CALCULUS UNLIMITED

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The Benjamin/Cummings Publishing Company, Inc.
Menlo Park, California • Reading, Massachusetts
London • Amsterdam • Don Mills, Ontario • Sydney

Sponsoring Editor: Susan A. Newman
Production Editor: Madeleine S. Dreyfack

To Alison, Christopher, Asha, and the future of calculus.

Library of Congress Cataloging in Publication Data

Marsden, Jerrold E.
Calculus unlimited.

Includes index.
1. Calculus. I. Weinstein, Alan, 1943- joint author. II. Title.
QA303.M3374 515 80-24446
ISBN 0-8053-6932-5

ABCDEFGHIJ-AL-83210

The Benjamin/Cummings Publishing Company, Inc.
2727 Sand Hill Road
Menlo Park, California 94025

Preface

Purpose

This book is intended to supplement our text, *Calculus* (Benjamin/Cummings, 1980), or virtually any other calculus text (see page vii, How To Use This Book With Your Calculus Text). As the title *Calculus Unlimited* implies, this text presents an alternative treatment of calculus using the method of exhaustion for the derivative and integral in place of limits. With the aid of this method, a definition of the derivative may be introduced in the first lecture of a calculus course for students who are familiar with functions and graphs.

Approach

Assuming an intuitive understanding of real numbers, we begin in Chapter 1 with the definition of the derivative. The axioms for real numbers are presented only when needed, in the discussion of continuity. Apart from this, the development is rigorous and contains complete proofs.

As you will note, this text has a more geometric flavor than the usual analytic treatment of calculus. For example, our definition of completeness is in terms of convexity rather than least upper bounds, and Dedekind cuts are replaced by the notion of a transition point.

Who Should Use This Book

This book is for calculus instructors and students interested in trying an alternative to limits. The prerequisites are a knowledge of functions, graphs, high school algebra and trigonometry.

How To Use This Book

Because the "learning-by-doing" technique introduced in *Calculus* has proved to be successful, we have adapted the same format for this book. The solutions to "Solved Exercises" are provided at the back of the book; however readers are encouraged to try solving each example before looking up the solution.

The Origin Of The Definition Of The Derivative

Several years ago while reading *Geometry and the Imagination*, by Hilbert and Cohn-Vossen (Chelsea, 1952, p. 176), we noticed a definition of the circle of curvature for a plane curve C. No calculus, as such, was used in this definition. This suggested that the same concept could be used to define the tangent line and thus serve as a limit-free foundation for the differential calculus. We introduced this new definition of the derivative into our class notes and developed it in our calculus classes for several years. As far as we know, the definition has not appeared elsewhere. If our presumption of originality is ill-founded, we welcome your comments.

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How To Use This Book With Your Calculus Text

There are two ways to use this book:

1. It can be used to take a second look at calculus from a fresh point of view after completion of a standard course.

2. It can be used simultaneously with your standard calculus text as a supplement. Since this book is theory oriented, it is meant for better students, although Chapter 1 is designed to be accessible to all students.

The table below shows the chapters of this book that can be used to supplement sections in some of the standard calculus texts.

Chapters in this Book	Corresponding Chapters in Standard Texts*									
	Ellis/ Gulick	Grossman	Leithold	Marsden/ Weinstein	Protter/ Morrey	Purcell	Salas/ Hille	Shenk	Swokowski	Thomas/ Finney
1	3.2	2.10	3.3	R-3	4.3	4.3	3.1	3.1	3.2	1.8
2	3.2	2.10	3.3	1.1	4.3	4.3	3.1	3.2	3.2	1.9
3	3.1	3.2	3.5	1.3	5.1	5.2	3.2	3.4	3.3	2.3
4	1.1	1.2	1.1	R-1	A-4	1.3	1.5	1.2	1.1	1.2
5	2.5	2.4/10.4	2.5/5.1	2.3	6.5	6.3	2.6/App. B	2.7/4.2	2.5/4.4	2.11/3.1
6	4.6	3.9	5.2-5.5	2.4	6.4	6.12	4.7	4.3	4.5	3.4
7	4.2	2.4/10.6	4.7	3.3	6.2	6.2	4.1	4.12	4.3	3.8
8	8.2	3.7/10.7	9.3/3.6	5.2	9.4/5.2	5.11/10.2	3.7	4.6	8.7	2.4
9	3.3	7.5	10.2	6.2	9.2	10.7	7.1	3.8	9.2	2.10
10	8.4	6.3	9.2-9.5	7.2	9.9	10.3	6.4	7.5	8.4	6.8
11	5.3	4.3	7.3	4.1	7.3	8.3	5.2	5.1	5.2	4.8
12	5.4	4.5	7.6	4.2	7.5	8.4	5.3	5.2	5.5	4.8
13	6.5	10.8	3.4	12.2	4.6	8.4	App. B	App. Ch. 5	App. II	Ch. 4/Summary

*See following list of references.

- R. Ellis and D. Gulick, *Calculus with Analytic Geometry* Harcourt, Brace, Jovanovich (1978)
- S. Grossman, *Calculus* Academic Press (1977)
- L. Leithold, *Calculus*, 3rd Ed., Harper and Row (1976)
- J. Marsden and A. Weinstein, *Calculus*, Benjamin/Cummings (1980)
- M. Protter and C. B. Morrey, *Calculus*, 4th Ed., Addison-Wesley (1980)
- E. J. Purcell *Calculus with Analytic Geometry*, 3rd Ed., Prentice-Hall (1978)
- S. L. Salas and E. Hille, *Calculus. One and Several Variables*, 3rd Ed., Wiley (1978)

- A. Shenk, *Calculus*, Goodyear (1977)
- E. W. Swokowski, *Calculus with Analytic Geometry*, Second Edition, Prindle Weber and Schmidt, (1980)
- G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, 5th Ed., Addison-Wesley (1979)

Preview For The Instructor

This preview is intended for those who already know calculus. Others should proceed directly to Chapter 1.

The method of exhaustion of Eudoxus and Archimedes may be summarized as follows: Having defined and computed areas of polygons, one determines the area of a curvilinear figure F using the principle that whenever P_1 and P_2 are polygons such that P_1 is inside F and F is inside P_2, then Area $(P_1) \leqslant$ Area $(F) \leqslant$ Area (P_2). This approach appears in modern mathematics in the form of Dedekind cuts, inner and outer measure, and lower and upper sums for integrals.

To apply the method of exhaustion to differentiation, we replace the relation of inclusion between figures by the relation of *overtaking* defined as follows.

Definition Let f and g be real-valued functions with domains contained in $\mathbb{R}$, and x_0 a real number. We say that *f overtakes g at x_0* if there is an open interval I containing x_0 such that

0. $x \in I$ and $x \neq x_0$ implies x is in the domain of f and g.
1. $x \in I$ and $x < x_0$ implies $f(x) < g(x)$.
2. $x \in I$ and $x > x_0$ implies $f(x) > g(x)$.

Given a function f and a number x_0 in its domain, we may compare f with the linear functions $l_m(x) = f(x_0) + m(x - x_0)$.

Definition Let f be a function defined in an open interval containing x_0. Suppose that there is a number m_0 such that:

1. $m < m_0$ implies f overtakes l_m at x_0.
2. $m > m_0$ implies l_m overtakes f at x_0.

Then we say that *f is differentiable at x_0*, and that m_0 is *the derivative* of f at x_0.

The following notion of *transition* occurs implicitly in both of the preceding definitions.

Definition Let A and B be sets of real numbers, and x_0 a real number. We say that x_0 is a *transition point from A to B* if there is an open interval I containing x_0 such that:

1. $x \in I$ and $x < x_0$ implies $x \in A$ and $x \notin B$.
2. $x \in I$ and $x > x_0$ implies $x \in B$ and $x \notin A$.

The preceding definition of the derivative is equivalent to the usual limit approach, as we shall prove in Chapter 13. However, it is conceptually quite different, and for students who wish a logically complete definition of the derivative, we believe that it is simpler and geometrically appealing.

Both Euclid and Archimedes probably employed the following definition of tangent: "the tangent line touches the curve, and in the space between the line and curve, no other straight line can be interposed".* This is in fact, a somewhat loose way of phrasing the definition of the derivative we have given here. Why, then, did Fermat, Newton, and Leibniz change the emphasis from the method of exhaustion to the method of limits? The reason must lie in the computational power of limits, which enabled Newton and Leibniz to establish the rules of calculus, in spite of the fact that limits were not clearly understood for at least another century. However, there is nothing to prevent one from carrying out the same program using the method of exhaustion. We shall do so in this book.

In teaching by this approach, one may begin by defining transition points (with examples like birth, freezing, and sunset) and then go on to define overtaking and the derivative. (One must emphasize the fact that, in the definition of a transition point, nothing is said about the number x_0 itself.) The notion of transition point occurs again in graphing, when we consider turning points and inflection points, so the computational techniques needed to determine overtaking are put to good use later.

Since the definition of the derivative is so close to that of the integral (a transition point between lower and upper sums), the treatment of the fundamental theorem of calculus becomes very simple.

For those courses in which the completeness of the real numbers is emphasized, the following version of the completeness axiom is especially well suited to the transitions approach.

*C. Boyer, *The History of the Calculus and Its Conceptual Development*, Dover (1959), p. 57.

Definition A set A of real numbers is convex if, whenever x and y are in A and $x < z < y$, then z is in A.

Completeness Axiom Every convex set of real numbers is an interval.

With this axiom, the proofs of "hard" properties of continuous functions, such as their boundedness on closed intervals and their integrability, are within the reach of most first-year students.

Although the transition-point approach has some computational disadvantages, it does enable one to present a logically complete definition, with geometric and physical motivation at the end of only one hour of lecture. (Chapter 1 is such a lecture.) Coupled with an early intuitive approach to limits for their computational power, this method allows one to delay rigorous limits until later in the course when students are ready for them, and when they are really needed for topics like L'Hôpital's rule, improper integrals, and infinite senes.

Limits are so important in mathematics that they cannot be ignored in any calculus course. It is tempting to introduce them early because they are simple to use in calculations, but the subtlety of the limit concept often causes beginning students to feel uneasy about the foundations of calculus. Transitions, in contrast, provide conceptually simple definitions of the derivative and integral, but they are quite complicated to use in calculations. Fortunately, one does not need to do many calculations directly from the definition. The great "machine" of Newton and Leibniz enables us to calculate derivatives by a procedure which is independent of the particular form of the definition being used.

Although our reasons for using the method of transitions stem mostly from trying to make calculus easier to learn, we have another reason as well. Many mathematicians have complained that calculus gives a distorted picture of modern mathematics, with total emphasis on "analysis." We hope that the use of transitions will partly answer this complaint. It gives a better balance between the various disciplines of mathematics and gives the student a more accurate picture of what modern mathematics is all about.

We have mentioned that the concept of transition is important in its own right, and we gave some nonmathematical examples involving sudden changes. Although the notion of transition is built into differential and integral calculus, the classical techniques of calculus (limits, the rules of calculus, and so on) have proven insufficient as a tool for studying many discontinuous phenomena. Such phenomena are of common occurrence in biology and the social sciences and include, for example, revolutions, birth, and death. They may all be described as transitions.

Contemporary mathematicians have been paying more and more attention to discontinuous phenomena and a geometric or qualitative description of nature. In biology and sociology this aspect is an important complement to a quantitative analysis. Our emphasis on transitions is inspired partly by the belief that this concept will play an increasingly important role in the applications of mathematics.*

*Of particular interest in this direction is the catastrophe theory introduced by the French mathematician René Thom in his book *Structural Stability and Morphogenesis* (Benjamin, 1975). Thom's book is somewhat philosophical in nature, but more concrete applications of catastrophe theory can be found in the books: E. C. Zeeman, *Catastrophe Theory: Selected Papers 1972-1977* (Addison-Wesley, 1977); T. Poston and I. Stewart, *Catastrophe Theory and its Applications* (Fearon-Pitman, 1978). The applications to biological and social sciences have received some rather sharp criticism; the main critical paper, by H. J. Sussmann and R. S. Zahler, is "Catastrophe theory as applied to the social and biological sciences: a critique," *Synthese* 37 (1978), 117-216. A general account of the controversy is given in A. Woodcock and M. Davis, *Catastrophe Theory* (Dutton, 1978).

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