1 The Derivative

This chapter gives a complete definition of the derivative assuming a knowledge of high-school algebra, including inequalities, functions, and graphs. The next chapter will reformulate the definition in different language, and in Chapter 13 we will prove that it is equivalent to the usual definition in terms of limits.

The definition uses the concept of change of sign, so we begin with this.

Change of Sign

A function is said to *change sign* when its graph crosses from one side of the x axis to the other. We can define this concept precisely as follows.

Definition Let f be a function and x_0 a real number. We say that f *changes sign from negative to positive at* x_0 if there is an open interval (a, b) containing x_0 such that f is defined on (a, b) (except possibly at x_0) and

$$f(x) < 0 \quad \text{if } a < x < x_0$$

and

$$f(x) > 0 \quad \text{if } x_0 < x < b$$

Similarly, we say that f *changes sign from positive to negative at* x_0 if there is an open interval (a, b) containing x_0 such that f is defined on (a, b) (except possibly at x_0) and

$$f(x) > 0 \quad \text{if } a < x < x_0$$

and

$$f(x) < 0 \quad \text{if } x_0 < x < b$$

Notice that the interval (a, b) may have to be chosen small, since a function which changes sign from negative to positive may later change back from positive to negative (see Fig. 1-1).

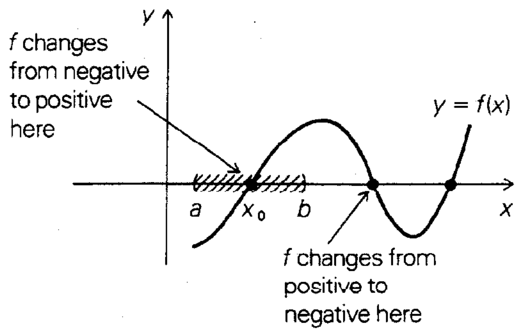

Fig. 1-1 Change of sign.

Worked Example 1 Where does $f(x) = x^2 - 5x + 6$ change sign?

Solution We factor f and write $f(x) = (x - 3)(x - 2)$. The function f changes sign whenever one of its factors does. This occurs at $x = 2$ and $x = 3$. The factors have opposite signs for x between 2 and 3, and the same sign otherwise, so f changes from positive to negative at $x = 2$ and from negative to positive at $x = 3$. (See Fig. 1-2).

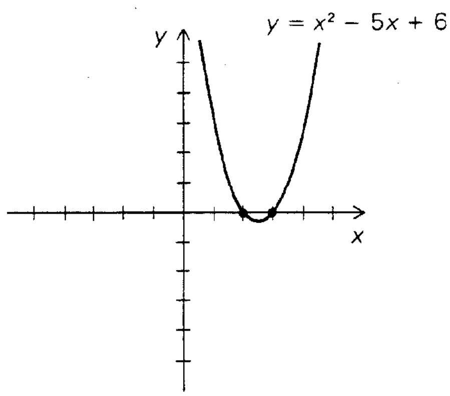

Fig. 1-2 $y = x^2 - 5x + 6$ **changes sign at** $x = 2$ **and** $x = 3$.

We can compare two functions, f and g, by looking at the sign changes of the difference $f(x) - g(x)$. The following example illustrates the idea.

Worked Example 2 Let $f(x) = \frac{1}{2}x^3 - 1$ and $g(x) = x^2 - 1$.

(a) Find the sign changes of $f(x) - g(x)$.

(b) Sketch the graphs of f and g on the same set of axes.

(c) Discuss the relation between the results of parts (a) and (b).

Solution

(a) $f(x) - g(x) = \frac{1}{2}x^3 - 1 - (x^2 - 1) = \frac{1}{2}x^3 - x^2 = \frac{1}{2}x^2(x - 2)$. Since the factor x appears twice, there is no change of sign at $x = 0$ (x^2 is positive both for $x < 0$ and for $x > 0$). There is a change of sign from negative to positive at $x = 2$.

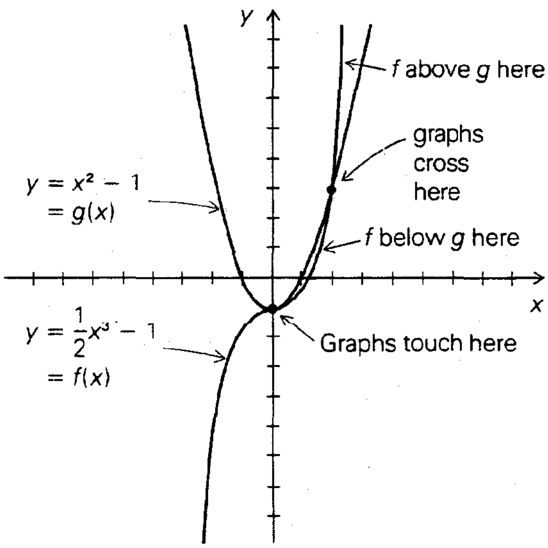

Fig. 1-3 $f - g$ changes sign when the graphs of f and g cross.

(b) See Fig. 1-3.

(c) Since $f(x) - g(x)$ changes sign from negative to positive at $x = 2$, we can say:

$$\text{If } x \text{ is near 2 and } x < 2, \text{ then } f(x) - g(x) < 0; \text{ that is, } f(x) < g(x).$$

and

$$\text{If } x \text{ is near 2 and } x > 2, \text{ then } f(x) - g(x) > 0; \text{ that is, } f(x) > g(x).$$

Thus the graph of f must cross the graph of g at $x = 2$, passing from below to above it as x passes 2.

Solved Exercises*

1. If $f(x)$ is a polynomial and $f(x_0) = 0$, must f necessarily change sign at x_0?
2. For which positive integers n does $f(x) = x^n$ change sign at zero?
3. If $r_1 \neq r_2$, describe the sign change at r_1 of $f(x) = (x - r_1)(x - r_2)$.

Exercises

1. Find the sign changes of each of the following functions:

   (a) $f(x) = 2x - 1$ (b) $f(x) = x^2 - 1$

   (c) $f(x) = x^2$ (d) $h(z) = z(z - 1)(z - 2)$

*Solutions appear in the Appendix.

2. Describe the change of sign at $x = 0$ of the function $f(x) = mx$ for $m = -2$, $0, 2$.

3. Describe the change of sign at $x = 0$ of the function $f(x) = mx - x^2$ for $m = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$.

4. Let $f(t)$ denote the angle of the sun above the horizon at time t. When does $f(t)$ change sign?

Estimating Velocities

If the position of an object moving along a line changes linearly with time, the object is said to be in *uniform motion*, and the rate of change of position with respect to time is called the *velocity*. The velocity of a uniformly moving object is a fixed number, independent of time. Most of the motion we observe in nature is not uniform, but we still feel that there is a quantity which expresses the rate of movement at any instant of time. This quantity, which we may call the *instantaneous velocity*, will depend on the time.

To illustrate how instantaneous velocity might be estimated, let us suppose that we are looking down upon a car C which is moving along the middle lane of a three-lane, one-way road. Without assuming that the motion of the car is uniform, we wish to estimate the velocity v_0 of the car at exactly 3 o'clock.

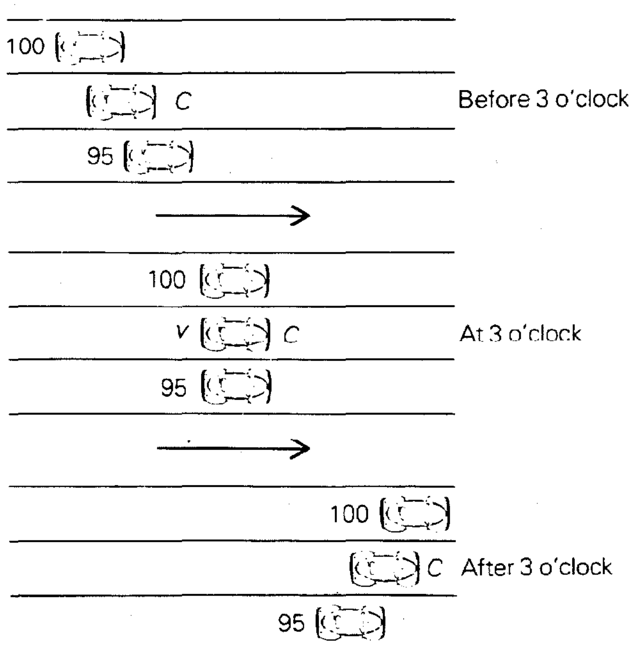

Fig. 1-4 The velocity of car C is between 95 and 100 kilometers per hour.

Suppose that we have the following information (see Fig. 1-4): A car which was moving uniformly with velocity 95 kilometers per hour was passed by car C at 3 o'clock, and a car which was moving uniformly with velocity 100 kilometers per hour passed car C at 3 o'clock.

We conclude that v_0 was at least 95 kilometers per hour and at most 100 kilometers per hour. This estimate of the velocity could be improved if we were to compare car x with more "test cars."

In general, let the variable y represent distance along a road (measured in kilometers from some reference point) and let x represent time (in hours from some reference moment). Suppose that the position of two cars traveling in the positive direction is represented by functions $f_1(x)$ and $f_2(x)$. Then car 1 passes car 2 at time x_0 if the function $f_1(x) - f_2(x)$, which represents the "lead" of car 1 over car 2, changes sign from negative to positive at x_0. (See Fig. 1-5.) When this happens, we expect car 1 to have a higher velocity than car 2 at time x_0.

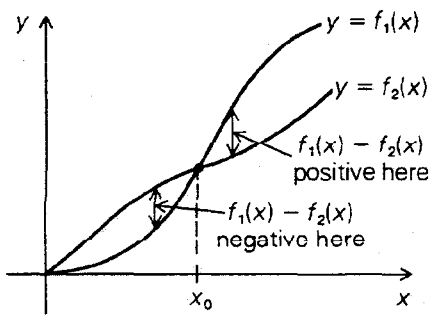

Fig. 1-5 $f_1 - f_2$ changes sign from negative to positive at x_0.

Since the graph representing uniform motion with velocity v is a straight line with slope v, we could estimate the velocity of a car whose motion is nonuniform by seeing how the graph of the function giving its position crosses straight lines with various slopes.

Worked Example 3 Suppose that a moving object is at position $y = f(x) = \frac{1}{2}x^2$ at time x. Show that its velocity at $x_0 = 1$ is at least $\frac{1}{2}$.

Solution We use a "test object" whose velocity is $v = \frac{1}{2}$ and whose position at time x is $\frac{1}{2}x$. When $x = x_0 = 1$, both objects are at $y = \frac{1}{2}$. When $0 < x < 1$, we have $x^2 < x$, so $\frac{1}{2}x^2 < \frac{1}{2}x$; when $x > 1$, we have $\frac{1}{2}x^2 > \frac{1}{2}x$. It follows that the difference $\frac{1}{2}x^2 - \frac{1}{2}x$ changes sign from negative to positive at 1, so the velocity of our moving object is at least $\frac{1}{2}$ (see Fig. 1-6).

Solved Exercise

4. Show that the velocity at $x_0 = 1$ of the object in Worked Example 3 is at most 2.

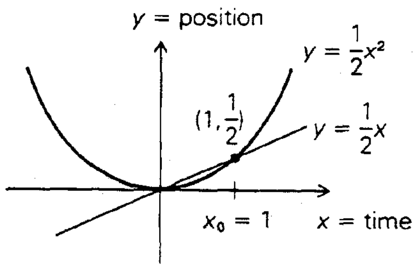

Fig. 1-6 The graph of $y = \frac{1}{2}x$ is above that of $y = \frac{1}{2}x^2$ when $0 < x < 1$ and is below when $x > 1$.

Exercise

5. How does the velocity at $x_0 = 1$ of the object in Worked Example 3 compare with $\frac{3}{4}$? With $\frac{3}{2}$?

Definition of the Derivative

While keeping the idea of motion and velocity in mind, we will continue our discussion simply in terms of functions and their graphs. Recall that the line through (x_0, y_0) with slope m has the equation $y - y_0 = m(x - x_0)$. Solving for y in terms of x, we find that this line is the graph of the linear function $l(x) = y_0 + m(x - x_0)$. We can estimate the "slope" of a given function $f(x)$ at x_0 by comparing $f(x)$ and $l(x)$, i.e. by looking at the sign changes at x_0 of $f(x) - l(x) = f(x) - [f(x_0) + m(x - x_0)]$ for various values of m. Here is a precise formulation.

Definition Let f be a function whose domain contains an open interval about x_0. We say that the number m_0 is the *derivative of f at x_0*, provided that:

1. For every $m < m_0$, the function

$$f(x) - [f(x_0) + m(x - x_0)]$$

changes sign from negative to positive at x_0.

2. For every $m > m_0$, the function

$$f(x) - [f(x_0) + m(x - x_0)]$$

changes sign from positive to negative at x_0.

If such a number m_0 exists, we say that f is *differentiable at* x_0, and we write $m_0 = f'(x_0)$. If f is differentiable at each point of its domain, we just say that f is *differentiable*. The process of finding the derivative of a function is called *differentiation*.

Geometrically, the definition says that lines through $(x_0, f(x_0))$ with slope less than $f'(x_0)$ cross the graph of f from above to below, while lines with slope greater than $f'(x_0)$ cross from below to above. (See Fig. 1-7.)

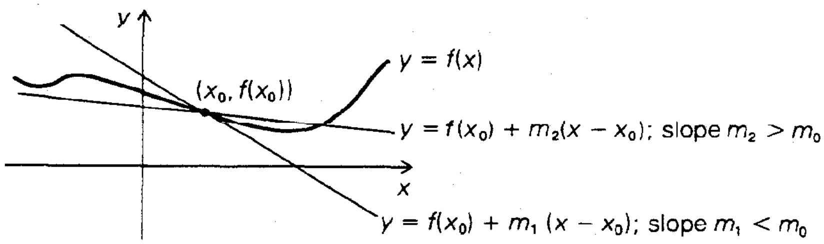

Fig. 1-7 Lines with slope different from m_0 cross the curve.

Given f and x_0, *the number* $f'(x_0)$ *is uniquely determined if it exists.* That is, at most one number satisfies the definition. Suppose that m_0 and $\bar{m}_0$ both satisfied the definition, and $m_0 \neq \bar{m}_0$; say $m_0 < \bar{m}_0$. Let $m = (m_0 + \bar{m}_0)/2$, so $m_0 < m < \bar{m}_0$. By condition 1 for $\bar{m}_0$, $f(x) - [f(x_0) + m(x - x_0)]$ changes sign from negative to positive at x_0, and by condition 2 for m_0, it changes sign from positive to negative at x_0. But it can't do both! Therefore $m_0 = \bar{m}_0$.

The line through $(x_0, f(x_0))$, whose slope is exactly $f'(x_0)$ is pinched, together with the graph of f, between the "downcrossing" lines with slope less than $f'(x_0)$ and the "upcrossing" lines with slope greater than $f'(x_0)$. It is the line with slope $f'(x_0)$, then, which must be tangent to the graph of f at (x_0, y_0). We may take this as our *definition* of tangency. (See Fig. 1-8.)

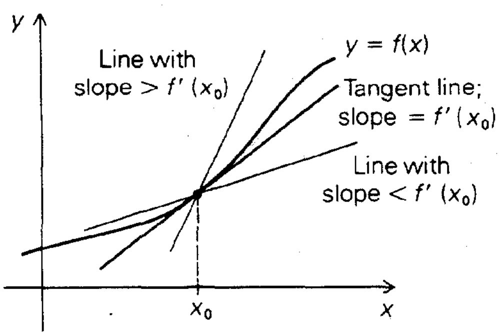

Fig. 1-8 The slope of the tangent line is the derivative.

Definition Suppose that the function f is differentiable at x_0. The line $y = f(x_0) + f'(x_0)(x - x_0)$ through $(x_0, f(x_0))$ with slope $f'(x_0)$ is called the *tangent line to the graph of f at* $(x_0, f(x_0))$.

Following this definition, we sometimes refer to $f'(x_0)$ as the *slope of the curve* $y = f(x)$ at the point $(x_0, f(x_0))$. Note that the definitions do not say anything about how (or even whether) the tangent line itself crosses the graph of a function. (See Problem 7 at the end of this chapter.)

Recalling the discussion in which we estimated the velocity of a car by seeing which cars it passed, we can now give a mathematical definition of velocity.

Definition Let $y = f(x)$ represent the position at time x of a moving object. If f is differentiable at x_0, the number $f'(x_0)$ is called the (instantaneous) *velocity* of the object at the time x_0.

More generally, we call $f'(x_0)$ the *rate of change* of y with respect to x at x_0.

Worked Example 4 Find the derivative of $f(x) = x^2$ at $x_0 = 3$. What is the equation of the tangent line to the parabola $y = x^2$ at the point $(3, 9)$?

Solution According to the definition of the derivative—with $f(x) = x^2$, $x_0 = 3$, and $f(x_0) = 9$—we must investigate the sign change at 3, for various values of m, of

$$\begin{aligned} f(x) - [f(x_0) + m(x - x_0)] &= x^2 - [9 + m(x-3)] \\ &= x^2 - 9 - m(x-3) \\ &= (x+3)(x-3) - m(x-3) \\ &= (x-3)(x+3-m) \end{aligned}$$

According to Solved Exercise 3, with $r_1 = 3$ and $r_2 = m - 3$, the sign change is:

1. From negative to positive if $m - 3 < 3$; that is, $m < 6$.
2. From positive to negative if $3 < m - 3$; that is, $m > 6$.

We see that the number $m_0 = 6$ fits the conditions in the definition of the derivative, so $f'(3) = 6$. The equation of the tangent line at $(3, 9)$ is therefore $y = 9 + 6(x - 3)$; that is, $y = 6x - 9$. (See Fig. 1-9.)

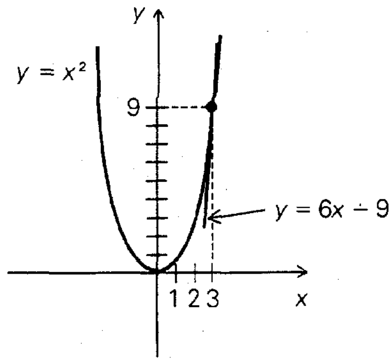

Fig. 1-9 The equation of the line tangent to $y = x^2$ at $x_0 = 3$ is $y = 6x - 9$.

Solved Exercises

5. Let $f(x) = x^3$. What is $f'(0)$? What is the tangent line at $(0, 0)$?

6. Let f be a function for which we know that $f(3) = 2$ and $f'(3) = \sqrt[5]{8}$. Find the y intercept of the line which is tangent to the graph of f at $(3, 2)$.

7. Let $f(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$ (the absolute value function). Show by a geometric argument that f is not differentiable at zero.

8. The position of a moving object at time x is x^2. What is the velocity of the object when $x = 3$?

Exercises

6. Find the derivative of $f(x) = x^2$ at $x = 4$. What is the equation of the tangent line to the parabola $y = x^2$ at $(4, 16)$?

7. If $f(x) = x^4$, what is $f'(0)$?

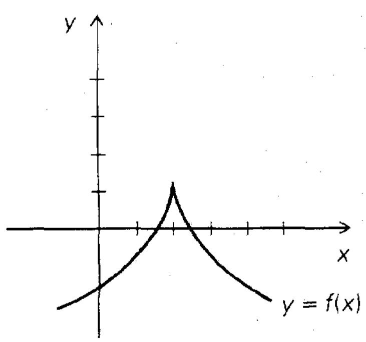

Fig. 1-10 *Where is f differentiable?* (See Exercise 9.)

8. The position at time x of a moving object is x^3. What is the velocity at $x = 0$?

9. For which value of x_0 does the function in Fig. 1-10 fail to be differentiable?

The Derivative as a Function

The preceding examples show how derivatives may be calculated directly from the definition. Usually, we will not use this cumbersome method; instead, we will use *differentiation rules*. These rules, once derived, enable us to differentiate many functions quite simply. In this section, we will content ourselves with deriving the rules for differentiating linear and quadratic functions. General rules will be introduced in Chapter 3.

The following theorem will enable us to find the tangent line to any parabola at any point.

Theorem 1 Quadratic Function Rule. Let $f(x) = ax^2 + bx + c$, where a, b, and c are constants, and let x_0 be any real number. Then f is differentiable at x_0, and $f'(x_0) = 2ax_0 + b$.

Proof We must investigate the sign changes at x_0 of the function

$$\begin{aligned} f(x) - [f(x_0) + m(x - x_0)] & \\ &= ax^2 + bx + c - [ax_0^2 + bx_0 + c + m(x - x_0)] \\ &= a(x^2 - x_0^2) + b(x - x_0) - m(x - x_0) \\ &= (x - x_0)[a(x + x_0) + b - m] \end{aligned}$$

The factor $[a(x + x_0) + b - m]$ is a (possibly constant) linear function whose value at $x = x_0$ is $a(x_0 + x_0) + b - m = 2ax_0 + b - m$. If $m < 2ax_0 + b$, this factor is positive at $x = x_0$, and being a linear function it is also positive when x is near x_0. Thus the product of $[a(x + x_0) + b - m]$ with $(x - x_0)$ changes sign from negative to positive at x_0. If $m > 2ax_0 + b$, then the factor $[a(x + x_0) + b - m]$ is negative when x is near x_0, so its product with $(x - x_0)$ changes sign from positive to negative at x_0.

Thus the number $m_0 = 2ax_0 + b$ satisfies the definition of the derivative, and so $f'(x_0) = 2ax_0 + b$.

Worked Example 5 Find the derivative at -2 of $f(x) = 3x^2 + 2x - 1$.

Solution Applying the quadratic function rule with $a = 3$, $b = 2$, $c = -1$, and $x_0 = -2$, we find $f'(-2) = 2(3)(-2) + 2 = -10$.

We can use the quadratic function rule to obtain quickly a fact which may be known to you from analytic geometry.

Worked Example 6 Suppose that $a \neq 0$. At which point does the parabola $y = ax^2 + bx + c$ have a horizontal tangent line?

Solution The slope of the tangent line through the point $(x_0, ax_0^2 + bx_0 + c)$ is $2ax_0 + b$. This line is horizontal when its slope is zero; that is, when $2ax_0 + b = 0$, or $x_0 = -b/2a$. The y value here is $a(-b/2a)^2 + b(-b/2a) + c = b^2/4a - b^2/2a + c = -(b^2/4a) + c$. The point $(-b/2a, -b^2/4a + c)$ is called the *vertex* of the parabola $y = ax^2 + bx + c$.

In Theorem 1 we did not require that $a \neq 0$. When $a = 0$, the function $f(x) = ax^2 + bx + c$ is linear, so we have the following corollary:

Corollary Linear Function Rule. *If $f(x) = bx + c$, and x_0 is any real number, then $f'(x_0) = b$.*

In particular, if $f(x) = c$, a constant function, then $f'(x_0) = 0$ for all x_0.

For instance, if $f(x) = 3x + 4$, then $f'(x_0) = 3$ for any x_0; if $g(x) = 4$, then $g'(x_0) = 0$ for any x_0.

This corollary tells us that the rate of change of a linear function is just the slope of its graph. Note that it does not depend on x_0: the rate of change of a linear function is *constant*. For a general quadratic function, though, the derivative $f'(x_0)$ does depend upon the point x_0 at which the derivative is taken. In fact, we can consider f' as a *new function*; writing the letter x instead of x_0, we have $f'(x) = 2ax + b$.

Definition Let f be any function. We define the function f', with domain equal to the set of points at which f is differentiable, by setting $f'(x)$ equal to the derivative of f at x. The function $f'(x)$ is simply called the *derivative* of $f(x)$.

Worked Example 7 What is the derivative of $f(x) = 3x^2 - 2x + 1$?

Solution By the quadratic function rule, $f'(x_0) = 2 \cdot 3x_0 - 2 = 6x_0 - 2$.

Writing x instead of x_0, we find that the derivative of $f(x) = 3x^2 - 2x + 1$ is $f'(x) = 6x - 2$.

> **Remark** When we are dealing with functions given by specific formulas, we often omit the function names. For example, we could state the result of Worked Example 7 as "the derivative of $3x^2 - 2x + 1$ is $6x - 2$."

Since the derivative of a function f is another function f', we can go on to differentiate f' again. The result is yet another function, called the *second derivative* of f and denoted by f''.

Worked Example 8 Find the second derivative of $f(x) = 3x^2 - 2x + 1$.

Solution We must differentiate $f'(x) = 6x - 2$. This is a linear function; applying the formula for the derivative of a linear function, we find $f''(x) = 6$. The second derivative of $3x^2 - 2x + 1$ is thus the constant function whose value for every x is equal to 6.

If $f(x)$ is the position of a moving object at time x, then $f'(x)$ is the velocity, so $f''(x)$ is the rate of change of velocity with respect to time. It is called the *acceleration* of the object.

We end with a remark on notation. It is not necessary to represent functions by f and independent and dependent variables by x and y; as long as we say what we are doing, we can use any letters we wish.

Worked Example 9 Let $g(a) = 4a^2 + 3a - 2$. What is $g'(a)$? What is $g'(2)$?

Solution If $f(x) = 4x^2 + 3x - 2$, we know that $f'(x) = 8x + 3$. Using g instead of f and a instead of x, we have $g'(a) = 8a + 3$. Finally, $g'(2) = 8 \cdot 2 + 3 = 19$.

Solved Exercises

9. Let $f(x) = 3x + 1$. What is $f'(8)$?
10. An apple falls from a tall tree toward the earth. After t seconds, it has fallen $4.9t^2$ meters. What is the velocity of the apple when $t = 3$? What is the acceleration?
11. Find the equation of the line tangent to the graph of $f(x) = 3x^2 + 4x + 2$ at the point where $x_0 = 1$.
12. For which functions $f(x) = ax^2 + bx + c$ is the second derivative equal to the zero function?

Exercises

10. Differentiate the following functions:

    (a) $f(x) = x^2 + 3x - 1$ (b) $f(x) = (x - 1)(x + 1)$
    (c) $f(x) = -3x + 4$ (d) $g(t) = -4t^2 + 3t + 6$

11. A ball is thrown upward at $t = 0$; its height in meters until it strikes the ground is $24.5t - 4.9t^2$ where the time is t seconds. Find:

    (a) The velocity at $t = 0, 1, 2, 3, 4, 5$.
    (b) The time when the ball is at its highest point.
    (c) The time when the velocity is zero.
    (d) The time when the ball strikes the ground.

12. Find the tangent line to the parabola $y = x^2 - 3x + 1$ when $x_0 = 2$. Sketch.

13. Find the second derivative of each of the following:

    (a) $f(x) = x^2 - 5$ (b) $f(x) = x - 2$
    (c) A function whose derivative is $3x^2 - 7$.

Problems for Chapter 1

1. Find the sign changes of:
   (a) $f(x) = (3x^2 - 1)/(3x^2 + 1)$ (b) $f(x) = 1/x$

2. Where do the following functions change sign from positive to negative?
   (a) $f(x) = 6 - 5x$ (b) $f(x) = 2x^2 - 4x + 2$
   (c) $f(x) = 2x - x^2$ (d) $f(x) = 6x + 1$
   (e) $f(x) = (x - 1)(x + 2)^2(x + 3)$

3. The position at time x of a moving object is x^3. Show that the velocity at time 1 lies between 2 and 4.

4. Let $f(x) = (x - r_1)^{n_1}(x - r_2)^{n_2} \cdots (x - r_k)^{n_k}$, where $r_1 < r_2 < \cdots < r_k$ are the roots of f and $n_1, \ldots, n_k$ are positive integers. Where does $f(x)$ change sign from negative to positive?

5. Using the definition of the derivative directly, find $f'(2)$ if $f(x) = 3x^2$.

6. If $f(x) = x^5 + x$, is $f'(0)$ positive or negative? Why?

7. Sketch each of the following graphs together with its tangent line at $(0, 0)$: (a) $y = x^2$ (b) $y = x^3$ (c) $y = -x^3$. Must a tangent line to a graph always lie on one side of the graph?

8. Find the derivative at $x_0 = 0$ of $f(x) = x^3 + x$.

9. Find the following derivatives:
   (a) $f(x) = x^2 - 2$; find $f'(3)$.
   (b) $f(x) = 1$; find $f'(7)$.
   (c) $f(x) = -13x^2 - 9x + 5$; find $f'(1)$.
   (d) $g(s) = 0$; find $g'(3)$.
   (e) $k(y) = (y + 4)(y - 7)$; find $k'(-1)$.
   (f) $x(f) = 1 - f^2$; find $x'(0)$.
   (g) $f(x) = -x + 2$; find $f'(3.752764)$.

10. Find the tangent line to the curve $y = x^2 - 2x + 1$ when $x = 2$. Sketch.

11. Let $f(x) = 2x^2 - 5x + 2$, $k(x) = 3x - 4$, $g(x) = \frac{3}{4}x^2 + 2x$, $l(x) = -2x + 3$, and $h(x) = -3x^2 + x + 3$.
    (a) Find the derivative of $f(x) + g(x)$ at $x = 1$.
    (b) Find the derivative of $3f(x) - 2h(x)$ at $x = 0$.
    (c) Find the equation of the tangent line to the graph of $f(x)$ at $x = 1$.
    (d) Where does $l(x)$ change sign from negative to positive?
    (e) Where does $l(x) - [k(x) \div k'(-1)](x + 1)$ change sign from positive to negative?

12. Find the tangent line to the curve $y = -3x^2 + 2x + 1$ when $x = 0$. Sketch.

13. Let R be any point on the parabola $y = x^2$. (a) Draw the horizontal line through R. (b) Draw the perpendicular to the tangent line at R. Show that the distance between the points where these lines cross the y axis is equal to $\frac{1}{2}$, regardless of the value of x. (Assume, however, that $x \neq 0$.)

14. Given a point $(\bar{x}, \bar{y})$, find a general rule for determining how many lines through the point are tangent to the parabola $y = x^2$.

15. If $f(x) = ax^2 + bx + c = a(x - r)(x - s)$ (r and s are the roots of f), show that the values of $f'(x)$ at r and s are negatives of one another. Explain this by appeal to the symmetry of the graph.

16. Let $f(t) = 2t^2 - 5t + 2$ be the position of object A and let $h(t) = -3t^2 + t + 3$ be the position of object B.
    (a) When is A moving faster than B?
    (b) How fast is B going when A stops?
    (c) When does B change direction?

17. Let $f(x) = 2x^2 + 3x + 1$.
    (a) For which values of x is $f'(x)$ negative, positive, and zero?
    (b) Identify these points on a graph of f.

18. How do the graphs of functions $ax^2 + bx + c$ whose second derivative is positive compare with those for which the second derivative is negative and those for which the second derivative is zero?

19. Where does the function $f(x) = -2|x - 1|$ fail to be differentiable? Explain your answer with a sketch.