9 The Trigonometric Functions

The theory of the trigonometric functions depends upon the notion of arc length on a circle, in terms of which radian measure is defined. It is possible to develop this theory from scratch, using the integral (just as for the logarithm), but intuition is sacrificed in this approach. At the expense of some rigor, we shall take for granted the properties of arc lengths of circular arcs and the attendant properties of π and the trigonometric functions.

The Derivative of sin θ and cos θ

We recall that if an arc length θ is measured along the unit circle in the x, y plane counterclockwise from $(1,0)$ to a point P, then the coordinates of P are $(\cos\theta, \sin\theta)$. (See Fig. 9-1.) The measure of the angle swept out is θ *radians*. The graphs of $\sin\theta$ and $\cos\theta$ are shown in Fig. 9-2.

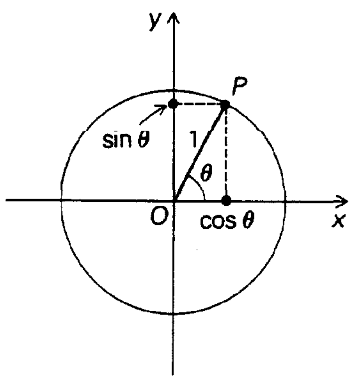

Fig. 9-1 The definition of $\cos\theta$ and $\sin\theta$.

The key to differentiating the trigonometric functions is the following lemma.

Lemma *For* $-\pi/2 < \theta < \pi/2, \theta \neq 0$, we have

$$1 - \frac{\theta^2}{2} < \cos\theta < \frac{\sin\theta}{\theta} < 1$$

Proof Refer to Fig. 9-3. Observe that for $0 < \theta < \pi/2$,

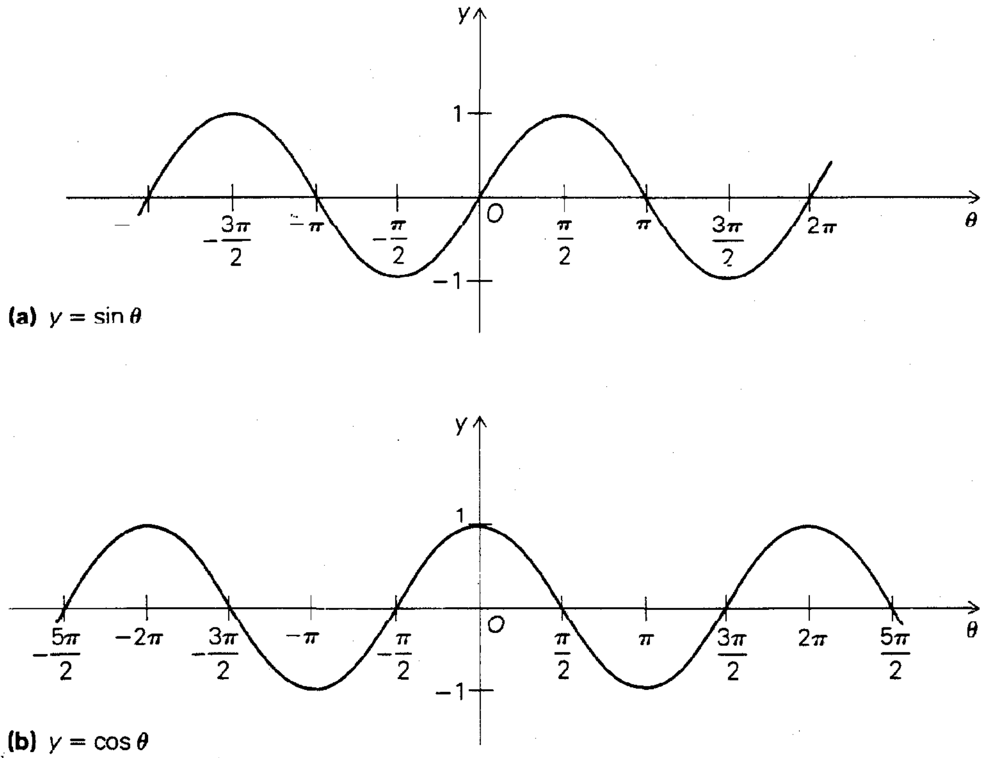

Fig. 9-2 Graphs of sine and cosine.

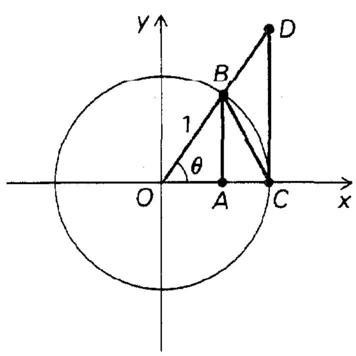

Fig. 9-3 Geometry used to determine $\sin'(0)$.

$$\text{area } \Delta OCB = \tfrac{1}{2}|OC| \cdot |AB| = \tfrac{1}{2}\sin\theta$$

and

$$\text{area } \Delta OCB < \text{area sector } OCB = \tfrac{1}{2}\theta$$

and

$$\text{area sector } OCB < \text{area } \Delta OCD = \tfrac{1}{2}|OC| \cdot |CD| = \tfrac{1}{2}\tan\theta .$$

Thus,

$$\frac{\sin\theta}{\theta} < 1 \quad \text{and} \quad \theta < \frac{\sin\theta}{\cos\theta}$$

and so

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

This inequality also holds for $\theta < 0$, since $\cos\theta$ and $(\sin\theta)/\theta$ are both unchanged if θ is replaced by $-\theta$.

Using the identity $\cos\theta = 1 - 2\sin^2(\theta/2)$, we get $\cos\theta > 1 - (\theta^2/2)$ and hence the lemma.

Theorem 1 *We have*

$$\sin'(0) = 1$$

and

$$\cos'(0) = 0$$

Proof For $m > 1$ we have

$$\frac{\sin\theta}{\theta} < 1 < m$$

and so

$$\sin\theta < m\theta \quad \text{if } \theta > 0$$

and

$$\sin\theta > m\theta \quad \text{if } \theta < 0$$

i.e., for $m > 1$, the line $y = m\theta$ (in the θ, y plane) overtakes $y = \sin\theta$ at $\theta = 0$.

Since $1 - (\theta^2/2)$ is continuous and equals 1 at $\theta = 0$, if $m < 1$, there is an interval I about 0 such that $1 - (\theta^2/2) > m$ if $\theta \in I$. Thus

$$\theta \in I \quad \text{and} \quad \theta > 0 \quad \text{implies} \quad \sin\theta > m\theta$$

and

$$\theta \in I \quad \text{and} \quad \theta < 0 \quad \text{implies} \quad \sin\theta < m\theta$$

i.e., for $m < 1$ the line $y = m\,\theta$ is overtaken by $y = \sin\theta$. Therefore, by definition of the derivative (Chapter 1), $\sin'(0) = 1$.

Rewriting the inequality $1 - (\theta^2/2) < \cos\theta < 1$ as $-\theta^2/2 < \cos\theta - 1 < 0$ and arguing in a similar manner, we get $\cos'(0) = 0$.

The formulas for the derivatives of $\sin\theta$ and $\cos\theta$ at a general point follow from the addition formulas and the chain rule:

$$\begin{aligned}\sin\theta &= \sin[\theta_0 + (\theta - \theta_0)] \\ &= \sin\theta_0 \cos(\theta - \theta_0) + \cos\theta_0 \sin(\theta - \theta_0)\end{aligned}$$

By the chain rule and Theorem 1, the right-hand side is differentiable in θ at $\theta = \theta_0$ with derivative

$$\sin\theta_0 \cos'(0) + \cos\theta_0 \sin'(0) = \cos\theta_0$$

Thus

$$\sin'\theta_0 = \cos\theta_0$$

In a similar way one shows that $\cos'\theta_0 = -\sin\theta_0$. We have proved the following result.

Theorem 2 *$\sin\theta$ and $\cos\theta$ are differentiable functions of θ with*

$$\sin'\theta = \cos\theta, \quad \cos'\theta = -\sin\theta$$

The rules of calculus now enable one to differentiate expressions involving $\sin\theta$ and $\cos\theta$.

Worked Example 1 Differentiate $(\sin 3x)/(1 + \cos^2 x)$.

Solution By the chain rule,

$$\frac{d}{dx}\sin 3x = 3\cos 3x$$

So, by the quotient rule,

$$\begin{aligned}\frac{d}{dx}\frac{\sin 3x}{1+\cos^2 x} &= \frac{(1+\cos^2 x)\cdot 3\cos 3x - \sin 3x \cdot 2\cos x(-\sin x)}{(1+\cos^2 x)^2} \\ &= \frac{3\cos 3x(1+\cos^2 x) + 2\cos x \sin x \cdot \sin 3x}{(1+\cos^2 x)^2}\end{aligned}$$

Now that we know how to differentiate the sine and cosine functions, we can differentiate the remaining trigonometric functions by using the rules of calculus. For example, the quotient rule gives:

$$\begin{aligned}\frac{d}{d\theta}\tan\theta &= \frac{\cos\theta\,(d/d\theta)\sin\theta - \sin\theta\,(d/d\theta)\cos\theta}{\cos^2\theta}\\ &= \frac{\cos\theta\cdot\cos\theta + \sin\theta\cdot\sin\theta}{\cos^2\theta}\\ &= \frac{1}{\cos^2\theta} = \sec^2\theta\end{aligned}$$

In a similar way, we see that

$$\frac{d}{d\theta}\cot\theta = -\csc^2\theta$$

Writing $\csc\theta = 1/\sin\theta$ we get

$$\csc'\theta = (-\sin'\theta)/(\sin^2\theta) = (-\cos\theta)/(\sin^2\theta) = -\cot\theta\csc\theta$$

and similarly

$$\sec'\theta = \tan\theta\sec\theta$$

Worked Example 2 Differentiate $\csc x\tan 2x$.

Solution Using the product rule and the chain rule,

$$\begin{aligned}\frac{d}{dx}\csc x\tan 2x &= \left(\frac{d}{dx}\csc x\right)(\tan 2x) + \csc x\left(\frac{d}{dx}\tan 2x\right)\\ &= -\cot x\cdot\csc x\cdot\tan 2x + \csc x\cdot 2\cdot\sec^2 2x\\ &= 2\csc x\sec^2 2x - \cot x\cdot\csc x\cdot\tan 2x\end{aligned}$$

Solved Exercises*

1. Differentiate:
   (a) $\sin x\cos x$ (b) $(\tan 3x)/(1+\sin^2 x)$ (c) $1-\csc^2 5x$
2. Differentiate $f(\theta) = \sin(\sqrt{3\theta^2+1})$.
3. Discuss maxima, minima, concavity, and points of inflection for $f(x) = \sin^2 x$. Sketch its graph.

*Solutions appear in the Appendix.

Exercises

1. Show that $(d/d\theta)\cos\theta = -\sin\theta$ can be derived from $(d/d\theta)\sin\theta = \cos\theta$ by using $\cos\theta = \sin(\pi/2 - \theta)$ and the chain rule.
2. Differentiate:
   (a) $\sin^2 x$ (b) $\tan(\theta + 1/\theta)$
   (c) $(4t^3 + 1)\sin\sqrt{t}$ (d) $\csc t \cdot \sec^2 3t$
3. Discuss maxima, minima, concavity, and points of inflection for $y = \cos 2x - 1$.

The Inverse Trigonometric Functions

In Chapter 8 we discussed the general concept of the inverse of a function and developed a formula for differentiating the inverse. Recall that this formula is

$$\frac{d}{dy}f^{-1}(y) = \frac{1}{(d/dx)f(x)}$$

where $y = f(x)$.

To apply these ideas to the sine function, we begin by using Theorem 1 (of Chapter 8) to locate an interval on which $\sin x$ has an inverse. Since $\sin' x = \cos x > 0$ on $(-\pi/2, \pi/2)$, $\sin x$ is increasing on this interval, so $\sin x$ has an inverse on the interval $[-\pi/2, \pi/2]$. The inverse is denoted $\sin^{-1} y$.* We obtain the graph of $\sin^{-1} y$ by interchanging the x and y coordinates. (See Fig. 9-4.)

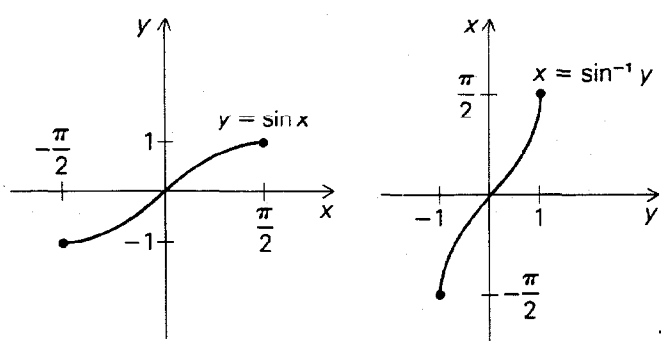

Fig. 9-4 The sine function and its inverse.

*Although the notation $\sin^2 y$ is commonly used to mean $(\sin y)^2$, $\sin^{-1} y$ does not mean $(\sin y)^{-1} = 1/\sin y$. Sometimes the notation arcsin y is used for the inverse sine function to avoid confusion.

The values of $\sin^{-1} y$ may be obtained from a table for $\sin x$. (Many pocket calculators can evaluate the inverse trigonometric functions as well as the trigonometric functions.)

Worked Example 3 Calculate $\sin^{-1} 1$, $\sin^{-1} 0$, $\sin^{-1}(-1)$, $\sin^{-1}(-\frac{1}{2})$, and $\sin^{-1}(0.342)$.

Solution Since $\sin \pi/2 = 1$, $\sin^{-1} 1 = \pi/2$. Similarly, $\sin^{-1} 0 = 0$, $\sin^{-1}(-1) = -\pi/2$. Also, $\sin(-\pi/6) = -\frac{1}{2}$, so $\sin^{-1}(-\frac{1}{2}) = -\pi/6$. Using a calculator, we find $\sin^{-1}(0.342) = 20°$.

We could have used any other interval on which $\sin x$ has an inverse, such as $[\pi/2, 3\pi/2]$, to define an inverse sine function; had we done so, the function obtained would have been different. The choice $[-\pi/2, \pi/2]$ is standard and is the most convenient.

Let us now calculate the derivative of $\sin^{-1} y$. By the formula for the derivative of an inverse,

$$\frac{d}{dy}\sin^{-1} y = \frac{1}{(d/dx)\sin x} = \frac{1}{\cos x}$$

where $y = \sin x$. However, $\cos^2 x + \sin^2 x = 1$, so $\cos x = \sqrt{1 - y^2}$. (The negative root does not occur since $\cos x$ is positive on $(-\pi/2, \pi/2)$.)

Thus

$$\frac{d}{dy}\sin^{-1} y = \frac{1}{\sqrt{1 - y^2}} = (1 - y^2)^{-1/2}, \quad -1 < y < 1$$

Notice that the derivative of $\sin^{-1} y$ is not defined at $y = \pm 1$ but is "infinite" there. This is consistent with the appearance of the graph in Fig. 9-4.

Worked Example 4 Differentiate $h(y) = \sin^{-1}(3y^2)$.

Solution From the chain rule, with $u = 3y^2$,

$$h'(y) = (1 - u^2)^{-1/2}\frac{du}{dy} = 6y(1 - 9y^4)^{-1/2}$$

Worked Example 5 Differentiate $f(x) = x \sin^{-1}(2x)$.

Solution Here we are using x for the variable name. Of course we can use any letter we please. By the product and chain rules,

$$f'(x) = \left(\frac{dx}{dx}\right)\sin^{-1}(2x) + x\frac{d}{dx}(\sin^{-1} 2x)$$

$$= \sin^{-1} 2x + 2x(1 - 4x^2)^{-1/2}$$

It is interesting to observe that, while $\sin^{-1} y$ is defined in terms of trigonometric functions, its derivative is an algebraic function, even though the derivatives of the trigonometric functions themselves are still trigonometric.

Solved Exercises

4. (a) Calculate $\sin^{-1}(\frac{1}{2})$, $\sin^{-1}(-\sqrt{3}/2)$, and $\sin^{-1}(2)$.
   (b) Simplify $\tan(\sin^{-1} x)$.
5. Calculate $(d/dx)(\sin^{-1} 2x)^{3/2}$.
6. Differentiate $\sin^{-1}(\sqrt{1-x^2})$, $0 < x < 1$. Discuss.

Exercises

4. What are $\sin^{-1}(0.3)$, $\sin^{-1}(2/\sqrt{3})$, $\sin^{-1}(\frac{3}{2})$, $\sin^{-1}(-\pi)$, and $\sin^{-1}(1/\sqrt{3})$?
5. Differentiate the indicated functions:
   (a) $(x^2 - 1)\sin^{-1}(x^2)$ (b) $(\sin^{-1} x)^2$
   (c) $\sin^{-1}[t/(t+1)]$ (domain = ?)
6. What are the maxima, minima, and inflection points of $f(x) = \sin^{-1} x$?
7. Show that $\cos x$ on $(0, \pi)$ has an inverse $\cos^{-1} x$ and

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

8. Show that $\tan^{-1} x$ is defined for all x, takes values between $-\pi/2$ and $\pi/2$, and

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

Problems for Chapter 9

1. Differentiate each of the following functions:
   (a) $f(x) = \sqrt{x} + \cos 3x$ (b) $f(x) = \sqrt{\cos x}$
   (c) $f(x) = (\sin^{-1} 3x)/(x^2 + 2)$ (d) $f(x) = (x^2 \cos^{-1} x + \tan x)^{3/2}$
   (e) $f(\theta) = \cot^{-1}(\sin\theta + \sqrt{\cos^2 3\theta + \theta^2})$
   (f) $f(r) = (r^2 + \sqrt{1 - r^2})/(r \sin r)$
2. Differentiate $\sec[\sin^{-1}(y - 2)]$ by (a) simplifying first and (b) using the chain rule right away.

3. (a) What is the domain of $\cos^{-1}(x^2 - 3)$? Differentiate.
   (b) Sketch the graph of $\cos^{-1}(x^2 - 3)$.
4. Show that $f(x) = \sec x$ satisfies the equation $f'' + f - 2f^3 = 0$.
5. Is the following correct: $(d/dx)\cos^{-1} x = (-1)(\cos^{-2} x)[(d/dx)\cos x]$?
6. Find a function $f(x)$ which is differentiable and increasing for all x, yet $f(x) < \pi/2$ for all x.
7. Find the inflection points of $f(x) = \cos^2 3x$.
8. Where is $f(x) = x \sin x + 2\cos x$ concave upward? Concave downward?
9. Prove that $y = \tan^{-1} x$ has an inflection point at $x = 0$.
10. A child is whirling a stone on a string 0.5 meter long in a vertical circle at 5 revolutions per second. The sun is shining directly overhead. What is the velocity of the stone's shadow when the stone is at the 10 o'clock position?
11. Prove that $f(x) = x - 1 - \cos x$ is increasing on $[0, \infty)$. What inequality can you deduce?