CHAPTER 3

CONSTITUTIVE THEORY

Constitutive theory gives functional form to the stress tensor, free energy, and heat flux vector in terms of the motion and temperature. This accomplishes two things. First, different general functional forms distinguish various broad classes of materials (elastic, fluid, materials with memory, etc.) and specific functional forms determine specific materials. Secondly, these additional relationships make the equations formally well-posed; that is, there are as many equations as unknowns.

In Section 3.1, we describe the features of constitutive theory in rather general terms. At this stage the functional form is very broad; the key objective is to isolate what should be a function of what. We can be more specific by making more hypotheses. Proceeding towards the traditional theories of elasticity, we specialize further in Section 3.2. We see that the hypothesis of locality restricts us to functions of the point values of the deformation gradient and temperature and their derivatives. The dependence on higher derivatives is then eliminated by way of the entropy production inequality. Finally, the assumption of invariance under rigid body motions—that is, material frame indifference—allows us to reduce the dependence on the deformation gradient $\boldsymbol{F}$ to dependence on the deformation tensor $\boldsymbol{C}$ (i.e., on the strain).

Section 3.3 treats constitutive theory using ideas of covariance and links this with our covariant treatment of energy balance in Section 2.3. The formula for the Cauchy stress $\boldsymbol{\sigma} = 2\rho(\partial\psi/\partial\boldsymbol{g})$ is shown to be equivalent to the standard formula $\boldsymbol{S} = 2\rho_{\text{Ref}}(\partial\Psi/\partial\boldsymbol{C})$. An application to the identification problem for thermoelastic and inelastic materials is given as well. Additional links with the Hamiltonian structure and conservation laws will be made in Chapter 5. The constitutive information is inserted into the basic balance laws in Section 3.4;

there we summarize the equations of a thermoelastic solid. Section 3.5 makes a further specialization by considering the restrictions that arise when the material possesses some symmetry. Special attention is given to isotropic solids.

3.1 THE CONSTITUTIVE HYPOTHESIS

From the previous chapter, we have a number of basic equations expressed in the material picture as follows:

(i) $\rho_{\text{Ref}} = \rho J$ (conservation of mass);

(ii) $\rho_{\text{Ref}} \dfrac{\partial V}{\partial t} = \text{DIV}\, \boldsymbol{P} + \rho_{\text{Ref}} \boldsymbol{B}$ (balance of momentum);

(iii) $\boldsymbol{S} = \boldsymbol{S}^T$ (balance of moment of momentum);

(iv) $\rho_{\text{Ref}} \dfrac{\partial E}{\partial t} + \text{DIV}\, \boldsymbol{Q} = \rho_{\text{Ref}} R + \boldsymbol{S} : \boldsymbol{D}$ (balance of energy);

(v) $\rho_{\text{Ref}} N \dfrac{\partial \Theta}{\partial t} + \dfrac{\partial \Psi}{\partial t} - \boldsymbol{S} : \boldsymbol{D} + \dfrac{1}{\Theta} \langle \mathbf{Q}, \text{GRAD}\, \Theta \rangle \leq 0$
(reduced dissipation inequality);

(vi) $E = \Psi + N\Theta$ (relation between internal energy and free energy).

These equations are formally ill-posed in the sense that there are not enough equations to determine the evolution of the system. The situation is analogous to Newton's second law $m\ddot{x} = \boldsymbol{F}$; one cannot solve this equation without specifying how $\boldsymbol{F}$ depends on x and $\dot{x}$.

Generally one treats the motion $\phi(X, t)$ and the temperature $\Theta(X, t)$ as the unknowns and attempts to solve for them from equations (ii) and (iv). Condition (i) determines ρ in terms of ρ_{Ref} (usually given) and ϕ, so we can eliminate condition (i). We often regard $\boldsymbol{B}$ and R as given externally. Thus, if we are going to determine ϕ and Θ, we must specify $\boldsymbol{S}$, $\boldsymbol{Q}$, and E as functions of ϕ and Θ. Since E is related to Ψ by (vi), we might also ask for N and Ψ to be functions of ϕ and Θ. These functions must be related in such a way that (iii) and (v) are satisfied. A particular functional form given to $\boldsymbol{S}$, $\boldsymbol{Q}$, N, and Ψ is supposed to characterize a particular material; these same functional forms are associated with any body made from that same material. As we shall see in the next section, the reduced dissipation inequality can be used in delineating the possible functional forms and their interrelationships. Section 3.3 shows how to derive the same basic relationships using covariance as an axiom in place of the second law of thermodynamics.

These ideas can be made more precise using the following terminology.

1.1 Definition Let r be a positive integer or $+\infty$. The set of all *past motions* up to time T is

$$\mathfrak{M}_T = \{\phi : \mathcal{B} \times (-\infty, T] \longrightarrow \mathcal{S} \mid \phi \text{ is a } C^r \text{ regular motion of } \mathcal{B} \text{ in } \mathcal{S} \quad \text{for} \quad -\infty < t \leq T\}$$

and the set of all *past temperature fields* up to time T is

$$\mathfrak{I}_T = \{\Theta : \mathcal{B} \times (-\infty, T] \longrightarrow (0, \infty) \mid \Theta \text{ is a } C^r \text{ function}\}.$$

Let $\mathcal{H} = \bigcup_{T \in \mathbb{R}} (\mathfrak{M}_T \times \mathfrak{I}_T \times \{T\})$, the set of *past histories*.

Here our notation is the same as in previous chapters; $\mathcal{B}$ is a reference configuration of the body and $\mathcal{S}$ is the containing space.

1.2 Definition By a *constitutive equation* for the second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ we mean a mapping

$$\hat{\boldsymbol{S}} : \mathcal{H} \longrightarrow S_2(\mathcal{B}),$$

where $S_2(\mathcal{B})$ denotes the space of C^s symmetric contravariant two-tensor fields on $\mathcal{B}$ (for some suitable positive integer s). The second Piola–Kirchhoff stress tensor *associated* with $\hat{\boldsymbol{S}}$, a motion $\phi(X, t)$, and a temperature field $\Theta(X, t)$ is

$$\boldsymbol{S}(X, t) = \hat{\boldsymbol{S}}(\phi_{[t]}, \Theta_{[t]}, t)(X),$$

where $\phi_{[t]} \in \mathfrak{M}_t$ and $\Theta_{[t]} \in \mathfrak{I}_t$ are defined to be ϕ and Θ restricted to $(-\infty, t]$.

Similarly, constitutive equations for $\boldsymbol{Q}$, Ψ, and N are defined to be mappings:

$$\hat{\boldsymbol{Q}} : \mathcal{H} \longrightarrow \mathfrak{X}(\mathcal{B}) \quad (\mathfrak{X} \text{ denotes the } C^s \text{ vector fields on } \mathcal{B}),$$

$$\hat{\Psi} : \mathcal{H} \longrightarrow \mathfrak{F}(\mathcal{B}) \quad (\mathfrak{F} \text{ denotes the } C^s \text{ scalar fields on } \mathcal{B}),$$

$$\hat{N} : \mathcal{H} \longrightarrow \mathfrak{F}(\mathcal{B}).$$

One sometimes sees the relationship between $\boldsymbol{S}$ and $\hat{\boldsymbol{S}}$ written as

$$\boldsymbol{S}(X, t) = \hat{\boldsymbol{S}}(\phi(X', t'), \Theta(X', t'), X, t).$$

This is a way of trying to say that $\hat{\boldsymbol{S}}$ is a function of the functions ϕ and Θ as a whole. For instance, $\hat{\boldsymbol{S}}$ can depend on spatial or temporal derivatives of ϕ and Θ.

So far the definition of constitutive equation is general enough to include rate effects, that is, dependence on velocities, and memory effects, that is, dependence on past histories. Such generality is important for the study of fluids, viscoelastic, or plastic materials. Our emphasis in this book is on the case of pure elasticity. To make this specialization, we eliminate rate and memory effects as follows.

1.3 Definition By a *thermoelastic constitutive equation* for $\boldsymbol{S}$ we mean a mapping

$$\hat{\boldsymbol{S}} : \mathcal{C} \times \mathfrak{I} \longrightarrow S_2(\mathcal{B}),$$

where $\mathcal{C} = \{\phi : \mathcal{B} \longrightarrow \mathcal{S} \mid \phi \text{ is a regular } C^r \text{ configuration}\}$, the configuration space, and $\mathfrak{I} = \{\Theta : \mathcal{B} \longrightarrow (0, \infty) \mid \Theta \text{ is a } C^r \text{ function}\}$. We assume that $\hat{\boldsymbol{S}}$ is differentiable (in the Fréchet sense; see Box 1.1 below).

The second Piola–Kirchhoff stress tensor *associated* with $\hat{\boldsymbol{S}}$, a configuration ϕ, and a temperature field Θ is given by

$$\boldsymbol{S}(X) = \hat{\boldsymbol{S}}(\phi, \Theta)(X).$$

If ϕ and Θ depend on time, we write

$$S(X, t) = \hat{S}(\phi_t, \Theta_t)(X),$$

where, as usual, $\phi_t(X) = \phi(X, t)$ and $\Theta_t(X) = \Theta(X, t)$.

Thermoelastic constitutive equations for $\boldsymbol{Q}$, Ψ, and N are defined similarly.

In the next section we shall impose requirements on our constitutive functions $\hat{S}$, $\hat{\boldsymbol{Q}}$, $\hat{N}$, and $\hat{\Psi}$ that will drastically simplify how they can depend on ϕ and Θ and that will establish key relationships between them.

Constitutive theory is given most conveniently in the material picture because the domain $\mathcal{B}$ of the functions remains fixed. However, it can all be done spatially as well. To do so properly, one needs considerations of covariance and dependence on the metric tensor $\boldsymbol{g}$ of $\mathcal{S}$. We shall take up this issue in Section 3.3.

Box. 1.1 ***The Fréchet Derivative***

In Section 1.1 we reviewed a bit of differential calculus in $\mathbb{R}^n$; the main point was familiarization with the idea that the derivative of a map $f: \mathcal{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at a point $x \in \mathcal{U}$ is a linear map $Df(x): \mathbb{R}^n \longrightarrow \mathbb{R}^m$. This box will generalize those ideas to Banach spaces. It will be in the nature of a review with a number of quite easy proofs omitted. (Consult standard texts such as Dieundonné [1960] or Lang [1971] for proofs.) However, in Section 4.1 we shall present a complete proof of the inverse mapping theorem for mappings between Banach spaces.

If $\mathfrak{X}$ and $\mathcal{Y}$ are Banach spaces, a linear mapping $A: \mathfrak{X} \longrightarrow \mathcal{Y}$ is called *bounded* if there is a constant $M > 0$ such that $\|Ax\|_{\mathcal{Y}} \leq M\|x\|_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$. Here $\|\cdot\|_{\mathcal{Y}}$ denotes the norm on $\mathcal{Y}$ and $\|\cdot\|_{\mathfrak{X}}$ that on $\mathfrak{X}$. (The subscripts are dropped if there is no danger of confusion.) A linear operator is bounded if and only if it is continuous. (The proof is not hard: if A is continuous at 0, then for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|Ax\|_{\mathcal{Y}} < \epsilon$ if $\|x\|_{\mathfrak{X}} < \delta$. Let $M = \epsilon/\delta$.) Let $\mathcal{B}(\mathfrak{X}, \mathcal{Y})$ denote the set of all bounded linear operators of $\mathfrak{X}$ to $\mathcal{Y}$. Define

$$\|A\| \equiv \|A\|_{\mathfrak{X},\mathcal{Y}} = \inf_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

A straightforward check shows that this makes $\mathcal{B}(\mathfrak{X}, \mathcal{Y})$ into a Banach space; we call the topology associated with the preceding norm the *norm topology*. If $\mathfrak{X}$ and $\mathcal{Y}$ are finite dimensional, $\mathcal{B}(\mathfrak{X}, \mathcal{Y})$ coincides with $\mathcal{L}(\mathfrak{X}, \mathcal{Y})$, the space of all linear mappings of $\mathfrak{X}$ to $\mathcal{Y}$.

Problem 1.1 For a linear map $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, show that $\|A\|$ is the largest eigenvalue of the symmetric operator A^TA.

1.4 Definition Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{U} \subset \mathcal{X}$ be open, and $f\colon \mathcal{U} \subset \mathcal{X} \longrightarrow \mathcal{Y}$. We say f is *differentiable* at $x_0 \in \mathcal{U}$ if there is a bounded linear operator $Df(x_0) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that for each $\epsilon > 0$ there is a $\delta > 0$ such that $\|h\|_{\mathcal{X}} < \delta$ implies

$$\|f(x_0 + h) - f(x_0) - Df(x_0)\cdot h\|_{\mathcal{Y}} \leq \epsilon \|h\|_{\mathcal{X}}.$$

(This uniquely determines $Df(x_0)$.)

We say that f is C^1 if it is differentiable at each point of $\mathcal{U}$ and if $x \mapsto Df(x)$ is continuous from $\mathcal{U}$ to $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, the latter with the *norm* topology. (Recall that in Euclidean spaces, $Df(x_0)$ is the linear map whose matrix in the standard bases is the matrix of partial derivatives of f.) Also note that if $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ then $DA(x) = A$.

The concept "f is of class C^r" ($0 \leq r \leq \infty$) is defined inductively. For example, f is C^2 if it is C^1 and $x \mapsto D^2f(x) \in \mathcal{B}(\mathcal{X}, \mathcal{B}(\mathcal{X}, \mathcal{Y}))$, the derivative of $x \mapsto Df(x)$, is norm continuous. The space $\mathcal{B}(\mathcal{X}, \mathcal{B}(\mathcal{X}, \mathcal{Y}))$ is isomorphic to $\mathcal{B}^2(\mathcal{X}, \mathcal{Y})$, the space of all continuous bilinear maps $b\colon \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Y}$. An isomorphism between these spaces is $b \mapsto \tilde{b}$, $\tilde{b}(x_1)\cdot x_2 = b(x_1, x_2)$. Thus $D^2f(x)$ is usually regarded as a bilinear map of $\mathcal{X} \times \mathcal{X}$ to $\mathcal{Y}$. Its value at $(u, v) \in \mathcal{X} \times \mathcal{X}$ will be denoted $D^2f(x)\cdot(u, v)$.

1.5 Proposition *If f is C^2, then $D^2f(x)$ is symmetric; that is,*

$$D^2f(x)\cdot(u, v) = D^2f(x)\cdot(v, u).$$

For $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, the matrix of $D^2f(x)$ is the matrix of second derivatives of f, so 1.5 generalizes the usual notion of symmetry of the second partial derivatives.

If f is a function of two (or more) variables, say $f\colon \mathcal{U} \subset \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}$, then the partial derivatives are denoted by $D_1 f$ and $D_2 f$ (or sometimes $D_{x_1} f$, etc.). If we identify $\mathcal{X}_1 \times \mathcal{X}_2$ with $\mathcal{X}_1 \oplus \mathcal{X}_2$, $(u, 0)$ with u, and $(0, v)$ with v, then we can write Df as the sum of its partial derivatives: $Df(x) = D_1 f(x) + D_2 f(x)$.

1.6 Definition Suppose $f\colon \mathcal{U} \subset \mathcal{Y} \longrightarrow \mathcal{Y}$ is differentiable. Define the *tangent* of f to be the map

$$Tf\colon \mathcal{U} \times \mathcal{X} \longrightarrow \mathcal{Y} \times \mathcal{Y} \qquad \text{given by} \qquad Tf(x, u) = (f(x), Df(x)\cdot u),$$

where $Df(x)\cdot u$ means $Df(x)$ applied to $u \in \mathcal{X}$ as a linear map.

From the geometric point of view developed in Chapter 1, T is more natural than D. If we think of (x, u) as a vector with base point x, then $(f(x), Df(x)\cdot u)$ is the image vector with its base point. See Figure 3.1.1. Another reason for favoring T is its behavior under composition, as given in the next theorem.

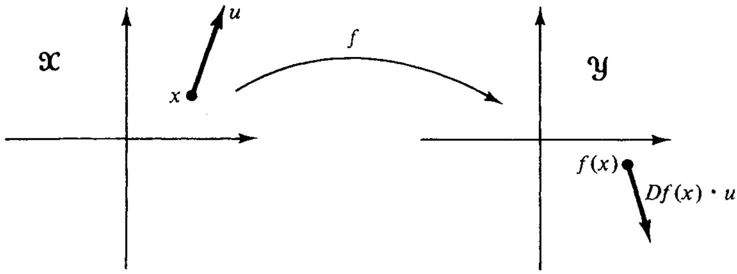

Figure 3.1.1

1.7 Proposition (*Chain Rule*) *Suppose* $f: \mathfrak{U} \subset \mathfrak{X} \longrightarrow \mathfrak{V} \subset \mathcal{Y}$ and g: $\mathfrak{V} \subset \mathcal{Y} \longrightarrow \mathcal{Z}$ *are* C^1 *maps. Then the composite* $g \circ f: \mathfrak{U} \subset \mathfrak{X} \longrightarrow \mathcal{Z}$ *is also* C^1 *and*

$$\boxed{T(g \circ f) = Tg \circ Tf.}$$

In terms of D, this formula is equivalent to the usual chain rule

$$\boxed{D(g \circ f)(x) \cdot u = Dg(f(x)) \cdot (Df(x) \cdot u).}$$

For a proof, see Dieudonné [1960], p. 145, or Marsden [1974a], p. 168. For the validity of this chain rule, f and g need only be differentiable.

We will now show how the derivative Df is related to the usual directional derivative. A C^1 curve in $\mathfrak{X}$ is a C^1 map from $\mathfrak{I}$ into $\mathfrak{X}$, $c: \mathfrak{I} \longrightarrow \mathfrak{X}$, where $\mathfrak{I}$ is an open interval of $\mathbb{R}$, Thus, for $t \in \mathfrak{I}$ we have $Dc(t) \in \mathfrak{B}(\mathbb{R}, \mathfrak{X})$, by definition. We identify $\mathfrak{B}(\mathbb{R}, \mathfrak{X})$ with $\mathfrak{X}$ by associating $Dc(t)$ with $Dc(t) \cdot 1$ (1 is the real number "one"). Let

$$\frac{dc}{dt}(t) = Dc(t) \cdot 1.$$

For $f: \mathfrak{U} \subset \mathfrak{X} \longrightarrow \mathcal{Y}$ of class C^1 we consider $f \circ c$, where $c: \mathfrak{I} \longrightarrow \mathfrak{U}$. It follows from the chain rule that we have the very useful formula

$$\boxed{Df(x) \cdot u = \frac{d}{dt}\{f(x + tu)\}\big|_{t=0}.}$$

1.8 Definition We call $Df(x) \cdot u$ the *directional derivative* of f in the direction of u. A map for which all the directional derivatives (defined by the preceding formula) exist at x is called *Gâteaux differentiable* at x.

On Euclidean space, d/dt defined this way coincides with the usual directional derivative. More specifically, suppose we have $f\colon \mathfrak{U} \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$ of class C^1. Now $Df(x)$ is a linear map from $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ and so it is represented by its components relative to the standard basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ of $\mathbb{R}^m$. By the above formula we see that

$$Df(u)\cdot \boldsymbol{e}_i = \left(\frac{\partial f^1}{\partial x^i}(u), \ldots, \frac{\partial f^n}{\partial x^i}(u)\right).$$

Thus $Df(u)$ is represented by the usual Jacobian matrix.

If we fix $x, y \in \mathfrak{X}$ and apply the fundamental theorem of calculus to the map $t \mapsto f(tx + (1 - t)y)$, assume f is C^1, and

$$\| Df(tx + (1 - t)y) \| \leq M,$$

we obtain the *mean value inequality*:

1.9 Proposition *If $f\colon \mathfrak{U} \subset \mathfrak{X} \longrightarrow \mathfrak{Y}$ is C^1, $x, y \in \mathfrak{U}$, the line joining x to y lies in $\mathfrak{U}$ and $\| Df(z) \|_{\mathfrak{X},\mathfrak{Y}} \leq M$ on this line, then*

$$\|f(x) - f(y)\|_{\mathfrak{Y}} \leq M \| x - y \|_{\mathfrak{X}}.$$

The tangent is very convenient for dealing with higher derivatives. For example, the rth-order chain rule is obtained inductively to be $T^r(g \circ f) = T^r g \circ T^r f$, while a corresponding statement in terms of D is a good deal more complicated. Higher derivatives also occur in Taylor's theorem:

1.10 Proposition (*Taylor's Theorem*) *Suppose $f\colon \mathfrak{U} \subset \mathfrak{X} \longrightarrow \mathfrak{Y}$ is C^{r+1}. Then (for h sufficiently small)*

$$\boxed{\begin{aligned} f(x + h) = f(x) &+ Df(x)\cdot h + \tfrac{1}{2}D^2 f(x)(h, h) + \cdots \\ &+ \frac{1}{r!}D^r f(x)\cdot(h, \ldots, h) + R_r(x, h), \end{aligned}}$$

where

$$R_r(x, h) = \frac{1}{r!}\int_0^1 (1 - t)^r D^{r+1} f(x + th)\cdot(h, \ldots, h)\, dt$$

satisfies $\| R_r(x, h) \|_{\mathfrak{Y}} \leq C \| h \|_{\mathfrak{X}}^{r+1}$ for $\| h \|_{\mathfrak{X}}$ small enough.

The proof proceeds in the same manner as in elementary calculus.

1.11 Proposition *Suppose $f\colon \mathfrak{X} \times \mathfrak{Y} \longrightarrow \mathfrak{Z}$ is a continuous bilinear mapping (i.e., $\|f(x, y)\|_{\mathfrak{Z}} \leq M \| x \|_{\mathfrak{X}} \| y \|_{\mathfrak{Y}}$ for some constant $M > 0$). Then f is C^∞ and*

$$Df(x, y)(u, v) = f(u, y) + f(x, v).$$

The verification of this *Leibniz rule* is straightforward.

Constitutive functions define mappings between Banach spaces of functions. To be able to use differential calculus on them we need to know how to differentiate them. The rest of this box will develop such skills. We begin with a simple example.

1.12 Example Let $\mathfrak{X}$ be the Banach space of continuous functions $\varphi\colon [0, 1] \longrightarrow \mathbb{R}$ with $\|\varphi\| = \sup_{x \in [0,1]} |\varphi(x)|$. Define

$$f\colon \mathfrak{X} \longrightarrow \mathfrak{X} \quad \text{by} \quad f(\varphi)(x) = [\varphi(x)]^2,$$

that is, $f(\varphi) = \varphi^2$. Then f is C^∞ and $Df(\varphi)\cdot\psi = 2\varphi\psi$.

To see this, note that the map $f_1\colon \varphi \mapsto (\varphi, \varphi)$ from $\mathfrak{X}$ to $\mathfrak{X} \times \mathfrak{X}$ is continuous linear and $f_2\colon \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{X}$; $(\varphi, \psi) \mapsto \varphi\cdot\psi$ is continuous bilinear. Now apply 1.11 and the chain rule to $f = f_2 \circ f_1$.

> ***Problem 1.2*** Let $\mathfrak{X}$ be the set of C^1 functions on [0, 1] and $\mathcal{Y}$ be the set of C^0 functions. Define $f\colon \mathfrak{X} \longrightarrow \mathcal{Y}$ by $f(\varphi) = (\varphi')^3$. Prove that f is C^∞ and find Df and $D^2 f$.

What about examples like $f(\varphi) = e^{\varphi'}$? The next theorem shows us how to differentiate such maps. Results of this type go back at least to Sobolev in the 1930s. (See also Abraham and Smale [1960].) For simplicity, we work in C^k spaces, but the same thing works in a variety of function spaces (such as Sobolev spaces $W^{s,p}$ that will be developed in Chapter 6). We stick to a special case that is relevant for elasticity.

1.13 Theorem (ω-*lemma*) *Let* $\bar{\Omega} \subset \mathbb{R}^n$ *be a region with smooth boundary; let* $\mathfrak{X}$ *be the Banach space of* C^k *maps*[1] $u\colon \bar{\Omega} \longrightarrow \mathbb{R}^m$; *let* $\mathcal{Y}$ *be the Banach space of* C^{k-1} *maps* $g\colon \bar{\Omega} \longrightarrow \mathbb{R}^p$ $(1 \le k < \infty)$. *Let*

$$W\colon \bar{\Omega} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}^p$$

be C^r $(r \geq k - 1 - l)$, *and define*

$$f\colon \mathfrak{X} \longrightarrow \mathcal{Y}, \qquad f(u)(x) = W(x, Du(x)).$$

Then f *is of class* C^l *and*

$$\boxed{\begin{aligned} &(Df(u)\cdot v)(x) = D_2 W(x, Du(x))\cdot[Dv(x)]; \\ &\textit{that is,} \quad (Df(u)\cdot v)^j = \frac{\partial W^j}{\partial(\partial u^i/\partial x^I)}\cdot\frac{\partial v^i}{\partial x^I}. \end{aligned}}$$

[1] A map $u\colon \bar{\Omega} = \Omega \cup \partial\Omega \longrightarrow \mathbb{R}^m$ is called C^k when it has a C^k extension to an open set containing $\bar{\Omega}$. The norm of u is defined by

$$\|u\| = \sup_{\substack{x \in \bar{\Omega} \\ 0 \le j \le k}} \|D^j u(x)\|$$

and this norm can be shown to make the C^k maps into a Banach space. (See Abraham and Robbin [1967], for example.)

Proof Induction reduces the argument to the case $r = 1$, $k = 1$. The following computation and the finite-demensional chain rule shows that f is Gâteaux differentiable with derivative as stated in the theorem:

$$[Df(u)\cdot v](x) = \frac{d}{d\epsilon} f(u + \epsilon v)(x)\,|_{\epsilon=0} = \frac{d}{d\epsilon} W(x, Du(x) + \epsilon Dv))\,|_{\epsilon=0}.$$

A straightforward uniform continuity argument shows that $u \mapsto Df(u) \in \mathcal{B}(\mathfrak{X}, \mathcal{Y})$ is norm continuous. The proof is now completed using the following:

1.14 Lemma *Let $f\colon \mathfrak{U} \subset \mathfrak{X} \to \mathcal{Y}$ be Gâteaux differentiable and assume $u \mapsto Df(u) \in \mathcal{B}(\mathfrak{X}, \mathcal{Y})$ is continuous. Then f is C^1.*

Proof By the fundamental theorem of calculus,

$$\begin{aligned}[f(u_0 + h) - f(u_0)] - Df(u_0)\cdot h &= \left[\int_0^1 \frac{d}{d\lambda} f(u_0 + \lambda h)\, d\lambda\right] - Df(u_0)\cdot h \\ &= \int_0^1 [Df(u_0 + \lambda h)\cdot h - Df(u_0)\cdot h]\, d\lambda.\end{aligned}$$

By continuity of Df, for any $\epsilon > 0$ there is a $\delta > 0$ such that $\| Df(u) - Df(u_0)\| < \epsilon$ if $\|u - u_0\| < \delta$. Then $\|h\| < \delta$ implies:

$$\begin{aligned}&\|f(u_0 + h) - f(u_0) - Df(u_0)\cdot h\| \\ &\quad\leq \int_0^1 \|Df(u_0 + \lambda h) - Df(x_0)\|\cdot\|h\|\, d\lambda \leq \epsilon\,\|h\|. \quad \blacksquare\end{aligned}$$

Since f is continuous and linear in W, this shows that in fact f is C^1 as a function of the pair (u, W).

Problem 1.3 Let f be defined by composition with W as above. Show that if we *assume* that f maps C^k to C^{k-1} and is C^l, then W is C^r, $r = k - 1 + l$ as well.

Box. 1.2 ***Summary of Important Formulas for Section 3.1***

Thermoelastic Constitutive Equation for $\boldsymbol{S}$

$\hat{S}\colon \mathcal{C} \times \mathfrak{I} \to S_2(\mathcal{B})$; $\quad S^{AB}(X, t) = \hat{S}^{AB}(\phi^a(X', t), \Theta(X', t), X)$

$S(X, t) = \hat{S}(\phi_t, \Theta_t)(X)$

for $\boldsymbol{Q}$:

$\hat{Q}\colon \mathcal{C} \times \mathfrak{I} \to \mathfrak{X}(\mathcal{B})$; $\quad Q^A(X, t) = \hat{Q}^A(\phi^a(X', t), \Theta(X', t), X)$

$Q(X, t) = \hat{Q}(\phi_t, \Theta_t)(X)$

for Ψ:	
$\hat{\Psi}: \mathcal{C} \times \mathfrak{J} \longrightarrow \mathfrak{F}(\mathcal{B})$;	$\Psi(X, t) = \hat{\Psi}(\phi^a(X', t), \Theta(X', t), X)$
$\Psi(X, t) = \hat{\Psi}(\phi_t, \Theta_t)(X)$	
for N:	
$\hat{N}: \mathcal{C} \times \mathfrak{J} \longrightarrow \mathfrak{F}(\mathcal{B})$;	$N(X, t) = \hat{N}(\phi^a(X', t), \Theta(X', t), X)$
$N(X, t) = \hat{N}(\phi_t, \Theta_t t)(X)$	
Derivative of a Function-Space Mapping Defined by Composition	
$f(u)(x) = W(x, Du(x))$	
$(Df(u)\cdot v)(x)$	$Df(u)\cdot v = \dfrac{\partial W}{\partial(\partial u^i/\partial x^j)} \cdot \dfrac{\partial v^i}{\partial x^j}$
$= D_2 W(x, Du(x))\cdot Dv(x)$	

3.2 CONSEQUENCES OF THERMODYNAMICS, LOCALITY, AND MATERIAL FRAME INDIFFERENCE

We now simplify the constitutive functions $\hat{S}$, $\hat{Q}$, $\hat{N}$, $\hat{\Psi}$ as follows. From an axiom of locality and the entropy production inequality, we shall deduce that $\hat{S}$ and $\hat{N}$ can be expressed in terms of $\hat{\Psi}$ and that $\hat{\Psi}$ depends only on points in $\mathcal{B}$, point values of F and Θ and not on ϕ or higher derivatives of F or Θ. Then, we postulate an axiom of material frame indifference to deduce that the dependence of $\hat{\Psi}$ on F is only through the Cauchy–Green tensor C.

2.1 Definition A constitutive function for thermoelasticity

$$\hat{\Psi}: \mathcal{C} \times \mathfrak{J} \longrightarrow \mathfrak{F}(\mathcal{B})$$

is called *local* if for any open set $\mathcal{U} \subset \mathcal{B}$ and $\phi_1, \phi_2 \in \mathcal{C}$, which agree on $\mathcal{U}$, and $\Theta_1, \Theta_2 \in \mathfrak{J}$, which agree on $\mathcal{U}$, then $\hat{\Psi}(\phi_1, \Theta_1)$ and $\hat{\Psi}(\phi_2, \Theta_2)$ agree on $\mathcal{U}$.

The idea of using locality as a basic postulate is due to Noll [1958]. It must be emphasized, however, that one sometimes may wish to impose *nonlocal* constraints, such as incompressibility (see Chapter 5). If the value of $\hat{\Psi}$ at X depends only on the values of ϕ and Θ and their derivatives up to order, say, k, at X, then $\hat{\Psi}$ is local. This is because knowledge of a mapping on an open set entails a knowledge of all its derivatives on that set. A constitutive function $\hat{\Psi}$ defined by a composition of this form is called a (nonlinear) *differential operator.* A simple example of a nonlocal operator is given by letting $\mathfrak{X} = C^0$ functions on [0, 1] and defining $f: \mathfrak{X} \longrightarrow \mathfrak{X}$ by $f(\phi)(x) = \int_0^x \phi(s)\, ds$.

While it is trivial that a differential operator is local, the converse is not so elementary. For linear operators, this is true and is due to Peetre [1959]. For nonlinear operators (satisfying some technical conditions), this is also true but is a deeper fact; see Epstein and Thurston [1979] (and for earlier versions, Dombrowski [1966] and Palais and Terng [1977]). Fortunately, we can bypass this theory in the present context, but it may be useful for higher-order materials.

We can summarize the situation as follows:

2.2 Axiom of Locality *Constitutive functions for thermoelasticity are assumed to be local.*

We next investigate the consequences of assuming the entropy production inequality. We wish to assume that it holds for all regular motions of the body. The momentum balance and energy balance are not taken into account, because any motion is consistent with them for a suitable choice of body force $\boldsymbol{B}$ and heat source R; that is, balance of momentum and energy *define* what $\boldsymbol{B}$ and R have to be. This is not unreasonable since we are supposed to be able to allow any choice of $\boldsymbol{B}$ and R.

2.3 Axiom of Entropy Production *For any (regular) motion of $\mathcal{B}$, the constitutive functions for thermoelasticity are assumed to satisfy the entropy production inequality:*

$$\rho_{\text{Ref}}\left(\hat{N}\frac{\partial \Theta}{\partial t} + \frac{\partial \hat{\Psi}}{\partial t}\right) - \hat{\boldsymbol{P}}:\frac{\partial \boldsymbol{F}}{\partial t} + \frac{1}{\Theta}\langle \hat{\boldsymbol{Q}}, \text{GRAD}\,\Theta\rangle \le 0.$$

2.4 Theorem (*Coleman and Noll* [*1963*]) *Suppose the axioms of locality and entropy production hold. Then $\hat{\Psi}$ depends only on the variables X, $\boldsymbol{F}$, and Θ. Moreover, we have*

$$\boxed{\hat{N} = -\frac{\partial \hat{\Psi}}{\partial \Theta} \quad \text{and} \quad \hat{\boldsymbol{P}} = \rho_{\text{Ref}}\, g^\sharp \frac{\partial \hat{\Psi}}{\partial \boldsymbol{F}}, \qquad \text{that is,} \quad \hat{P}_a{}^A = \rho_{\text{Ref}} \frac{\partial \hat{\Psi}}{\partial F^a{}_A},}$$

and the entropy production inequality reduces to

$$\langle \boldsymbol{Q}, \text{GRAD}\,\Theta\rangle \le 0.$$

Remarks (1) One can allow the constitutive functions to depend on derivatives of higher order than the first derivative $\boldsymbol{F}$ and still be consistent with the entropy production inequality provided one postulates the existence of higher-order stresses. This is the "multipolar" or "higher-order" theory. (See Green and Rivlin [1964b].)

(2) The precise meaning of "$\hat{\Psi}$ depends only on the variables X, $\boldsymbol{F}$, and Θ" is that for any configuration $\phi\colon \mathcal{B} \to \mathcal{S}$, there is a mapping (also denoted $\hat{\Psi}$) $\hat{\Psi}\colon \mathcal{B} \times \mathcal{L}_\phi(T\mathcal{B}, T\mathcal{S}) \times \mathbb{R} \to \mathbb{R}$ such that $\hat{\Psi}(\phi, \Theta)(X) = \hat{\Psi}(X, \boldsymbol{F}(X), \Theta(X))$, where $\mathcal{L}_\phi(T\mathcal{B}, T\mathcal{S})$ denotes the bundle over $\mathcal{B}$ of (1, 1) two-point tensors; that is, the fiber of $\mathcal{L}_\phi(T\mathcal{B}, T\mathcal{S})$ at $X \in \mathcal{B}$ consists of all linear maps $\boldsymbol{F}\colon T_X\mathcal{B} \to T_{\phi(X)}\mathcal{S}$.

(3) If $\mathcal{S}$ is a linear space and if one is willing to identify all the tangent spaces T_x with $\mathcal{S}$, then one can identify $\mathcal{L}_\phi(T\mathcal{B}, T\mathcal{S})$ with $\mathcal{L}(T\mathcal{B}, \mathcal{S})$, the bundle of linear maps of $T_X\mathcal{B}$ to $\mathcal{S}$. In this way, the apparent dependence on ϕ itself disappears.

(4) If $\hat{\Psi}$ is independent of X, the body is called *homogeneous*. This is a special case of a material symmetry, a topic treated in Section 3.5.

Proof of 2.4 (See Gurtin [1972b].) Given $\hat{\Psi}: \mathfrak{C} \times \mathfrak{I} \to \mathfrak{F}$, we consider a motion $\phi_t \in \mathfrak{C}$, a temperature field $\Theta_t \in \mathfrak{I}$, and form the composition $f(t) = \hat{\Psi}(\varphi_t, \Theta_t) \in \mathfrak{F}$. By the chain rule,

$$\frac{df}{dt} = D_\phi \hat{\Psi} \cdot V + D_\Theta \hat{\Psi} \cdot \dot{\Theta},$$

where $D_\phi \hat{\Psi}$ is the partial derivative of $\hat{\Psi}$ in the Frechet sense (see Box 1.1 of this Chapter). Note that for an arbitrary function $k(X, t)$, $\dot{k} \equiv \partial k/\partial t$.

Substituting the preceding equation into the entropy production inequality gives

$$\rho_{\mathrm{Ref}} \hat{N} \dot{\Theta} + \rho_{\mathrm{Ref}}(D_\phi \hat{\Psi} \cdot V + D_\Theta \hat{\Psi} \cdot \dot{\Theta}) - \hat{P}: \frac{\partial F}{\partial t} + \frac{1}{\Theta}\langle Q, \mathrm{GRAD}\, \Theta \rangle \leq 0.$$

By assumption, this holds for all processes (ϕ_t, Θ_t). First of all, choose ϕ_t independent of time t; then we must have

$$\rho_{\mathrm{Ref}}(\hat{N} \dot{\Theta} + D_\Theta \hat{\Psi} \cdot \dot{\Theta}) + \frac{1}{\Theta}\langle Q, \mathrm{GRAD}\, \Theta \rangle \leq 0.$$

Suppose $\rho_{\mathrm{Ref}}(\hat{N} \dot{\Theta} + D_\Theta \hat{\Psi} \cdot \dot{\Theta})$ did not vanish for some ϕ and all Θ_t. Then we can alter Θ_t to a new one $\tilde{\Theta}_t$ so that $\tilde{\Theta}_{t_0} = \Theta_{t_0}$ and $\dot{\tilde{\Theta}}_{t_0} = \alpha \dot{\Theta}_{t_0}$, where α is any prescribed constant. We can then choose the constant α to violate the assumed inequality. Therefore, we deduce our first identity

$$\rho_{\mathrm{Ref}}(\hat{N} \dot{\Theta} + D_\Theta \hat{\Psi} \cdot \dot{\Theta}) = 0. \tag{1}$$

Similarly, fixing Θ and altering ϕ_t we get a second identity

$$\rho_{\mathrm{Ref}} D_\phi \hat{\Psi} \cdot V = \hat{P}: \dot{F}. \tag{2}$$

Consider now this second identity. Fix $X_0 \in \mathfrak{B}$ and fix Θ. Let ϕ_0 and ϕ_1 be two configurations with $\phi_0(X_0) = \phi_1(X_0)$ and $F_0(X_0) = F_1(X_0)$. In a chart on $\mathcal{S}$, let $\phi(X, t) = \phi_0(X) + t(\phi_1(X) - \phi_0(X))$ define a motion ϕ for X near X_0 and small t. Extend ϕ outside this neighborhood in a regular but otherwise arbitrary way. (This can be done by first extending the governing velocity field, for example.) By the axiom of locality, the way in which this extension is done does not affect $\hat{\Psi}$ in the neighborhood. This motion has a velocity field V_t and a deformation gradient F_t that satisfies

$$V_t(X_0) = \phi_1(X_0) - \phi_0(X_0) = \mathbf{0} \quad \text{and} \quad \dot{F}_t(X_0) = \mathbf{0}.$$

[Note that we *cannot* conclude that $(D_\phi \hat{\Psi} \cdot V_t)(X_0) = 0$ from $V_t(X_0) = \mathbf{0}$ alone, since $D_\phi \hat{\Psi} \cdot V_t$ is a linear operator on V_t and could depend on higher derivatives of V_t, for example.] Thus, from the second identity (2) above,

$$(\rho_{\mathrm{Ref}} D_\phi \hat{\Psi} \cdot V_t)(X_0) = 0.$$

Therefore,
$$\frac{d}{dt}[\rho_{\mathrm{Ref}}(X_0)\hat{\Psi}(\phi_t, \Theta)](X_0) = 0$$

and so
$$\rho_{\mathrm{Ref}}(X_0)[\hat{\Psi}(\phi_1, \Theta)](X_0) = \rho_{\mathrm{Ref}}(X_0)[\hat{\Psi}(\phi_0, \Theta)](X_0).$$

Thus, $\hat{\Psi}$ depends on ϕ only through F. In a similar way, the first identity implies that $\hat{\Psi}$ depends only on the point values of Θ. Substituting this information back into the two identities (and using the results of Box 1.1) yields

$$\rho_{\text{Ref}}\left(\hat{N}\dot{\Theta} + \frac{\partial \hat{\Psi}}{\partial \Theta}\dot{\Theta}\right) = 0 \quad \text{and} \quad \rho_{\text{Ref}} \frac{\partial \hat{\Psi}}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} = \boldsymbol{P} : \dot{\boldsymbol{F}}.$$

Arbitrariness of $\dot{\Theta}$ and $\dot{F}$ then yields the stated identities. ∎

The relationships

$$P_a{}^A = \rho_{\text{Ref}} \frac{\partial \hat{\Psi}}{\partial F^a{}_A} \quad \text{and} \quad N = -\frac{\partial \hat{\Psi}}{\partial \Theta}$$

can be used to simplify the first law (energy balance). Indeed, balance of energy reads

$$\rho_{\text{Ref}} \frac{\partial \hat{E}}{\partial t} + \text{DIV}\, \boldsymbol{Q} = \boldsymbol{P} : \boldsymbol{F} + \rho_{\text{Ref}} R.$$

Write $\hat{E} = \hat{\Psi} + \hat{N}\Theta$ so that

$$\frac{\partial E}{\partial t} = \frac{\partial \hat{\Psi}}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \frac{\partial \hat{\Psi}}{\partial \Theta}\dot{\Theta} + \frac{\partial \hat{N}}{\partial t}\Theta + \hat{N}\dot{\Theta} = \frac{\partial \hat{\Psi}}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \frac{\partial \hat{N}}{\partial t}\Theta$$

using $\partial\hat{\Psi}/\partial\Theta = -N$. Substituting this expression into balance of energy and using $\rho_{\text{Ref}}\,(\partial\hat{\Psi}/\partial \boldsymbol{F}) = \boldsymbol{P}$ yields $\rho_{\text{Ref}}\Theta(\partial N/\partial t) + \text{DIV}\, \boldsymbol{Q} = \rho_{\text{Ref}} R$. In summary, we have proved:

2.5 Proposition *Assuming the identities in 2.4, balance of energy reduces to*

$$\boxed{\rho_{\text{Ref}}\Theta \frac{\partial N}{\partial t} + \text{DIV}\, \boldsymbol{Q} = \rho_{\text{Ref}} R.}$$

For pure elasticity (ignoring all thermal effects), the above constitutive conclusions may be obtained from balance of energy alone. Indeed, if $\boldsymbol{Q}$ and R are omitted from balance of energy, then we must have $\rho_{\text{Ref}}(\partial E/\partial t) = \boldsymbol{P} : \dot{\boldsymbol{F}}$ for all processes. Thus $\partial E/\partial \boldsymbol{F}$ and $\boldsymbol{P}$ must balance. We list the axioms special to this situation as follows:

Axiom 0. $\hat{E}: \mathcal{C} \longrightarrow \mathfrak{F}$ is a given differentiable map.
Axiom 1. $\hat{E}$ is local.
Axiom 2. There is a map $\hat{\boldsymbol{P}}: \mathcal{C} \longrightarrow$ two-point (0, 2) tensors such that for all motions $\phi_t \in \mathcal{C}$, balance of energy holds:

$$\rho_{\text{Ref}} \frac{\partial \hat{E}}{\partial t} = \boldsymbol{P} : \dot{\boldsymbol{F}}.$$

2.6 Theorem *Under Axioms 0, 1, and 2, $\hat{E}$ depends only on the point values of X and $\boldsymbol{F}$ and we have the identity*

$$\boxed{\rho_{\text{Ref}} g^\sharp \frac{\partial \hat{E}}{\partial \boldsymbol{F}} = \hat{\boldsymbol{P}}, \qquad \textit{that is,} \quad P_a{}^A = \rho_{\text{Ref}} \frac{\partial \hat{E}}{\partial F^a{}_A}.}$$

This is proved by the same techniques that were used to prove 2.4.

We now return to the thermoelastic context. From 2.4 we can draw no conclusion about the dependence of $\boldsymbol{Q}$ on ϕ and Θ; it could conceivably depend on many derivatives. It is often *assumed* that $\hat{\boldsymbol{Q}}$ depends only on the point values of X, $\boldsymbol{C}$, Θ, and $\nabla\Theta$; in this case, we say we have a *grade (1, 1)* material. In any case, one can generally say the following:

2.7 Proposition *$\hat{\boldsymbol{Q}}$ vanishes when its argument* GRAD Θ *vanishes.*

Proof Fixing all other arguments, let $f(\alpha) = \langle \hat{\boldsymbol{Q}}(\alpha \text{ GRAD } \Theta), \text{GRAD } \Theta\rangle$. Thus $\alpha f(\alpha) \leq 0$, so f changes sign at $\alpha = 0$. Since f is continuous, $f(0) = 0$. Thus $\langle \hat{\boldsymbol{Q}}(0), \text{GRAD } \Theta\rangle = 0$, so $\hat{\boldsymbol{Q}}(0) = \boldsymbol{0}$. ∎

Taylor's theorem implies that for GRAD Θ small, $\boldsymbol{Q}$ is well approximated by a matrix that is negative semi-definite times GRAD Θ: fixing all arguments but GRAD Θ,

$$\hat{\boldsymbol{Q}}(\text{GRAD } \Theta) = \boldsymbol{K}\cdot\text{GRAD } \Theta,$$

where

$$\boldsymbol{K} = \int_0^1 \frac{\partial \hat{\boldsymbol{Q}}}{\partial(\text{GRAD } \Theta)}(s \text{ GRAD } \Theta)\, ds$$

by the fundamental theorem of calculus. The inequality $\langle \hat{\boldsymbol{Q}}, \text{GRAD } \Theta\rangle \leq 0$ means $\boldsymbol{K}$ is negative semi-definite—that is, dissipative. If $\boldsymbol{K}$ were assumed constant one would recover the Fourier law (see Example 3.6, Chapter 2).

2.8 Example (Rigid Heat Conductor) The rigid heat conductor (cf. 3.6, Chapter 2) makes the assumption that the motion is fixed, say ϕ = identity, for all time. We also assume that $\boldsymbol{Q}$ depends only on X, Θ, and GRAD Θ. Then the evolution of Θ in time is determined from balance of energy:

$$\rho_{\text{Ref}}\Theta\frac{\partial \hat{N}}{\partial t} + \text{DIV } \hat{\boldsymbol{Q}} = \rho_{\text{Ref}}R.$$

Since $\hat{N}$ depends only on X and Θ, we get

$$\begin{aligned}\left(\rho_{\text{Ref}}\Theta\frac{\partial \hat{N}}{\partial \Theta}\right)\frac{\partial \Theta}{\partial t} &= -(\hat{Q}^A)_{|A} + \rho_{\text{Ref}}R \\ &= -\left(\frac{\partial \hat{Q}^A}{\partial \Theta_{|B}}\right)\Theta_{|B|A} - \frac{\partial \hat{Q}^A}{\partial \Theta}\Theta_{|A} + \rho_{\text{Ref}}R - R_I.\end{aligned} \tag{3}$$

where $R_I = \text{DIV}_X\hat{\boldsymbol{Q}}$, the divergence of $\hat{\boldsymbol{Q}}$ with Θ and GRAD Θ fixed. As was observed above, the matrix $\partial\hat{Q}^A/\partial\Theta_{|B}$ is negative semi-definite. We also *assume* the scalar function $\partial N/\partial\Theta = -\partial^2\hat{\Psi}/\partial\Theta^2$ (= specific heat at constant volume) is positive, so the equation is formally parabolic. (3) is the general form of a nonlinear heat equation. Notice, finally, that positivity of $\partial^2\hat{\Psi}/\partial\Theta^2$ means that $\hat{\Psi}$ is a convex function of Θ.[2]

[2]The theory of monotone operators is applicable here [if $\rho_{\text{Ref}}\Theta(\partial\hat{N}/\partial\Theta)$ is not a constant, one presumably uses a weighted norm] to yield an existence and uniqueness theorem for $\Theta(X, t)$ for all time, given Θ at time $t = 0$ and given $\hat{\boldsymbol{Q}}$, $\hat{N}$. The extra assumption of positivity of $\partial^2\Psi/\partial\Theta^2$ may be regarded as formally equivalent to well-posedness of the equations (see Crandall and Nohel [1978] for details).

Now we turn to our final constitutive axiom, material frame indifference.

2.9 Axiom of Material Frame Indifference *Let $\hat{\Psi}$ be a thermoelastic constitutive function satisfying the above axioms, so that $\hat{\Psi}$ is a function of X, $\boldsymbol{F}$, and Θ. Assume that if $\xi\colon \mathcal{S} \longrightarrow \mathcal{S}$ is a regular, orientation-preserving map taking x to x' and $T\xi$ is an isometry from $T_x\mathcal{S}$ to $T_{x'}\mathcal{S}$, then*

$$\boxed{\hat{\Psi}(X, \boldsymbol{F}, \Theta) = \hat{\Psi}(X, \boldsymbol{F}', \Theta), \qquad \textit{that is,} \quad \hat{\Psi}(X, \boldsymbol{F}, \Theta) = \hat{\Psi}(X, \xi_*\boldsymbol{F}, \Theta),}$$

where $\boldsymbol{F}\colon T_X\mathcal{B} \longrightarrow T_x\mathcal{S}$, $\boldsymbol{F}'\colon T_X\mathcal{B} \longrightarrow T_{x'}\mathcal{S}$, and $\boldsymbol{F}' = T\xi\cdot\boldsymbol{F} = \xi_*\boldsymbol{F}$. See Figure 3.2.1.

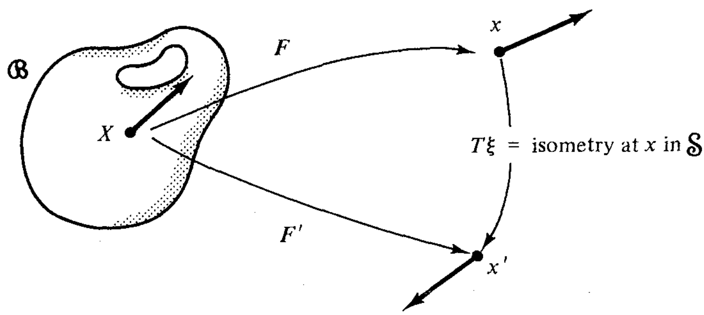

Figure 3.2.1

Stated loosely, this axiom means that our constitutive functions are invariant under rotations of the ambient space $\mathcal{S}$ in which our body moves. One sometimes says that $T\xi$ "rotates observer frames." One can interpret ξ either actively (superposed motion) or passively (change of framing) as explained in Boxes 3.1 and 3.2 of Chapter 1.

2.10 Theorem *Let $\hat{\Psi}$ satisfy the above axioms. Then $\hat{\Psi}$ depends only on X, $\boldsymbol{C}$, and Θ. By abuse of notation we shall write $\hat{\Psi}(X, \boldsymbol{C}, \Theta)$.*

Remarks (1) Equally well, we could say $\hat{\Psi}$ is a function only of X, Θ, and the right stretch tensor $\boldsymbol{U} = \boldsymbol{C}^{1/2}$ (see 3.11 and 3.13, Chapter 1).

(2) By abuse of notation we shall write $\boldsymbol{C}$ for both $\boldsymbol{C}$ and $\boldsymbol{C}^\flat$ when there is little danger of confusion.

(3) One can also state as an axiom the corresponding transformation law for the first Piola–Kirchhoff stress $\hat{\boldsymbol{P}}$; that is, for any ξ as above, $\xi_*\{\hat{\boldsymbol{P}}(X, \boldsymbol{F}, \Theta)\} = \hat{\boldsymbol{P}}(X, \xi_*\boldsymbol{F}, \Theta)$, where $\xi_*\hat{\boldsymbol{P}}$ means push-forward the spatial index of $\hat{\boldsymbol{P}}$. This axiom is a consequence of the axiom for $\hat{\Psi}$. In Euclidean space, material frame indifference for the stress reads:

$$\boxed{R^a{}_b\hat{P}^{bA}(X, F^c{}_C, \Theta) = \hat{P}^{aA}(X, R^c{}_bF^b{}_C, \Theta)}$$

for any proper orthogonal matrix $R^a{}_b$.

(4) Note that in $\mathbb{R}^3$, material frame indifference is the same as objectivity for orthogonal transformations (see Box 6.1, Chapter 1, for a discussion of objectivity).

Proof of 2.10 Suppose $F_1: T_X\mathcal{B} \longrightarrow T_{x_1}\mathcal{S}$ and $F_2: T_X\mathcal{B} \longrightarrow T_{x_2}\mathcal{S}$ and $F_1^TF_1 = F_2^TF_2$; that is, F_1 and F_2 give rise to the same C tensor. We have to show that $\hat{\Psi}(X, F_1, \Theta) = \hat{\Psi}(X, F_2, \Theta)$. Then $\hat{\Psi}(X, C, \Theta)$ will be well defined if we let it be this common value.

Choose a regular map $\xi: \mathcal{S} \longrightarrow \mathcal{S}$ with $\xi(x_1) = x_2$ and such that $T\xi(x_1)F_1 = F_2$. This is possible since F_1 and F_2 are assumed to be invertible. The assumption $F_1^TF_1 = F_2^TF_2$ implies that $T\xi(x_1)$ is an isometry. Indeed,

$$\begin{aligned}\langle T\xi(x_1)\cdot F_1 \cdot V_1, T\xi(x_1)\cdot F_1 \cdot V_2\rangle &= \langle F_2 \cdot V_1, F_2 \cdot V_2\rangle \\ = \langle F_2^TF_2V_1, V_2\rangle &= \langle F_1^TF_1V_1, V_2\rangle = \langle F_1V_1, F_1V_2\rangle.\end{aligned}$$

Therefore, by our axiom, $\hat{\Psi}(X, F_1, \Theta) = \hat{\Psi}(X, F_2, \Theta)$. ∎

The last part of the proof can also be seen in coordinates as follows:

$$\begin{aligned}C_{AB} &= g_{ab}F^a_{1A}F^b_{1B} && (C = F_1^TF_1)\\ &= g_{ab}F^a_{2A}F^b_{2B} && (F_1^TF_1 = F_2^TF_2)\\ &= g_{ab}\frac{\partial\xi^a}{\partial x^c}F^c_{1A}\frac{\partial\xi^b}{\partial x^d}F^d_{1B} && (T\xi\cdot F_1 = F_2).\end{aligned}$$

Thus, comparing the first and third lines, $g_{ab} = g_{cd}(\partial\xi^c/\partial x^a)(\partial\xi^d/\partial x^b)$ (evaluated at x_1) so ξ is an isometry—that is, leaves the metric tensor g_{ab} invariant.

Problem 2.1 What becomes of this proof for shells?

Problem 2.2 Show that even though $\hat{P}$ is materially frame indifferent, $\partial\hat{P}/\partial t$ need not be.

If we write the axiom of entropy production in terms of S and C by writing

$$P: \frac{\partial F}{\partial t} = S: D = \tfrac{1}{2}S: \frac{\partial C}{\partial t}$$

and repeat the argument of 2.4, Chapter 3 (or 2.6, Chapter 3 for pure elasticity), we find, instead of $\hat{P} = \rho_{\text{Ref}}g^\sharp(\partial\hat{\Psi}/\partial F)$, the important identity

$$S = 2\rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial C}.$$

This identity may also be proved by the chain rule as follows: regarding C as a function of F through $C = F^TF$—that is, $C_{AB} = g_{ab}F^a{}_AF^b{}_B$—the chain rule gives

$$S = F^{-1}\hat{P} = F^{-1}\rho_{\text{Ref}}g^\sharp\frac{\partial\hat{\Psi}}{\partial F} = F^{-1}\rho_{\text{Ref}}g^\sharp\frac{\partial\hat{\Psi}}{\partial C}\frac{\partial C}{\partial F}.$$

Noting that

$$\frac{\partial C_{AB}}{\partial F^a{}_C} = g_{ab}\delta^C{}_A F^b{}_B + g_{ca}F^c{}_A\delta^C{}_B,$$

we get

$$\begin{aligned}\hat{S}^{DC} &= \rho_{\text{Ref}}(F^{-1})^D{}_d \frac{\partial\hat{\Psi}}{C_{AB}}\frac{\partial C_{AB}}{\partial F^a{}_C} g^{ad}\\ &= \rho_{\text{Ref}}(F^{-1})^D{}_d \frac{\partial\hat{\Psi}}{C_{AB}}(g_{ab}\delta^C{}_A F^b{}_B + g_{ca}F^c{}_A\delta^C{}_B)g^{ad}\\ &= \rho_{\text{Ref}}(F^{-1})^D{}_d \frac{\partial\hat{\Psi}}{\partial C_{AB}}(\delta^d{}_b\delta^C{}_A F^b{}_B + \delta^d{}_c\delta^C{}_B F^c{}_A)\\ &= \rho_{\text{Ref}}(F^{-1})^D{}_b F^b{}_B \frac{\partial\hat{\Psi}}{C_{CB}} + \rho_{\text{Ref}}(F^{-1})^D{}_c F^c{}_A \frac{\partial\hat{\Psi}}{\partial C_{AC}}\\ &= \rho_{\text{Ref}}\delta^D{}_B \frac{\partial\hat{\Psi}}{\partial C_{CB}} + \rho_{\text{Ref}}\delta^D{}_A \frac{\partial\hat{\Psi}}{\partial C_{AC}}\\ &= \rho_{\text{Ref}}\left(\frac{\partial\hat{\Psi}}{\partial C_{CD}} + \frac{\partial\hat{\Psi}}{\partial C_{DC}}\right) = 2\rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial C_{DC}}.\end{aligned}$$

Thus, we have:

2.11 Proposition *Under the axioms for constitutive theory listed above, the second Piola–Kirchhoff stress is given by*

$$\boxed{\boldsymbol{S} = 2\rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial \boldsymbol{C}}: \qquad \text{that is,} \quad S^{AB} = 2\rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial C_{AB}}.}$$

Problem 2.3 (J. Ball; cf. Wang and Truesdell [1973]). Show that frame indifference of $\boldsymbol{P}$ and symmetry of $\boldsymbol{S}$ implies frame indifference of Ψ. Also, show that frame indifference of Ψ implies symmetry of $\boldsymbol{S}$ [assuming $\boldsymbol{P} = \rho_{\text{Ref}}(\partial\hat{\Psi}/\partial \boldsymbol{F})$]. [*Hint*: Let $\boldsymbol{Q}$ be a proper orthogonal matrix; write $\boldsymbol{Q} = e^{\boldsymbol{\Omega}}$, where Ω is skew, and join $\boldsymbol{F}$ to $\boldsymbol{QF}$ along the curve $\boldsymbol{F}(t) = e^{t\boldsymbol{\Omega}}\boldsymbol{F}$ $(0 \le t \le 1)$. Consider $(d/dt)\hat{\Psi}(\boldsymbol{F}(t))$.]

So far we have formulas for the first and second Piola–Kirchhoff stress tensors in terms of the free energy $\hat{\Psi}$. What about the Cauchy stress tensor $\boldsymbol{\sigma}$? Since $\boldsymbol{C} = \phi^*\boldsymbol{g}$, we can regard $\boldsymbol{C}$ as a function of the point values of $\boldsymbol{F}$ and the spatial metric $\boldsymbol{g}$. Therefore, $\hat{\Psi}$ becomes a function of $X, \boldsymbol{F}(X), \boldsymbol{g}(x)$, and $\Theta(X)$. Set

$$\hat{\psi}(x, \boldsymbol{F}(X), \boldsymbol{g}(x), \Theta(X)) = \hat{\Psi}(X, \boldsymbol{C}(\boldsymbol{F}(X), \boldsymbol{g}(x)), \Theta(X)).$$

By the chain rule,

$$\frac{\partial\hat{\psi}}{\partial g_{ab}} = \frac{\partial\hat{\Psi}}{\partial C_{AB}}\frac{\partial C_{AB}}{\partial g_{ab}} = \frac{\partial\hat{\Psi}}{\partial C_{AB}}F^a{}_A F^b{}_B = \left(\phi_*\frac{\partial\hat{\Psi}}{\partial \boldsymbol{C}}\right)^{ab}.$$

Therefore,

$$\sigma^{ab} = \frac{1}{J}(\phi_* S)^{ab} = \frac{1}{J}\left(\phi_*\left(2\rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial C}\right)\right)^{ab} = 2\rho\left(\phi_*\frac{\partial\hat{\Psi}}{\partial C}\right)^{ab} = 2\rho\,\frac{\partial\hat{\psi}}{\partial g_{ab}}.$$

Thus, we have proved:

2.12 Proposition *Defining the function $\hat{\psi}$ as above, the equation $S = 2\rho_{\text{Ref}}(\partial\hat{\Psi}/\partial C)$ is equivalent to*

$$\boxed{\sigma = 2\rho\,\frac{\partial\hat{\psi}}{\partial g}.}$$

Thus, the equation for the Cauchy stress that we derived in Box 3.3, Chapter 2, is in agreement with and indeed is equivalent to the standard deduction 2.11 in constitutive theory. (See Box 2.1 below for the relation between $\partial\hat{\psi}/\partial g$ and $\partial\hat{e}/\partial g$.) As we noted in Box 3.3, Chapter 2, the formula in Proposition 2.12 was first given (in a slightly different from) by Doyle and Ericksen [1956].

Problem 2.4 Consider a stored energy function of the form $\hat{E} = h(J)$. Verify, using $P = \rho_{\text{Ref}} g^\sharp(\partial\hat{E}/\partial F)$ and $\sigma = 2\rho(\partial\hat{E}/\partial g)$ that the same formula for the Cauchy stress results: $\sigma = \rho h' g^\sharp$.

Box 2.1 ***Entropy and Temperature as Conjugate Variables***

Recall that given a Lagrangian $L(q^i, \dot{q}^j)$, one defines the momentum by $p_j = \partial L/\partial\dot{q}^j$ and the energy by $H(q^i, p_j) = p_j\dot{q}^j - L(q^j, \dot{q}^j)$, assuming $p_j = \partial L/\partial\dot{q}^j$ defines a legitimate change of variables $(q^i, \dot{q}^j) \mapsto (q^i, p_j)$. The relationship $N = -\partial\hat{\Psi}/\partial\Theta$ is analogous to $p_j = \partial L/\partial\dot{q}^j$.

Suppose $\hat{\Psi}$ is regarded as a function of ϕ and Θ, and the change of variables from (ϕ, Θ) to (ϕ, N) given by $\Theta \mapsto \Theta(\phi, N)$ is legitimate (invertible). Then the formula

$$\hat{E} = \Theta N + \hat{\Psi} \tag{1}$$

for the internal energy is a partial Legendre transform, so $\hat{E}$ is now a function of ϕ and N. (If $\hat{\Psi}$ is $-L$ above, then H is $\hat{E}$ so an overall sign is off; but this is convention.) We claim that our formula

$$\hat{P}_a{}^A = \rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial F^a{}_A} \quad \text{(variables } \phi, \Theta) \tag{2}$$

is equivalent to

$$\hat{P}_a{}^A = \rho_{\text{Ref}}\frac{\partial\hat{E}}{\partial F^a{}_A} \quad \text{(variables } \phi, N). \tag{3}$$

Indeed, from (1), with Θ a function of (ϕ, N), we get

$$\frac{\partial \hat{E}}{\partial F^a{}_A} = \frac{\partial \Theta}{\partial F^a{}_A} N + \left(\frac{\partial \hat{\Psi}}{\partial F^a{}_A} + \frac{\partial \hat{\Psi}}{\partial \Theta}\frac{\partial \Theta}{\partial F^a{}_A}\right).$$

The first and last terms cancel since $N = -\partial\hat{\Psi}/\partial\Theta$, so $\partial\hat{E}/\partial F^a{}_A = \partial\hat{\Psi}/\partial F^a{}_A$. Thus (2) and (3) are equivalent.

In the same way we have the spatial form, $\partial\hat{\psi}/\partial \boldsymbol{g} = \partial\hat{e}/\partial \boldsymbol{g}$, where $\hat{\psi}$ is a function of $\boldsymbol{g}$, ϕ and θ and $\hat{e}$ is a function of $\boldsymbol{g}$, ϕ and η. Thus, Proposition 2.12 is consistent with the results of Box 3.3, Chapter 2; in that box, if constitutive assumptions are added, the basic variables should be $\boldsymbol{g}$, ϕ and η.

Problem 2.5 Prove that $\Theta = \partial\hat{E}/\partial N$ and $\theta = \partial\hat{e}/\partial\eta$.

The fact that Θ and N are analogous to velocity and momentum indicates that results of symplectic geometry are relevant in thermodynamics. See Box 6.1, Chapters 1 and 5, and, for example, Oster and Perelson [1973].

Box 2.2 ***Summary of Important Formulas for Section 3.2***

If the constitutive function $\hat{\Psi}$ satisfies the axioms of locality, entropy production, and material frame indifference, then $\hat{\Psi}$ is a function only of X, $\boldsymbol{C}$, and Θ.

Constitutive Identities

$$\hat{N} = -\frac{\partial\hat{\Psi}}{\partial\Theta}$$

$$\hat{\boldsymbol{S}} = 2\rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial \boldsymbol{C}} \qquad \hat{S}^{AB} = 2\rho_{\text{Ref}}\frac{\partial\hat{\Psi}}{\partial C_{AB}}$$

$$\hat{\boldsymbol{P}} = \rho_{\text{Ref}}\boldsymbol{g}^\sharp\frac{\partial\hat{\Psi}}{\partial \boldsymbol{F}} = \boldsymbol{F}\hat{\boldsymbol{S}} \qquad \hat{P}^{aB} = \rho_{\text{Ref}}g^{ab}\frac{\partial\hat{\Psi}}{\partial F^b{}_A} = F^a{}_B\hat{S}^{BA}$$

$$\hat{\boldsymbol{\sigma}} = 2\rho\frac{\partial\hat{\psi}}{\partial \boldsymbol{g}} \qquad \hat{\sigma}^{ab} = 2\rho\frac{\partial\hat{\psi}}{\partial g_{ab}}$$

Entropy Production Inequality

$\langle\hat{\boldsymbol{Q}}, \text{GRAD}\,\Theta\rangle \le 0 \qquad \hat{Q}^A\Theta_{|A} \le 0$

$\hat{\boldsymbol{Q}}$ vanishes when its argument GRAD Θ does.

Balance of Energy ($\hat{E} = \hat{\Psi} + \hat{N}\Theta$)

$$\rho_{\text{Ref}}\Theta\frac{\partial\hat{N}}{\partial t} + \text{DIV}\,\hat{\boldsymbol{Q}} = \rho_{\text{Ref}}R \qquad \rho_{\text{Ref}}\Theta\frac{\partial\hat{N}}{\partial t} + \hat{Q}^A{}_{|A} = \rho_{\text{Ref}}R$$

Rigid Heat Conductor ($\boldsymbol{C} \equiv \boldsymbol{0}$)	
$\left(\rho_{\mathrm{Ref}}\Theta\,\dfrac{\partial\hat{N}}{\partial\Theta}\right)\dfrac{\partial\Theta}{\partial t} + \mathrm{DIV}\,\hat{\boldsymbol{Q}} = \rho_{\mathrm{Ref}}R$	$\left(\rho_{\mathrm{Ref}}\Theta\,\dfrac{\partial\hat{N}}{\partial\Theta}\right)\dfrac{\partial\Theta}{\partial t} = -\left(\dfrac{\partial\hat{Q}^A}{\partial\Theta_{\mid B}}\right)\Theta_{\mid B\mid A} - \dfrac{\partial\hat{Q}^A}{\partial\Theta}\Theta_{\mid A} + \rho_{\mathrm{Ref}}R - R_I$
Material Frame Indifference	
If ξ is an isometry, then $\xi_*\hat{\boldsymbol{P}}(X, \boldsymbol{F}, \Theta) = \hat{\boldsymbol{P}}(X, \xi_*\boldsymbol{F}, \Theta)$. This implies $\hat{\boldsymbol{P}}$ depends only on X, $\boldsymbol{C}$, and Θ.	For any proper orthogonal matrix $R^a{}_b$, $R^a{}_b\hat{P}^{bA}(X, F^c{}_G, \Theta) = \hat{P}^{aA}(X, R^c{}_bF^b{}_G, \Theta)$.
Legendre Transformation: $\Theta \mapsto \Theta(\phi, N)$	
$\dfrac{\partial\hat{\Psi}}{\partial\boldsymbol{F}} = \dfrac{\partial\hat{E}}{\partial\boldsymbol{F}}, \quad \dfrac{\partial\hat{\Psi}}{\partial\boldsymbol{C}} = \dfrac{\partial\hat{E}}{\partial\boldsymbol{C}}, \quad \dfrac{\partial\hat{\psi}}{\partial\boldsymbol{g}} = \dfrac{\partial\hat{e}}{\partial\boldsymbol{g}}$	$\dfrac{\partial\hat{\Psi}}{\partial F^a{}_A} = \dfrac{\partial\hat{E}}{\partial F^a{}_A}, \quad \dfrac{\partial\hat{\Psi}}{\partial C_{AB}} = \dfrac{\partial\hat{E}}{\partial C_{AB}}, \quad \dfrac{\partial\hat{\psi}}{\partial g_{ab}} = \dfrac{\partial\hat{e}}{\partial g_{ab}}$

3.3 COVARIANT CONSTITUTIVE THEORY

We shall now obtain the principal theorems of constitutive theory by using a different set of axioms. The idea is to reproduce the theorems of Section 3.2 from a point of view that is covariant, not relying on the rigid Euclidean structure of $\mathbb{R}^3$. (The results therefore directly generalize to manifolds, but this is not our main motivation.) In Section 3.2 and in Box 3.3, Chapter 2, we derived the basic relationship between stress and free energy: $\boldsymbol{\sigma} = 2\rho(\partial\hat{\psi}/\partial\boldsymbol{g})$. This relationship enables us to obtain a covariant description of constitutive theory, as well as of balance of energy. As in the previous section, the hypotheses here are intended to be relevant for thermoelasticity. For other continuum theories they must be modified using more general assumptions as indicated in Section 3.1.

We shall begin by discussing pure elasticity. This will involve combining Theorem 2.6 with a covariant version of material frame indifference. Secondly, we discuss thermoelasticity. Covariance assumptions allow one to make the conclusions of the Coleman–Noll Theorem (2.4) from balance of energy alone, *without* invoking the Clausius–Duhem inequality.

3.1 Covariant Constitutive Axioms (*Elasticity*) *Let* $\hat{E}\colon \mathcal{C} \longrightarrow \mathfrak{F}$ *be a given differentiable map. Assume:*

1. $\hat{E}$ *is local.*
2. *There is a map* $\hat{\boldsymbol{P}}\colon \mathcal{C} \longrightarrow$ *two-point tensors, such that for all motions* ϕ_t,

balance of energy holds (see 3.5 of Chapter 2):

$$\rho_{\text{Ref}} \frac{\partial \hat{E}}{\partial t} = \hat{\boldsymbol{P}} : \dot{\boldsymbol{F}}.$$

3. *Let* $\boldsymbol{g}$ *be a given metric on* $\mathcal{S}$ *and let* $\mathcal{O}_g$ *denote the set of metrics of the form* $\xi_*\boldsymbol{g}$*, where* $\xi\colon \mathcal{S} \longrightarrow \mathcal{S}$ *is a diffeomorphism. Assume there is a map* $\hat{\hat{E}}\colon \mathcal{C} \times \mathcal{O}_g \longrightarrow \mathcal{F}$ *such that*

$$\hat{E}(\phi) = \hat{\hat{E}}(\xi \circ \phi, \xi_* \boldsymbol{g})$$

for all diffeomorphisms $\xi\colon \mathcal{S} \longrightarrow \mathcal{S}$. *(Taking* $\xi =$ *identity, note that* $\hat{E}(\phi) = \hat{\hat{E}}(\phi, \boldsymbol{g})$.*)*

Axioms 1 and 2 were discussed in the previous section. Axiom 3 may be interpreted as follows: think of ξ as being either a coordinate chart for $\mathcal{S}$, or a framing at one instant (see Box 4.1, Chapter 1). Axiom 3 states that one can compute $\hat{E}(\phi)$ in terms of the representative of ϕ and the representative of the metric in the framing ξ.

The following remarks are intended to make Axiom 3 plausible:

3.2 Thought Experiment to Justify Axiom 3 Imagine that ξ is a (nonlinear) coordinate chart. The representation of a configuration ϕ is $\xi \circ \phi$. (For example, if ϕ is given in Cartesian coordinates, ξ might be the coordinate change to spherical coordinates and $\xi \circ \phi$ becomes ϕ written in spherical coordinates.) How can the observer ξ determine the internal energy in the configuration he sees to be $\xi \circ \phi$? He could, for example, perform some experiment to see how much work this configuration can do, such as in Figure 3.3.1. The measurement of the magnitude of velocity uses the observers metric—namely, $\xi_*\boldsymbol{g}$. Thus, the internal energy also should require a knowledge of $\xi \circ \phi$ and $\xi_* \boldsymbol{g}$ for its measurement.

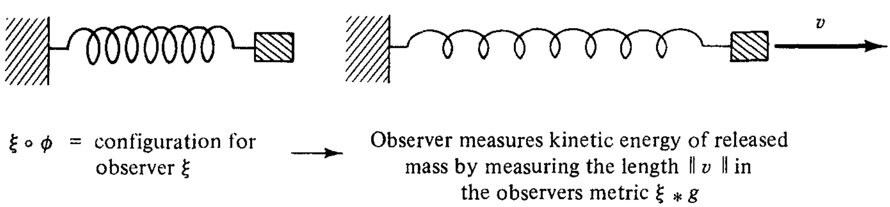

Figure 3.3.1 An observer ξ measuring the energy in a configuration.

3.3 Theorem *Assume Axioms* 1, 2, *and* 3. *Then*

(i) $\hat{E}$ *depends on* ϕ *only through the point values of* $\boldsymbol{C} = \phi^*\boldsymbol{g}$ *and*

(ii) $2\rho_{\text{Ref}}(\partial \hat{E}/\partial \boldsymbol{C}) = \boldsymbol{S}$*; in paticular,* $\boldsymbol{S}$ *is symmetric.*

Proof Suppose that $\phi_1^*\boldsymbol{g} = \phi_2^*\boldsymbol{g}$. Letting $\xi = \phi_2 \circ \phi_1^{-1}$, we see that $\xi_*\boldsymbol{g} = \boldsymbol{g}$ and so

$$\hat{E}(\phi_1) = \hat{\hat{E}}(\xi \circ \phi_1, \xi_* \boldsymbol{g}) = \hat{\hat{E}}(\phi_2, \boldsymbol{g}) = \hat{E}(\phi_2).$$

Thus $\hat{E}$ depends on ϕ only through C. Axiom 2 shows that $\hat{E}$ depends only on the point values of C and that $2\rho_{\text{Ref}}(\partial\hat{E}/\partial C) = S$ holds, by the same argument as in Theorem 2.4 (cf. Theorem 2.6). ∎

Axiom 3 is equivalent to the usual form of material frame indifference. Indeed, if $\hat{E}$ depends only on $\phi^*g = C$, then $\hat{\hat{E}}(\xi\circ\phi, \xi_*g) = \hat{E}(\phi)$ is well defined, for $(\xi\circ\phi)^*(\xi_*g) = \phi^*g = C$; so from $\xi\circ\phi$ and ξ_*g we can construct C.

Once $\hat{E}$ is a function of C, it is reasonable to ask how it depends on the representation of the reference configuration. In order to form a scalar from C, the metric G on $\mathcal{B}$ is needed. Relative to various representations $\Xi: \mathcal{B} \longrightarrow \mathcal{B}$ of $\mathcal{B}$, G will look different. In fact, under such a change of reference configuration, G changes to Ξ_*G and ϕ changes to $\phi\circ\Xi^{-1}$, so C changes to Ξ_*C. The assertion that $\hat{E}$ changes correspondingly is the content of the next concept.

3.4 Definition We say that $\hat{E}$ is *materially covariant* if there is a function

$$\check{E}: \mathcal{O}_G \times \mathcal{C} \longrightarrow \mathcal{F}$$

such that
$$\hat{E}(\phi) = \check{E}(G, \phi)$$
and
$$\check{E}(\Xi^*G, \phi\circ\Xi) = \check{E}(G, \phi)\circ\Xi$$
for every diffeomorphism $\Xi: \mathcal{B} \longrightarrow \mathcal{B}$.

When these axioms hold, we say that $\hat{E}$ is a *tensorial function* of G and C. Material covariance is essentially equivalent to isotropy of the material. The precise situation is given in Section 5.

The overall situation is indicated in Figure 3.3.2.

Problem 3.1 Define the *neo-hookean* constitutive function by

$$E(G, C) = \lambda_1^2 + \cdots + \lambda_n^2 - n,$$

where $n = \dim \mathcal{B} = \dim \mathcal{S}$ and $\lambda_1, \ldots, \lambda_n$ are the principal stretches—that is, eigenvalues of $\sqrt{C}$ using the metric G. Show that this is a materially covariant energy function satisfying Axioms 1–3.

Next we discuss the role of covariance in thermoelasticity. There are numerous ways of modifying the axioms in Section 3.2 to take into account the extra information covariance gives (see Problems 3.3, 3.4, and 3.5). We shall bypass the Clausius–Duhem inequality and derive all the relations in 2.4 by making reasonable covariance assumptions and assuming balance of energy. *As in the covariant approach in Box* 3.3, *Chapter* 2, *we focus not on all processes, but rather on all transformations of a given process.* Comparing the original and transformed process enables us to cancel out the heat source term R in balance of energy. This was the principal obstacle previously and the reason for the success of the entropy production inequality in 2.4.

If one has obtained the identities $\hat{N} = -\partial\hat{\Psi}/\partial\Theta$ and $P_a{}^A = \rho_{\text{Ref}}(\partial\hat{\Psi}/\partial F^a{}_A)$ by other means, then the entropy production inequality is *equivalent* to $\langle Q, \text{GRAD}\,\Theta\rangle \leq 0$.

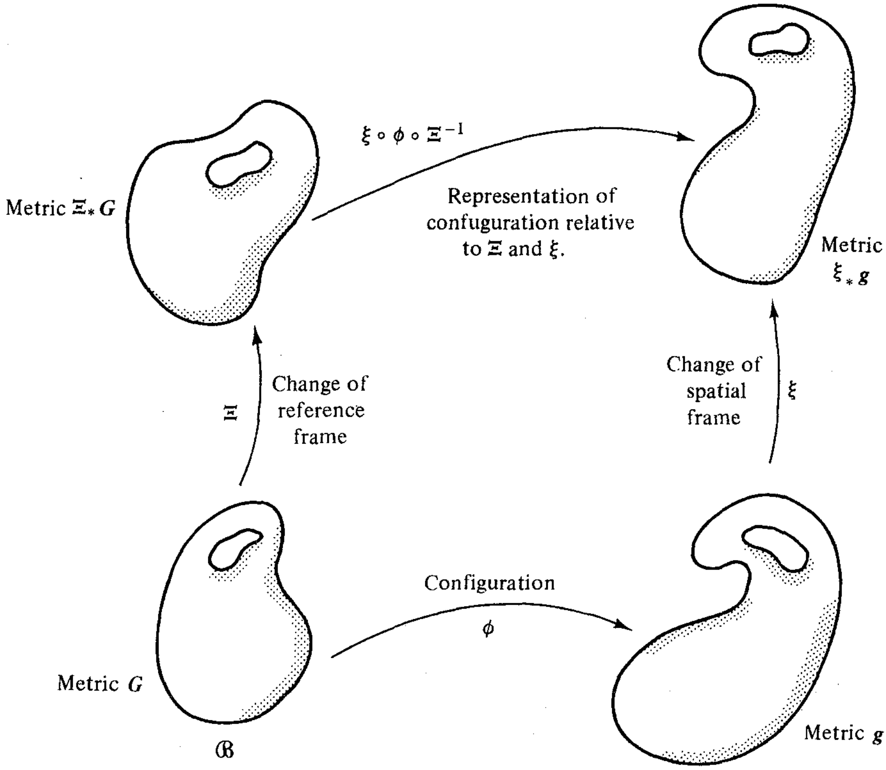

Figure 3.3.2

Our axioms involve not only coordinate changes $\xi: \mathcal{S} \longrightarrow \mathcal{S}$ of space, but also temperature rescalings (i.e., representations of the temperature in various units). This is expressed in terms of the space of monotone increasing diffeomorphisms $r: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. For our purposes it is enough to consider linear rescalings.[3]

3.5 Covariant Constitutive Axioms (*Thermoelasticity*) *Assume* $\hat{\Psi}$, $\hat{\boldsymbol{P}}$, $\hat{\boldsymbol{Q}}$, *and* $\hat{N}$ *are given thermoelastic constitutive functions satisfying:*

(1) *The axiom of locality.*

[3]We could have expressed our axioms in terms of a metric on $\mathcal{S} \times \mathbb{R}^+$ and transformations of the whole space $\mathcal{S} \times \mathbb{R}^+$. This may be important for theories with internal variables, for example, but for thermoelasticity, direct temperature rescalings are easier to understand. The relationship to metrics is as follows: In one dimension a diffeomorphism $r: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ can be reconstructed from $r_*\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the standard metric on $\mathbb{R}$. If we identify metrics with positive functions, then $\boldsymbol{\delta} = 1$ and $r_*\boldsymbol{\delta} = (dr/dx)^2$, so $r(x) = \int^x \sqrt{r_*\boldsymbol{\delta}}\, dx +$ constant. Here $r_*\boldsymbol{\delta}$ plays the same role as ξ_*g.

(2) *For a given process ϕ_t, Θ_t, balance of energy*[4] *holds:*

$$\rho_{\mathrm{Ref}} \frac{\partial E}{\partial t} + \mathrm{DIV}\, \boldsymbol{Q} = \boldsymbol{P}: \dot{\boldsymbol{F}} + \rho_{\mathrm{Ref}} R,$$

where $E = \Psi + N\Theta$.

(3) *There is a map $\hat{\hat{\Psi}}: \mathcal{C} \times \mathcal{O}_g \times \mathfrak{I} \times \mathbb{R}^+ \longrightarrow \mathfrak{F}$ such that for any diffeomorphism $\xi: \mathcal{S} \longrightarrow \mathcal{S}$ and any $r: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$,*

$$\hat{\Psi}(\phi, \Theta) = \hat{\hat{\Psi}}(\xi \circ \phi, \xi_* g, r\Theta, r)$$

(4) *For curves $\xi_t: \mathcal{S} \longrightarrow \mathcal{S}$ and $r_t(x) \in \mathbb{R}^+$, assume that $\phi_t' = \xi_t \circ \phi_t$ and $\Theta_t' = r_t \Theta_t$ satisfy balance of energy where we demand the transformation properties in 4.12, Chapter 2, except that Ψ and N should transform as scalars, $\boldsymbol{Q}_t' = r_t \xi_{t*} \boldsymbol{Q}_t$ and $(R_t' - \Theta_t'(\partial N_t'/\partial t)) = r_t \cdot (R_t - \Theta_t(\partial N_t/\partial t))$. (This latter transformation formula accounts for the "apparent heat supply" due to entropy production and is analogous to the transformation formula for body forces in 4.12, Chapter 2, in which there are "apparent body forces" due to the velocity and acceleration of ξ_t.)*

3.6 Theorem *Under these assumptions, $\hat{\Psi}$ depends only on the point values of $\boldsymbol{C}$ and Θ and we have*

$$\boxed{\hat{N} = -\frac{\partial \hat{\Psi}}{\partial \Theta}, \qquad \boldsymbol{S} = 2\rho_{\mathrm{Ref}} \frac{\partial \hat{\Psi}}{\partial \boldsymbol{C}} \quad \left(\text{or equivalently, } P_a{}^A = \rho_{\mathrm{Ref}} \frac{\partial \hat{\Psi}}{\partial F^a{}_A}\right).}$$

Proof Using the notation of 2.4, balance of energy reads:

$$\rho_{\mathrm{Ref}}\left(D_\phi \hat{\Psi} \cdot V + D_\Theta \hat{\Psi} \cdot \dot{\Theta} + \frac{\partial \hat{N}}{\partial t}\Theta + \hat{N}\dot{\Theta}\right) + \mathrm{DIV}\, \boldsymbol{Q} = \boldsymbol{P}: \dot{\boldsymbol{F}} + \rho_{\mathrm{Ref}} R.$$

If we write down the corresponding expression for the primed quantities at $\xi =$ identity, $\dot{\xi} = w$, $r = 1$, $\dot{r} = u$, and subtract we are led to the identity

$$\rho_{\mathrm{Ref}}(D_\phi \hat{\Psi} \cdot w + D_\Theta \hat{\Psi} \cdot (u\Theta) + \hat{N} \cdot (u\Theta)) = \boldsymbol{P}: \nabla w.$$

Thus we get the two identities

$$\rho_{\mathrm{Ref}}(D_\phi \hat{\Psi} \cdot w) = \boldsymbol{P}: \nabla w \tag{1}$$

and

$$D_\Theta \hat{\Psi} \cdot (u\Theta) + \hat{N} \cdot (u\Theta) = 0 \tag{2}$$

Since ∇w and $u = \dot{r}$ are arbitrary, the identities stated in the Theorem and the dependence on point values now follows by the same argument used to analyze (1) and (2) in the proof of Theorem 2.4. Axiom 3 implies material frame indifference; so, as usual, $\hat{\Psi}$ depends only on $\boldsymbol{C}$. ∎

Problem 3.2 (J. Ball) Show that the entropy production inequality for all subregions of $\mathcal{B}$ follows from that for all of $\mathcal{B}$ and covariance under all superposed motions ξ_t.

---

[4]We have written balance of energy in a simplified form in which balance of momentum has been used. Of course one could go back to the primitive form of balance of energy and *derive* balance of momentum using Box 3.3, Chapter 2.

Problem 3.3 (M. Gurtin) Derive the constitutive identities by assuming an appropriate transformation property of the rate of entropy production

$$\Gamma = \rho_{\mathrm{Ref}}(N\dot{\Theta} + \dot{\Psi}) - \boldsymbol{S}{:}\boldsymbol{D} + \frac{1}{\Theta}\langle \boldsymbol{Q}, \nabla\Theta\rangle.$$

Problem 3.4 Show that preservation of the Clausius–Duhem inequality under superposed motions $\xi_t(x)$ and temperature rescalings $r_t(x)$ gives a monotonicity condition on $\boldsymbol{Q}$. Is this new information?

Box 3.1 ***The Duhamel–Neumann Hypothesis***[5]

The purpose of this box is to prove a decomposition of the rate of deformation tensor, which generalizes the "Duhamel–Neumann hypothesis" (Sokolnikoff [1956], p. 359). This decomposition, often made on an ad hoc basis, has proven to be very useful in the identification of constitutive functions. We shall confine ourselves to thermoelasticity, although more general theories using internal variables can be used as well. We assume the existence of a free energy Ψ, a function of the Cauchy–Green tensor C_{AB}, and the temperature Θ; that is, i.e. $\Psi = \hat{\Psi}(C_{AB}, \Theta)$. We also assume the usual relationship between the second (symmetric) Piola–Kirchhoff stress tensor S^{AB} and $\hat{\Psi}$, derived in this and the previous sections, namely,

$$S^{AB} = 2\rho_{\mathrm{Ref}}\frac{\partial\hat{\Psi}}{\partial C_{AB}} \tag{1}$$

Our derivation of the Duhamel–Neumann hypothesis depends on recasting this into an *equivalent* spatial form, and relating the free energy $\hat{\psi}$ in spatial coordinates and the *Kirchhoff stress* $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ (= Jacobian of the deformation $\times$ Cauchy stress), namely,

$$\boldsymbol{\tau} = 2\rho_{\mathrm{Ref}}\frac{\partial\hat{\psi}}{\partial \boldsymbol{g}}, \qquad \text{that is,} \quad \tau^{ab} = 2\rho_{\mathrm{Ref}}\frac{\partial\hat{\psi}}{\partial g_{ab}}, \tag{2}$$

where we take $\hat{\psi}$ to be a function of the spatial metric $\boldsymbol{g}$, the (left) Cauchy–Green tensor $\boldsymbol{c}$ and the temperature θ. We have derived (2) in the first section of this chapter and showed its equivalence to (1). The idea now is to manipulate (2) using the Lie derivative and a Legendre transform. In particular, we recall that the rate of deformation tensor $d_{ab} = \frac{1}{2}(v_{a|b} + v_{b|a})$ is given in terms of the Lie derivative by

$$\boldsymbol{d} = \tfrac{1}{2}\boldsymbol{L}_v \boldsymbol{g}, \tag{3}$$

where $\boldsymbol{v}$ denotes the spatial velocity field of a given motion and $\boldsymbol{L}_v$ denotes Lie differentiation with respect to $\boldsymbol{v}$ (see Section 1.6). For a

[5]The results of this box were obtained in collaboration with K. Pister.

scalar function f of position and time, $L_v f = \dot{f} = \partial f/\partial t + v \cdot \nabla f$ is the material derivative. By the chain rule,

$$L_v \frac{\partial \hat{\psi}}{\partial g} = \frac{\partial^2 \hat{\psi}}{\partial g^2} L_v g + \frac{\partial^2 \hat{\psi}}{\partial g\, \partial c} L_v c + \frac{\partial^2 \hat{\psi}}{\partial g\, \partial \theta} \dot{\theta}.$$

By definition (3) and the simple identity $L_v c = \mathbf{0}$, we get

$$L_v \frac{\partial \hat{\psi}}{\partial g} = 2 \frac{\partial^2 \hat{\psi}}{\partial g^2} d + \frac{\partial^2 \hat{\psi}}{\partial g\, \partial \theta} \dot{\theta}. \tag{4}$$

Since $\dot{\rho}_{\mathrm{Ref}} = 0$, combining (4) and (3) yields the following:

3.7 Proposition

$$\boxed{L_v \tau = \rho_{\mathrm{Ref}}(\mathsf{a} \cdot 2d + m\dot{\theta}),} \tag{5}$$

where

$$\mathsf{a} = 2 \frac{\partial^2 \hat{\psi}}{\partial g^2} \tag{6}$$

is the tangential mechanical stiffness *and*

$$m = 2 \frac{\partial^2 \hat{\psi}}{\partial g\, \partial \theta} \tag{7}$$

are the thermal stress coefficients. *Using indices, Equations* (*5*), (*6*), *and* (*7*) *read:*

$$(L_v \tau)^{ab} = \rho_{\mathrm{Ref}}(2\mathsf{a}^{abcd}\, d_{cd} + m^{ab}\dot{\theta}), \tag{5$'$}$$

$$\mathsf{a}^{abcd} = 2 \frac{\partial^2 \hat{\psi}}{\partial g_{ab}\, \partial g_{cd}}, \tag{6$'$}$$

and

$$m^{ab} = 2 \frac{\partial^2 \hat{\psi}}{\partial g_{ab}\, \partial \theta}. \tag{7$'$}$$

For isothermal deformations we can regard θ as absent; so (5) reduces to $L_v \tau = \rho_{\mathrm{Ref}} \cdot \mathsf{a}(2d)$, an incremental form of the mechanical constitutive relation.

Now we perform a Legendre transformation to obtain the inverse form of (5). Define the *complementary free energy* by

$$\begin{aligned} \chi &= \hat{\chi}(\tau, c, \theta) \\ &= \frac{1}{\rho_{\mathrm{Ref}}} \operatorname{tr}(\tau \cdot g) - 2\hat{\psi}(g, c, \theta), \end{aligned} \tag{8}$$

We are assuming that the change of variables from g to τ defined by (2) is invertible.[6]

[6]This can be justified (locally at least) if we make the usual hypothesis that the material is strongly elliptic; the result then follows from the inverse function theorem. See Chapters 4, 5, and 6 for the necessary background.

In formula (8), $\operatorname{tr}(\boldsymbol{\tau}\cdot \boldsymbol{g}) = \operatorname{trace}(\tau^{ab}g_{bc}) = \tau^{ab}g_{ab} = \boldsymbol{\tau} : \boldsymbol{g}$, the full contraction of $\boldsymbol{\tau}$ and $\boldsymbol{g}$. We then have

$$\rho_{\text{Ref}}\frac{\partial \hat{\chi}}{\partial \boldsymbol{\tau}} = \boldsymbol{g} + \operatorname{tr}\left(\boldsymbol{\tau}\cdot\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\tau}}\right) - 2\rho_{\text{Ref}}\frac{\partial \hat{\psi}}{\partial \boldsymbol{g}}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\tau}},$$

that is,

$$\rho_{\text{Ref}}\frac{\partial \hat{\chi}}{\partial \tau^{ab}} = g_{ab} + \tau^{cd}\frac{\partial g_{cd}}{\partial \tau^{ab}} - 2\rho_{\text{Ref}}\frac{\partial \hat{\psi}}{\partial g_{cd}}\frac{\partial g_{cd}}{\partial \tau^{ab}}.$$

By (2), the last two terms cancel, leaving

$$\boldsymbol{g} = \rho_{\text{Ref}}\frac{\partial \hat{\chi}}{\partial \boldsymbol{\tau}}. \tag{9}$$

Operating on (9) with the Lie derivative gives

$$2\boldsymbol{d} = \rho_{\text{Ref}}\frac{\partial^2 \hat{\chi}}{\partial \boldsymbol{\tau}^2}\cdot \boldsymbol{L}_v\boldsymbol{\tau} + \rho_{\text{Ref}}\frac{\partial^2 \hat{\chi}}{\partial \boldsymbol{\tau}\,\partial \theta}\dot{\theta} \tag{10}$$

by (3) and $\boldsymbol{L}_v\boldsymbol{c} = 0$. Define the *material compliance tensors*

$$\mathsf{r} = \frac{\partial^2 \hat{\chi}}{\partial \boldsymbol{\tau}^2} \quad\text{and}\quad \boldsymbol{s} = \frac{\partial^2 \hat{\chi}}{\partial \boldsymbol{\tau}\,\partial \theta} \tag{11}$$

so that (10) yields:

3.8 Theorem

$$\boxed{2\boldsymbol{d} = \rho_{\text{Ref}}(\mathsf{r}\cdot \boldsymbol{L}_v\boldsymbol{\tau} + \boldsymbol{s}\dot{\theta})} \tag{12}$$

In componential form these read

$$\mathsf{r}_{abcd} = \frac{\partial^2 \hat{\chi}}{\partial \tau^{ab}\,\partial \tau^{cd}} \quad\text{and}\quad s_{ab} = \frac{\partial^2 \hat{\chi}}{\partial \tau^{ab}\,\partial \theta} \tag{11$'$}$$

and

$$2d_{ab} = \rho_{\text{Ref}}\mathsf{r}_{abcd}(\boldsymbol{L}_v\boldsymbol{\tau})^{cd} + \rho_{\text{Ref}}s_{ab}\dot{\theta}, \tag{12$'$}$$

where

$$(\boldsymbol{L}_v\boldsymbol{\tau})^{cd} = \frac{\partial \tau^{cd}}{\partial t} - \tau^{ad}v^c{}_{|a} - \tau^{ca}v^d{}_{|a}.$$

Equation (12) has the following interpretation:
The total deformation rate = the mechanical rate + the thermal rate. This generalizes the Duhamel–Neumann hypothesis for infinitesimal (linearized) thermoelasticity.

3.9 Remarks (1) Differentiating (9) in $\boldsymbol{g}$, we have

$$\mathbf{I} = \rho_{\text{Ref}}\frac{\partial}{\partial \boldsymbol{g}}\left(\frac{\partial \hat{\chi}}{\partial \boldsymbol{\tau}}\right) = \rho_{\text{Ref}}\frac{\partial^2 \hat{\chi}}{\partial \boldsymbol{\tau}^2}\cdot\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{g}},$$

where $\mathbf{I}$ is the fourth-order Kroneker delta with components $\delta^{ef}_{ab} = 1$ if $(e, f) = (a, b)$ and 0 otherwise. Substituting from (3), (6), and (11), we get

$$\mathsf{I} = \left(\rho_{\mathrm{Ref}} \frac{\partial^2 \hat{\chi}}{\partial \boldsymbol{\tau}^2}\right) 2\rho_{\mathrm{Ref}} \frac{\partial^2 \hat{\psi}}{\partial g^2} = (\rho_{\mathrm{Ref}} \mathsf{r}) \cdot (\rho_{\mathrm{Ref}} \mathsf{a}),$$

that is,

$$\delta_{ab}^{ef} = \rho_{\mathrm{Ref}}^2 \mathsf{r}_{abcd} \mathsf{a}^{cdef}.$$

In other words, $\rho_{\mathrm{Ref}} \mathsf{r}$ and $\rho_{\mathrm{Ref}} \mathsf{a}$ are inverse tensors.

(2) As explained in Box 6.1, Chapter 1, various stress rates in common use can be related to the Lie derivative of tensors associated to $\boldsymbol{\tau}$ or $\boldsymbol{\sigma}$. The Lie derivative of $\boldsymbol{\tau}$ (the "Truesdell stress rate", see Box 6.1, Chapter 1) is the most useful for the present context.

Problem 3.5 Do all this materially.

Box 3.2 Summary of Important Formulas for Section 3.3

Internal Energy Under a Change of Spatial Frame

$$E(\phi) = \hat{\hat{E}}(\xi \circ \phi, \xi_* g), \qquad \xi : \mathcal{S} \to \mathcal{S}$$

Material Covariance (Change of Reference Frame)

$$\check{E}(\Xi^* G, \phi \circ \Xi) = \check{E}(G, \phi) \circ \Xi$$

Covariance

Covariance of energy balance under superposed motions and temperature recalings implies

$$\hat{N} = -\frac{\partial \hat{\Psi}}{\partial \Theta} \quad \text{and} \quad \hat{S} = 2\rho_{\mathrm{Ref}} \frac{\partial \hat{\Psi}}{\partial C}.$$

Kirchhoff Stress

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \qquad\qquad \tau^{ab} = J\sigma^{ab}$$

Mechanical Stiffness and Thermal Stress Tensors

$$\mathsf{a} = 2\frac{\partial^2 \hat{\psi}}{\partial g^2}, \quad \boldsymbol{m} = 2\frac{\partial^2 \hat{\psi}}{\partial g\, \partial \theta} \qquad \mathsf{a}^{abcd} = 2\frac{\partial^2 \hat{\psi}}{\partial g_{ab}\, \partial g_{cd}}, \quad m^{ab} = 2\frac{\partial^2 \psi}{\partial g_{ab}\, \partial \theta}$$

Stress Rate

$$\boldsymbol{L}_v \boldsymbol{\tau} = \rho_{\mathrm{Ref}}(\mathsf{a} \cdot 2\boldsymbol{d} + \boldsymbol{m}\dot{\theta}) \qquad\qquad (\boldsymbol{L}_v \boldsymbol{\tau})^{ab} = \rho_{\mathrm{Ref}}(2\mathsf{a}^{abcd} d_{cd} + m^{ab}\dot{\theta})$$

Complementary Free Energy

$$\hat{\chi}(\boldsymbol{\tau}, c, \theta) = \frac{1}{\rho_{\mathrm{Ref}}} \boldsymbol{\tau} : g - 2\hat{\psi}(g, c, \theta) \qquad \hat{\chi} = \frac{1}{\rho_{\mathrm{Ref}}} \tau^{ab} g_{ab} - 2\hat{\psi}$$

Compliance Tensors

$$\mathsf{r} = \frac{\partial^2 \hat{\chi}}{\partial \boldsymbol{\tau}^2}, \quad s = \frac{\partial^2 \hat{\chi}}{\partial \tau\, \partial \theta} \qquad\qquad \mathsf{r}_{abcd} = \frac{\partial^2 \hat{\chi}}{\partial \tau^{ab}\, \partial \tau^{cd}}, \quad s_{ab} = \frac{\partial^2 \hat{\chi}}{\partial \tau^{ab}\, \partial \theta}$$

Duhamel–Neumann Relation

$$2\boldsymbol{d} = \rho_{\mathrm{Ref}}(\mathsf{r} \cdot \boldsymbol{L}_v \boldsymbol{\tau} + s\dot{\theta}) \qquad\qquad 2d_{ab} = \rho_{\mathrm{Ref}}(r_{abcd}(\boldsymbol{L}_v \boldsymbol{\tau})^{cd} + s_{ab}\dot{\theta})$$

3.4 THE ELASTICITY TENSOR AND THERMOELASTIC SOLIDS

The basic equations of motion for a continuum were derived in Section 2.2:

$$\rho_{\mathrm{Ref}}\boldsymbol{A} = \rho_{\mathrm{Ref}}\boldsymbol{B} + \mathrm{DIV}\,\boldsymbol{P}, \qquad \text{that is,} \quad \rho_{\mathrm{Ref}}\left(\frac{\partial V^a}{\partial t} + \gamma^a_{bc}V^bV^c\right) = \rho_{\mathrm{Ref}}B^a + P^{aA}{}_{|A}.$$

If we use the constitutive hypothesis for a thermoelastic material, $\boldsymbol{P}$ will be a function $\hat{\boldsymbol{P}}$ of X, $\boldsymbol{F}$, and Θ. Then we can compute DIV $\hat{\boldsymbol{P}}$ by the chain rule, as follows:

$$\mathrm{DIV}\,\hat{\boldsymbol{P}} = \frac{\partial\hat{\boldsymbol{P}}}{\partial\boldsymbol{F}}\cdot\nabla_X F + \mathrm{DIV}_X\,\hat{\boldsymbol{P}} + \frac{\partial\hat{\boldsymbol{P}}}{\partial\Theta}\cdot\nabla_X\Theta,$$

where $\mathrm{DIV}_X\,\hat{\boldsymbol{P}}$ means the divergence of $\hat{\boldsymbol{P}}$ holding the variables $\boldsymbol{F}$ and Θ constant. In coordinates, this equation reads as follows:

$$(\mathrm{DIV}\,\hat{\boldsymbol{P}})^a = \frac{\partial\hat{P}^{aA}}{\partial F^b{}_B}F^b{}_{B|A} + \left(\frac{\partial\hat{P}^{aA}}{\partial X^A} + \hat{P}^{aA}\Gamma^B_{AB} + \hat{P}^{aA}\gamma^b_{bc}F^c{}_A\right) + \frac{\partial\hat{P}^{aA}}{\partial\Theta}\frac{\partial\Theta}{\partial X^A},$$

where we have written out $\mathrm{DIV}_X\,\hat{\boldsymbol{P}}$ explicitly using the formula for the covariant derivative of a two-point tensor from Section 1.4, Chapter 1.[7] The covariant derivative of the deformation gradient is

$$F^b{}_{B|A} = \frac{\partial^2\phi^b}{\partial X^A\,\partial X^B} + \frac{\partial\phi^e}{\partial X^B}\gamma^b_{ec}\frac{\partial\phi^c}{\partial X^A} - \frac{\partial\phi^b}{\partial X^C}\Gamma^C_{AB}.$$

Thus the leading term in DIV $\hat{\boldsymbol{P}}$ containing second derivatives of ϕ is

$$\frac{\partial\hat{\boldsymbol{P}}^{aA}}{\partial F^b{}_B}\frac{\partial^2\phi^b}{\partial X^A\,\partial X^B} = \frac{1}{2}\left(\frac{\partial\hat{\boldsymbol{P}}^{aA}}{\partial F^b{}_B} + \frac{\partial\hat{\boldsymbol{P}}^{aB}}{\partial F^b{}_A}\right)\frac{\partial^2\phi^b}{\partial X^A\,\partial X^B}.$$

The term $\partial\hat{P}^{aA}/\partial F^b{}_B$ involved in this calculation is closely related to one of the basic tensors in elasticity theory:

4.1 Definition Let $\hat{\boldsymbol{P}}$ be a constitutive function for the first Piola–Kirchhoff stress for thermoelasticity, depending on the point values of X, $\boldsymbol{F}$, and $\boldsymbol{\theta}$. The (*first*) *elasticity tensor* is the two-point tensor $\mathbf{A}$ defined by

$$\boxed{\mathbf{A} = \frac{\partial\hat{\boldsymbol{P}}}{\partial\boldsymbol{F}}, \qquad \text{that is,} \quad \mathsf{A}^{aA}{}_b{}^B = \frac{\partial\hat{\boldsymbol{P}}^{aA}}{\partial\boldsymbol{F}^b{}_B}.}$$

We shall also write $\mathbf{A}^\flat$ for $\mathbf{A}$ with its first index lowered—that is, $\mathsf{A}_a{}^A{}_b{}^B$—and shall write $\mathbf{A}_s$ for $\mathbf{A}$ symmetrized on its large indices; its components are written as follows:

---

[7]Actually there are technical subtleties involved here in understanding this calculation in non-Euclidean spaces. $\hat{\boldsymbol{P}}$ is a mapping of vector bundles: $\hat{\boldsymbol{P}}$: $\mathcal{E} \longrightarrow \mathcal{F}$, where $\mathcal{E}$ is the bundle over $\mathcal{B}$ whose fiber at $X \in \mathcal{B}$ is the direct sum of the space of linear maps of $T_X\mathcal{B}$ to $T_{\phi(X)}\mathcal{S}$ and the real numbers ($\mathbb{R}$), and $\mathcal{F}$ is the bundle of two-point tensors over ϕ. The fiber derivative $\partial\hat{\boldsymbol{P}}/\partial\boldsymbol{F}$ is well defined, but to write $\mathrm{DIV}_X\,\hat{\boldsymbol{P}} = \mathrm{tr}(\nabla_X\hat{\boldsymbol{P}})$, one has to put a connection on $\mathcal{E}$ and $\mathcal{F}$, compute $\nabla\boldsymbol{P}$, and take its horizontal part. This process is equivalent to the computation given in the text.

$$\mathsf{A}^{a(A}{}_b{}^{B)} = \tfrac{1}{2}(\mathsf{A}^{aA}{}_b{}^B + \mathsf{A}^{aB}{}_b{}^A).$$

The tensor $\mathbf{A}_s{}^\flat$ is defined similarly.

Problem 4.1 Prove that $\mathbf{A}^\flat = \partial\hat{\boldsymbol{P}}^\flat/\partial F$, where $\hat{\boldsymbol{P}}^\flat$ is the tensor with components $\hat{P}_a{}^A$: that is, prove that

$$\mathsf{A}_a{}^A{}_b{}^B = \frac{\partial P_a{}^A}{\partial F^b{}_B}.$$

To exploit material frame indifference and balance of moment of momentum it is useful to work with the second Piola–Kirchhoff stress tensor. This leads to the following.

4.3 Definition Let S be a constitutive function for the second Piola–Kirchhoff stress depending on X, C, and Θ, as in Section 3.2. Then the tensor on the body $\mathcal{B}$ defined by

$$\boxed{\mathbf{C} = \frac{\partial S}{\partial C}, \qquad \text{that is,} \quad \mathsf{C}^{ABCD} = \frac{\partial \hat{S}^{AB}}{\partial C_{CD}},}$$

is called the (*second*) *elasticity tensor* or the *elasticities.*

Notice that $\mathbf{C}$ is a fourth-order tensor on $\mathcal{B}$; that is, it is not a two-point tensor. In the previous two sections, we saw that S and $\hat{P}$ are related to the free energy $\hat{\Psi}$ by $\hat{S}^{AB} = 2\rho_{\text{Ref}}(\partial\hat{\Psi}/\partial C_{AB})$ and $\hat{P}_a{}^A = \rho_{\text{Ref}}(\partial\hat{\Psi}/\partial F^a{}_A)$; so we get the following important formula:

4.4 Proposition

$$\text{(a)} \quad \mathsf{C}^{ABCD} = 2\rho_{\text{Ref}} \frac{\partial^2 \hat{\Psi}}{\partial C_{AB}\,\partial C_{CD}},$$

and so we have the symmetries: $\mathsf{C}^{ABCD} = \mathsf{C}^{BACD} = \mathsf{C}^{ABDC} = \mathsf{C}^{CDAB}$.

$$\text{(b)} \quad \mathsf{A}_a{}^A{}_b{}^B = \rho_{\text{Ref}} \frac{\partial^2 \hat{\Psi}}{\partial F^a{}_A\,\partial F^b{}_B},$$

so we have the symmetry $\mathsf{A}_a{}^A{}_b{}^B = \mathsf{A}_b{}^B{}_a{}^A$.

We can relate $\mathbf{A}$ and $\mathbf{C}$ using the formula $P = FS$, that is, $P^{aA} = F^a{}_B S^{BA}$. The following proposition contains the relevant computations.

4.5 Proposition *The following formulas hold in general coordinates:*

$$\boxed{\begin{aligned} &\text{(a)} \quad \mathsf{A}^{aA}{}_b{}^B = 2\mathsf{C}^{CADB} F^c{}_D F^a{}_C g_{cb} + \hat{S}^{AB}\delta^a_b \\ &\text{(b)} \quad \mathsf{A}_a{}^A{}_b{}^B = 2\mathsf{C}^{CADB} F^c{}_D F^d{}_C g_{cb} g_{da} + \hat{S}^{AB} g_{ab} \end{aligned}}$$

Proof The chain rule applied to $P^{aA} = F^a{}_B S^{BA}$ gives

$$\frac{\partial \hat{P}^{aA}}{\partial F^b{}_B} = \frac{\partial \hat{S}^{CA}}{\partial C_{DE}} \frac{\partial C_{DE}}{\partial F^b{}_B} F^a{}_C + \hat{S}^{CA} \frac{\partial F^a{}_C}{\partial F^b{}_B}$$

From $C_{DE} = F^d{}_D F^c{}_E g_{dc}$, we have

$$\frac{\partial C_{DE}}{\partial F^b{}_B} = \delta^B{}_D F^c{}_E g_{bc} + F^d{}_D \delta^B{}_E g_{db}.$$

Substituting,

$$\frac{\partial \hat{P}^{aA}}{\partial F^b{}_B} = \mathsf{C}^{CADE}(\delta^B{}_D F^c{}_E g_{bc} + F^d{}_D \delta^B{}_E g_{db}) F^a{}_C + \hat{S}^{CA} \delta^a{}_b \delta^B{}_C$$

$$= \mathsf{C}^{CABE} F^c{}_E F^a{}_C g_{bc} + \mathsf{C}^{CADB} F^d{}_D F^a{}_C g_{db} + \hat{S}^{BA} \delta^a{}_b.$$

Using the symmetries $\mathsf{C}^{CABE} = \mathsf{C}^{CAEB}$ and $S^{AB} = S^{BA}$, (a) follows. Part (b) follows by lowering the first index. ▮

The tensor $\mathbf{A}^\flat$ is not necessarily symmetric in each pair of indices AB and ab separately, but only when both pairs are *simultaneously* transposed. In three dimensions it is easy to see that the tensors with this symmetry form (pointwise) a space of dimension 45. However the dimension of the space of tensors with the symmetries of the $\mathbf{C}$ tensor is only 21. Thus in the second elasticity tensor there is less to keep track of, in principle.

Problem 4.2 Write out a set of 21 independent components of $\mathbf{C}$ explicitly.

Often $\hat{S}$ or $\hat{P}$ are taken as primitive objects and are not necessarily assumed to be derived from a free energy function $\hat{\Psi}$. Notice that even if we don't assume $\hat{\Psi}$ exists, C^{ABCD} is always symmetric in AB and CD separately and 4.5 holds. From the Poincaré lemma (Box 7.2, Chapter 1) or basic vector calculus, together with the observation that the set of variables C_{AB} is an open convex cone, we see that the symmetry condition $\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB}$ is *equivalent* to the existence of a free energy function. Also notice that this symmetry is in turn equivalent to 4.4(b). We summarize:

4.6 Proposition *If the axiom of entropy production is (temporarily) dropped, the following assertions are equivalent:*

(i) *There is a function $\hat{\Psi}$ of X, C, and Θ such that*

$$\hat{S}^{AB} = 2\rho_{\text{Ref}} \frac{\partial \hat{\Psi}}{\partial C_{AB}}.$$

(ii) $\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB}$.

(iii) $\mathsf{A}_a{}^A{}_b{}^B = \mathsf{A}_b{}^B{}_a{}^A$.

One sometimes refers to either of these three conditions as defining *hyperelasticity*. Further discussion of these points occurs in Chapter 5.

4.7 Notation In case thermal effects are ignored—that is, if Θ is omitted—we are in the case of isothermal hyperelasticity. We shall mean this if we just say "elasticity" in the future. In this case the free energy $\hat{\Psi}$ coincides with the internal energy $\hat{E}$ and is sometimes denoted W and is called the *stored energy function.*

Thus W will be a function of (X, C) and so

$$\hat{P} = \rho_{\text{Ref}} g^{\sharp} \frac{\partial W}{\partial F}, \qquad \hat{S} = 2\rho_{\text{Ref}} \frac{\partial W}{\partial C}, \quad \text{etc.}$$

The elasticity tensor $\mathbf{C}$ measures how the stresses S are changing with the measures of strain C. In principle, these are measurable for a given material. However, in general, the fact that S is a nonlinear function of C makes this problem of *identification* difficult or impossible in practice. In specific situations, the Duhamel–Neumann hypothesis is sometimes used (see Box 3.1). In the linearized approximation, identification involves measuring 21 numbers—that is, the 21 independent components of $\mathbf{C}$. We shall discuss the process of linearization in Chapter 4.

Now we return to the equations of motion for a thermoelastic material, insert the first elasticity tensor, and formulate the basic boundary value problems. The following notation will be convenient.

4.8 Definition The vector

$$B_I = \mathrm{DIV}_X \hat{P}, \qquad \text{that is,} \quad B_I{}^a = \frac{\partial \hat{P}^{aA}}{\partial X^A} + \hat{P}^{aA}\Gamma^B_{AB} + \hat{P}^{bA}\gamma^a_{bc}F^c{}_A,$$

is called the *resultant force due to inhomogeneities.* (B_I is a function of X, F, and Θ.)

The basic equation of motion reads

$$\rho_{\text{Ref}} \dot{V} = \rho_{\text{Ref}} B + \mathrm{DIV}\, P = \rho_{\text{Ref}} B + \mathbf{A} \cdot \nabla_X F + B_I + \frac{\partial \hat{P}}{\partial \Theta} \cdot \mathrm{GRAD}\, \Theta.$$

This equation is usually thught of as governing the evolution of the configuration ϕ and is coupled to the equation of energy balance that is usually thought of as governing the evolution of Θ. In addition to these evolution equations, some boundary conditions must be imposed. For each of ϕ and Θ there are three types in common use:

4.9 Definition

(I) **Boundary conditions for ϕ:**

(a) *displacement*—ϕ is prescribed on $\partial\mathcal{B}$, the boundary of $\mathcal{B}$;
(b) *traction*—the tractions $\langle P, N \rangle^a = P^{aA}N_A$ are prescribed on $\partial\mathcal{B}$; or
(c) *mixed*—ϕ is prescribed on a part ∂_d of $\partial\mathcal{B}$ and $\langle \hat{P}, N \rangle$ on part ∂_τ of $\partial\mathcal{B}$ where $\partial_d \cap \partial_\tau = \varnothing$ and $\overline{\partial_d \cup \partial_\tau} = \partial\mathcal{B}$.

(II) **Boundary conditions for Θ:**

(a) *prescribed temperature*—Θ is prescribed on $\partial\mathcal{B}$ (= Dirichlet boundary conditions);
(b) *prescribed flux*—$\langle Q, N \rangle$ is precribed on $\partial\mathcal{B}$ (= Neumann boundary conditions); or
(c) *mixed*—Θ is prescribed on a part ∂_Θ of $\partial\mathcal{B}$ and $\langle Q, N \rangle$ on another part ∂_f of $\partial\mathcal{B}$ where $\partial_\Theta \cap \partial_f = \varnothing$, $\overline{\partial_\Theta \cup \partial_f} = \partial\mathcal{B}$.

Notice that the conditions I(b) and II(b) are, in general, nonlinear boundary conditions because $\hat{\boldsymbol{P}}$ and $\hat{\boldsymbol{Q}}$ are nonlinear functions of $\boldsymbol{F}$ and Θ.

Prescribing the traction $\langle \boldsymbol{P}, \boldsymbol{N} \rangle$ to be constant is an example of *dead loading* (see Figure 3.4.1). The reason for this is that $\langle \boldsymbol{P}, \boldsymbol{N} \rangle$ is actually a traction vector that assigns to $X \in \mathcal{B}$ a vector attached to the new configuration point $\phi(X, t) = x$. This traction vector is assigned in advance, independent of x. Notice, therefore, that the dead loading boundary condition requires $\mathcal{S}$ to be, for example, a linear space for it to make sense, since it implicitly identifies vectors at different points.

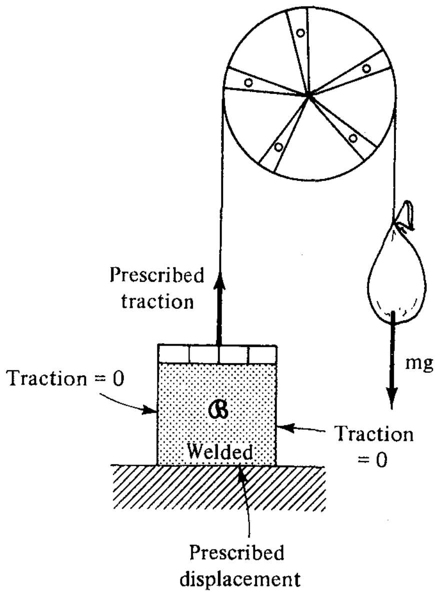

Figure 3.4.1

Problem 4.4 Formulate spatial versions of these boundary conditions.

4.10 Definition By the *initial boundary value problem for thermoelasticity*, we mean the problem of finding $\phi(X, t)$ and $\Theta(X, t)$ such that

(i) $\rho_{\text{Ref}} \dot{\boldsymbol{V}} = \mathbf{A} \cdot \nabla_X \boldsymbol{F} + \rho_{\text{Ref}} \boldsymbol{B} + \boldsymbol{B}_I + (\partial \hat{\boldsymbol{P}} / \partial \Theta) \cdot \text{GRAD}\, \Theta$,
(ii) $\rho_{\text{Ref}} \Theta\, (\partial N / \partial t) + \text{DIV}\, \hat{\boldsymbol{Q}} = \rho_{\text{Ref}} R$,
(iii) boundary conditions (I) and (II) hold, and
(iv) ϕ, $\boldsymbol{V}$, and Θ are given at $t = 0$ (*initial conditions*),

where $\hat{\Psi}$ is a given constitutive function depending on X, $\boldsymbol{C}$, and Θ and $\hat{N}$, $\hat{\boldsymbol{P}}$, $\mathbf{A}$, and $\boldsymbol{B}_I$ are given in terms of it as above, where $\boldsymbol{B}$, ρ_{Ref}, and R are given and $\hat{\boldsymbol{Q}}$ is a given function of X, $\boldsymbol{C}$, Θ, and GRAD Θ satisfying $\langle \hat{\boldsymbol{Q}}, \text{GRAD}\, \Theta \rangle \leq 0$.

If the motions are not sufficiently differentiable and shocks can develop, these conditions have to be supplemented by appropriate discontinuity conditions.

Notice that in (ii),

$$\rho_{\text{Ref}} \frac{\partial \hat{N}}{\partial t} = \rho_{\text{Ref}} \frac{\partial \hat{N}}{\partial C} \frac{\partial C}{\partial t} + \rho_{\text{Ref}} \frac{\partial \hat{N}}{\partial \Theta} \frac{\partial \Theta}{\partial t} = -\rho_{\text{Ref}} \frac{\partial}{\partial C} \frac{\partial \hat{\Psi}}{\partial \Theta} \cdot 2D - \rho_{\text{Ref}} \frac{\partial^2 \hat{\Psi}}{\partial \Theta^2} \frac{\partial \Theta}{\partial t}$$
$$= -\frac{\partial S}{\partial \Theta} : \boldsymbol{D} - \rho_{\text{Ref}} \frac{\partial^2 \hat{\Psi}}{\partial \Theta^2} \frac{\partial \Theta}{\partial t},$$

where $\boldsymbol{D}$ is the rate of deformation tensor.

4.11 Definition If we omit Θ, the corresponding equation (ii), the term $(\partial \hat{P}/\partial \Theta) \cdot \text{GRAD}\, \Theta$ in (i), and the boundary and initial conditions for Θ, the resulting problem for determination of ϕ is called the *initial boundary value problem for elasticity* (or hyperelasticity).

In Chapter 6 we shall study the initial boundary value problem posed here in some detail, as well as the corresponding static problem.

4.12 Definition The *boundary value problem for thermoelastostatics* consists of finding ϕ and Θ as functions of X alone such that

(i) $\text{DIV}\, \hat{\boldsymbol{P}} + \rho_{\text{Ref}} \boldsymbol{B} = \mathbf{A} \cdot \nabla_X F + \rho_{\text{Ref}} \boldsymbol{B} + \boldsymbol{B}_l + (\partial \hat{P}/\partial \Theta) \cdot \text{GRAD}\, \Theta = \mathbf{0}$,
(ii) $\text{DIV}\, \hat{Q} = \rho_{\text{Ref}} R$, and
(iii) boundary conditions (I) and (II) hold.

The *boundary value problem for elastostatics* consists of finding a (regular) deformation ϕ such that (i) holds (with the Θ term omitted) and boundary conditions (I) hold.

The static problem is, of course, obtained from the dynamic one by dropping time derivatives. There is a simple basic necessary condition the traction must satisfy in order that the static problem be soluble (assuming, as usual, regularity).

4.13 Proposition *The prescribed tractions $\langle \hat{\boldsymbol{P}}, N \rangle$ in $I(b)$ on $\partial \mathcal{B}$ must satisfy the necessary condition*

$$\int_{\partial \mathcal{B}} \langle \hat{\boldsymbol{P}}, N \rangle \, dA + \int_{\mathcal{B}} \rho_{\text{Ref}} \boldsymbol{B} dV = \mathbf{0} \quad (\textit{using Euclidean coordinates in } \mathbb{R}^n)$$

if the traction boundary value problem for (thermo-) elastostatics has a (regular) solution. Similarly the prescribed fluxes in $II(b)$ must satisfy

$$\int_{\partial \mathcal{B}} \langle \hat{Q}, N \rangle \, dA - \int_{\mathcal{B}} \rho_{\text{Ref}} R \, dV = 0.$$

Proof This follows from $\text{DIV}\, \hat{\boldsymbol{P}} + \rho_{\text{Ref}} \boldsymbol{B} = \mathbf{0}$ by integration over $\mathcal{B}$ and use of the divergence theorem, etc. ∎

There are more general boundary conditions that are compatible with the principle of virtual work (see Box 3.1, Chapter 2). The interested reader should consult Antman and Osborne [1979].

All of the above can be formulated in terms of the spatial picture and it is sometimes useful to do so. We give some of the key ideas in the following:

4.14 Definition Define the *spatial elasticity tensors* a and c by these coordinate formulas:

$$\mathsf{a}^{ac}{}_b{}^d = \frac{1}{J} F^c{}_A F^d{}_B \mathsf{A}^{aA}{}_b{}^B,$$

$$\mathsf{c}^{abcd} = \frac{2}{J} F^a{}_A F^b{}_B F^c{}_C F^d{}_D \mathbf{C}^{ABCD},$$

that is, a and c are push-forwards and Piola transforms of $\mathbf{A}$ and $\mathbf{C}$ on each large index.

The following relations then follow from the corresponding material results and the Piola identity.

4.15 Proposition *The following hold:*

(a) $\mathsf{a}^{ac}{}_b{}^d = \sigma^{cd}\delta^a{}_b + \mathsf{c}^{aced}g_{eb}$, *that is,* $\mathsf{a}^{acbd} = \sigma^{cd}g^{ab} + \mathsf{c}^{acbd}$.

(b) a *and* c *have these symmetries:*

$$\mathsf{a}^{acbd} = \mathsf{a}^{bdac},$$

$$\mathsf{c}^{abcd} = \mathsf{c}^{bacd} = \mathsf{c}^{abdc} = \mathsf{c}^{cdab}.$$

(c) *If $U^a{}_B$ is a two-point tensor field over ϕ, then*

$$(\mathsf{A}^{aA}{}_b{}^B U^b{}_B)_{|A} = J(\mathsf{a}^{ad}{}_b{}^e u^b{}_e)_{|d},$$

where $u^d{}_b = (F^{-1})^B{}_b U^d{}_B$ is the push-forward of $\boldsymbol{U}$.

The tensors a and c play an important role in the theory of linearization (see Chapter 4).

The equations of motion in the spatial picture

$$\rho \boldsymbol{a} = \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b}$$

become

$$\rho a^a = \mathsf{e}^{ab}{}_d{}^B F^d{}_{B|b} + (\operatorname{div}_x \boldsymbol{\sigma})^a + \rho b^a,$$

where e is the tensor given by

$$\mathsf{e}^{ab}{}_d{}^B = \frac{\partial \sigma^{ab}}{\partial F^d{}_B}$$

and div_x is the divergence holding F constant. The reader may wish to work out the relationship between e and a. The tensor c is *not* given by $\partial\sigma^{ab}/\partial c_{cd}$, and $\sigma^{ab} \neq 2\rho(\partial\psi/\partial c_{ab})$. Rather, the correct relations are

$$\mathsf{c}^{abcd} = \frac{\partial \sigma^{ab}}{\partial g_{cd}} \quad \text{and} \quad \sigma^{ab} = 2\rho \frac{\partial \hat{\psi}}{\partial g_{ab}} \quad \text{(see Section 3.3).}$$

We conclude this section with a brief remark on incompressible elasticity, Here one imposes the constraint that ϕ be volume preserving—that is, $J = 1$. For instance, such a condition is often imposed on rubber. (The Mooney–Rivlin–Ogden constitutive assumption for rubber is given in the next section.) The condition of incompressibility is perhaps best understood in terms of Hamiltonian systems with constraints, and we shall discuss this point of view in Chapter 5. For now we merely remark that this condition introduces a Lagrange multiplier into the equation as follows: we replace

$$\boldsymbol{\sigma} \text{ by } \boldsymbol{\sigma} - p\boldsymbol{g}^{\sharp}, \qquad \text{that is, } \sigma^{ab} \text{ by } \sigma^{ab} - pg^{ab},$$

where p is an unknown function, the *pressure*, to be determined by the condition of incompressibility. In terms of the first Piola–Kirchhoff tensor $\boldsymbol{P}$, we replace $\hat{\boldsymbol{P}}$ from our constitutive theory by $\hat{\boldsymbol{P}} - JPF^{-1}$, where P is a function of (X, t) to be determined by $J = 1$. We emphasize that in an initial boundary value problem, P becomes an unknown and it depends on ϕ in a non-local way, as in fluid mechanics (see Hughes and Marsden [1976] and Ebin and Marsden [1970]).

Box 4.1 ***Summary of Important Formulas for Section 3.4***

Elasticity Tensors

$$\mathsf{A} = \frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{F}}, \quad \mathsf{C} = \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{C}} \qquad \mathsf{A}^{aA}{}_{b}{}^{B} = \frac{\partial \hat{P}^{aA}}{\partial F^{b}{}_{B}}, \quad \mathsf{C}^{ABCD} = \frac{\partial \hat{S}^{AB}}{\partial C_{CD}}$$

Symmetries of Elasticity Tensors

C is symmetric in its first and second slots, and in its third and fourth slots: $\quad \mathsf{C}^{ABCD} = \mathsf{C}^{BACD} = \mathsf{C}^{ABDC}$

If $\hat{\boldsymbol{P}}, \hat{\boldsymbol{S}}$ are derived from an internal energy function (hyperelasticity): $\quad \mathsf{C}^{ABCD} = \mathsf{C}^{CDAB}, \quad \mathsf{A}_{a}{}^{A}{}_{b}{}^{B} = \mathsf{A}_{b}{}^{B}{}_{a}{}^{A}$

Relationship Between A *and* C

$$\mathsf{A} = 2\mathsf{C}\cdot \boldsymbol{F}\cdot \boldsymbol{F}\cdot \boldsymbol{g} + \hat{\boldsymbol{S}} \otimes \boldsymbol{\delta} \qquad \mathsf{A}^{aA}{}_{b}{}^{B} = 2\mathsf{C}^{CADB}F^{c}{}_{D}F^{a}{}_{C}g_{cb} + \hat{S}^{AB}\delta^{a}{}_{b}$$

Stored Energy Function (*for Isothermal Hyperelasticity*)

If Θ is absent, $\hat{E} = \hat{\Psi}$ is often denoted by W; it depends on X, $\boldsymbol{C}$:

$$\hat{\boldsymbol{P}} = \rho_{\text{Ref}}\boldsymbol{g}^{\sharp}\frac{\partial W}{\partial \boldsymbol{F}} \qquad \hat{P}^{aA} = \rho_{\text{Ref}}g^{ab}\frac{\partial W}{\partial F^{b}{}_{A}}$$

$$\hat{\boldsymbol{S}} = 2\rho_{\text{Ref}}\frac{\partial W}{\partial \boldsymbol{C}} \qquad \hat{S}^{AB} = 2\rho_{\text{Ref}}\frac{\partial W}{\partial C_{AB}}$$

Resultant Force due to Inhomogeneities

$$\boldsymbol{B}_{I} = \text{DIV}_{X}\,\hat{\boldsymbol{P}} \qquad B_{I}^{a} = \frac{\partial \hat{P}^{aA}}{\partial X^{A}} + \hat{P}^{aA}\Gamma^{B}_{AB} + \hat{P}^{bA}\gamma^{a}_{bc}F^{c}{}_{A}$$

Boundary Conditions for ϕ:

(a) *displacement*—ϕ given on $\partial\mathcal{B}$;
(b) *traction*—$\langle P, N\rangle^a = P^{aA} N_A$ given on $\partial\mathcal{B}$; or
(c) *mixed*—ϕ given on ∂_d, traction on ∂_τ, where $\partial_d \cap \partial_\tau = \varnothing$ and $\overline{\partial_d \cup \partial_\tau} = \partial\mathcal{B}$.

Boundary Conditions for Θ:

(a) *temperature*—Θ given on $\partial\mathcal{B}$;
(b) *flux*—$\langle Q, N\rangle = Q^A N_A$ given on $\partial\mathcal{B}$; or
(c) *mixed*—Θ given on ∂_Θ, $\langle Q, N\rangle$ given on ∂_f, where $\partial_\Theta \cap \partial_f = \varnothing$, and $\overline{\partial_\Theta \cup \partial_f} = \partial\mathcal{B}$.

Evolution Equation for ϕ for a Thermoelastic Solid (remove Θ for elasticity)

$$\rho_{\mathrm{Ref}}\dot{V} = \mathbf{A}\cdot\nabla_X F + \rho_{\mathrm{Ref}}B + B_I + \frac{\partial \hat{P}}{\partial \Theta}\cdot \mathrm{GRAD}\,\Theta$$

$$\rho_{\mathrm{Ref}}\frac{\partial^2\phi^a}{\partial t^2} + \rho_{\mathrm{Ref}}\frac{\partial\phi^b}{\partial t}\frac{\partial\phi^c}{\partial t}\gamma^a_{bc} = \mathsf{A}^{aA}{}_b{}^B\phi^b{}_{|A|B} + \rho_{\mathrm{Ref}}B^a + B^a_I + \frac{\partial\hat{P}^{aA}}{\partial\Theta}\frac{\partial\Theta}{\partial X^A}$$

Evolution Equation for Θ for a Thermoelastic Solid

$$-\rho_{\mathrm{Ref}}\frac{\partial^2\hat{\Psi}}{\partial\Theta^2}\frac{\partial\Theta}{\partial t} - \frac{\partial\hat{S}}{\partial\Theta}: D = \rho_{\mathrm{Ref}} - \mathrm{DIV}\,\hat{Q}$$

$$-\rho_{\mathrm{Ref}}\frac{\partial^2\hat{\Psi}}{\partial\Theta^2}\frac{\partial\Theta}{\partial t} - \frac{\partial\hat{S}^{AB}}{\partial\Theta}\cdot D_{AB} = \rho_{\mathrm{Ref}}R - \hat{Q}^A{}_{|A}$$

Initial Boundary Value Problem for Thermoelastodynamics (ignore Θ for elastodynamics)

Find $\phi(X, t)$, $\Theta(X, t)$ satisfying their respective evolution equations, boundary conditions, and with ϕ, $\partial\phi/\partial t$ and Θ given at $t = 0$.

Boundary Value Problem for Thermoelastostatics (ignore Θ for elastostatics)

Find $\phi(X)$, $\Theta(X)$ such that the right-hand sides of the above evolution equations are zero, and the boundary conditions are satisfied.

Necessary Conditions on the Boundary Data for Thermoelastostatics (ignore Θ for elastostatics)

If the traction boundary condition for ϕ and the flux boundary condition for Θ are used, then in a Cartesian frame

$$\int_{\partial\mathcal{B}} \langle\hat{P}, N\rangle\, dA + \int_{\mathcal{B}} \rho_{\mathrm{Ref}} B\, dV = \mathbf{0}$$

and

$$\int_{\partial\mathcal{B}} \langle\hat{Q}, N\rangle\, dA - \int_{\mathcal{B}} \rho_{\mathrm{Ref}}\, R\, dV = 0$$

The Spatial Elasticity Tensors

$$\mathsf{a} = \frac{1}{J}\phi_*\mathsf{A} \qquad \mathsf{a}^{ac}{}_b{}^d = \frac{1}{J}F^c{}_A F^d{}_B \mathsf{A}^{aA}{}_b{}^B$$

$\mathbf{c} = 2\frac{1}{J}\phi_*\mathbf{C}$	$\mathsf{c}^{abcd} = 2\frac{1}{J}F^a{}_AF^b{}_BF^c{}_CF^d{}_D\mathsf{C}^{ABCD}$
$\mathbf{a} = \mathbf{c}\cdot g + \boldsymbol{\sigma} \otimes \delta$	$\mathsf{a}^{ac}{}_b{}^d = \mathsf{c}^{aced}g_{eb} + \sigma^{cd}\delta^a{}_b$
Symmetries	$\mathsf{a}^{acbd} = \mathsf{a}^{bdac}$, $\mathsf{c}^{abcd} = \mathsf{c}^{bacd} = \mathsf{c}^{abdc} = \mathsf{c}^{cdab}$
Piola Identity DIV$(\mathbf{A}\cdot U) = J\,\mathrm{div}(\mathbf{a}\cdot u)$, where $u = \phi_*U$	$(\mathsf{A}^{aA}{}_b{}^BU^b{}_B)_{\mid A} = J(\mathsf{a}^{ac}{}_b{}^du^b{}_d)_{\mid c}$

3.5 MATERIAL SYMMETRIES AND ISOTROPIC ELASTICITY

In Sections 3.2 and 3.3 we proved that various locality, thermodynamic, or covariance assumptions simplified the functional form of the constitutive functions. In this section we investigate further simplifications when the material has some symmetries. In particular, we are concerned with isotropic materials, and within that class shall give the St. Venant–Kirchhoff and Mooney–Rivlin–Ogden materials as examples. We work primarily with the free energy $\hat{\Psi}$, since the other constitutive functions and the elasticity tensors are derived from it.

5.1 Definition A *material symmetry* for $\hat{\Psi}$ at a point $X_0 \in \mathcal{B}$ is a linear isometry $\boldsymbol{\lambda}: T_{X_0}\mathcal{B} \to T_{X_0}\mathcal{B}$ (i.e., $\boldsymbol{\lambda}$ preserves the inner product G_{X_0}) such that

$$\hat{\Psi}(X_0, C, \Theta) = \hat{\Psi}(X_0, \boldsymbol{\lambda}^*C, \Theta),$$

where C is an arbitrary symmetric positive-definite 2-tensor at X_0 and $\boldsymbol{\lambda}^*C$ is its transformation (pull-back) under $\boldsymbol{\lambda}$. The set of all material symmetries of $\hat{\Psi}$ at X_0 is denoted $\mathfrak{S}_{X_0}(\hat{\Psi})$ as is called the *material symmetry group* of $\hat{\Psi}$ at X_0. Similarly, $\boldsymbol{\lambda}$ is a *material symmetry for* $\hat{Q}$ at X_0 if

$$\hat{Q}(X_0, \boldsymbol{\lambda}^*C, \Theta, \boldsymbol{\lambda}^*\nabla\Theta) = \boldsymbol{\lambda}^*\hat{Q}(X_0, C, \Theta, \nabla\Theta).$$

($\hat{Q}$ is assumed to be grade (1, 1).) The set of such symmetries is denoted $\mathfrak{S}_{X_0}(\hat{Q})$. The *material symmetries for a thermoelastic material* are the simultaneous symmetries for $\hat{\Psi}$ and $\hat{Q}$. For pure elasticity, we look only at the symmetries of the internal energy $\hat{E}$ (i.e., $\hat{W}$). We shall write $\mathfrak{S}_{X_0} = \mathfrak{S}_{X_0}(\hat{\Psi}) \cap \mathfrak{S}_{X_0}(\hat{Q})$ for the material symmetries and understand $\mathfrak{S}_{X_0}$ to mean $\mathfrak{S}_{X_0}(\hat{E})$ for pure elasticity.

Relative to a coordinate system $\{X^A\}$ on $\mathcal{B}$ we write the components of $\boldsymbol{\lambda}$ as $\lambda^A{}_B$ so that if $V \in T_{X_0}\mathcal{B}$ has components V^A, then $(\boldsymbol{\lambda}\cdot V)^A = \lambda^A{}_BV^B$. The relationship $\hat{\Psi}(X_0, C, \Theta) = \hat{\Psi}(X_0, \boldsymbol{\lambda}^*C, \Theta)$ in coordinates reads

$$\hat{\Psi}(X_0, C_{AB}, \Theta) = \hat{\Psi}(X_0, C_{CD}\lambda^C{}_A\lambda^D{}_B, \Theta).$$

Similarly, the coordinate version for the symmetry condition on $\hat{Q}$ reads

$$\hat{Q}^E(X_0, C_{CD}\lambda^C{}_A\lambda^D{}_B, \Theta, \Theta_{|A}\lambda^A{}_B) = (\boldsymbol{\lambda}^{-1})^E{}_F\hat{Q}^F(X_0, C_{AB}, \Theta, \Theta_{|B}).$$

It is easily checked that $\mathfrak{S}_{X_0}$ is a group. It is a subgroup of the group $GL(T_{X_0}\mathcal{B})$ of invertible linear transformations of $T_{X_0}\mathcal{B}$ to itself, and in fact a subgroup of $O(T_{X_0}\mathcal{B})$, the orthogonal linear transformations of $T_{X_0}\mathcal{B}$ to itself (orthogonal with respect to the inner product $\boldsymbol{G}_{X_0}$). Furthermore, one can show that $\mathfrak{S}_{X_0}$ is a Lie group; that is, it is a smooth manifold and group multiplication and inversion are smooth maps.[8]

5.2 Definition If $\boldsymbol{\lambda}(s)$ is a smooth curve in $\mathfrak{S}_{X_0}$ with $\boldsymbol{\lambda}(0) =$ identity, the linear transformation $\boldsymbol{\xi}: T_{X_0}\mathcal{B} \longrightarrow T_{X_0}\mathcal{B}$ defined by

$$\boldsymbol{\xi}(V) = \frac{d}{ds}\boldsymbol{\lambda}(s)\cdot V_{|s=0}$$

is called an *infinitesimal material symmetry*. The collection of all such $\boldsymbol{\xi}$ is called the *Lie algebra* of $\mathfrak{S}_{X_0}$ and is denoted $\mathcal{A}_{X_0}$. The *Lie bracket* of $\boldsymbol{\xi} \in \mathcal{A}_{X_0}$ and $\boldsymbol{\eta} \in \mathcal{A}_{X_0}$ is given by

$$[\boldsymbol{\xi}, \boldsymbol{\eta}](V) = \boldsymbol{\xi}(\boldsymbol{\eta}(V)) - \boldsymbol{\eta}(\boldsymbol{\xi}(V)).$$

5.3 Example Let $\mathcal{B} = \mathbb{R}^3$ and let X_0 be the origin. Suppose $\mathfrak{S}_{X_0}$ consists of rotations about the z-axis. Then $\mathfrak{S}_{X_0}$ is one-dimensional, parametrized by the angle of rotation. A curve in $\mathfrak{S}_{X_0}$ is

$$\boldsymbol{\lambda}(s) = \begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the corresponding infinitesimal symmetry is given by

$$\boldsymbol{\xi}(V) = -\boldsymbol{k} \times V \quad (\boldsymbol{k} = (0, 0, 1)).$$

If $\mathfrak{S}_{X_0}$ is the full rotation group $O(3)$ and if the infinitesimal generators $\boldsymbol{\xi}$ are identified with the axes $\boldsymbol{\omega}$ about which they are rotations, that is, if

$$\boldsymbol{\xi}(V) = -\boldsymbol{\omega} \times V,$$

then the Lie algebra is identified with $\mathbb{R}^3$ and the Lie bracket with the cross product. (The reader should verify these statements.) ∎

Now we proceed to investigate the consequences of a material symmetry group on the constitutive functions.

5.4 Proposition *For $\boldsymbol{\lambda} \in \mathfrak{S}_{X_0}(\hat{\Psi})$, we have:*

(i) $\hat{N}(X_0, \boldsymbol{\lambda}^*C, \Theta) = \hat{N}(X_0, C, \Theta)$,

---

[8]We shall not require a background in Lie group theory for what follows, although such knowledge may be helpful. A concise summary of some of the important facts may be found in Abraham and Marsden [1978], Sect. 4-1.

(ii) $\hat{S}(X_0, \boldsymbol{\lambda}^*C, \Theta) = \boldsymbol{\lambda}^*\hat{S}(X_0, C, \Theta)$, and
(iii) $\mathbf{C}(X_0, \boldsymbol{\lambda}^*C, \Theta) = \boldsymbol{\lambda}^*\mathbf{C}(X_0, C, \Theta)$.

Proof This follows by differentiation of the relationship $\hat{\Psi}(X_0, \boldsymbol{\lambda}^*C, \Theta) = \hat{\Psi}(X_0, C, \Theta)$ with respect to Θ and to C, respectively. For instance, differentiation with respect to C and using $\hat{S} = 2\rho_{\text{Ref}}(\partial\hat{\Psi}/\partial C)$ gives

$$\hat{S}(X_0, \boldsymbol{\lambda}^*C, \Theta)\cdot\boldsymbol{\lambda}^* = \hat{S}(X_0, C, \Theta),$$

that is,

$$\hat{S}^{AB}(X_0, \boldsymbol{\lambda}^*C, \Theta)\lambda^C{}_A\lambda^D{}_B = \hat{S}^{CD}(X_0, C, \Theta).$$

Thus

$$\hat{S}^{AB}(X_0, \lambda^*C, \Theta) = (\lambda^{-1})^A{}_C(\lambda^{-1})^B{}_D\hat{S}^{CD}(X_0, C, \Theta).$$

This yields (ii) of our proposition. ∎

The infinitesimal versions of 5.4 lead to differential identities that are useful in studying a given symmetry. Some preparatory remarks are needed. Recall that if $\boldsymbol{\phi}_t$ is a motion and $\mathbf{t}$ a time-independent tensor field, $(d/dt)\phi_t^*\mathbf{t} = \phi_t^*\mathfrak{L}_v\mathbf{t}$ defines the Lie derivative. If $\phi_t(X_0) = X_0$ for all t, then $v(X_0) = \mathbf{0}$ and $\mathfrak{L}_v\mathbf{t}$ does not involve spatial derivatives of $\mathbf{t}$ and depends only on the first derivative of $\boldsymbol{v}$. If $\boldsymbol{\lambda}(s)$ is a curve in $GL(T_{X_0}\mathcal{B})$, and $\mathbf{t}$ is a tensor at X_0, then $(d/ds)\boldsymbol{\lambda}(s)^*\mathbf{t}_{|s=0} = \mathfrak{L}_\xi\mathbf{t}$ is defined; if $\boldsymbol{\lambda}(s) = T\phi_s(X_0)$, then $\boldsymbol{\xi} = Dv(X_0)$ and the two concepts agree. Now we are ready to linearize 5.4.

5.5 Theorem *For each $\boldsymbol{\xi} \in \mathcal{A}_{X_0}$, the following identities must hold at X_0:*

(i) $\hat{S}(X_0, C, \Theta)\cdot\mathfrak{L}_\xi C = 0,$

(ii) $\dfrac{\partial\hat{N}}{\partial C}(X_0, C, \Theta)\cdot\mathfrak{L}_\xi C = 0,$

(iii) $\mathbf{C}(X_0, C, \Theta)\cdot\mathfrak{L}_\xi C = \mathfrak{L}_\xi\hat{S}(X_0, C, \Theta)$
(*the contraction in the left-hand side is in the last two slots of* $\mathbf{C}$), *and*

(iv)
$$\frac{\partial\hat{Q}}{\partial C}(X_0, C, \Theta, \text{GRAD}\,\Theta)\cdot\mathfrak{L}_\xi C + \frac{\partial\hat{Q}}{\partial(\text{GRAD}\,\Theta)}(X_0, C, \Theta, \text{GRAD}\,\Theta)\cdot\mathfrak{L}_\xi(\text{GRAD}\,\Theta) = \mathfrak{L}_\xi\hat{Q}(X_0, C, \Theta, \text{GRAD}\,\Theta).$$

Proof Let $\boldsymbol{\xi}$ be tangent to the curve $\boldsymbol{\lambda}(s) \in \mathfrak{S}_{X_0}$ at $s = 0$. Then differentiation of

$$\hat{\Psi}(X_0, \boldsymbol{\lambda}(s)^*C, \Theta) = \hat{\Psi}(X_0, C, \Theta)$$

in s at $s = 0$ using the chain rule and the definition of the Lie derivative immediately yields (i). The others are similarly obtained. ∎

We shall not study the general implications of these identities, nor the classification of possible symmetry groups; the interested reader should consult Wang and Truesdell [1973]. Instead, we specialize immediately to the isotropic case.

5.6 Definition Let $\mathcal{B}$ be a simple thermoelastic body in $\mathbb{R}^3$. We say $\mathcal{B}$ is *isotropic* at $X_0 \in \mathcal{B}$ if $\mathfrak{S}_{X_0} \supset SO(T_{X_0}\mathcal{B}) = SO(3)$, the group of proper orthogonal 3×3 matrices. A material is *isotropic* if it is isotropic at every point.

In Section 3.3 we defined $\hat{\Psi}$ to be materially covariant if, for all diffeomorphisms $\Xi: \mathcal{B} \longrightarrow \mathcal{B}$, we have

$$\Xi^*\{\hat{\Psi}(G, C, \Theta)\} = \hat{\Psi}(\Xi^*G, \Xi^*C, \Xi^*\Theta)$$

(the base points X are suppressed since C, Θ are now fields, and G is included).

5.7 Proposition *Suppose $\hat{\Psi}$ is materially covariant. Then for each $X_0 \in \mathcal{B}$,*

$$\mathfrak{S}_{X_0}(\hat{\Psi}) \supset SO(T_{X_0}\mathcal{B}).$$

In particular, any materially covariant body in $\mathbb{R}^3$ is isotropic.

Proof Let $\boldsymbol{\lambda} \in SO(T_{X_0}\mathcal{B})$. We claim that there is a diffeomorphism $\Xi: \mathcal{B} \longrightarrow \mathcal{B}$ such that $\Xi(X_0) = X_0$ and $D\Xi(X_0) = \boldsymbol{\lambda}$. Indeed, we can write $\boldsymbol{\lambda} = \exp(\boldsymbol{\xi})$ for $\boldsymbol{\xi}$ a skew 3×3 matrix; find a vector field V on $\mathcal{B}$ such that $V(X_0) = \mathbf{0}$ and $DV(X_0) = \boldsymbol{\xi}$. We can do this in local coordinates and extend V to be arbitrary outside this neighborhood, say zero near $\partial\mathcal{B}$. Let Ξ_t be the flow of V and let $\Xi = \Xi_1$. By uniqueness of solutions to differential equations, $D\Xi_t(X_0) = \exp(t\boldsymbol{\xi})$, so Ξ has the required properties. At X_0, the relation $\Xi^*(\hat{\Psi}(G, C, \Theta)) = \hat{\Psi}(\Xi^*G, \Xi^*C, \Xi^*\Theta)$ then reduces to $\hat{\Psi}(X_0, C, \Theta) = \hat{\Psi}(X_0, \boldsymbol{\lambda}^*C, \Theta)$, so λ is a material symmetry. ∎

Roughly speaking, this proposition states that if one makes up a free energy function in a fully covariant or "tensorial" manner out of G, C, and Θ, then it necessarily will be isotropic. To describe non-isotropic materials then requires some "non-tensorial" constructions or the introduction of additional variables.

By definition, an isotropic free energy constitutive function $\hat{\Psi}$ is to be a "rotationally invariant" function of the argument C. Since C is symmetric, it can be brought to diagonal form by an orthogonal transformation, so $\hat{\Psi}$ is a function only of the eigenvalues of C; that is, $\hat{\Psi}$ depends only on the principal stretches. Since the eigenvalues are reasonably complicated functions of C, it is sometimes convenient to use the invariants of C.

5.8 Definition The *invariants* of a symmetrix matrix C (in an inner product space) are defined by

$$I_1(C) = \operatorname{tr} C, \qquad I_2(C) = \det C \operatorname{tr} C^{-1} \quad \text{and} \quad I_3(C) = \det C.$$

5.9 Proposition *The invariants of C are related to the coefficients in the characteristic polynomial $P(\nu)$ of C as follows:*

$$P(\nu) = \nu^3 - I_1(C)\nu^2 + I_2(C)\nu - I_3(C).$$

In terms of the (eigenvalues) ν_1, ν_2, ν_3, we have

$$I_1(C) = \nu_1 + \nu_2 + \nu_3, \qquad I_2(C) = \nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3, \quad \text{and} \quad I_3(C) = \nu_1\nu_2\nu_3$$

(these are the elementary symmetric functions of ν_1, ν_2, ν_3). Moreover, the following formula for I_2 holds:

$$I_2(C) = \tfrac{1}{2}[\operatorname{tr} C)^2 - \operatorname{tr}(C^2)]$$

This is verified by using an orthonormal basis in which C is diagonal, noting that I_1, I_2, I_3 are rotationally invariant functions.

Since ν_1, ν_2, ν_3 completely determine the characteristic polynomial and hence the invariants, and vice versa, we obtain the following:

5.10 Proposition *The following are equivalent:*

(a) *A scalar function f of C is invariant under orthogonal transformations.*
(b) *f is a function of the invariants of C.*
(c) *f is a symmetric function of the principal stretches.* ∎

Thus for isotropic materials we can regard $\hat{\Psi}$ as a function of X, I_1, I_2, I_3, and Θ. Note that the number of arguments in the C variable is thus reduced from 6 to 3. The invariants are covariant scalar functions of C, so we obtain from 5.7 and 5.10 the following:

5.11 Corollary *A body is materially covariant if and only if it is isotropic.*

Next we compute the second Piola–Kirchhoff stress tensor in terms of this data.

5.12 Theorem *For isotropic materials, the following constitutive relation holds:*

$$\boxed{\hat{S} = \alpha_0 G + \alpha_1 C + \alpha_2 C^2, \qquad \text{that is,} \quad \hat{S}_{AB} = \alpha_0 G_{AB} + \alpha_1 C_{AB} + \alpha_2 C_A{}^D C_{DB},}$$

where the α_i's are scalar functions of X, the invariants of C, and Θ.

Proof We have $\hat{S} = 2\rho_{\text{Ref}}(\partial\hat{\Psi}/\partial C)$. By the chain rule,

$$\frac{\partial\hat{\Psi}}{\partial C} = \frac{\partial\hat{\Psi}}{\partial I_1}\frac{\partial I_1}{\partial C} + \frac{\partial\hat{\Psi}}{\partial I_2}\frac{\partial I_2}{\partial C} + \frac{\partial\hat{\Psi}}{\partial I_3}\frac{\partial I_3}{\partial C}.$$

Now $I_1(C) = C_{AB}G^{AB}$, so $\partial I_1/\partial C = G^\sharp$, that is, $\partial I_1/C_{AB} = G^{AB}$. Next we calculate the derivative of I_3.

5.13 Lemma

$$\frac{\partial I_3}{\partial C} = (\det C)\cdot C^{-1}$$

Proof From the definition of the determinant,

$$\det C = \epsilon^{BCD} C_{A_1B} C_{A_2C} C_{A_3D},$$

where $\epsilon^{BCD} = \pm 1$ depending on whether (B, C, D) is an even or odd permutation of (1, 2, 3) and where (A_1, A_2, A_3) is a fixed even permutation of (1, 2, 3). Thus

$$\frac{\partial}{\partial C_{A_1B}}(\det C) = \epsilon^{BCD}C_{A_2C}C_{A_3D} = \epsilon^{BCD}C_{A_1E}C_{A_2C}C_{A_3D}(C^{-1})^{A_1E}$$
$$= (\det C)\delta^B{}_E C^{A_1E} = (\det C)(C^{-1})^{A_1B} \quad \blacksquare$$

The derivative of I_2 is given by

$$\frac{\partial I_2}{\partial C} = \left(\frac{\partial}{\partial C}\det C\right)\operatorname{tr} C^{-1} + \det C\frac{\partial}{\partial C}\operatorname{tr} C^{-1}$$
$$= (C^{-1}\det C \operatorname{tr} C^{-1}) - (\det C)\operatorname{tr}\frac{\partial C^{-1}}{\partial C}.$$

To carry on we need to compute $\partial(C^{-1})/\partial C$:

5.14 Lemma

$$\frac{\partial C^{-1}}{\partial C}\cdot H = -C^{-1}\cdot H\cdot C^{-1}, \qquad \text{that is,} \quad \frac{\partial (C^{-1})^{AB}}{\partial C_{CD}} = -(C^{-1})^{AC}(C^{-1})^{DB}.$$

Proof Differentiate the identity $C\cdot C^{-1} = Id$ in the direction H to get

$$H\cdot C^{-1} + C\cdot\frac{\partial C^{-1}}{\partial C}\cdot H = 0,$$

which gives the result. $\blacksquare$

We now note that

$$\operatorname{tr}\left(\frac{\partial C^{-1}}{\partial C}\right)^{CD} = -(C^{-1})^{AC}(C^{-1})^D{}_A, \qquad \text{that is,} \quad \operatorname{tr}\left(\frac{\partial C^{-1}}{\partial C}\right) = -C^{-2}.$$

Substitution of these formulas yields:

$$\hat{S} = 2\rho_{\mathrm{Ref}}\left[\frac{\partial\hat{\Psi}}{\partial I_1}G^\sharp + \left(\frac{\partial\hat{\Psi}}{\partial I_2}I_2 + \frac{\partial\hat{\Psi}}{\partial I_3}I_3\right)C^{-1} - \frac{\partial\hat{\Psi}}{\partial I_2}I_3C^{-2}\right].$$

The Cayley–Hamilton theorem from linear algebra tells us that C satisfies its characteristic equation:

$$C^3 - I_1(C)C^2 + I_2(C)C - I_3(C) = 0.$$

Thus

$$C^{-1} = \frac{1}{I_3(C)}\{C^2 - I_1(C)C + I_2(C)\}$$

and

$$C^{-2} = \frac{1}{I_3(C)}\{C - I_1(C) + I_2(C)C^{-1}\}$$
$$= \frac{1}{I_3(C)}\left\{\frac{I_2(C)}{I_3(C)}C^2 + \left(Id - \frac{I_1(C)I_2(C)}{I_3(C)}\right)C + \left(\frac{I_2(C)}{I_3(C)} - I_1(C)\right)\right\}.$$

Inserting these expressions into the above formula for $\hat{S}$ yields the desired conclusion. $\blacksquare$

Recalling that $B = C^{-1}$, notice that we have also proved the following:

5.15 Corollary

$$\hat{S} = 2\rho_{\mathrm{Ref}}\left\{\frac{\partial\hat{\Psi}}{\partial I_1}G^\sharp + \left(\frac{\partial\hat{\Psi}}{\partial I_2} + \frac{\partial\hat{\Psi}}{\partial I_3}I_3\right)B - \frac{\partial\hat{\Psi}}{\partial I_2}I_3B^2\right\}$$

In the spatial picture one can write $\sigma = \beta_0 g + \beta_1 b + \beta_2(b^2)$, where b is the Finger deformation tensor (see Section 1.3).

5.16 Corollary *For isotropic materials, the (second) elasticity tensor* $\mathbf{C}$ *has the following component form.*

$$\begin{aligned}\mathsf{C}^{ABCD} &= \gamma_1\cdot G^{AB}G^{CD} + \gamma_2\cdot\{C^{AB}G^{CD} + G^{AB}C^{CD}\} + \gamma_3\{(C^2)^{AB}G^{CD} + G^{AB}(C^2)^{CD}\}\\ &\quad + \gamma_4\cdot C^{AB}C^{CD} + \gamma_5\cdot\{(C^2)^{AB}C^{CD} + C^{AB}(C^2)^{CD}\} + \gamma_6\cdot(C^2)^{AB}(C^2)^{CD}\\ &\quad + \gamma_7\{G^{AC}G^{BD} + G^{BC}G^{AD}\} + \gamma_8\{G^{AC}C^{BD} + G^{BC}C^{AD} + G^{AD}C^{BC}\\ &\quad + G^{BD}C^{AC}\},\end{aligned}$$

where $\gamma_1, \ldots, \gamma_8$ *are scalar functions of* X, *the invariants of* C *and, if the material is thermoelastic,* Θ.

Proof We differentiate the expression for $\hat{S}$ and proceed as in the proof of 5.12. The details, including the verification of the equality of the various coefficients in the expression for $\mathbf{C}$ using symmetry of the second derivatives of $\hat{\Psi}$, will be left to the reader. (It is straightforward, although tedious.) ∎

Isotropy reduces the number of pointwise independent components in S from 6 to 3 and for $\mathbf{C}$, from 21 to 8.

Problem 5.1 In plane strain dim $\mathcal{B} = 2$ and dim $\mathcal{S} = 2$. How many independent components does $\mathbf{C}$ have? Compute C^{ABCD} and $\mathsf{A}^{A}{}_{a}{}^{Bb}$ in terms of $\hat{\Psi}$. Compare with Knowles and Sternberg [1977], Formula (1.17). (It will be necessary to translate the notation.)

We give, finally, two examples of constitutive relations for purely elastic isotropic materials. The first was proposed by St. Venant and Kirchhoff around 1860.

5.17 Example If $\hat{S}$ is a linear function of E and describes an isotropic material, then it has the form

$$\hat{S} = \lambda(\mathrm{tr}\,E)G^\sharp + 2\mu\,E^\sharp$$

for λ and μ functions of X. [*Warning*: This does *not* lead to linear equations of motion! Fosdick and Serrin [1979] show that $\hat{P}$ cannot be a linear function of F and be materially frame indifferent.] One calls $\lambda + \frac{2}{3}\mu$ the *bulk modulus*. The elasticity tensor is, as in our computations above,

$$\mathsf{C}^{ABCD} = \lambda G^{AB}G^{CD} + \mu(G^{AC}G^{BD} + G^{AD}G^{CB}).$$

Thus, in 5.15, $\lambda = \gamma_1$, $\mu = \gamma_7$, and the other γ_i are zero.

Problem 5.2 Find $W(F)$ for this example and verify material covariance.

5.18 Example The previous example is only appropriate in the small strain regime. It is used in the derivation of the von Karmen equations by Ciarlet [1983]. The following example, due to Mooney and Rivlin and Ogden, is often used to model rubber. Again, the material is isotropic. It is convenient to express the stored energy function W as a symmetric function of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ (the eigenvalues of $C^{1/2}$). The form proposed by Ogden [1972] is as follows:

$$W = \sum_{i=1}^{M} a_i(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3)$$
$$+ \sum_{j=1}^{N} b_j((\lambda_2\lambda_3)^{\beta_j} + (\lambda_3\lambda_2)^{\beta_j} + (\lambda_1\lambda_2)^{\beta_j} - 3) + h(\lambda_1\lambda_2\lambda_3),$$

where a_i, b_j are positive constants, $\alpha_i \geq 1$, $\beta_j \geq 1$, and h is a convex function of one variable. The term "3" is a normalization constant such that the first two terms vanish when there is no deformation. The special case when $M = N = 1$, $\alpha_1 = \beta_1 = 2$, and $h = 0$ is called the *Mooney–Rivlin material*; W may then be written

$$\begin{aligned} W &= a_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + b_1((\lambda_2\lambda_3)^2 + (\lambda_3\lambda_1)^2 + (\lambda_1\lambda_2)^2 - 3) \\ &= a_1(I_1 - 3) + b_1(I_2 - 3), \end{aligned}$$

where I_i are the principal invariants of C. The further special case $W = a_1(I_1 - 3)$ is called a *neo-Hookean material.*

Box 5.1 ***Summary of Important Formulas for Section 5.1***

Material Symmetry at X_0

$\boldsymbol{\lambda}: T_{X_0}\mathcal{B} \longrightarrow T_{X_0}\mathcal{B}$ is orthogonal and

(i) $\hat{\Psi}(X_0, \boldsymbol{\lambda}^*C, \Theta) = \hat{\Psi}(X_0, C, \Theta)$ (i) $\hat{\Psi}(_0, \lambda^C{}_A C_{CD} \lambda^D{}_B, \Theta) = \hat{\Psi}(X_0, C_{AB}, \Theta)$

(ii) $\hat{Q}(X_0, \lambda^*C, \Theta, \lambda^*(\text{GRAD}\,\Theta)) = \boldsymbol{\lambda}^*\hat{Q}(X_0, C, \Theta, \text{GRAD}\,\Theta)$ (ii) $\hat{Q}^E(X_0, \lambda^C{}_A C_{CD} \lambda^D{}_B \Theta, (\boldsymbol{\lambda}^{-1})^A{}_B \Theta^{|B}) = (\boldsymbol{\lambda}^{-1})^E{}_F \hat{Q}^F(X_0, C_{AB}, \Theta, \Theta^{|A})$

Infinitesimal Material Symmetry at X_0

$\boldsymbol{\xi} = \dfrac{d}{ds}\boldsymbol{\lambda}_s$ $\xi^A{}_B = \dfrac{\partial}{\partial s}\lambda^A{}_B(s)_{|s=0}$

where $\boldsymbol{\lambda}_s$ is a curve of material symmetries at X_0, such that $\lambda_0 = Id$.

Transformation of $\hat{S}$ under a material symmetry $\boldsymbol{\lambda}$ at X_0

$\hat{S}(X_0, \boldsymbol{\lambda}^*C, \Theta) = \boldsymbol{\lambda}^*\hat{S}(X_0, \Theta)$ $\hat{S}^{AB}(X_0, \lambda^E{}_C C_{EF} \lambda^F{}_D, \Theta) = (\boldsymbol{\lambda}^{-1})^A{}_E (\boldsymbol{\lambda}^{-1})^B{}_F \cdot \hat{S}^{EF}(X_0, C_{CD}, \Theta)$

Infinitesimal Symmetry Identities

$\hat{S}(X_0, C, \Theta)\cdot\mathfrak{L}_\xi C = 0$ $\qquad$ $\hat{S}^{AB}(X_0, C_{CD}, \Theta)\cdot\{\xi^C{}_A(X_0)C_{CB} + \xi^C{}_B(X_0)C_{AC}\} = 0$

$\mathbf{C}(X_0, C, \Theta)\cdot\mathfrak{L}_\xi C = \mathfrak{L}_\xi\hat{S}(X_0, C, \Theta)$ $\qquad$ $\mathsf{C}^{ABCD}\cdot(\xi^E{}_C C_{ED} + \xi^E{}_D C_{EC}) = -\hat{S}^{EB}\xi^A{}_E - \hat{S}^{AE}\xi^B{}_E$

Invariants of C

$I_1 = \operatorname{tr} C$ $\qquad$ $I_1 = C^A{}_A$

$I_2 = \det C \operatorname{tr} C^{-1}$ $\qquad$ $I_2 = (\det(C_{AB}))(C^{-1})^A{}_A$

$I_3 = \det C$ $\qquad$ $I_3 = \det(C_{AB})$

Second Piola–Kirchhoff Stress Tensor for an Isotropic Material

$S^\flat = \alpha_0 G + \alpha_1 C + \alpha_2 C^2$ $\qquad$ $S_{AB} = \alpha_0 G_{AB} + \alpha_1 C_{AB} + \alpha_2 C_A{}^D C_{DB}$

where $\alpha_1, \alpha_2, \alpha_3$ are scalar functions of X, the invariants of C and Θ.

$$S = 2\rho_{\text{Ref}}\left\{\frac{\partial\hat{\Psi}}{\partial I_1}G^\sharp + \left(\frac{\partial\hat{\Psi}}{\partial I_2}I_2 + \frac{\partial\hat{\Psi}}{\partial I_3}I_3\right)B - \frac{\partial\hat{\Psi}}{\partial I_2}I_3 B^2\right\}$$

$$S^{AB} = 2\rho_{\text{Ref}}\{\hat{\Psi}_{,I_1}G^{AB} + (\hat{\Psi}_{,I_2}I_2 + \hat{\Psi}_{,I_3}I_3)B^{AB} - \hat{\Psi}_{,I_2}I_3 B^{AD}B_D{}^B\}$$

Elasticity Tensor for Isotropic Materials

$$\begin{aligned}\mathsf{C}^{ABCD} = {}&\gamma_1 G^{AB}G^{CD} + \gamma_2\{C^{AB}G^{CD} + G^{AB}C^{CD}\} + \gamma_3\{C^{AE}C_E{}^B G^{CD}\\ &+ G^{AB}C^{CE}C_E{}^D\} + \gamma_4 C^{AB}C^{CD} + \gamma_5\{C^{AE}C_E{}^B C^{CD}\\ &+ C^{AB}C^{CE}C_E{}^D\} + \gamma_6 C^{AE}C_E{}^B C^{CF}C_F{}^D + \gamma_7\{G^{AC}G^{BD}\\ &+ G^{BC}G^{AD}\} + \gamma_8\{G^{AC}C^{BD} + G^{BC}C^{AD} + G^{AD}C^{BC}\\ &+ G^{BD}C^{AC}\}\end{aligned}$$

Stress For St. Venant–Kirchhoff Material

$\hat{S} = \lambda(\operatorname{tr} E)G^\sharp + 2\mu E^\sharp$ $\qquad$ $S^{AB} = \lambda E^D{}_D G^{AB} + 2\mu E^{AB}$