off $(\lambda_i, 0)$ where $\lambda_i \neq \lambda_1$ are unstable; however, no proof is known.

Benard model. In this model if $\lambda = \alpha g(T_0-T_1)h^3/\nu^2$, $\sigma = (\alpha^2+\beta^2)^{1/2}$, α, β like on page 318, it is known that for some $\lambda_1(\sigma)$, $\lambda \in [0, \lambda_1]$, $\omega = 0$ is the only solution of the stationary problem.

For this model the bifurcation picture is determined by the spectrum of $K = A^{-1/4}M(v^0)A^{-3/4}$, where v^0 is given in Cartesian coordinates by

$$v_0 = 0, \quad P_0(x_3) = -\frac{gh^3}{\nu^2}(x_3 + \alpha(T_0-T_1)), \quad \theta_0(x_3) = -x_3.$$

In order to obtain simple eigenvalues, Judovich introduces even solutions $u(-x) = (-u_1(x), -u_2(x), u_3(x))$, $\theta(-x) = \theta(x)$, $q(-x) = -q(x)$, and he shows in his articles, <u>On the origin of convection</u> (Judovich [6]) and <u>Free convection and bifurcation</u> (1967) the following theorem.

<u>Theorem 4.6</u>. i) <u>The Bénard problem possesses for approximately all</u> α <u>and</u> β <u>countably many simple positive characteristic values</u> λ_i. <u>Furthermore</u> $(\lambda_i, 0) \in \mathbb{R} \times D(A)$ <u>is a bifurcation point</u>.

ii) <u>If</u> n, α, β <u>are chosen according to the note on</u> pg. 318, <u>the branches emanating from</u> $(\lambda_i, 0)$ <u>are doubly periodic</u>, <u>rolls</u>, <u>hexagons</u>, <u>rectangles</u>, <u>and triangles</u>.

iii) <u>If</u> λ_1 <u>denotes the smallest characteristic value</u>, <u>then the nontrivial solution branches to the right of</u> λ_1 <u>and permits the parametrization</u> $\omega(\lambda) = \pm(\lambda-\lambda_1)^{1/2}F(\lambda)$ <u>where</u> $F: R \to D(A)$ <u>is holomorphic in</u> $(\lambda-\lambda_1)^{1/2}$.

It is interesting to observe that since the characteristic values are determined only by σ, we may consider differentials α, β with the same value of σ, and to note that we have solutions of every possible cell structure emanating from each bifurcation point.

<u>Stability</u>.

About this topic I am going to give a short description of the principal results.

It is known that the basic solution loses stability for some $\lambda_c \in (0, \lambda_1]$, λ_1 as in the past section. Under the assumption that $\lambda_c = \lambda_1$, and λ_1 simple, the nontrivial solution branch emanating from $(\lambda_1, 0)$ gains stability for $\lambda > \lambda_1$ and is unstable for $\lambda < \lambda_1$. This result can be derived using Leray-Schauder degree or by analytic perturbation methods.

Precisely if we take $V = v_0 + \omega^+$ where ω^+ is a stationary solution of (SP) V is called stable if for $\tilde{A}(\lambda) = A + \lambda M(V)$, $\omega = 0$ is stable in sense with respect to strict solutions in $D(A^\beta)$, $3/4 < \beta < 1$. (Strict solution in the sense of Kato-Fujita of the article), and V is called unstable if it is not stable in $\mathcal{J}^T$. If we consider that the nontrivial solution branch $(\lambda(\alpha), V(\alpha))$ in $R \times \mathcal{J}^T$, $|\alpha| \leq 1$ $(\lambda(0), V(0)) = (\lambda_1 v_0)$ can be written in the form

$$\begin{aligned} \alpha(\lambda) &= \pm c_1 |\lambda - \lambda_1|^{1/r} \\ V(\lambda) &= v_0 + |\lambda - \lambda_1|^{1/r} F(\lambda) \end{aligned} \qquad r \in N$$

where $F\colon R \to D(A)$ is analytic in $(\lambda - \lambda_1)^{1/r}$ and $F(\lambda_1) \neq 0$ (valid conditions for Bénard's and Taylor's models (Theorem 4.6 and Corollary 4.4), we have the following Theorem or Lemma 5.6.

Lemma 5.6. Assume $\lambda_c = \lambda_1$ and $\operatorname{Re} \mu \geq \alpha > 0$ for all nonvanishing μ in the spectrum of $\sigma(\tilde{A}(\lambda_1, v_0))$. Let λ_1 be a simple characteristic value of $K(v_0)$, 0 a simple eigenvalue of $\tilde{A}(\lambda_1, v_0)$.

Then, if λ is restricted to a suitable neighborhood of λ_1

i) v_0 is stable for $\lambda < \lambda_1$ and unstable for $\lambda > \lambda_1$

ii) $V(\lambda)$ is stable for $\lambda > \lambda_1$ and unstable for

$\lambda < \lambda_1$.

For the Bénard's model, the assumptions of Lemma 5.6 are satisfied for fixed n, α, β; λ_1 is a simple characteristic value of $K(v_0)$ by Theorem 4.6 and $\lambda_c = \lambda_1$ follows from Lemma 4.5 of the article.

For the Taylor's model only the simplicity of λ_1 as a characteristic value of $K(v_0)$ is known. The simplicity of $\mu = 0$ in $\sigma(\tilde{A}\lambda_1, v_0)$ is an open problem.

However, we may give the following theorem.

Theorem 5.7. i) For the Bénard's problem, every solution with a given cell pattern (fixed n, α, β) exists in some right neighborhood of λ_1 and is asymptotically stable in $D(A^\beta)$, $\beta \in (3/4,1)$. The basic solution v_0 is asymptotically stable for $\lambda < \lambda_1$ and unstable for $\lambda > \lambda_1$.

ii) For the Taylor's problem, let the assumptions of Lemma 5.6 on the spectrum of $\tilde{A}(\lambda; v_0)$ be valid. Then for every period (σ fixed) $V(\lambda)$ is asymptotically stable if it exists for $\lambda > \lambda_1$, and is unstable if it exists for $\lambda < \lambda_1$.

Finally, we remark that these results can also be obtained using the invariant manifold approach. (See, for example, Exercise 4.3). That this is possible was noted already by Ruelle-Takens [1] in their elegant and simple proof of Velte's theorem. We also note that Prodi's basic results relating the spectral and stability properties of the Navier-Stokes equations are contained in the smoothness properties of the flow from Section 9 and the results of Section 2A.