STEADY FLOW TO TILE DRAINS ABOVE AN IMPERVIOUS LAYER — A THEORETICAL STUDY

by

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Report No. KH-R-9

January 1965

ACKNOWLEDGEMENT

The work reported herein was originally a portion of a thesis submitted to the University of Auckland, New Zealand in partial fulfillment of the requirements for the degree of Master of Engineering. The major part of the computation was completed as a laboratory research project at the California Institute of Technology, Pasadena, California. The author is indebted to Fletcher Industries, and the University of Auckland, Auckland, New Zealand for financial assistance during 1962 and the California Institute of Technology for the use of its computing facilities. Numerous suggestions and assistance by Dr. C. M. Segedin (University of Auckland) and Dr. N. H. Brooks (California Institute of Technology) are gratefully acknowledged. The assistance of Mrs. Pat Rankin, Mr. F. C. Cammack and Mr. Paul Kochendorfer in the preparation of the manuscript is also gratefully acknowledged.

SUMMARY

The Problem.

Existing analytical solutions for vertical influent seepage flow to an infinite series of tile drains underlain by an impervious layer do not adequately consider the free surface nor give solutions for the drain diameter (see Figure 1).

The theoretical investigations described herein basically sought to find a method whereby an engineer could find the required diameter, depth, and spacing of an adequate tile drainage system when given the ratio of infiltration to soil hydraulic conductivity and the depth of the impervious layer.

Investigations.

From a study of the available literature it was decided the problem would be exceedingly difficult to solve exactly but it seemed as if an approximate solution of good accuracy could be obtained by approximating the shape of the impervious layer, where the flow is sluggish, and satisfying the conditions exactly at the free surface, a critical region of flow. Previously published solutions invariably adopt the opposite viewpoint, with a consequent loss in accuracy. The solution to the problem when the impervious layer is at infinite depth had been solved exactly using the rather laborious hodograph technique. This work employs to advantage an image method developed by Davison and Rosenhead (1940).*

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* See Appendix D for list of references.

Results.

Detailed results are obtained herein for the condition when no water stands over the tile lines and these are given in a graphical form which enables a ready computation of drain depth diameter and spacing when the field data are known (ratio of rate of infiltration to hydraulic conductivity and the depth of the impervious layer). A comparison with field results shows that the solutions are probably as accurate as could be desired.

The case which occurs when water stands over the tile lines has not been extensively tabulated for reasons explained in the report.

Conclusions.

A method is given whereby adequate tile drainage systems may be designed on a long term basis. The design curves given are not for intermittent unsteady flows such as irrigation schemes (a more difficult problem) but should prove exceedingly useful for engineers concerned with land reclamation schemes and marsh drainage.

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CHAPTER 1

INTRODUCTION

1.0 Introductory Note

The problem of land drainage has long been recognized and numerous endeavors have been made to find the universal solution to all land drainage problems.

Perhaps the greatest advances were made by one Joseph Elkington, whose fame as a drainage engineer had become so widespread that, to quote from, "Treatise on Practical Drainage of Land 1844", by Henry Hutchinson, ". . . . on the 10th of June, 1795, a motion by the President in the House of Commons (England), that a sum of money ($2,800) should be granted to him as an inducement to discover his mode of Draining." However, Mr. Joseph Elkington failed to disclose his secret, the result of which was an early act of "business spying", for to quote from Nicholson (1942) "A group of noblemen despatched a Mr. John Johnstone, a Surveyor, to watch the progress of Elkington's works, and report the result to them, which he subsequently did in his publication, 'An Account of the Mode of Draining Land, according to the System Practised by Mr. Joseph Elkington.' (Richard Phillips, London; 1801.)"

Although the theory of land drainage has come some way from Mr. Joseph Elkington's day the principles are still the same and the analysis and experimental work done on the problem since those times has, in the main, been an attempt to give a quantitative basis for design and thus place land drainage on a sound economic basis.

1.1 Review of the Literature

An extensive review of the literature of the steady flow theory of land drainage is given in Appendix A. A division has been made into theoretical and experimental studies. These two divisions have been further subdivided according to the type of analytical or experimental study. However, the papers directly connected with this study will be mentioned at this stage.

The major contribution to this work comes from a paper by Davison and Rosenhead (1940) on the application of the method of images to free surface problems. The essential point of the method of images is that the real or imaginary part of a potential function along any boundary may be made constant by reflecting the flow region in the boundary thus creating an image. Of course, whether it is the real or imaginary part which is held constant depends on the sign of the image. The method also depends on the fact that complex potential functions may be defined which have either their real or imaginary part constant on a free surface, e.g. $W = w - iKz = \varphi + Ky + i(\psi - Kx)$ has its real part constant on a free surface (K the hydraulic conductivity, φ and ψ the velocity potential and stream function respectively).

Another major contribution comes initially from the work of Vedernikov (1939). Vedernikov designated the drain as a line sink in the flow field and the drain perimeter as the line of constant potential about the line sink which satisfies the condition $\varphi + Ky = 0$ at its uppermost point. This corresponds to a drain running full with a small air gap at its uppermost point. The idea has been used extensively since then, notably

by Gustafsson (1946), Engelund (1951), Van Deemter (1949), and more recently by Watson (1960). It is rather surprising that the work done by Vedernikov has been repeated so often by others and generally with far more effort involved!

Numerous attempts (see Appendix A) have been made at providing a solution to the problem of steady state land drainage when an impervious layer is present but all known attempts to date have relied upon approximations being made at the free surface, a critical region of flow, and for this reason their accuracy is perhaps questionable.

1.2 Object and Scope of the Present Work

It will be readily perceived from a perusal of Appendix A that there is at present no true analytical solution to the problem of a series of tile drains overlying an impervious layer and receiving their flow from a steady inflow through the ground surface. The object of this work is an endeavor to fill this gap in the known theory of drainage as indicated by the review of the literature.

It may well be asked what exactly constitutes a solution. From the engineer's point of view this can, without a doubt, be stated to be the required diameter, depth and spacing of a series of drains when the ratio of rate of infiltration to hydraulic conductivity and depth of the impervious layer are known. The correct solution for when the impervious layer is at infinite depth has already been given by Vedernikov (1939) and many others.

The aim of this report is therefore to provide design curves whereby an engineer can design an adequate long term drainage system in the most economical manner.

CHAPTER 2

SOLUTION OF THE PROBLEM

2.0 Method of Solution.

The flow configuration for a series of tile drains above an impervious layer is that shown diagrammatically in Figure 1. From this figure it is easily seen that there is a streamline along the impervious layer and it is from this fact that the essential idea for the method of solution to the problem comes. For suppose there were two drains, one above the other, the upper drain receiving its flow in the normal manner by precipitation infiltrating through the surface the lower drain receiving its flow from an artesian source of equal intensity to the infiltration. In this case a dividing streamline will exist between the upper and lower drains. This dividing streamline can then be taken to represent a pseudo-impervious layer, that is, as far as the upper drain is concerned it is not aware whether there is an impervious layer below it or another drain as has been postulated. In this way the essential mathematical difficulty associated with the problem as depicted in Figure 1 is overcome. The mathematical problem to be solved now becomes that depicted in Figure 2.

It is obvious that there is one difference however. The dividing streamline will no longer be straight as it is for a true impervious layer but this should not unduly influence the accuracy of the solutions as the flow in the neighborhood of the impervious layer is rather torpid. The essential point is that the conditions on the free surface, a critical region of flow, can be exactly satisfied.

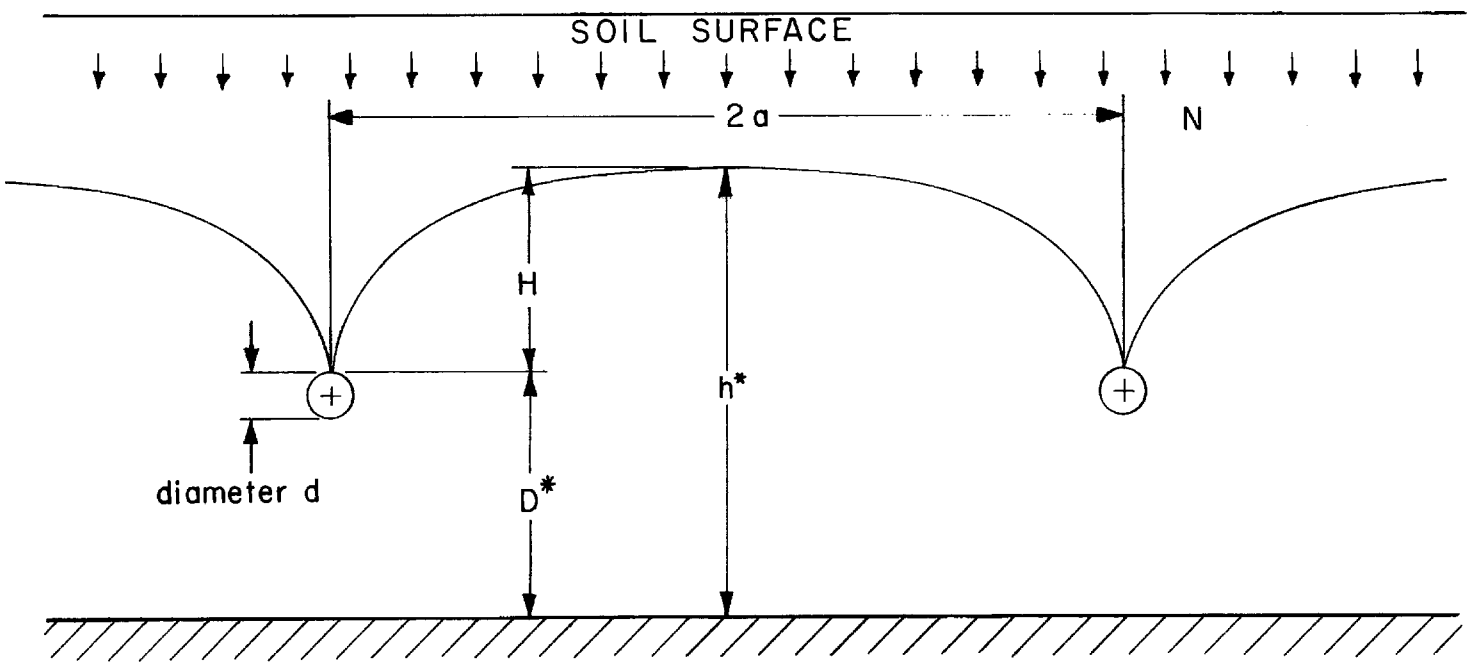

Figure 1 - The flow geometry for an optimum drainage system

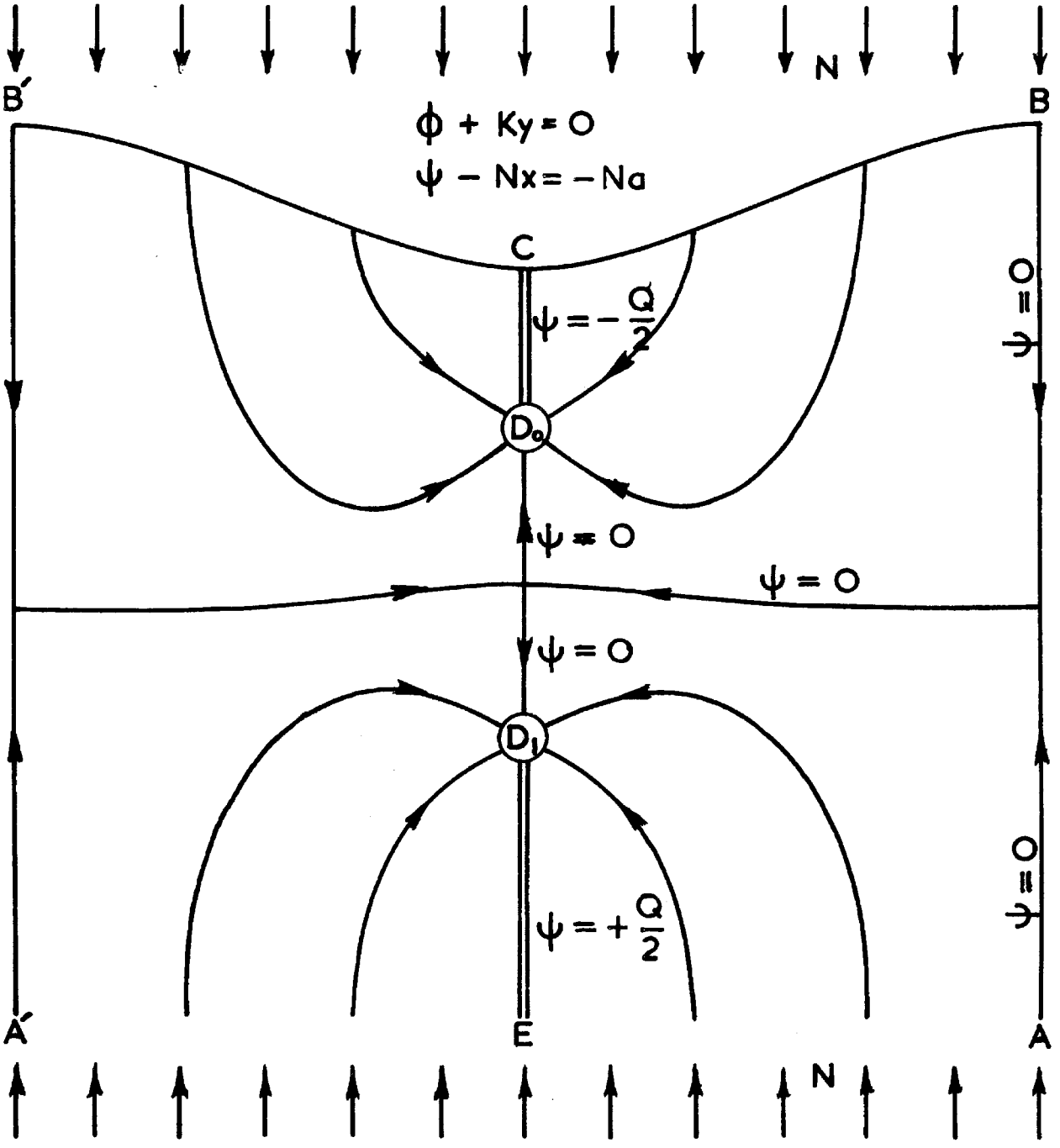

Figure 2 - The mathematical model

The mathematical details of the solution are given in Appendix B which is a reprint of a paper by the author in the Journal of Geophysical Research, Volume 69, No. 16, August 15, 1964.

2.1 The Drain Diameter.

An essential point too often neglected in the study of drainage systems is the drain diameter. Vedernikov (1939), Engelund (1950), Childs (1951) and Watson (1960) have shown that the drain diameter plays an important part in determining the shape of the free surface when the impervious layer is at infinite depth and the same thing can be expected when the drains overlie an impervious layer. Engelund (1950) and Watson (1960) have shown that there exists a diameter of drain for which the water table is at its lowest. This has been termed the "optimum condition" of draining by Watson and the associated drain diameter, the "optimum" diameter. This choice of drain diameter is based solely on the flow pattern outside the drain, and does not necessarily imply adequate hydraulic capacity inside.

In an optimum drainage system the water table lowering is at a maximum, the drain diameter is at its largest effective size and no water stands over the tile lines. Engelund has further shown that the maximum angle of inclination of the water table to the horizontal is 90° and that this occurs immediately above the drain. In a sub-optimum drainage system the diameter of the drain is smaller than the optimum diameter and water stands over the tile lines. It would then appear that in the design of a drainage system the most conservative design would be an optimum system.

For the purposes of analysis drain perimeters are normally taken as lines of constant potential, which corresponds to a drain running full. Furthermore, if a small air gap exists at the top of the drain then the potential at the uppermost point of the drain satisfies the phreatic condition $\varphi + Ky = 0$ where y is the elevation, φ the potential and K the hydraulic conductivity of the medium. The drain can also be designed for running empty in which case the drain perimeter is a surface of seepage, i.e. all points such that $\varphi + Ky = 0$. As would be expected the drain running empty has a higher infiltration capacity and this is reflected by the analysis in showing that a drain running "empty" need not be as large in diameter as a drain running full.

CHAPTER 3

RESULTS AND APPLICATIONS

3.0 Classification of Results.

The results can be classified in accordance with the type of drainage system, i.e. optimum or sub-optimum or whether the drain is running full or empty. For this report it has been decided only to give results for optimum condition in the belief that they are the more conservative for design, as explained below.

It is shown in Appendix B that if a drainage system is an optimum one and no water stands over the tile lines then as the rate of infiltration increases the drainage system then becomes a sub-optimum one at the new rate of infiltration. Furthermore, although the water may stand relatively deep over the tile lines the maximum height of the water table does not rise too significantly. Thus an optimum drainage system has some capacity to absorb infiltration rates higher than the design value without significant flooding.

The results given are further subdivided into the two categories for drains running empty and drains running full. The conservative engineer will no doubt use the results for the drains running full.

It should be noted here that the drain diameters given have no relation to the hydraulic capacity of the tile lines. This is something that can only be determined once a spacing and length is decided upon and the maximum anticipated flow computed. It may well be that hydraulic considerations dictate what size of tile should be used in any particular location.

The computation details for the plotting of the curves are given in Appendix C.

3.1 Spacing and Depth of Drains.

Figure 3 gives a graph of $2a/h^*$ against D^*/h^* (see Figure 1 for notation) for various values of N/K, the ratio of rate of infiltration, N, to hydraulic conductivity K. The full line crossing the curves gives the limit for the drain on the impervious layer when the diameter is taken for the drain running empty, the broken line the limit when the drain is running full.

The normal procedure in designing a drainage system is to decide how far below the surface the water table should be kept. Once this decision is made and the depth below the surface of the impervious layer is known then h^* is fixed. Hence with N/K known, the curves in Figure 3 give alternative spacings and depths. For example suppose h^* were 10 feet, N/K = 0.04 then from Figure 3 alternatives are:

2a	$2a/h^*$	D^*/h^*	D^*
60	6	0.700	7.00
80	8	0.565	5.65
100	10	0.420	4.20
120	12	0.240	2.40

Note that if the drains were placed wider than 130 feet i.e. $2a/h^* = 13$ then flooding would occur, regardless of the drain size used.

These curves are only valid if they are used with the correct diameter of drain as given by Figures 4 and 5.

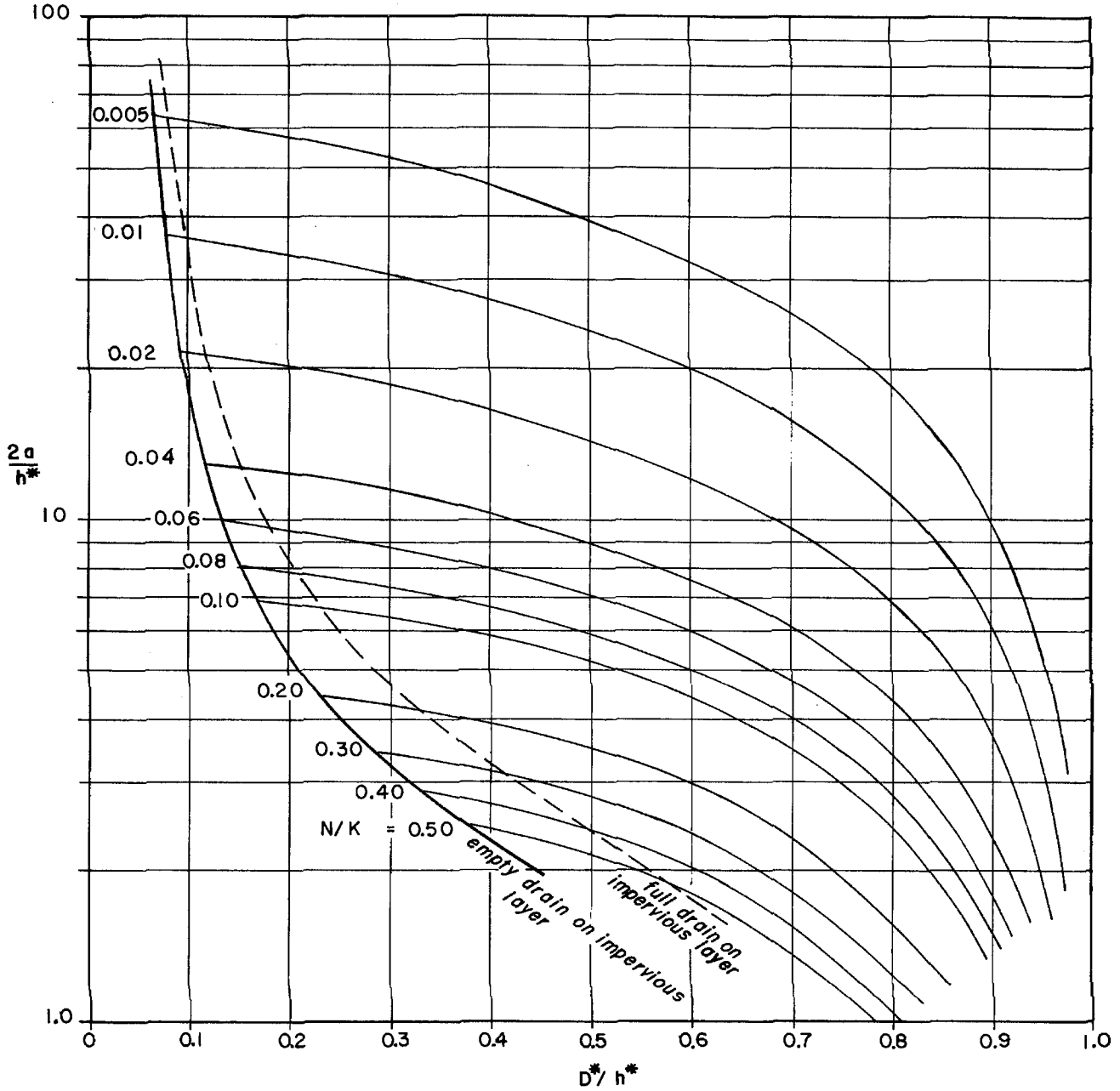

Figure 3 - The depth and spacing of drains for a range of N/K, for optimum drain diameters

3.2 Drain Diameters.

As stated above two choices of diameter are possible depending upon whether the drain is running full or "empty". Figure 4 gives the drain diameters required if the drains are running mostly full. Consider the example from above with $h^* = 10$ feet, $N/K = 0.04$.

2a	$2a/h^*$	D^*/h^*	D^*	d_f/h^*	d_f (ft)	d_f (ins.)
60	6	0.700	7.00	0.0445	0.445	5.3 inches
80	8	0.565	5.65	0.060	0.60	7.2 inches
100	10	0.420	4.20	0.0765	0.765	9.2 inches
120	12	0.240	2.40	0.104	1.04	12.5 inches

This table shows that if it is desired to have a spacing of 60 feet then the drains should be 7 feet above the impervious layer and almost 6 inches in diameter; however for a spacing of 100 feet the drains should be 4.2 feet above the layer and 9 inches in diameter (or more). These figures are on the basis of the drains running full.

Should it be desired to design the system for the drains running empty then Figure 5 gives the diameters. Consider the example as above.

2a	$2a/h^*$	D^*/h^*	D^*	d_e/h^*	d_e (ft)	d_e (ins.)
60	6	0.700	7.00	0.0333	0.333	4 inches
80	8	0.565	5.65	0.0460	0.460	5.5 inches
100	10	0.420	4.20	0.0580	0.580	7
120	12	0.240	2.40	0.0730	0.730	9

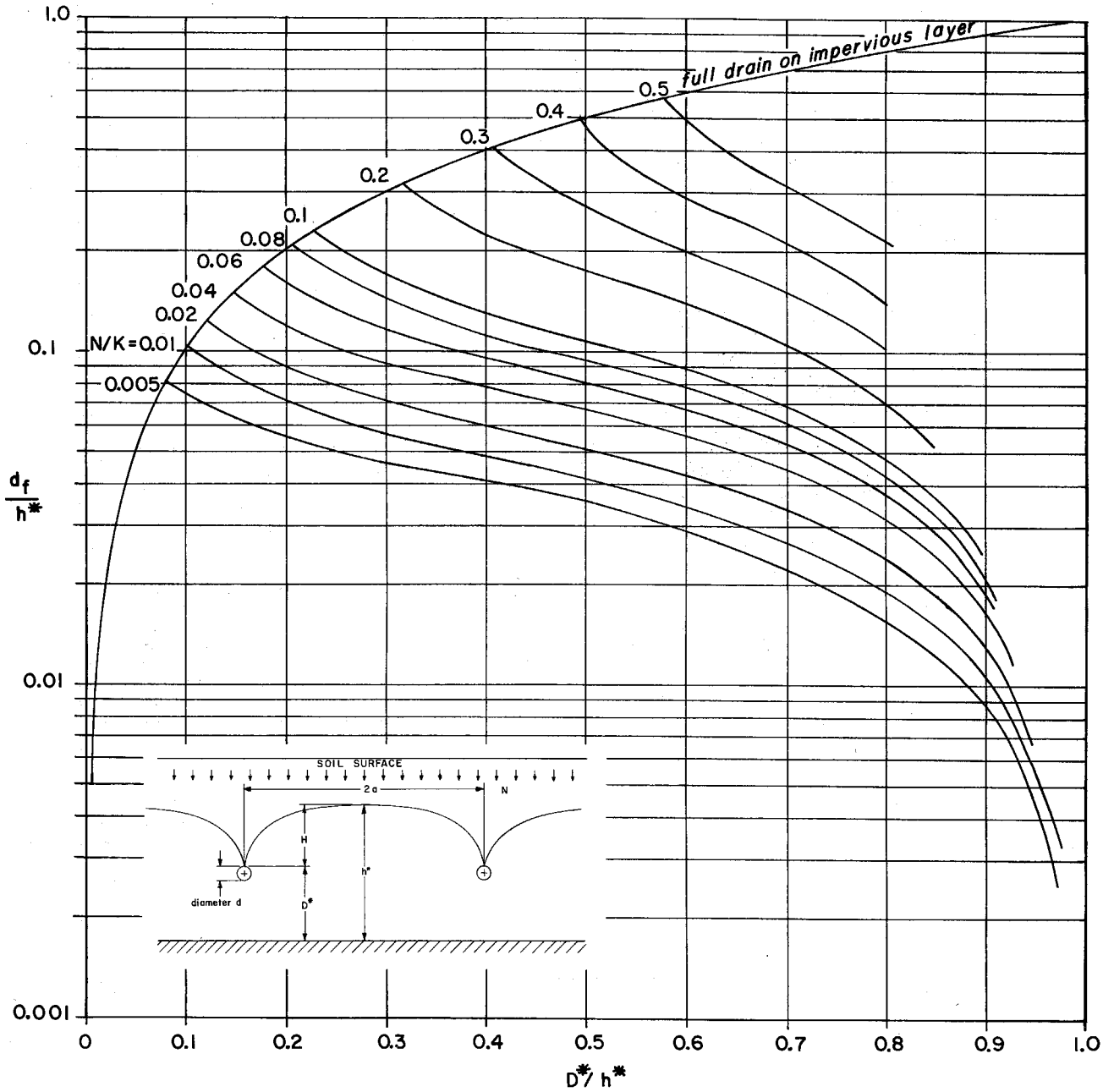

Figure 4 - The optimum diameter of drain, flowing full, as a function of the depth of the drain for a range of N/K

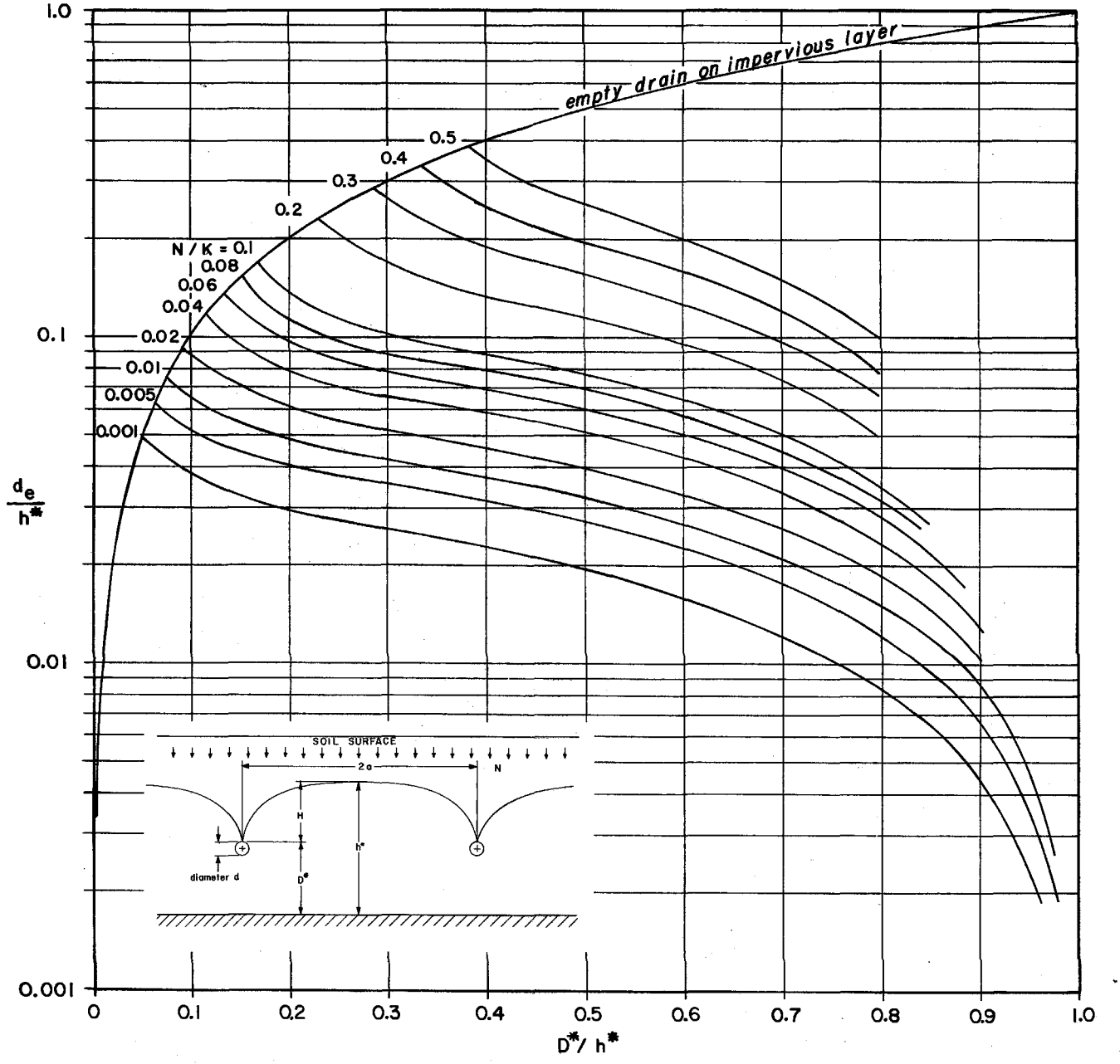

Figure 5 - The optimum diameter of drain, flowing empty, as a function of the depth of the drain for a range of N/K

3.3 Special Cases.

Two limiting cases are the impervious layer at infinite depth and the drain in contact with the impervious base.

Figure 6 is a graph of H/2a versus $D^*/2a$ for a range of values of N/K (see Figure 1 for the notation). It is seen that H/2a rapidly tends to a stable asymptotic value as $D^*/2a$ becomes large i.e. the impervious layer becomes very deep in comparison with the drain spacing. For example with N/K = 0.01 and $D^*/2a > 0.25$ then little change occurs in the water table height, similarly, when N/K - 0.50 little change occurs for $D^*/2a > 0.3$. This gives credence to the well-known rule 'For $D^*/2a > 0.3$, forget the impervious layer' (Collis-George and Youngs (1958)).

The limiting values of H/2a are plotted against N/K in Figure 7 for the two cases, infinite depth and drain resting on the impervious layer. They are to be used in conjunction with Figure 8 which gives the optimum drain diameters for both these limiting cases when the drains are both empty and full.

Thus when the impervious layer is either very shallow or very deep these two curves may be used instead of Figures 3, 4 and 5.

3.4 Applications and Limitations.

Since the curves given come from a steady state analysis and do not take account of periods of high intensity precipitation they are therefore only good for long term design. However, by designing the system as an optimum the engineer can be sure that the drainage system is capable of handling a somewhat higher than design intensity without worry of serious

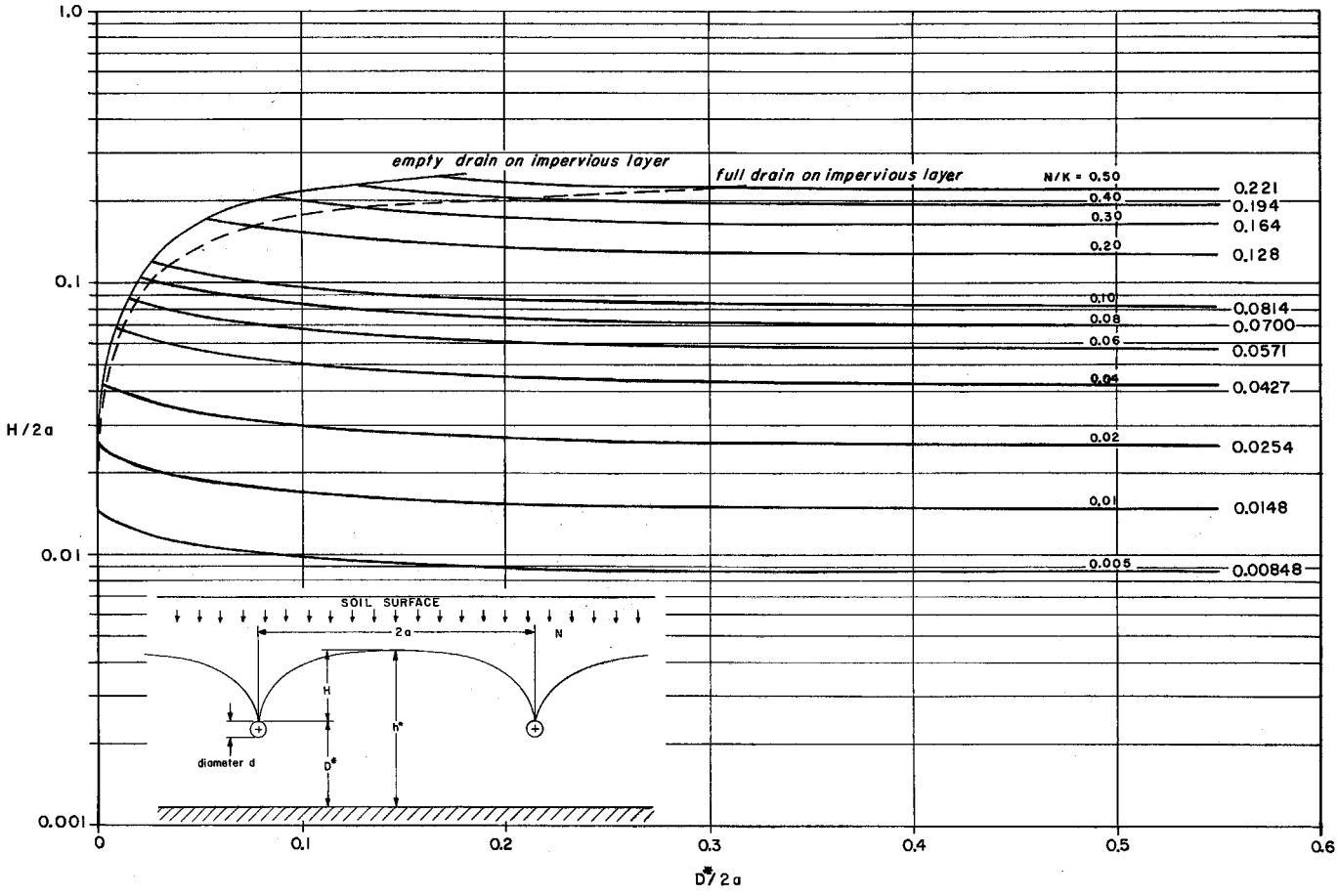

Figure 6 - The water table height as a function of drain depth with optimum size drains

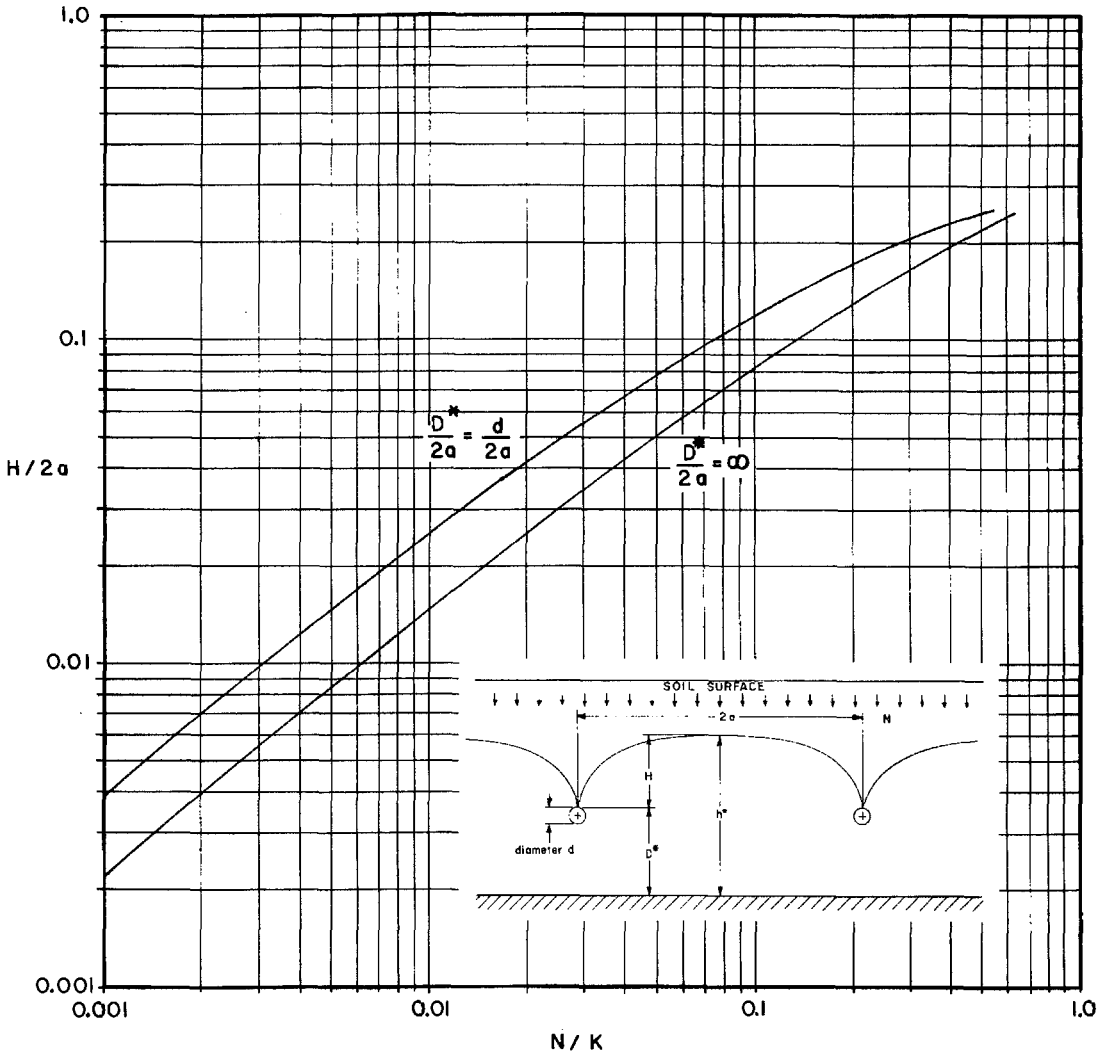

Figure 7 - The water table height for limiting cases as the optimum condition

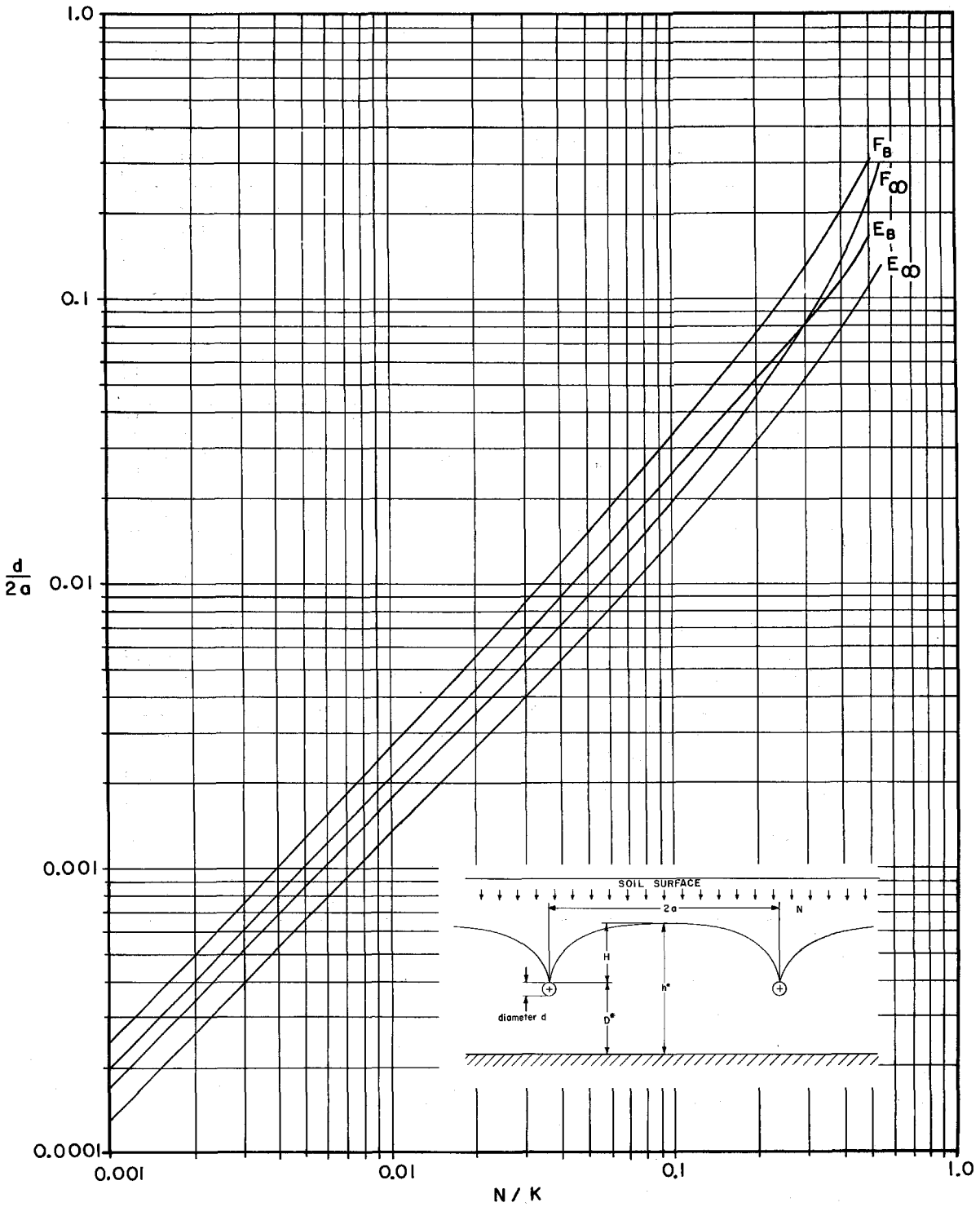

Figure 8 - The drain diameter for the limiting cases.
(F_B, E_B - full and empty on the impervious layer,
F_∞, E_∞ - full and empty impervious layer very deep)

overloading and consequent flooding. The curves are therefore best employed for long term design, say where the mean annual infiltration is known. For this reason the curves given are probably best employed for land reclamation schemes. The design of a drainage system to handle intermittent recharge e.g. irrigation, is another exceedingly difficult problem.

As has been previously stated (see Appendix B also) the mathematical model least represents the actual physical situation when the drains are closest to the impervious layer, for it is here that the shape of the pseudo-impervious layer becomes least straight and the greatest distortion occurs in the drain shape. However, it is believed that confidence may still be placed in the accuracy of the results to within the degree that the isotropic homogeneous model represents any given physical situation.

One further point is that the analysis has completely ignored the presence of a capillary zone. For an evaluation of how this affects the results of such an analysis the reader is referred to Watson (1960).

APPENDIX A

A-1. Review of the Theoretical Literature.

A review of this literature is best considered under two headings. Firstly, what will be termed quasi-analytical solutions, depending on basic assumptions in addition to Darcy's law and secondly, true analytical solutions depending only upon the validity of Darcy's law.

A-1.1. Quasi-analytical solutions.

There have been numerous attempts, since the time Darcy established his law, to derive a formula which would give the required drain spacing, depth, and sometimes, diameter, for any given soil and rate of influx of water. Most of these endeavors rest upon the parallel flow assumption of Du Puit (1863) and the first, according to Gustafsson (1946) and Zunker (1923), was by Colding (1872). Various others, Rothe (1924), Kozeny (1932), Hooghoudt (1937) and Aronovici and Donnan (1946) also derived this solution independently. Rothe and Kozeny both maintained that there was no flow to the drains below drain level even though, according to Russell (1934), Penniuk (1907) had proved the falsity of this experimentally. All these solutions proved the existence of an elliptical phreatic surface between adjacent drains.

Another interesting attempt at a solution was presented by Spöttle (1911) who, it seems, forgot the equation of continuity and had the velocity of flow to the drain being constant from the point of supply right to the drain surface, an assumption that could be justified except for flow near

the drain. Strangely enough, a method very similar to Spöttle's was presented by Walker (1952) who assumed a unit hydraulic gradient, unwittingly it would seem. Hore and Kidder (1954) considered Walker's formula for a practical case where field data was available. They found it gave a spacing 50% greater than the actual spacing and then reached the conclusion that it was a good formula since the drains in place were inadequate anyway!

Tokutaro (1939) used an image system for solving flow to drains from a ponded water table condition and also gave a copy of Kozeny's (1931) proof of an elliptical water table in flow to wide flat drains on an impermeable layer, the problem considered later by Engelund (1951). He stated that the ellipse becomes a parabola when water is standing over the drains and from area considerations calculated working tables for lowering times. Muskat (1946), postulating a horizontal groundwater table, also dealt with image systems (p. 189) and gave an approximate solution (after Farr and Gardner (1933)) for finding the spacing of drain tiles when there is an artesian seepage flow. The work has been superseded by the later work of Engelund (1951).

As previously mentioned, Aronovici and Donnan (1946) employ the Du Puit postulate to arrive at a formula for spacing and depth but as is typical of all the earlier work on drain tiles, no significance whatsoever is placed on the drain size. The junior author a year later, Donnan (1947), gave experimental comparisons with his formula and got 10% accuracy, after making some corrections for the flow in the capillary fringe. Slater (1950) developed this same formula, $S = 4P\,(b^2-a^2)/Q$, by taking the slope

from the highest point of the water table to the drain and calling this the hydraulic gradient where:

b - is depth from highest point of water table to pan.

a - is depth from drain to pan.

P - hydraulic conductivity.

S - spacing and Q maximum flow.

Jaeger (1946) presented an original theory of critical depth of flow to tile drains. He stated that this critical depth over the tile is associated with the maximum possible flow to the drain placed at any given depth. However, his treatment rests upon the Du Puit postulate. In a later work, Jaeger (1956), he approximated the flow to a tile located above an impermeable layer by assuming the flow boundary to be a line from the bottom of the drain to a point on the impermeable layer midway between the tile lines. In other words halfway between the reasoning of Rothe and Kozeny and the actual state of flow. This assumption is definitely a dubious one, as will be shown in later work (see Appendix B).

Ferris (1950) was concerned with flow from a confined aquifer to a drain and again based his reasoning on the Du Puit postulate. Van Schilfgarde et al (1956) in their critical review dealt with his efforts in a few lines.

Baittinger (1953a) (1953b) gave another empirical formula which was again independent of drain diameter but he conceded that his formula did not consider the depth of any impervious stratum and that past experience was the best way of designing any drainage system. Other proponents of the insignificant diameter were Beauchamp and Fasken (1955) who gave specifications for depth and spacing depending on the soil type and apparently thought that the drain size did not matter.

The American Society of Agricultural Engineers (1953) presented a draught specification for their members' perusal and comments; it also based spacings and depths on soil type, district and crop and hence is not much use to the drainage engineer outside the specified areas. However, there is an excellent section for the practicing engineer on specifications for any job put up for tender. Sutton (1960), who also presented the Donnan formula again, had an excellent section of the planning required for installation of a drainage system.

The so-called Glover formula for non-steady state solutions was presented by Dumm (1954). He gave two forms, one where the lowering is moderate and the other for large lowerings; the formula only applied where the height of the water table is small compared to the depth of any impermeable barrier below the drain.

A-1.2. Analytical solutions.

Steady state solutions for land drainage problems really began with the publication by Vedernikov (1939), of his solution for an infinite series of drains located in a semi-infinite permeable continuum. Vedernikov's solution used the result of a previous paper, Vedernikov (1936), which in turn depended on the use of the Zhukovsky (1923) potential function and the velocity hodograph of Davison (1932). Using Schwarz-Christoffel transformations, Vedernikov succeeded in solving the problem that was later published by Gustafsson (1946) and extended by Engelund (1951) and van Deemter (1949). He even anticipated, in a general sort of way, the later refinement of Childs (1959) in including the capillary fringe in his analysis. Numerical results in agreement with Engelund's both for depth and diameter of drains are included.

The normal source quoted for the hodograph methods is two German papers by Hamel (1934) and Hamel and Gunther (1935), which apply the method to seepage through dams. Childs (1943, 1945a) indicated how the hodograph is drawn for the drainage problem but never actually obtained a solution. It remained for Gustafsson (1946) to apply the correct transformations to this velocity hodograph. In a magnificent paper including both theory and model studies he considered the ponded water table with and without the presence of an impermeable layer and also with a highly permeable layer below the drains. These solutions, given in closed form in terms of elliptic functions, were also given in a series of papers by Kirkham, (1940b, 1945, 1949, 1951), who avoided elliptic functions and summed 44 terms of a series solution. Gustafsson also dealt with flow on a sloping field with and without an impermeable layer. He applied the velocity hodograph method to the special case of a series of drains when the maximum angle of inclination of the phreatic surface is equal to 90 degrees, the optimum condition. Childs (1946), indicated the deficiencies in Gustafsson's results with regard to drain size.

Engelund, (1951), in a fine publication dealt with three problems; the first, previously considered by Kozeny (1932), concerns flow to wide flat drains forming an elliptical phreatic surface. The second was an extension of Gustafsson's work to include sub-optimal conditions, i. e. when the maximum angle of inclination of the water table is less than 90 degrees and water is standing over the drains, and the third was flow from a ponded water table through an anisotropic medium to a slit drain.

Van Deemter (1949, 1950) also used hodograph methods but a somewhat more general hodograph than Engelund's. As well as influent seepage from the surface, he also included an artesian flow, the dividing streamline thus forming the equivalent of a curved impermeable layer with the drain half embedded in it; this device was also used by Kirkham (1947) for the ponded condition. Relaxation solutions to a few drainage problems are also included. Childs (1959) extended the analysis to include the capillary fringe and Watson (1960) used van Deemter's solutions to carry out an investigation into the diameter of drains. The perimeter of the drain was stated to be an equipotential line which was not quite circular and Watson investigated how the upper and lower radii varied with depth of drain and rate of influx.

Hooghoudt (1940) used a form of image system to give approximate solutions in the presence of an impermeable layer but his solutions become less accurate as the impermeable layer comes close to the surface. Kirkham (1940a) applied results of his analysis to some sand tank experiments. Kirkham (1950) not only gave solutions to the problem treated by Gustafsson (1946) but also dealt with flow to "gappy drains" from a ponded water table; flow to perforated drains was considered by Kirkham and Schwab (1951a, 1951b). Kirkham (1954) applied image system methods to flow from above and below and compared this with the experimental results of Harding and Wood (1941). In all these papers he gives an estimate of the time required for a depth of ponded water to disappear. Kirkham (1958) in an attempt to deal with a curved water table does in fact find a solution for the greatest height of the water table. Tile diameter does not greatly

affect the results he asserted, as it is contained logarithmically in the result. A rationalized basis was given for the Du Puit-Forcheimer parallel flow theory for the case of wide drain spacing and flat water table. Kozeny (1953) also referred to the ponded solution of Gustafsson (1946).

Numerous numerical methods have been used for calculating solutions, relaxation techniques in particular. van Deemter (1949) appears to have been one of the earliest and he considered the phreatic surface. Luthin (1950) applied the method to a leaching problem and Luthin and Gaskell (1950) applied relaxation techniques to the same problem that Kirkham (1949) considered, i. e., flow of ponded water to a series of drains with no impermeable layer. Visser (1954) presented nomogram solutions to the drainage problem with phreatic surface and impermeable layer, stating that these were based on the relaxation work of Boumans (1953) and Ernst (1954). These nomograms probably present the best data yet published for the practising engineer.

Drainage of Agricultural Lands, (1957), is a comprehensive summary of drainage theory to the date of publication. It contains chapters written by the leaders in the various fields. The book is in five parts:

I Physics of land drainage by Dr. E. C. Childs gives a complete coverage of infiltration theories and the general physics associated with groundwater flow.

II Theory of land drainage contains sections by van Schilfgarde, Engelund, Kirkham, Peterson and Maasland, and covers Du Puit postulates, mathematical analysis and anisotropy.

III Engineering aspects of land drainage deals with hydraulics of tile lines and ditches.

IV Drainage investigation methods.

V Land drainage in relation to soils and crops is concerned with agricultural aspects of drainage.

The book would be indispensable to the engineer or agricultural field officer advising on drainage installations.

Ivitskiĭ (1954) gave a whole series of formulae for drain depth, spacing and diameter. Unfortunately, all the references are to Russian publications which are not available. The formulae appear to have been obtained analytically although some of the constants would appear to be empirical.

Polubarinova and Falkovich (1951), although mainly concerned with seepage problems in general, outlined methods for solution of problems which are applicable to tile drainage work. They have offered a solution to a tile drainage problem but the reasoning is far from lucid.

Van Schilfgarde et al (1956) presented a critical review of mathematical theories of land drainage up to 1956. They dealt with the theories of Glover (Dumm (1954)), Ferris (1950), Spöttle (1911), Walker (1952), Hooghoudt (1940) and van Deemter (1950). Field, Kirkham and De Zeeuw (1952), and experimental, Childs (1943), results are given and compared with the various analyses.

Peck (1960) and Swartzendruber and Kirkham (1956a, 1956b), while not directly concerned with the mathematical analysis of any land drainage problem have brought out two rather interesting points. Peck

dealt with the influence of barometric pressure on the ground water level. He presented a theoretical solution and experimental results, which correlate with the theory rather well. He has shown that a normal change in air pressure from a high to a low can influence the water table by as much as 3 1/2 inches and this is something investigators have overlooked when recording test results. The second paper dealt with the amount of flow which can occur in the capillary fringe when flow is considered along an impermeable layer. The influence of the impermeable layer is large in this and with a high ($h^*/2a = 0.1$) impermeable layer the capillary fringe flow can be as high as 170% of the base flow.

Two recent attempts at obtaining a solution are by Hammad (1962) and Dagan (1964). Both of these solutions employ approximations at the free surface, a critical region of flow, whilst satisfying the boundary conditions at the impervious layer exactly. Dagan's solution is probably the more accurate of the two and is quite likely it is satisfactory for water tables of low slope. Hammad's solution has been shown List (1962) to deviate from Watson's (1960) exact solution, when the impervious layer is at infinite depth, by at least 30% indicating the approximations made may not be as justifiable as at first appears.

The paper by Davison and Rosenhead (1940), which can be regarded as giving a very significant contribution to the mathematical methods of handling land drainage, has not received the attention it deserves. The method involved is extremely powerful in that problems with a phreatic surface can be considered simply and easily without recourse to the velocity hodograph method and what is perhaps more significant, it enables problems

with vertical equipotential lines to be handled, although series solutions are obtained for this type of problem, List (1961).

A-2. Review of the Experimental Literature.

As the analysis of land drainage problems has been drawn out over a long period it is not surprising that there has been a volume of experimental work published. Work has been carried out in both the field and the laboratory; laboratory methods have been usually either a sand tank model or some form of electrical analogy. The Hele-Shaw apparatus has also been used with excellent results. A review of the literature is best considered under these catagories.

A-2.1. Field tests.

Two early investigators in the U.S.A. were Weir (1928) and Neal (1934). Weir dealt with a series of observations on a set of drains but did not attempt to provide any empirical rules; he offered the conclusion that flow above the phreatic surface is vertical (see later). Neal, on the other hand, carried out an extensive investigation in an attempt to find an empirical formula governing drain depth and spacing. In common with most American investigators following this system, he did not consider the drain size, although he was apparently aware of this since he stated that German formulae were not applicable since the Germans used smaller drains. He provided nomograms for the depth and spacing but based these on a fixed rainfall, leaving them dependent on a somewhat dubiously defined soil type.

About this time Englehardt (1930) carried out what Kirkham and De Zeeuw (1952) claimed to be the most extensive recording of drain discharge and rainfall that had ever been done.

Evans et al (1950) spent some time trying to determine soil permeability, as part of a drainage investigation, by observing the rate of loss from a pond. Kirkham and De Zeeuw (1952) have published a set of observational data obtained in the Netherlands. The site they chose was underlain by a relatively impermeable peat layer and hence they have a set of data which is readily applicable to the problem undertaken in this work. Van Schilfgarde et al (1954) described an outdoor field laboratory constructed in Iowa; there seemed to have been numerous problems encountered in the construction and the project appears to have been subsequently abandoned, judging by the lack of publications.

Goins (1956) included some observational data on the effect of various storms on the flow rate of several different tile systems, it does not seem to have contributed much to the theory of tile drainage; similarly the paper by Goins and Taylor (1959), who measured the rate of recession of the water table for a few drain systems. Isherwood and Pillsbury (1958) also measured the rate of recession of the water table in a series of drains, data which was later used by Isherwood (1959), to compare with relaxation solutions.

Talsma and Haskew (1959) measured the rate of recession of the water table on several plots underlain by an impermeable layer and compared their results with the theoretical results of Kirkham (1958), Hooghoudt (1940) and the non-steady state formula of Glover as reported by Dumm (1954). They stated the concept of an impermeable layer was not easily recognised in the field and that soil heterogeneity required caution in the field and that soil heterogeneity required caution in the use of drain spacing formulae.

They found Hooghoudt's formula favorable and the others satisfactory under the conditions laid down by the assumptions.

A-2.2 Sand Tank Models.

The sand tank has been a popular method of carrying out drain tile experiments, some investigators considering the ponded condition and others the phreatic surface. Difficulties arise in the latter from the existence of the capillary fringe and it has been found necessary to use sufficiently large sand grains to keep this at a minimum.

The first recorded use of a sand tank model seems to have been by Farr and Gardner (1933), who considered flow from an artesian basin to a series of drain tiles. Kirkham, (1939) and (1940a), followed this up and carried out experiments on the influence of permeability and compared streamlines from experiment and analysis. Kirkham (1940b) applied the same image system methods as Muskat (1937) to the ponded flow to drains located above an impermeable layer and he again made a comparison with sand tank experiments. Harding and Wood (1941) used the sand tank model to consider flow from an artesian basin and Kirkham (1945) presented the analysis behind his previous work, (1940b) and compared the results with those of Harding and Wood.

Gustafsson (1946) used both sand tank and Hele-Shaw apparatus to confirm his analytical results. Aronovici and Donnan (1946) attempted to justify a Du Puit approach. Bouwer (1955) was interested in the comparison of longitudinal to lateral draining when the ground is sloping. He used the nomogram results of Visser (1954). Keller and Robinson (1959) were more concerned with interceptor drains and carried out a large scale model study

comparing the results with previously developed quasi-analytical results attributed to Glover, but stated by Maasland (1959b) to be an old Du Puit result reported by Dachler (1936).

Childs (1953) reported the construction of a new drainage laboratory at Cambridge. Collis-George and Youngs (1958) employed this tank to determine the effect of an impermeable layer on the spacing of drains, and also used an electrical analogy method. Good agreement was reached between the hydraulic and electrical results. The conclusion was reached that the impermeable floor had no effect on the hydraulics of drainage for the values of N/K given when $(h^{*}-h.)/2a>0.15$. They dealt only with the theoretically optimum drain size which of necessity was Engelund's computed result.

Ede (1958) used a hydraulic sand tank for making rapid comparisons between various drain installation types. Discharge data for various gapped, spaced and perforated drains were presented. He stated that where $N/K<0.01$ then no account need be taken of the drain diameter but the diameter does effect the intake capacity and hence drains should always be covered with gravel or clinker to ensure maximum efficiency.

Childs and Youngs (1958) were concerned with the model analysis of flow to "gappy" drains, i.e. relatively impermeable tile drains laid with gaps between tile drains. Engelund's results were dealt with fully and the results for a "gappy" system compared.

A-2.3 Electrical Analogy Methods.

According to Luthin (1953) it was Slichter (1899), who pointed out the analogy between Ohm's Law and Darcy's Law and thus suggested the idea of employing electrical analogy methods to solve groundwater flow problems.

A prolific worker in this field has been Dr. E. C. Childs at Cambridge. In one paper, (1943), he used the method to solve various cases of steady influent seepage to a series of drain tiles located above an impermeable layer. While drawings of the equipotential lines were given, it is a pity that no attempt was made to collate the solutions in graphical form. Childs (1945a), dealt with the influence of test bore holes on the phreatic surface in any field investigation. He found that bore holes could cause appreciable perturbation of the water table shape and in fact lead to erroneous field recordings. Childs (1945b) was concerned with determining the verticality of flow in the capillary fringe. Electrical analogy methods using Teledeltos graphite paper were employed and the conclusion reached ". . . . That in soil with sensibly uniform pore sizes the streamlines are truly vertical thereby justifying the assumptions made in previous work."

Two non-steady problems were considered by Childs (1947) and Childs and O'Donnell (1951). The former paper dealt with the case of a falling water table and the author reached the conclusion that the whole water table falls almost uniformly over the greater part of its descent until the height at the drain is very little above the level of the drain itself. He showed Gustafsson's solution only applied to a particular case, the so called "optimum" condition. Childs and O'Donnell (1951), attacked the problem of a rising water table, but to quote the authors, "The technical difficulties of this particular experiment were such that the work described appears to have reached the limit of usefulness of the analogue method." As in the other papers of Childs, no graphs or design curves were given.

Edwards (1956) carried out a series of experiments solving cases of flow to drains in anisotropic, homogeneous media. The difficulties associated with anisotropy were surmounted by using a distorted model. The proximity of an impermeable layer was considered and graphs were drawn comparing various ratios of permeability with isotropic results.

Youngs (1959) presented graphs comparing electric analogy results for drains overlying an infinitely permeable layer with those given by Engelund (1951) for a semi-infinite homogeneous soil mass. The results were given in the form of a correction factor to be applied to the water table height midway between the drains. The results were also compared with those for flow over an impermeable layer, Collis-George and Youngs (1958). Only optimum drain sizes were considered.

Kirkham and Schwab (1951b) used analogue methods to test their theory of flow to perforated drains (1951a). The results obtained were in good agreement with theory.

Luthin (1953) used a somewhat different electrical analogue; instead of the Teledeltos paper of Childs et al he employed a resistance network which he has stated instantaneously relaxed a system to the same degree of accuracy as a numerical relaxation solution. The author developed the theory and dealt with the accuracy of the results; solutions obtained were compared with the relaxation solutions of Luthin and Gaskell (1950). Problems were considered of flow to tile lines located in inhomogeneous media; i. e. layers of varied permeability. The conclusion was reached that the maximum flow was when the tile line was laid in the more permeable stratum. For instance, when the tile was located in a layer with 5. 5 times

the hydraulic conductivity of the soil above, then for the ponded condition, the tiles will have to deal with twice the flow that would occur in a homogeneous soil of hydraulic conductivity equal to the upper layer.

JOURNAL OF GEOPHYSICAL RESEARCH VOL. 69, NO. 16 AUGUST 15, 1964

APPENDIX B

The Steady Flow of Precipitation to an Infinite Series of Tile Drains above an Impervious Layer

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Abstract. A solution is given to the problem of the steady flow of precipitation to an infinite series of tile drains above an impervious layer. The free surface boundary conditions are exactly satisfied, but the shape of the impervious layer is approximated. Capillary forces are neglected. The theoretical results are compared with limited field measurements, and good agreement is found. The solutions become less accurate when the drains are very close to the impervious layer. The limiting cases are shown.

Introduction. Analytical solutions to the problem of flow to an infinite series of tile drains in a semi-infinite, homogeneous, isotropic, permeable medium have been available for some years. The earliest appears to have been given by *Vedernikov* [1939]. Numerous others have also presented solutions, notably *Gustafsson* [1946], *van Deemter* [1949], and *Engelund* [1951]. *Childs* [1959] extended van Deemter's analysis to include the capillary fringe, and *Watson* [1960] dealt with the drain size.

The more difficult problem of an infinite series of tile drains above an impervious layer has resisted solution. *Hooghoudt* [1940] presented an approximate solution which has been critically reviewed by *van Schilfgarde et al.* [1956] and compared with field data by *Talsma and Haskew* [1959]. *Kirkham* [1958] attempted a Fourier series approach which was also reviewed by Talsma and Haskew. *Hammad* [1962] has given an approximate solution. The ponded condition has been handled by *Gustafsson* [1946] and *Kirkham* [1949]. A set of nomograms presented by *Visser* [1954] from the relaxation work of Boumans and Ernst form an excellent set of design data, with the limitation that drain sizes are not given.

A significant paper has been either overlooked completely or dismissed with scant mention. *Davison and Rosenhead* [1940], whose methods will be employed to advantage in this communication, presented what is probably the most powerful method yet given for handling problems of two-dimensional drainage with a free surface boundary. The solution of the problem when the impervious layer is at infinite depth becomes very straightforward. The presence of an impervious layer, however, still poses analytical difficulties. For this reason, a device is employed which simplifies the analysis but limits the application of the results. The limiting cases are shown.

The problem. The flow to a set of tile drains above an impervious layer is found to be that given in Figure 1. Generally, the aim in designing such a system is to prevent the water table from rising above a certain depth from the surface. The problem is to determine the diameter d, depth h, and spacing $2a$, of the drains when the depth of the impervious layer, h^*, and the ratio of rate of infiltration to hydraulic conductivity N/K are known.

The analytical difficulties associated with the presence of the impervious layer can be overcome if an image drain is placed below the impervious layer, as in Figure 2. The upper drain D_0 receives its flow Q in the normal way by a constant seepage of intensity N through the surface. The image drain D_1 receives an equal flow from an artesian source, also of intensity N, at infinite depth. There will be a streamline dividing these two flows and it will be shown that this streamline is sufficiently straight to form a pseudo-impervious layer below the upper drain. It is believed that it is more important to satisfy the boundary conditions at the free surface exactly, rather than at the impervious

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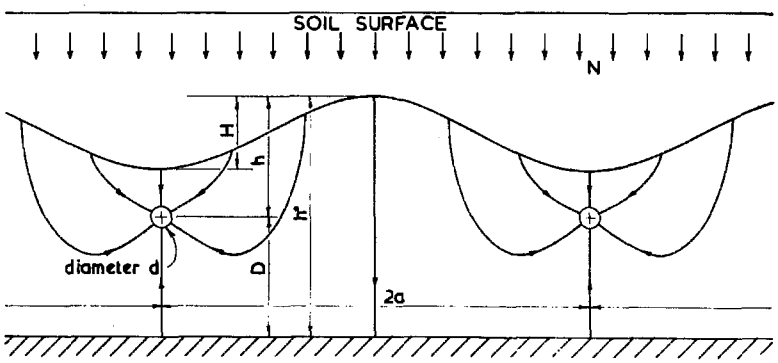

Fig. 1. Flow geometry.

layer, a rather torpid region of flow. A similar device was used by van Deemter, although he considered only a single drain.

Since the flow is incompressible, there is a stream function ψ, such that

$$u = \partial\psi/\partial y \qquad v = -\partial\psi/\partial x$$

where u and v are the horizontal and vertical components of velocity, respectively. Darcy's law may be written in the form

$$u = -K\frac{\partial}{\partial x}\left(\frac{p}{\rho g} + y + c\right)$$

$$v = -K\frac{\partial}{\partial y}\left(\frac{p}{\rho g} + y + c\right)$$

where K is the hydraulic conductivity, p is the pressure and ρ the density of water, and C is a constant. Note that $K = kg/\nu$ where k is the permeability, ν the kinematic viscosity of water, and g the gravitational field strength. If ϕ is defined as

$$\phi = -K\left(\frac{p}{\rho g} + y - \frac{p_0}{\rho g}\right) \qquad (1)$$

where for convenience the constant is chosen as $-p_0/\rho g$, with p_0 the atmospheric pressure, u and v are then given by

$$u = \partial\phi/\partial x \qquad v = \partial\phi/\partial y$$

It then follows that

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \partial\phi/\partial y = -\partial\psi/\partial x$$

In other words, ϕ and ψ are conjugate functions of a potential flow, so that it is possible to write

$$w(z) = \phi + i\psi \qquad (z = x + iy)$$

The problem is solved by finding a parametric relationship between w and z in terms of $\zeta(= \xi + i\eta)$, a subsidiary complex variable.

It is found convenient [*Davison and Rosenhead*, 1940] to define two potential functions, W_1 and W_2,

$$W_1 = w - iNz \qquad W_2 = w - iKz$$

so that

$$w = (KW_1 - NW_2)/(K - N) \qquad (2a)$$

$$z = (W_1 - W_2)/i(K - N) \qquad (2b)$$

where K is the hydraulic conductivity of the medium.

The condition on the free surface is that the pressure should be atmospheric, and thus it

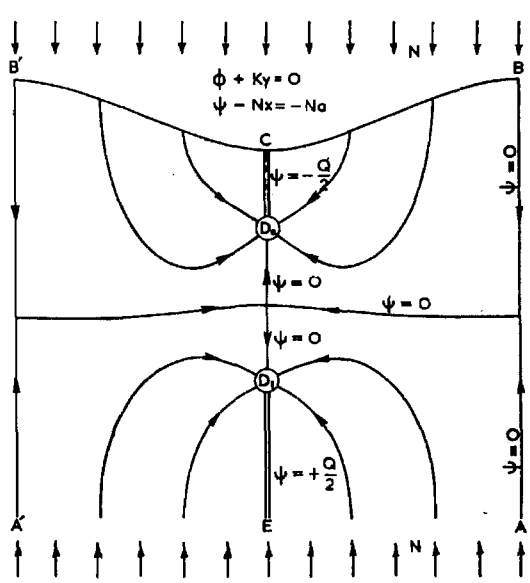

Fig. 2. Mathematical model.

follows from (1) that $\phi = -Ky$ on the free surface. But $\phi + Ky$ is the real part of W_2, and therefore on the free surface Re $W_2 = 0$. Since there is a uniform inflow through the surface, the condition Im $W_1 = \psi - Nx = -Na$ is also satisfied. Furthermore, along the vertical streamlines dividing the flow to adjacent drains, Im W_2 = a constant and Im $W_1 = 0$. Thus, either the real or the imaginary parts of both W_1 and W_2 are known over all the boundary. The functions W_1 and W_2 may therefore be found parametrically in terms of a complex variable $\zeta = \xi + i\eta$ by an application of two-dimensional potential theory. It then becomes possible to find w and z in terms of the parameter ζ by using (2*a*) and (2*b*).

The mathematical analysis of the problem is given in the next section. Those readers less interested in these details may well skip without loss of continuity to the section headed 'Results.'

Solution. The flow system to analyze is that shown in Figure 2, and for the purposes of mathematical representation the drains are taken to be line sinks of strength $Q = 2Na$. It will be shown later how this representation is reconciled with a physical drain.

The boundary conditions on W_1 and W_2 on the right-hand side of Figure 2 are as follows.

AB	Im $W_1 = -Na$	and	Im $W_2 = -Ka$
BC	Im $W_1 = -Na$	and	Re $W_2 = 0$
CD_0	Im $W_1 = -Q/2$	and	Im $W_2 = -Q/2$
D_0D_1	Im $W_1 = 0$	and	Im $W_2 = 0$
D_1E	Im $W_1 = +Q/2$	and	Im $W_2 = +Q/2$

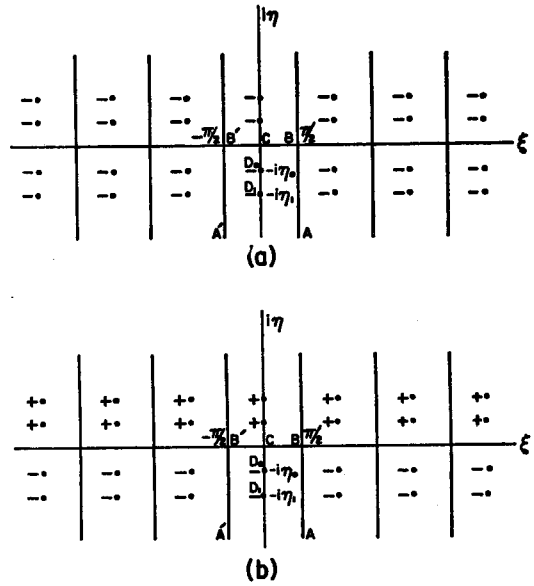

Fig. 3. (*a*) Image system for W_1 in the ζ plane; (*b*) image system for W_2 in the ζ plane.

The region of flow on both sides of the drain is now mapped into the semi-infinite strip $|\xi| \leq \pi/2$, $\eta \leq 0$, in the subsidiary ζ plane. The function W_1 will then be of the form

$$W_1 = -\frac{Q}{2\pi}\,[\ln(\zeta + i\eta_0)(\zeta + i\eta_1)]$$

+ terms in ζ that are regular in the strip where the upper and lower drain coordinates in the ζ plane are $(0, -i\eta_0)$, $(0\ -i\eta_1)$, respectively. The boundary conditions may now be satisfied by erecting an infinite series of negative images in both the upper and lower half-planes as in Figure 3*a*.

Summing all the images leads to a sufficiently general form for W_1 $(=w - iNz)$:

$$W_1 = -\frac{Q}{2\pi}\,\{\ln\,[\sin(\zeta + i\eta_0)\sin(\zeta - i\eta_0) \cdot \sin(\zeta + i\eta_1)\sin(\zeta - i\eta_1)] + Di\zeta + Ei + F\}$$

The real constants D, E, and F are found by satisfying the boundary conditions on W_1, and if BB' is taken as datum level, i.e., $\phi = 0$, $y = 0$, the function is found to be

$$W_1 = -\frac{Q}{2\pi}\left\{\ln\left[\frac{\sin(\zeta + i\eta_0)\sin(\zeta - i\eta_0)\sin(\zeta + i\eta_1)\sin(\zeta - i\eta_1)}{\cosh^2\eta_0\cosh^2\eta_1}\right] + i\pi\right\}$$

It is seen that this has the correct logarithmic singularities at $\zeta = -i\eta_0$, $\zeta = -i\eta_1$.

W_2 may also be found in terms of ζ, the image system being as in Figure 3*b*.

$$W_2 = -\frac{Q}{2\pi}\left\{\ln\left[\frac{\sin(\zeta + i\eta_0)\sin(\zeta + i\eta_1)}{\sin(\zeta - i\eta_0)\sin(\zeta - i\eta_1)}\right] + 2\,\frac{(K + N)}{N}\,i\zeta - i\pi\right\}$$

It is immediately seen that Re $W_2 = 0$ for ζ real, the free surface. The outer boundary conditions on W_2 can also be shown to be satisfied quite readily.

Equations 2*a* and 2*b* may now be used to write w and z in terms of the parameter ζ, where Q has been put equal to $2Na$.

$$w = -\frac{Na}{\pi}\left\{\left(\frac{K+N}{K-N}\right)(\ln[\sin(\zeta - i\eta_0) \cdot \sin(\zeta - i\eta_1)] + i(\pi - 2\zeta)) + \ln[\sin(\zeta + i\eta_0)\sin(\zeta + i\eta_1)] - \frac{2K}{K-N}\ln\cosh\eta_0\cosh\eta_1\right\} \quad (3)$$

$$z = \frac{2a\zeta}{\pi}\left(\frac{K+N}{K-N}\right) - \frac{2aN}{K-N} + \frac{2aiN}{\pi(K-N)} \cdot \ln\left[\frac{\sin(\zeta - i\eta_0)\sin(\zeta - i\eta_1)}{\cosh\eta_0\cosh\eta_1}\right] \quad (4)$$

The problem is therefore solved, in principle.

Interpretation of the solution. It is shown that on the free surface ζ is real, and so with the use of (4) it will be possible to write the equation of the free surface parametrically in terms of ξ. The parametric equations are

$$\frac{\pi x}{2a} = \mu\xi - \lambda[\tan^{-1}(\tan\xi\coth\eta_0) + \tan^{-1}(\tan\xi\coth\eta_1)] \quad (5)$$

$$\frac{\pi y}{2a} = \frac{\lambda}{2} \cdot \ln\left[\frac{(\cosh 2n_0 - \cos 2\xi)(\cosh 2\eta_1 - \cos 2\xi)}{4\cosh^2\eta_0\cosh^2\eta_1}\right] \quad (6)$$

where $\mu = (K+N)/(K-N)$ and $\lambda = N/(K-N)$.

The coordinate of the upper drain D_0 in the ζ plane is $(0 - i\eta_0)$, and hence from (4) the height of the water table, h, is given by

$$\frac{\pi h}{2a} = \mu\eta_0 - \lambda\ln\left[\frac{\sinh 2\eta_0\sinh(\eta_0 + n_1)}{\cosh\eta_0\cosh\eta_1}\right] \quad (7)$$

The lowest point of the free surface, i.e., immediately above the drain, is given by $\zeta = (0 + i0)$ and hence the water-table lowering, H, can be found from (6) to be

$$\frac{\pi H}{2a} = \lambda\ln[\coth\eta_0\coth\eta_1] \quad (8)$$

Straightness of the pseudo-impervious layer. To determine the degree of confidence with which these solutions can be applied it will be necessary to determine the straightness of the pseudo-impervious layer. If the depth of the dividing streamline is found at three points, namely $\xi = 0$, $\pi/4$ and $\pi/2$, this will give a fair estimate of the shape.

Along $\xi = 0$ the dividing streamline depth can be determined from the depth of the stagnation point. For stationary values of ϕ, $\partial\phi/\partial\eta = 0$, and thus from (3) the ζ-plane coordinate is given by the solution of

$$\mu(\coth(\eta - \eta_0) + \coth(\eta - \eta_1) + 2) + \coth(\eta + \eta_0) + \coth(\eta + \eta_1) = 0 \quad (9)$$

If the root of (9) is given by $\eta = -\alpha$, the depth of the pseudo-impervious layer immediately below the drain D_0 found from (4) is

$$\frac{\pi h_\alpha^*}{2a} = \mu\alpha - \lambda\ln\left[\frac{\sinh(\alpha + \eta_0)\sinh(\alpha + \eta_1)}{\cosh\eta_0\cosh\eta_1}\right] \quad (10)$$

Along $\xi = \pi/2$, the depth of the stagnation point can be found similarly by determining the value of η for which ϕ has a stationary value. This leads to the equation

$$\mu(\tanh(\eta - \eta_0) + \tanh(\eta - \eta_1) + 2) + \tanh(\eta + \eta_0) + \tanh(\eta + \eta_1) = 0 \quad (11)$$

If (11) has the solution $\eta = -\gamma$, the depth of the pseudo-impervious layer midway between the drains is given by substituting $\zeta = \pi/2 - i\gamma$ in (4):

$$\frac{\pi h_\gamma^*}{2a} = \mu\gamma - \lambda\ln\left[\frac{\cosh(\gamma + \eta_0)\cosh(\gamma + \eta_1)}{\cosh\eta_0\cosh\eta_1}\right] \quad (12)$$

Along $\xi = \pi/4$, the depth of the dividing streamline is determined by finding the value of η at which $\psi = 0$ when $\zeta = \pi/4 + i\eta$. The equation for the solution is

$$\mu[\mathrm{gd}2(\eta - \eta_0) + \mathrm{gd}2(\eta - \eta_1) + \pi] + \mathrm{gd}2(\eta + \eta_0) + \mathrm{gd}2(\eta + \eta_1) = 0 \quad (13)$$

where gd is the Gudermannian function, gd $(x) = 2 \tan^{-1} (\tanh x/2)$. If the solution of (13) is $\eta = -\beta$, the depth of the pseudo-impervious layer is given by

$$\frac{\pi h_\beta^*}{2a} = \mu\beta - \lambda \ln \left[\frac{\cosh^{1/2} 2(\beta + \eta_0) \cosh^{1/2} 2(\beta + \eta_1)}{2 \cosh \eta_0 \cosh \eta_1} \right] \tag{14}$$

If η_0 and η_1 are known, it is possible to find $h_\alpha^*/2a$, $h_\beta^*/2a$, $h_\gamma^*/2a$, and the abscissa corresponding to $h_\beta^*/2a$. The latter comes from substituting $\zeta = \pi/4 - i\beta$ into (4) and taking the real part, when

$$\frac{\pi x_\beta}{2a} = \frac{\pi\mu}{2} + 2\lambda[\tan^{-1} \coth (\beta + \eta_0) + \tan^{-1} \coth (\beta + \eta_1)] \tag{15}$$

An estimate of the shape of the pseudo-impervious layer is then available.

Drain diameter and the optimum condition. In the above analysis a drain was taken as a line sink. However, at the relatively shallow depths to which a drain is placed this idealization is not correct, and it will be necessary to define the actual drain surface.

If a drain is assumed to be running full, the drain perimeter must be an equipotential line; i.e., ϕ = constant. It is convenient to assume [*Childs*, 1943; *Engelund*, 1951] that the drain, although running full, has a small air gap at the uppermost point on the perimeter. This implies that the potential line taken for the drain perimeter must satisfy the atmospheric condition Re $W_2 = \phi + Ky = 0$ at the uppermost point. That such a potential line does exist can readily be shown; in fact, there are two values of η for which Re $W_2 = 0$ along $\xi = 0$, $|\eta| < |\eta_0|$. One of these is obviously $\eta = 0$, the free surface itself; the other occurs between $\eta = 0$ and $\eta = -\eta_0$. If the transformation $\eta = -\eta_0 + \delta\eta$ is made in Re $W_2 = 0$, the solutions become $\delta\eta = \eta_0$ and $\delta\eta_r$, where $\delta\eta$ may now be regarded as the upper radius of the drain in the ζ plane.

The two drain sizes possible, then, are a small one given by the root $\delta\eta_r$ and a larger one by the root η_0. As η_0 decreases, however, the drain becomes shallower and the water-table lowering, H, increases until the water table intersects the uppermost point of the drain. In this case Re W_2 has coincident roots with the value η_0; i.e., the radius of the upper drain in the ζ plane becomes η_0. This corresponds to Engelund's case of $\beta = 90°$, i.e., the maximum angle of inclination of the water table to the horizontal being 90°, and has been referred to [*Watson*, 1960] as the 'optimum condition' of draining. The water-table lowering is then at a maximum and the water-table height midway between the drains is at a minimum. The optimum condition may be found by letting $\delta\eta$ tend to η_0 in the function

$$\partial(\text{Re } W_2)/\partial(\delta\eta) = 0$$

It is readily shown that this condition is

$$\coth \eta_0 + \coth \eta_1 = \mu/\lambda \tag{16}$$

A quick check on the validity of this result is given by letting η_1 tend to infinity, i.e., placing the impervious layer at infinite depth, so that the optimum condition becomes

$$\tanh \eta_0 = N/K$$

Equation 8 then becomes

$$\pi H/2a = \lambda \ln \coth \eta_0$$

and for $N/K = 0.1$, $H/2a = 0.08144$, which is the result quoted by *Engelund* [1951, p. 44].

For the optimum condition the radius of the upper drain will be given by $(h_0 - H)/2a$, where $h_0/2a$ is the optimum depth of drain. For suboptimum conditions it is necessary to solve Re $W_2 = 0$ for $\delta\eta_r$.

The drain perimeter may be chosen in two ways, either with the drain almost full, as above, or with the drain quite empty and the perimeter a surface of seepage, i.e., all points such that $\phi + Ky = 0$. Since a drain system is generally designed for the drains running quite full, this is the only case that will be considered here. To calculate the lower radius for the drain running full it will be necessary to calculate the value of ϕ_A, the potential satisfying the condition Re $W_2 = 0$ and find where this intersects the vertical axis below the drain. If the lower radius of the drain in the ζ plane is assumed to be $\delta\eta_f$, it can be shown that the equation for solution is, in the optimum case,

$$(1 + \mu) \ln [\sinh \eta_0 \sinh \eta_1] + 2\mu(\eta_0 + \delta\eta_f)$$
$$= \mu \ln [\sinh (2\eta_0 + \delta\eta_f) \sinh (\eta_1 + \eta_0 + \delta\eta_f)]$$
$$+ \ln [\sinh \delta\eta_f \sinh (\eta_1 - \eta_0 - \delta\eta_f)] \qquad (17)$$

The lower radius of the drain is then found by substituting $\zeta = 0 - i(\eta_0 + \delta\eta_f)$ into (4), taking the imaginary part, and subtracting $h_0/2a$; the result is

$$\frac{\pi r_f}{2a} = \mu\delta\eta_f + \lambda$$
$$\cdot \ln \left[\frac{\sinh 2\eta_0 \sinh (\eta_0 + \eta_1)}{\sinh (2\eta_0 + \delta\eta_f) \sinh (\eta_1 + \eta_0 + \delta\eta_f)}\right]$$

The sum of the upper and lower radii of the drain is then taken to be the diameter.

For the suboptimum condition, the upper radius of the drain can be determined by solving for $\delta\eta_r$ from Re $W_2 = 0$ along $\xi = 0$. The drain radius is then given by

$$\frac{\pi r_r}{2a} = \mu\delta\eta_r + \lambda$$
$$\cdot \ln \left[\frac{\sinh (2\eta_0 - \delta\eta_r) \sinh (\eta_0 + \eta_1 - \delta\eta_r)}{\sinh 2\eta_0 \sinh (\eta_0 + \eta_1)}\right]$$

The potential ϕ satisfying Re $W_2 = 0$ can then be found, and the drain perimeter is defined by the equipotential having this value of ϕ. If ϕ intersects the y axis below the drain it may be computed to give the lower radius of the drain, the sum of the drain upper and lower radii being taken as diameter.

Computation of results. Optimum condition: if $h_\alpha^*/2a$ and N/K are known, (9), (10), and (16) may be regarded as simultaneous equations in α, η_0, and η_1. Once α, η_0, and η_1 are determined, it becomes possible to find $h_0/2a$ and $H/2a$ from (7) and (8), respectively. Furthermore, it is possible to solve (17) for $\delta\eta_f$ and thus compute $r_f/2a$ from (18).

A high-speed digital computer was programmed to handle the computations for the following data:

$$h_\alpha^*/2a = 0.40, \quad 0.25, \quad 0.20, \quad 0.15$$
$$N/K = 0.100, \quad 0.050, \quad 0.010, \quad 0.001$$

thus giving a total of 16 solutions. Obviously, the limiting case for the lower radius is given when $\delta\eta_f = \alpha - \eta_0$ in the ζ plane or $r_f = h_\alpha^* - h_0$ in the physical plane, in which case the drain touches the impervious layer.

To determine the straightness of the impervious layer, (11) and (13) are solved for γ and β, respectively, and $h_\beta^*/2a$ and $h_\gamma^*/2a$ are calculated from (12) and (14). The abscissa is obviously α for h_γ^* and is given by (15) for h_β^*. These values were computed using the following data:

$$h_\alpha^*/2a = 0.40, \quad 0.25, \quad 0.15$$
$$N/K = 0.100, \quad 0.010$$

The results are plotted in Figure 7.

Suboptimum condition: the suboptimum condition of drainage is characterized by water standing over the drains and is brought about by installing a drain smaller than the optimum diameter required for the particular value of N/K.

Solutions for various values of N/K are obtained by selecting values of η_0 larger than that for the optimum condition and then solving for α and η_1 from (9) and (10) by postulating N/K and $h_\alpha^*/2a$; $h/2a$ may then be obtained from (7). The drain diameter may then be computed as explained above.

Results. Before presenting the results, some discussion of the drain diameter and how it is defined is in order.

For the mathematical analysis, the drains were represented by line sinks of strength $Q = 2Na$. However, because the drains are at relatively shallow depth, some reconciliation must be made with the fact that the drains have a physical diameter. This is accomplished by assuming that a drain is running full, whereupon its perimeter must be a line of constant potential; i.e., $\phi = \phi_A$ is constant. If it is further assumed that the drain has a small air gap at its uppermost point y_A on the perimeter, the potential line chosen as the drain perimeter must satisfy the free surface condition $\phi_A + Ky_A = 0$. It is obvious from (1) that at least one such potential line must exist, namely, the one intersecting the lowest point of the free surface immediately above the drain. But there is also another which passes between the line sink and the free surface. It is this one which will be chosen as the drain perimeter. Note that this potential line will not necessarily be circular; however, it is possible to determine its upper and lower radii,

the sum of which may be taken as a measure of the drain diameter. The difference between the upper and lower radii may also be taken as a measure of the distortion from a circle. This is shown in Figure 8 and will be discussed later.

To return to the potential line which defines the drain perimeter, under certain conditions only one such potential line occurs when the two mentioned above have become coincident. Physically this means that where there were two alternative drain sizes to give the same water-table lowering there is now one, and it is found that this drain size gives the maximum over-all lowering of the water table, and furthermore, no water stands over the drains. *Childs* [1943] has shown that this corresponds to the maximum angle of inclination of the water table to the horizontal, 90°, and that it occurs immediately above the drain. It has been called the 'optimum condition' of draining by *Watson* [1960]. Thus it is seen that for the optimum-sized drain for which the over-all water-table lowering is a maximum there is no water standing over the drains. Therefore, any drainage system which has the drains running almost full and has no water standing over the drains is an optimum one. It can be argued that any drainage system with a steady water table and no water standing over the drains is in fact close to an optimum one, although the size of drain installed may be larger than the optimum and not running full.

The drain system is called suboptimum if the drain installed is less than the optimum diameter, in which case water will stand over the drains.

The results are then given in two categories, for optimum and suboptimum conditions. The limited computing facilities available have precluded presenting a full set of results for suboptimum conditions, but sufficient results are given to indicate the trends.

It is convenient both from a computational and a design point of view to take $h^*/2a$ and N/K as the solution parameters. $h^*/2a$ is known from the depth below the surface of the impervious layer and the depth below the surface at which it is desired to maintain the highest point of the water table. N/K is determined from field investigations. Thus, with $h^*/2a$ and N/K known, it is possible to determine the maximum height at the water table in the optimum condition ($h_o/2a$) from the results plotted in Figure 4. Note how close the $h^*/2a = 0.40$ lies to Engelund's solution for $h^*/2a = \infty$, i.e., the impervious layer at infinite depth. This indicates that the influence of the impervious layer is almost negligible at this depth. The optimum drain diameter $d_o/2a$ is given in the results plotted in Figure 5. Solutions for the suboptimum condition for $N/K = 0.10$ are given in Figure 6. From this it is seen how the water table rises when a suboptimum drain is installed.

This can perhaps be illustrated by a numerical example. With $N/K = 0.1$ and $h^*/2a = 0.25$, the optimum-sized drain can be determined from Figure 5 and the water-table height midway between drains from Figure 4. From Figure 6, when $d/d_o = 0.25$, h_o/h is seen to be 0.909. When $2a = 15.2$ meters, the results given in Table 1 are obtained.

It is seen that with a drain one-quarter the optimum-sized drain the maximum height has risen only 15 cm, but there is now 73 cm of water standing over the drains (this is computed from equation 8).

As stated in the introduction, the results are limited by the fact that, although the boundary

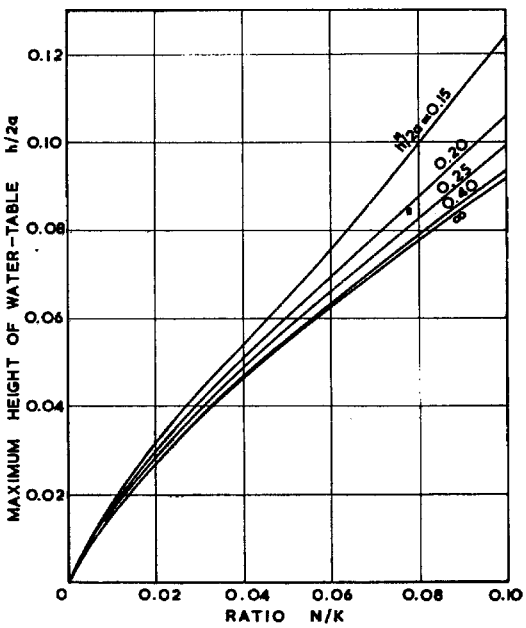

Fig. 4. Maximum water-table heights as a function of N/K for the optimum conditions of draining.

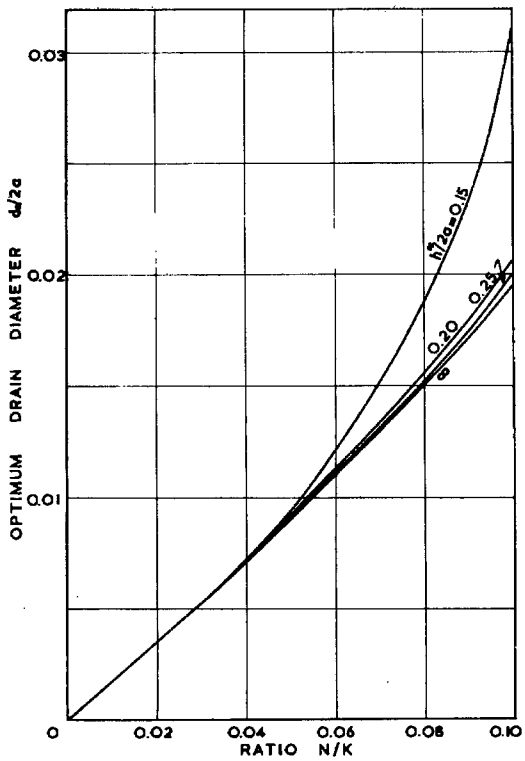

Fig. 5. Drain diameter as a function of N/K for the optimum condition of draining.

condition on the free surface is exactly satisfied, the shape of the impervious layer is approximate. It is felt that this approximation is relatively unimportant, as the flow in the neighborhood of the impervious layer is rather sluggish compared with, say, the flow through the free surface. To give some indication of the approximations being made, the depth of the pseudo-impervious layer was calculated at three points, namely, (i) immediately below the drain, $h_\alpha^*/2a$; (ii) midway between drains, $h_\gamma^*/2a$; (iii) at approximately one-quarter the distance from a drain to an adjacent drain $h_\beta^*/2a$. From this an estimate may be made of the shape. The results are plotted in Figure 7, and it is seen that the shape of the pseudo-impervious layer, though not linear, is not very curved. As would be expected, the discrepancy is greatest midway between the drains and thus where the flow has its lowest velocity. The percentage variation as a function of the distance from the drain to the layer is plotted in Figure 8, where it is seen that the greatest inaccuracy occurs when the drain is close to the layer (also in Table 2).

TABLE 1. Comparison of Water Table Heights for Two Different Drain Sizes with $h_\alpha^*/2a = 0.25$, $N/K = 0.10$, $2a = 15.3$ meters

d/d_0	d, cm	h, m	$h - H$, m	η_0	η_1
1.00	30.40	1.52	0.168	0.10308	1.07369
0.25	7.62	1.67	0.732	0.2000	1.07199

The other failing of the theory which limits the results is that, as the drain comes close to

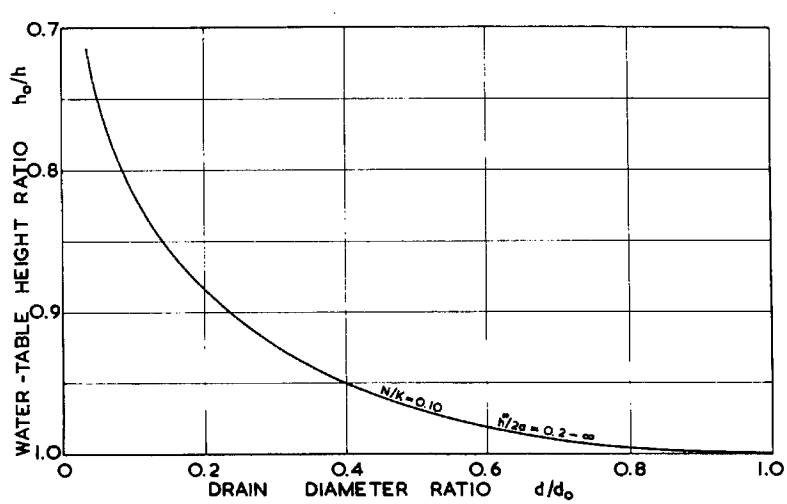

Fig. 6. Suboptimum condition solutions for $N/K = 0.1$ (h_0 and d_0 are optimum solutions).

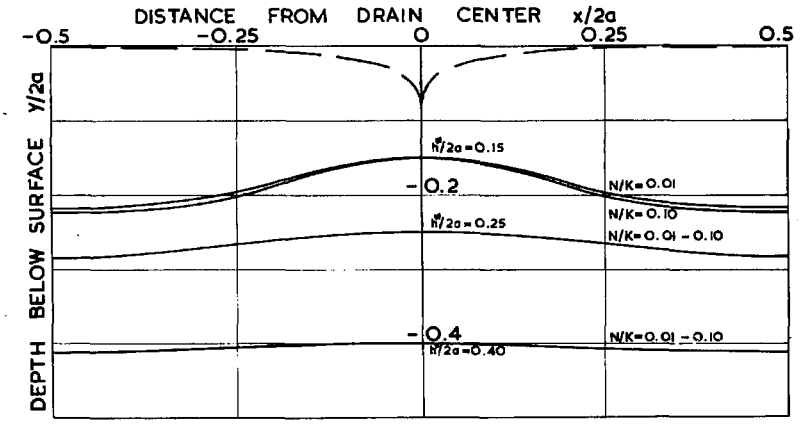

Fig. 7. Shape of the pseudo-impervious layer at optimum conditions of drawing.

the pseudo-impervious layer, its shape becomes distorted. As mentioned above, a measure of this distortion is the difference between the upper and lower radii. This is also plotted in Figure 8 as a function of the distance between the drain center and the impervious layer. The curve for $N/K = 0.10$ shows how the lower radius takes on large values near the layer.

It is thus shown that limits on theory do exist, and some cognizance must be given to them in any application of these results. The comment may be made that the straightness of the pseudo-impervious layer could be improved by increasing the intensity of the artesian flow. This would mean another parameter in the solutions which is not justified in the light of the limited comparison of theoretical results with field results made below.

Comparison with field results. Kirkham and De Zeeuw [1952] reported that they found no water standing over the tile lines in their field investigations, and it may therefore be legitimately assumed, in the light of the discussion above, that they were in fact dealing with optimum drainage systems. A comparison with the theoretical results given here is therefore pos-

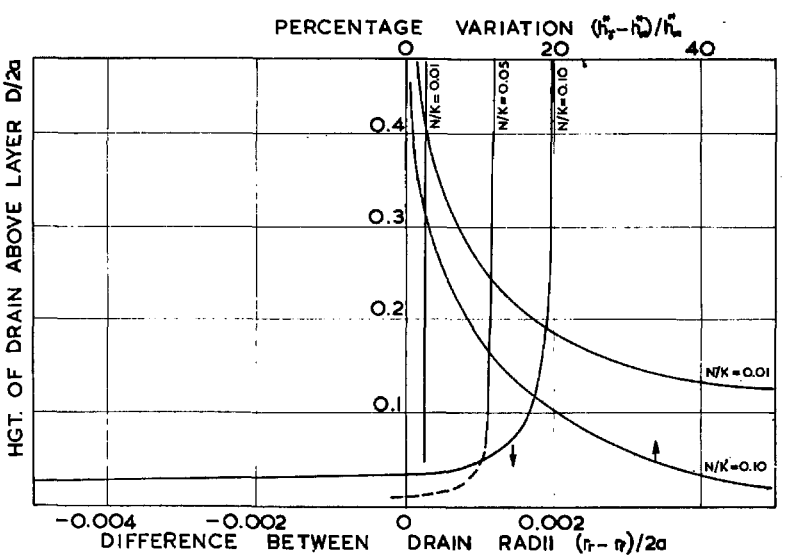

Fig. 8. Limiting solutions as shown by drain shape and the shape of the pseudo-impervious layer.

TABLE 2. Deviation of the Pseudo-Impervious Layer from a Straight Line

ξ	N/K	$h_a^*/2a$ 0.40	0.25	0.15
0	0.10	0	0	0
$\pi/4$	0.10	1.60	7.84	33.02
$\pi/2$	0.10	2.99	13.22	49.10
0	0.01	0	0	0
$\pi/4$	0.01	1.69	7.77	28.59
$\pi/2$	0.01	3.14	13.16	43.87

sible. Table 4 gives this comparison. Since the values of $h_a^*/2a$ are less than 0.15 and thus not readily available from the curves given, separate calculations have been made to obtain the theoretical results (Table 3). Kirkham and De Zeeuw took the bottom of the tile lines as their datum, and thus the results quoted here have been corrected to the center lines of the tiles. The theoretical results are also plotted in Figure 9 along with the curve given by Kirkham and De Zeeuw in their Figure 6.

The actual shape of the water table in the field for 16-meter tile spacing, as given in Kirkham and De Zeeuw's Figure 7, is compared in Figure 10 with the theoretical shape, as given by (5) and (6).

It is seen that the theoretical results are in very good agreement with the field results quoted, the theory tending to predict a slightly higher water table. The variation in the shape of the theoretical free surface from the field results (Figure 10) has a ready explanation. The maximum variation is seen to occur at depth of 50 cm below the surface. Kirkham and De Zeeuw give, in their Figure 5, the distribution of permeability with depth in the field, and it is seen in this figure that the maximum values of permeability occur at a depth of 50 cm. Since a

TABLE 3. Solutions of Equations 9, 10, and 15 Using Field Data from *Kirkham and De Zeeuw* [1952]

$2a$	h^*	$h^*/2a$	N/K	η_0	α	η_1
8	119.6	0.1495	0.032	0.03280	0.4184	0.6464
10	132.7	0.1327	0.032	0.03315	0.3596	0.5272
12	142.8	0.1190	0.032	0.03331	0.3102	0.4669
16	164.5	0.1030	0.032	0.03400	0.2499	0.3699

TABLE 4. Comparison of Theoretical and Field Results, $h/2a$, Computed from Equation 7 Using Results Given in Table 3

$2a$, meters	$h/2a$	h_T, cm	h_F, cm	Error, %
8	0.0451	36.1	34.1	+5.9
10	0.0479	47.9	47.2	+1.5
12	0.0489	58.7	57.3	+2.4
16	0.0510	81.6	79.0	+3.3

mean permeability was taken for the computation of the theoretical results, it is not surprising that the maximum variation from the field results occurs at this depth of 50 cm. A higher-than-mean permeability would tend to give a greater lowering at that point.

Conclusions. Solutions have been given for the steady flow to a series of tile drains above an impervious layer. Although the solution is approximate, the approximation involves the sluggish region of flow in the neighborhood of the impervious layer midway between the tile lines and not, as in previously published solutions, the more critical region of flow at the free surface. The limited field results quoted confirm the validity of the solutions and thereby justify the approximations.

The solutions are limited in that their ac-

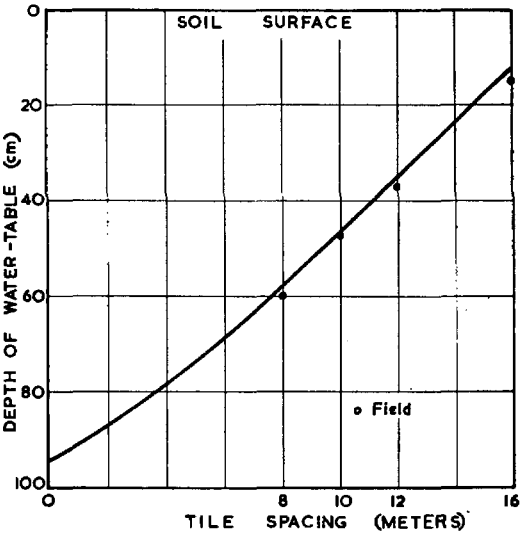

Fig. 9. Comparison of theoretical water-table heights with field results given by *Kirkham and De Zeeuw* [1952].

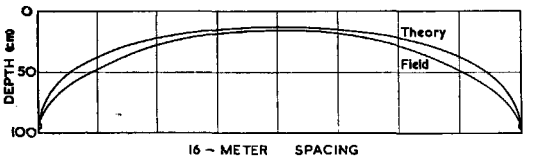

Fig. 10. Comparison of theoretical water-table shape with field shape is given by *Kirkham and De Zeeuw* [1952].

curacy becomes questionable when the drains are very close to the impervious layer, which implies that the distortion of the drain shape and the pseudo-impervious layer become too consequential to ignore.

The water-table height for any given N/K ratio and depth of impervious layer depends on the size of drain installed and the spacing. If the drain is greater than the optimum size, no water will stand over the tile lines, and the height of the water table midway between the drains will be at a minimum. If drains smaller than the optimum size are laid, the height of the water table midway between the drains will not rise very significantly, but water will stand over the drains. If this can be tolerated, there would appear to be a definite economic advantage in using suboptimum-sized drains. It would be important in this case, however, to employ the maximum anticipated value of N/K in the design, to avoid overloading the drainage system.

Acknowledgments. I wish to express my appreciation for the suggestions and assistance given by Dr. C. M. Segedin of the School of Engineering, University of Auckland.

Financial assistance in the form of the J. C. Fletcher Scholarship is gratefully acknowledged.

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(Manuscript received March 6, 1964;
revised June 12, 1964.)

APPENDIX C

COMPUTATION DETAILS

In order to determine the drain depth and spacing it is necessary to solve the following three equations simultaneously for η^*, η_1 and $h^*/2a$ given N/K and η_o.

$$\frac{h^*}{2a} = \frac{\eta^*}{\pi}\left(\frac{K+N}{K-N}\right) - \frac{N}{\pi(K-N)} \ln \frac{\sinh(\eta^*+\eta_o)\cdot\sinh(\eta^*+\eta_1)}{\cosh\eta_o\cdot\cosh\eta_1} \tag{1}$$

$$\coth\eta_o + \coth\eta_1 = \frac{K+N}{N} \tag{2}$$

$$\left(\frac{K+N}{K-N}\right)\left(\coth(\eta^*+\eta_o) + \coth(\eta^*+\eta_1) - 2\right)$$

$$+ \coth(\eta^*-\eta_o) - \coth(\eta^*-\eta_1) = 0 \tag{3}$$

Now it is known that when the impervious layer is at infinite depth $\eta_1 \to \infty$ this implies the limiting value of η_o is given by $\tanh\eta_o = N/K$; and when the drain sits on the impervious layer

$$\coth\eta_o = \frac{K+N}{2N}$$

so this gives the two limiting values of η_o for each N/K ratio. Thus a range of η_o between these two limits can be chosen and η_1 found by equation (2). It is then necessary to solve (3) by Newton's Method for η^*

and then substitute N/K, η_0 , η^* , η_1 into (1) to get $h^*/2a$.

H/2a is then computed from equation (4)

$$\frac{H}{2a} = \frac{N}{\pi (K - N)} \ln (\coth \eta_0 \cdot \coth \eta_1) \tag{4}$$

Now $D^*/2a = h^*/2a - H/2a$

and it becomes possible to compute $2a/h^*$ and D^*/h^* for each value of N/K by using the range of η_0 between the two limiting values it takes when the drain is on the impervious layer and when the impervious layer is at infinite depth.

Drain Diameter.

The upper radius of the drain is easily computed by first computing h/2a from equation (5)

$$\frac{h}{2a} = \frac{\eta_0}{\pi} \left(\frac{K+N}{K-N}\right) - \frac{N}{\pi (K - N)} \ln \frac{\sinh 2\eta_0 \ \sinh (\eta_0 + \eta_1)}{\cosh \eta_0 \cdot \cosh \eta_1} \tag{5}$$

and then

$$\frac{r_r}{2a} = \frac{h}{2a} - \frac{H}{2a}$$

The lower radius is computed in two ways depending on whether the drain is empty or full.

a) Drain Full.

First of all the potential at the uppermost point on the perimeter is computed. This is given by

$$\varphi = \ln (\sinh \eta_0 \ \sinh \eta_1) + \left(\frac{K - N}{K + N}\right) \ln (\sinh \eta_0 \sinh \eta_1) + 2\eta_0 \tag{6}$$

The floor radius when full is then obtained by solving

$$F(\delta\eta_f) = \left(\frac{K-N}{K+N}\right) \ln \left[\sinh \delta\eta_f \cdot \sinh(\eta_1 - \eta_o - \delta\eta_f)\right]$$

$$+ \ln \sinh(2\eta_o + \delta\eta_f) \sinh(\eta_o + \eta_1 + \delta\eta_f)$$

$$- \varphi - 2\delta\eta_f$$

$$= 0 \tag{7}$$

for $\delta\eta_f$. This is easily accomplished using Newton's Method:

$$\delta\eta_f(2) = \delta\eta_f(1) - \frac{F(\delta\eta_f(1))}{F'(\delta\eta_f(1))} .$$

Some care is needed, however, in making the first guess in order that the solution on the correct branch is found.

The floor radius is then computed from equation (18) in Appendix A.

A similar procedure is used to find the lower diameter when the drain is empty and then the diameter is merely the sum of the respective radii.

The computations were programmed for an IBM 7090/7094 in The California Institute of Technology, Booth Computing Center. Total computation including "debugging" was approximately 30 minutes.

APPENDIX D

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