A Geometric Theorem for Approximate Disk Covering Algorithms *

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Abstract

We present a basic theorem in combinatorial geometry that leads to a family of approximation algorithms for the the geometric disk covering problem. These algorithms exhibit constant approximation factors, with a wide range of their choices. This flexibility allows to achieve a running time that compares favourably with those of existing procedures.

1 Introduction and Main Results

In this paper we present a basic theorem in combinatorial geometry and discuss its application to the well known problem of computing a geometric disk covering [1].

The theorem states as follows:

Theorem 1 *Consider a square lattice where the distance between two neighboring lattice vertices is one. Call a disk of fixed radius r , centered at a lattice vertex, a grid disk. The number $\mathcal{N}$ of grid disks that are necessary and sufficient to cover any disk of radius r placed on the plane, is given by:*

- *CASE 1. For* $r < \frac{\sqrt{2}}{2}$, $\mathcal{N}$ *does not exist.*
- *CASE 2. For* $\frac{\sqrt{2}}{2} \leq r < \frac{\sqrt{10}}{4}$, $\mathcal{N} = 6$.
- *CASE 3. For* $\frac{\sqrt{10}}{4} \leq r < 1$, $\mathcal{N} = 5$.
- *CASE 4. For* $1 \leq r < \frac{5\sqrt{2}}{4}$, $\mathcal{N} = 4$.
- *CASE 5. For* $r \geq \frac{5\sqrt{2}}{4}$, $\mathcal{N} = 3$.

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Application of this theorem to the geometric disk covering problem is in order. Let us try to find the minimum number of disks of prescribed radius r, which cover a given set of n points in the plane. It is known that this problem is strongly NP-complete [6], therefore we are interested in finding approximate solutions that run in polynomial time and have a bounded error ratio.

Application of a simple greedy heuristic, originally developed for the set covering problem, leads to a worst case error ratio bounded by $(1 + \ln n)$ and this bound is tight, as discussed in Section 4.

Approximation algorithms that have a constant worst case error ratio have been developed and are summarized in Table 1. Hochbaum and Maass [8] [9] present a polynomial approximation scheme applying a shifting technique originally proposed by Baker [2] in the context of planar graphs. By using this technique, they obtain a family of algorithms with a worst case error ratio of $(1 + \epsilon)$, with $0 < \epsilon \leq 3$, and a running time that is polynomial when ϵ is fixed. The degree of the polynomial is however nine if $\epsilon = 3$ and grows, as ϵ approaches zero, according to the sequence: $\{9, 19, 51 \ldots, 2\lceil l\sqrt{2}\rceil^2 + 1, \ldots\}$, making their strategy not practical for even small values of the number of points to cover. [1] Additional efforts were then made to reduce the running time. Feder and Greene [5] and, independently, Gonzalez [7] considered related problems and obtained similar results that can be applied to the disk covering problem. Their approach leads to a family of algorithms with a worst case error ratio of $(1 + \epsilon)$, with $0 < \epsilon \leq 1$, and a running time that is polynomial when ϵ is fixed. The degree of the polynomial is thirteen if $\epsilon = 1$ and grows, as ϵ approaches zero, according to the sequence: $\{13, 19, 31 \ldots, 6\lceil l\sqrt{2}\rceil + 1, \ldots\}$. [2] Their procedure also leads to an additional, faster algorithm, with a worst case error ratio of eight and a running time of $O(n + n \log S)$, where $S \leq n$ is the number of disks in the optimal solution. We also mention Brönnimann and Goodrich [3], who present an $O(n^3 \log n)$ algorithm that leads to a constant, yet not determined, worst case error ratio for the disk covering problem.

Application of our quoted theorem leads to another family of algorithms, that also exhibit constant worst case error ratios, with a running time that is linear in the number of points to cover. Our simple strategy is to find a covering of the points by placing disks only at the vertices of a mesh. The error ratio for this strategy is given by the number of grid disks that are required, in the worst case, to cover all the points that a single disk could cover in the optimal solution. This value is provided by Theorem 1. By combining this grid strategy with the shifting strategy of Hochbaum and Maass, we obtain a family of algorithms that have an error ratio of $(3 + \epsilon)$, with $0 < \epsilon \leq 21$ and a running time that is linear when ϵ is fixed. The slope of the linear function increases as ϵ approaches zero.

Inspection of Table 1 shows that algorithms number three and number five exhibit the most promising running times. It is noted that algorithm number five, at variance of the other, allows a convenient flexibility in the choice of the approximation. In this way, the size of the slope factor K is controlled, leading to convenient running times at the expense

[1] According to [8] and [9], the degree of the polynomials grows according to the sequence: $\{5, 17, 37, \ldots, 4l^2 + 1, \ldots\}$. That is because a ceiling operation is missing in their derivation. The corrected sequence appears in [7].

[2] The running time reported in [7] is a polynomial of fixed degree. That is because there are some typos in that paper. Correct values for both time complexity and approximation appear in Table 1 and have been verified in a private communication with the author.

	Author	*Approximation*	*Running Time*	*Year*
1	Hochbaum and Maass	$\left(1+\frac{1}{l}\right)^2$	$O\left(l^2 \lceil l\sqrt{2}\rceil^2 \, n^{2\lceil l\sqrt{2}\rceil^2+1}\right)$	1985
2	Gonzalez / Feder and Greene	$\left(1+\frac{1}{l}\right)$	$O\left(6l\lceil l\sqrt{2}\rceil \, n^{6\lceil l\sqrt{2}\rceil+1}\right)$	1991
3	Gonzalez / Feder and Greene	8	$O(n + n\log S)$	1991
4	Brönnimann and Goodrich	$O(1)$	$O(n^3 \log n)$	1995
5	This Paper	$\alpha\left(1+\frac{1}{l}\right)^2$	O(K n)	2000

Table 1: **Approximation Algorithms for Disk Covering.** *Symbol l is an integer parameter, n is the number of points to cover, $S \leq n$ is the number of disks in the optimal solution, α is a parameter in the set $\{3,4,5,6\}$, K is a constant value that depends on α and on l; an analytical expression for K is given in Section 2 and its values for given approximation factors are depicted in Figure 11 (Appendix B).*

of the corresponding approximation. On the contrary, the running time of of number three exhibits a slope that cannot be controlled and that is proportional to $\log S \sim \log n$ when the points to cover are very sparse.

Finally, we mention some applications for geometric disk covering. This problem arises in the area of locating emergency facilities such that all potential customers are within a reasonable small radius around the facility, as well as in the area of wireless computing, where a wireless network is usually modeled as a set of disks (base stations) that need to cover a given set of wireless clients [12].

The paper is organized as follows. Section 2 focuses on the application of Theorem 1: it reviews the shifting strategy and the Shifting Lemma of Hochbaum and Maass, it combines it with our grid strategy and presents our main computational result. Section 3 presents a proof for Theorem 1. Section 4 discusses the approximation given by a simple greedy heuristic. Section 5 draws conclusions and discusses some future work. Appendix A completes Section 3, collecting part of the proof. Appendix B completes section 2, taking into account that our algorithm runs on a finite grid.

Throughout the paper, the following notation is used: OPA denotes the solution set delivered by algorithm $\mathcal{A}$ and its cardinality is denoted by $|OPA|$. Similarly, an optimal solution set is denoted by OPT and its cardinality by $|OPT|$. The approximation factor of an algorithm $\mathcal{A}$ is its worst case error, denoted by $AF_{\mathcal{A}}$, and is defined as the supremum of the ratio $|OPA|/|OPT|$, over all problem instances.

2 The Algorithm

Our algorithm is a combination of a shifting strategy and of a grid strategy. We first summarize the shifting strategy. A more detailed discussion appears in [8] and [9].

2.1 The Shifting Strategy

We want to cover a set of n points in the plane with a minimal number of disks of given radius r. The shift algorithm $S(\mathcal{A})$, defined for a given local algorithm $\mathcal{A}$, works as follows. Let the shifting parameter be l. We subdivide the plane into vertical strips of width $D = 2r$.

Groups of l consecutive strips are considered. Each group is itself a thicker strip of width $l \times D$. Let $\mathcal{A}$ be an algorithm that delivers a solution within each group. By applying algorithm $\mathcal{A}$ to each of the groups and then considering the union of all disks used, we find a feasible solution. We then repeat the same strategy after shifting all the groups by the length D. Since each group is a strip $l \times D$ wide, we can repeat the shift a total of $l-1$ times, choosing the feasible solution of minimum cardinality as the final best covering.

The following lemma is proven in [8].

Lemma 1 (The Shifting Lemma)

$$AF_{S(\mathcal{A})} \leq AF_{\mathcal{A}} \left(1 + \frac{1}{l}\right). \tag{1}$$

On the plane, the shifting strategy can be applied twice. We first cut the plane into vertical strips of width $l \times D$. Then, in order to cover the points in each such a strip, we apply the shifting strategy in the other dimension. Thus, we cut the considered strip into squares of side length $l \times D$. Let $\mathcal{B}$ the local algorithm that delivers a solution within a square. By repeatedly applying the shifting lemma, we have: $AF_{S(S(\mathcal{B}))} \leq AF_{\mathcal{B}} \left(1 + \frac{1}{l}\right)^2$.

2.2 The Grid Strategy

The algorithm of Hochbaum and Maass finds a solution within the square of side length $l \times D$ by exhaustive enumeration. They note that with $\lceil l\sqrt{2}\rceil^2$ disks of diameter D, they can cover the square of side length $l \times D$ compactly. Thus, the number of disks to cover points in the square does not exceed $\lceil l\sqrt{2}\rceil^2$. Furthermore, any disk that covers at least two of the given points can be assumed, without loss of generality, to have two of these points on its border. Thus, the number of possible disks positions is finite. By checking all possible arrangements of a maximum of $\lceil l\sqrt{2}\rceil^2$ disks, they find an optimal covering within the square. Using this strategy, $AF_{\mathcal{B}} = 1$.

Our grid strategy constrains the covering disks to be placed at the vertices of a mesh. A sub-optimal covering of the points in the square is found by checking all possible arrangements of a maximum of $\lceil l\sqrt{2}\rceil^2 - 1$ disks centered at mesh vertices or by using the compact covering arrangement of $\lceil l\sqrt{2}\rceil^2$ disks. Using this strategy $AF_{\mathcal{B}} \in \{3, 4, 5, 6\}$, as stated by the following Corollary to Theorem 1:

Corollary 1 *Consider a square lattice where the distance between two neighboring lattice vertices is one. The approximation factor resulting from covering a set of n points in the plane using disks of radius r only centered at lattice vertices is:*

$$AF \leq \begin{cases} 3 & \text{if} \quad & & r & \geq & \frac{5\sqrt{2}}{4} \\ 4 & \text{if} \quad 1 & \leq & r & < & \frac{5\sqrt{2}}{4} \\ 5 & \text{if} \quad \frac{\sqrt{10}}{4} & \leq & r & < & 1 \\ 6 & \text{if} \quad \frac{\sqrt{2}}{2} & \leq & r & < & \frac{\sqrt{10}}{4} \end{cases} \tag{2}$$

and is undefined for $r < \frac{\sqrt{2}}{2}$.

Proof.
Consider the optimal covering OPT of the given n points. Call a minimal covering obtained by constraining the covering disks to be centered at lattice vertices a grid covering. The approximation factor is the number of disks in the grid covering that are required, in the worst case, to cover all the points covered by a single disk in the optimal set OPT. Hence, the corollary immediately follows from Theorem 1. $\square$

Consider now a finite lattice placed on top of a square of side length $l \times D$. Let the finite lattice be large enough that we can apply Corollary 1 to determine the approximation factor resulting from covering a set of points in the square using disks centered only at lattice vertices. Let p be the number of vertices of this lattice and $\alpha \in \{3, 4, 5, 6\}$ its approximation factor. Let $K_{(l,p)} = l^2 \sum_{i=1}^{\lceil l\sqrt{2}\rceil^2-1} \binom{p}{i}\, i$. We state our main computational result as follows:

Theorem 2 *Let $p, \alpha, K_{(l,p)}$ as defined above. There is a linear time approximation algorithm $\mathcal{A}_l$ such that for every given natural number $l \geq 1$, the algorithm $\mathcal{A}_l$ delivers a cover of the n given points by disks of given diameter D in $O(K_{(l,p)}\, n)$ steps, with approximation factor $\leq \alpha \left(1 + \frac{1}{l}\right)^2$.*

Proof.
Let $\overline{n}$ be the number of given points to cover that are in a square of side length $l \times D$. Since with $\lceil l\sqrt{2}\rceil^2$ disks we can cover the square compactly, by restricting the search for an optimal cover to disks placed on a grid, we have to consider arrangements of no more than $\lceil l\sqrt{2}\rceil^2 - 1$ grid disks, over p possible disk positions. That is because if we cannot cover all $\overline{n}$ given points using $\lceil l\sqrt{2}\rceil^2 - 1$ grid disks, we can abandon the grid and use the compact covering arrangement. In order to check whether an arrangement of disks is a feasible cover of the $\overline{n}$ points in the square, we need to determine for each point whether it is within a distance at most $r = D/2$ from one of the centers. Therefore, assuming we can determine the distance between two points in the plane with the necessary precision in one step, the number of steps needed to find a minimal cover in a square is at most:

$$\sum_{i=1}^{\lceil l\sqrt{2}\rceil^2-1} \binom{p}{i} i\, \overline{n}\,. \tag{3}$$

By summing over all the squares, we have a number of steps $\sum_{i=1}^{\lceil l\sqrt{2}\rceil^2-1} \binom{p}{i} i\, n$. The two nested applications of the shifting strategy add another factor l^2 to our global time bound, that becomes: $l^2 \sum_{i=1}^{\lceil l\sqrt{2}\rceil^2-1} \binom{p}{i}\, i\, n = K_{(l,p)}\, n$ steps. The bound on the approximation factor immediately follows from the Shifting Lemma and Corollary 1. $\square$

Note that a closed form expression for the number of lattice vertices p, is given in Appendix B.

3 Proof of Theorem 1

In this section we prove Theorem 1, whose application has been discussed in the previous section.

Proof of CASE 1.
For $r < \frac{\sqrt{2}}{2}$ the grid disks do not cover the plane compactly, since they do not cover the

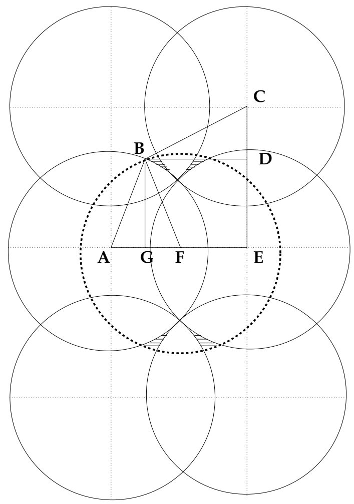

Figure 1: **CASE 2, necessary condition.** *The distance between the two lattice vertices A and E is 1. Therefore,* $\overline{AF} = \overline{FE} = \frac{1}{2}$. *Since* $\overline{AB} = \overline{BF} = r$, $\overline{AG} = \overline{GF} = \frac{1}{4}$.

centers of the lattice squares. It follows that any number of grid disks is not sufficient to cover any disk that covers the center of a lattice square.□

Proof of CASE 2.

The necessary condition is proven by showing that there exists a disk that requires six grid disks to be covered. The sufficient condition is proven using a tiling argument: we first show that there exists a triangle ABC, such that any disk centered inside triangle ABC is covered by six grid disks and then we show that, by symmetry, we can tile the entire plane using triangles that have this property.

Necessary Condition. Consider $r \geq \frac{\sqrt{2}}{2}$ and a disk centered halfway between two neighboring lattice vertices. Such disk is the dashed disk depicted in Figure 1. We have that the six (solid) grid disks depicted in Figure 1 are necessary to completely cover the dashed disk if $\overline{BC} > r$, i.e. when the four shaded areas in Figure 1 are not null. By repeatedly applying the Pythagorean theorem, we have:

$$\overline{BC} = \sqrt{\left(1 - \sqrt{r^2 - \frac{1}{16}}\right)^2 + \frac{9}{16}}\,; \tag{4}$$

imposing $\overline{BC} > r$, we obtain $r < \frac{\sqrt{10}}{4}$. □

Sufficient Condition. Call $\mathcal{A}$ the area covered by the six grid disks in Figure 2. Any point inside triangle ABC has distance greater than r from the border of area $\mathcal{A}$. Therefore, a disk can be centered inside triangle ABC and be covered by the six grid disks depicted in Figure 2. By symmetry, we can tile the plane with triangles inside which disks can be centered and covered by six grid disks. □

Proof of CASE 3.

Necessary Condition. Consider $r \geq \frac{\sqrt{10}}{4}$ and a disk centered at a distance ϵ to the left from halfway between two neighboring lattice vertices (dashed disk in figure 3). By the

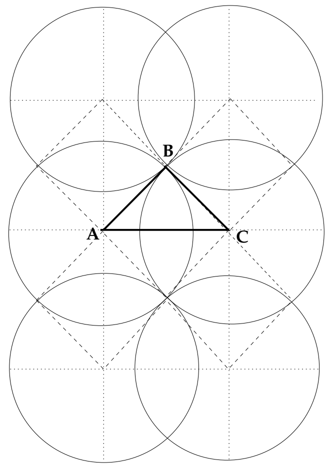

Figure 2: **CASE 2, sufficient condition.** *Any disk centered inside triangle ABC is covered by the six grid disks.*

same reasoning of CASE 2, we have that the five (solid) grid disks depicted in Figure 3 are necessary to completely cover the dashed disk, if $\overline{BC} > r$. By repeatedly applying the Pythagorean theorem, we have in this case:

$$\overline{BC} = \sqrt{\left[1 - \sqrt{r^2 - \left(\frac{1}{4} - \frac{\epsilon}{2}\right)^2}\right]^2 + \left(\frac{3}{4} + \frac{\epsilon}{2}\right)^2} \; ; \tag{5}$$

imposing $\overline{BC} > r$ we obtain:

$$\sqrt{\frac{\epsilon^2 + \epsilon}{2} + \frac{5}{8}} > r \; . \tag{6}$$

By symmetry we can restrict the ϵ range: $0 < \epsilon < \frac{1}{2}$ and for any $r < 1$ inequality (6) is verified. □

Proof (sufficient condition). Call $\mathcal{A}$ the area covered by the five grid disks in Figure 4. Any point inside triangle AOB has distance greater than r from the border of area $\mathcal{A}$. Therefore, a disk can be centered inside triangle AOB and be covered by the five grid disks depicted in Figure 4. By symmetry, we can tile the plane with triangles inside which disks can be centered and covered by five grid disks. □

Proofs for the two remaining cases are more involved and are included in Appendix A.

4 When is Greedy so bad?

Having discussed families of algorithms for the geometric disk covering problem that have constant approximation, one could wonder what would the approximation factor be, for the

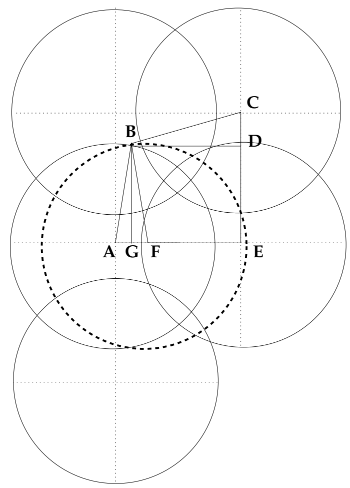

Figure 3: **CASE 3, necessary condition.** *Point F is the center of the dashed disk and is shifted by* ϵ *to the left from halfway between the lattice vertices A and E. Therefore, we have:* $\overline{AF} = \frac{1}{2} - \epsilon, \overline{FE} = \frac{1}{2} + \epsilon$.

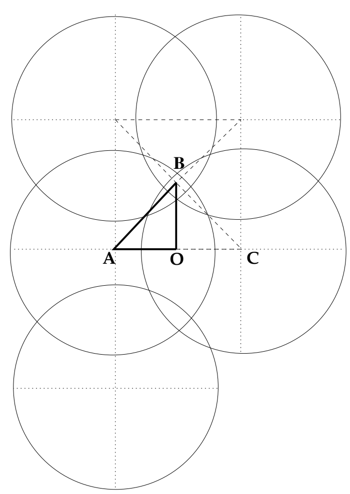

Figure 4: **CASE 3, sufficient condition.** *Point O is halfway between lattice vertices A and C. Any disk centered inside triangle AOB is covered by the five grid disks.*

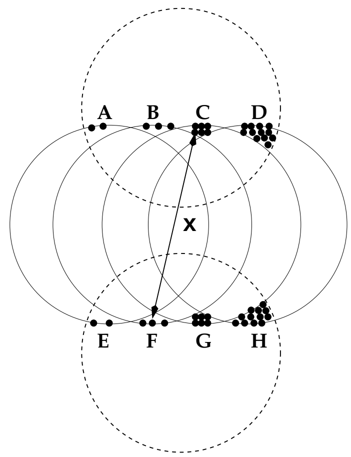

Figure 5: **The Greedy Algorithm.** *The optimal solution is given by the two dashed disks, while the greedy algorithm selects the k solid disks.*

simple greedy heuristic that selects at each iteration the disk that covers the largest number of points. This algorithm has been extensively studied in the context of the set covering problem. The first paper that analyzes its performance was written by Johnson [10] and appeared in 1974. The problem starts with a universal set of n elements and a collection of m sets, and seeks the smallest number of these sets, such that their union is the universal set of all elements. The worst case error for the greedy solution to set covering is bounded by $(1 + \ln d)$, where d is the largest set size [11]. Johnson, in his 1974 paper, showed that this bound is tight, by explicitly giving a construction of a problem for which the greedy algorithm achieves an error of at least $O(\ln d)$. The greedy algorithm is also considered to be optimal for the set covering problem. In fact Feige [4] recently proved that for any $\epsilon > 0$, there is no $(\ln n - \epsilon)$ approximation algorithm, unless some likely complexity theoretic conjecture fails ($NP \neq DTIME(n^{O(\log \log n)})$. Hence, not only the greedy, but any other algorithm, could never achieve a constant worst case approximation.

On the contrary, geometric disk covering can be approximated within a constant and a reasonable question to ask is what is the approximation factor of the greedy approach in this case. The upper bound of $\ln d$ on the approximation factor still holds for geometric disk covering, because this problem is a restriction of the set covering problem: the plane represents the universal set, the n points to cover represent the elements of the sets and the number $m \leq 2\binom{n}{2}$ of possible disks placements represent the possible sets to choose from. However, in order to show that this bound is tight and therefore prove that the greedy algorithm does not achieve a constant error ratio, one needs to give a geometric construction that achieves the upper bound, as Johnson did for set covering.

The authors are not aware of any such geometric construction in the literature and Johnson's original one cannot be applied to the geometric disk covering problem. We describe a slightly different construction that follows Johnson's original idea. A similar construction, but given for set covering, appeared in [3].

We place points on the plane in such a way that the greedy algorithm places $k \sim \ln d$

disks on the plane to cover them, instead of two optimal ones. The points are placed in small clusters, as depicted in Figure 5. We have $2, 3, 3 \cdot 2, \ldots 3 \cdot 2^{k-3}, 3 \cdot 2^{k-2}$ points in $A, B, C, D, \ldots$ respectively, and similarly in $E, F, G, H, \ldots$. Each cluster of points occupies an infinitesimal area. The distance $X = \overline{BE}, \overline{CF}, \overline{DG}, \ldots$, as well as $X = \overline{AF}, \overline{BG}, \overline{CH}, \ldots$, is chosen larger than the diameter of a disk. Therefore, a disk that covers clusters D and H, cannot cover cluster C nor G; and similarly for the other clusters. Accordingly, the greedy algorithm places the disks as depicted in Figure 5 with a solid line, since at each iteration they cover the largest number of points. In particular, at each iteration they cover one more point than any of the two dashed line disks.

Clearly, the approximation ratio for this example is $k/2$. Since $d = 3 \cdot 2^{k-1}$, $k = \log_2 d + 1 - \log_2 3$, we have $k/2 \sim \ln d$. Since k can be taken arbitrarily large, it follows that the approximation factor for the greedy algorithm applied to disk covering is at least $O(\ln d)$.

5 Conclusions

We have presented a basic theorem in combinatorial geometry that leads to a family of approximation algorithms for the geometric disk covering problem. The wide range of choices for the approximation allows to achieve competitive running times. As a side result, we have shown that the greedy algorithm approximation factor of $\ln d$ is tight for the geometric disk covering problem.

In the future, we plan to generalize our Theorem to other lattice structures (e.g. hexagonal) and check if by using such lattices, better performance guarantees could be achieved. We also plan to study a related problem inspired by wireless computing. That is the problem of placing a minimum number of disks of given radius r on the plane in such a way that they cover a set of given points and the resulting graph that has the centers of the disks as vertices and vertices joined by an edge if the corresponding disks intersect, is connected.

Acknowledgments

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A Theorem 1, CASE 4 and CASE 5

In this appendix we present proofs for the two cases of Theorem 2 that are omitted in section 3.

Proof of CASE 4.

Necessary Condition. Consider $r \geq 1$ and a disk placed at the center of a lattice square. If we place the grid disks as depicted in the right section of Figure 6, four grid disks are always necessary to cover the dashed disk placed at the center of a lattice square, because:

$$\overline{EF} = \overline{DF} > \overline{OF} = r \; ; \tag{7}$$

the same holds whenever two grid disks are centered at neighboring lattice vertices or at lattice vertices on the same diagonal of a lattice square. If we examine the other remaining possible placement of grid disks, depicted in the left section of Figure 6, we have that four grid disks are necessary only if $\overline{BC} = \overline{CD} > r$. By the Pythagorean theorem, we have in this case:

$$\overline{BC} = \sqrt{\left(r - \frac{\sqrt{2}}{2}\right)^2 + 2} \tag{8}$$

imposing $\overline{BC} > r$ we obtain $r < 5\frac{\sqrt{2}}{4}$. □

Sufficient Condition. Consider the left section of Figure 7. The four grid disks cover any disk centered inside the shaded area $ABCE$. This area is defined by the triangle ABC and the circle centered at point P. This circle is the locus of the centers of the disks to be covered that touch point P. Any disk centered inside the remaining area ACE is not covered by the four grid disks. Consider now the right section of Figure 7, the four (solid) grid disks cover in this case any disk centered inside the shaded area $CDEF$. This area is defined by triangle BCD and by the circle centered at point Q. This circle is the locus of the centers of the disks to be covered that touch point Q. Such circle passes by point E, therefore, adjoining the two shaded areas $ABCE$ and $CDEF$, we fully cover the area of triangle BCD. Any disk centered inside triangle BCD is covered by four grid disks: the four depicted in the left section or the four depicted in the right section of Figure 7. By symmetry, the same holds for triangle ABD and we can tile the plane with triangles inside which disks can be centered and covered by four grid disks. □

Proof of CASE 5.

Necessary Condition. By symmetry, any two disks with the same radius, centered far apart by an arbitrary $\epsilon > 0$, cover less than half of each other's perimeter, therefore, any grid disk must cover less than half of the perimeter of any other disk not centered at a lattice point. It follows that any two grid disks must cover less than the entire perimeter of any other disk not centered at a lattice point. Hence, any disk not centered at a lattice point requires at least three grid disks to be covered. □

Sufficient Condition. It is enough to prove this condition when r is minimum, therefore, we carry out calculations fixing $r = \frac{5\sqrt{2}}{4}$. We consider three different placements of three grid disks on the lattice and show that they are enough to cover any disk arbitrarily placed on the plane. Consider the upper left section of Figure 8. Taking point O as the origin of

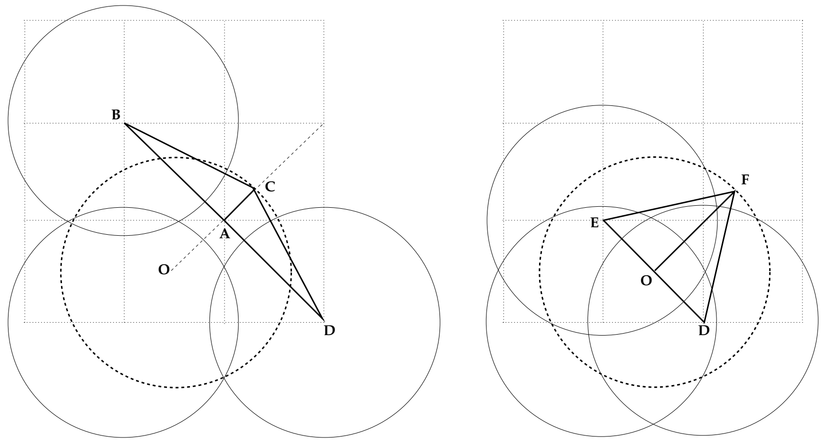

Figure 6: **CASE 4, necessary condition.** *Point O is at the center of a lattice square. In the left section of the figure, the dashed disk centered at point O requires one more grid disks to be covered, if $\overline{BC} > r$. In the right section of the figure, the dashed disk centered at point O always requires one more grid disk to be covered.*

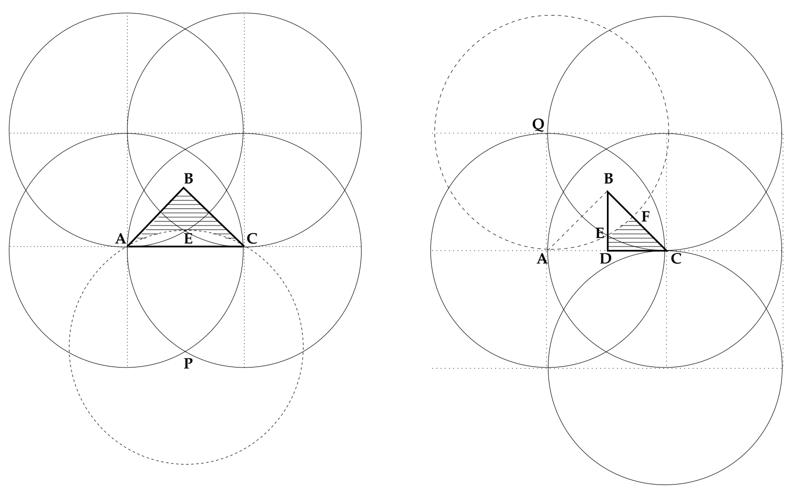

Figure 7: **CASE 4, sufficient condition.** *Four grid disks cover any disk centered inside the shaded areas. Intersecting the two shaded areas, we obtain triangle BCD.*

the coordinate system, the three grid disks are placed at points: $(-1,0),(-1,2),(1,0)$. The coordinates of point P are calculated by applying the Pythagorean theorem to triangle AOP obtaining: $P = \left(0, -\frac{\sqrt{34}}{4}\right)$. Therefore, the locus of the centers of the disks to be covered that touch point P, given by the circle centered at point P, is defined by the equation:

$$x^2 + \left(y + \frac{\sqrt{34}}{4}\right)^2 = r^2 ; \tag{9}$$

this circle passes by point A and intersects segment $\overline{BH}$ at point $C = \left(-\frac{1}{2}, \frac{\sqrt{46}-\sqrt{34}}{4}\right)$. It is easy to see that any disk centered inside the shaded area ABC is covered by the three grid disks centered at points: $(-1,0),(-1,2),(1,0)$.

We now consider the three grid disks depicted in the upper right section of Figure 8, centered at points: $(-1,0),(0,1),(0,-1)$. First, let us focus on point Q. Its coordinates are calculated by applying the Pythagorean theorem to triangle OQR, obtaining: $Q = \left(\frac{\sqrt{34}}{4}, 0\right)$. Since $\overline{HQ} = \overline{HO} + \overline{OQ} = \frac{1}{2} + \frac{\sqrt{34}}{4} > r$, the locus of the centers of the disks to be covered that touch point Q, given by the circle centered at point Q, does not intersect triangle ABH. Now, let us focus on point P'. The coordinates of point P' are calculated by intersecting the two grid circles:

$$\begin{cases} (x+1)^2 + y^2 &= r^2 \\ x^2 + (y-1)^2 &= r^2 , \end{cases} \tag{10}$$

obtaining: $P' = (\frac{-2-\sqrt{21}}{4}, \frac{2+\sqrt{21}}{4})$. Therefore, the locus of the centers of the disks to be covered that touch point P', given by the circle centered at point P', is defined by the equation:

$$\left(x + \frac{2+\sqrt{21}}{4}\right)^2 + \left(y - \frac{2+\sqrt{21}}{4}\right)^2 = r^2 ; \tag{11}$$

this circle passes by point A and intersects segment $\overline{BH}$ at point $C' = \left(-\frac{1}{2}, \frac{2+\sqrt{21}-\sqrt{29}}{4}\right)$. It is easy to see that any disk centered in the shaded area $AC'H$ is covered by the three grid disks centered at points: $(-1,0),(0,1),(0,-1)$.

Intersecting the two circles centered at points P and P', defined by equations (9) and (11) respectively, we obtain two points:

$$\begin{aligned} (x_0, y_0) &= (-1, 0) \\ (x_1, y_1) &= \left(\frac{2-\sqrt{21}}{4}, \frac{2+\sqrt{21}-\sqrt{34}}{4}\right) \end{aligned}$$

depicted in the left section of Figure 9. By the coordinates of these points, (x_1, y_1) is placed inside triangle ABH. Hence, any disk centered inside the shaded area given by the intersection of the two disks depicted in the left section of Figure 9 is not covered by neither the three grid disks depicted in the upper left section or the upper right section of Figure 8.

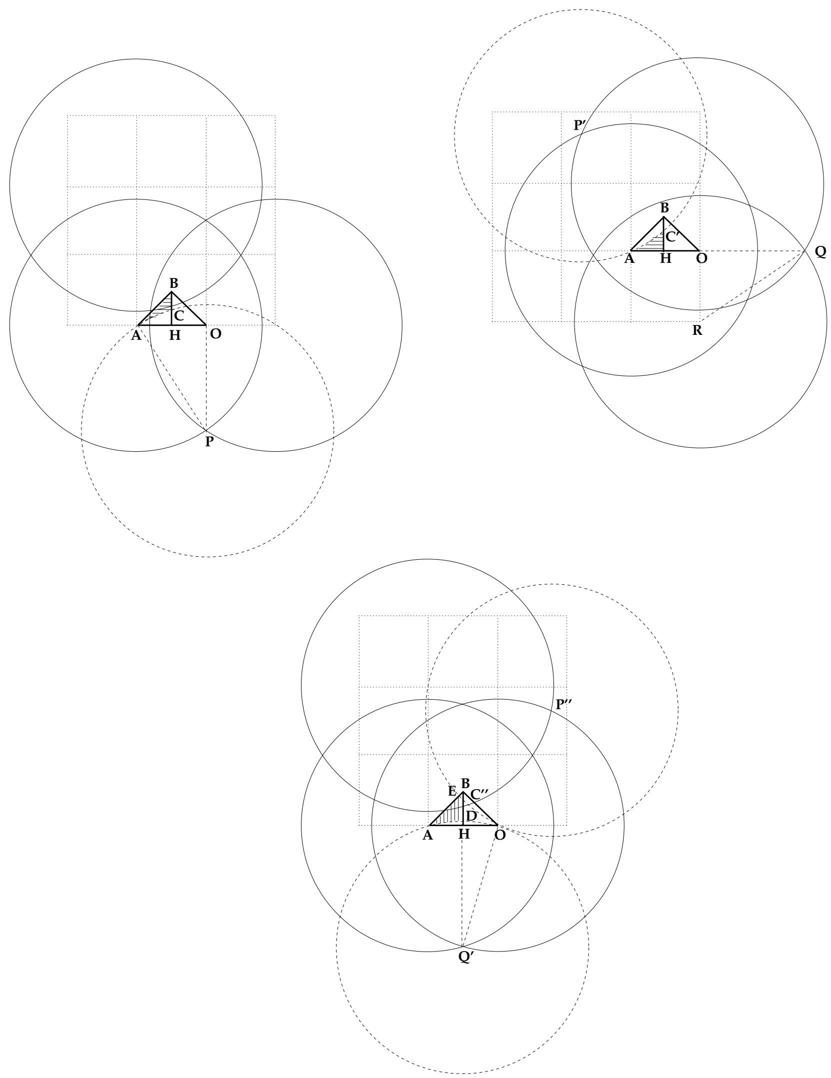

Figure 8: **CASE 5, sufficient condition.** *Three grid disks cover any disk centered inside the shaded areas. Intersecting the three shaded areas, we obtain triangle ABH.*

In order to cover such a disk, we consider the three grid disks depicted in the lower section of Figure 8, placed at points: $(-1,0),(-1,2),(0,0)$. In this case, the coordinates of point Q' are calculated by applying the Pythagorean theorem to triangle $Q'OH$, thus obtaining: $Q'=\left(-\frac{1}{2},-\sqrt{\frac{23}{8}}\right)$. Therefore, the locus of the centers of the disks to be covered that touch point Q', given by the circle centered at Q', is defined by the equation:

$$\left(x+\frac{1}{2}\right)^2+\left(y+\sqrt{\frac{23}{8}}\right)^2=r^2\ ; \tag{12}$$

this circle passes by points A and O and intersects segment $\overline{BH}$ at point $D=\left(-\frac{1}{2},\frac{5\sqrt{2}-\sqrt{46}}{4}\right)$. The coordinates of point P'' are calculated by intersecting the two grid circles:

$$\begin{cases} (x+1)^2+(y-2)^2 & =r^2 \\ x^2+y^2=r^2\ , \end{cases} \tag{13}$$

thus obtaining: $P''=\left(\frac{\sqrt{6}-1}{2},\frac{\sqrt{6}+4}{4}\right)$. Therefore, the locus of the centers of the disks to be covered that touch point P'', given by the circle centered in P'', is defined by the equation:

$$\left(x-\frac{\sqrt{6}-1}{2}\right)^2+\left(y-\frac{\sqrt{6}+4}{4}\right)^2=r^2\ ; \tag{14}$$

this circle passes by point O and intersects segment $\overline{BH}$ at point $C''=\left(-\frac{1}{2},\frac{4+\sqrt{6}-\sqrt{26}}{4}\right)$. It is easy to see that any disk centered inside the shaded area $AEC''D$ is covered by the three grid disks centered at points: $(-1,0),(-1,2),(0,0)$. In order to check that this placement of grid disks covers any disk centered inside the shaded area given by the intersection of the two disks depicted in the left section of Figure 9, we intersect the two circles centered at points P' and Q', defined by equations (11) and (12) respectively, obtaining two points:

$$\begin{aligned} (x_0,y_0) &= (-1,0) \\ (x_2,y_2) &= \left(-\frac{\sqrt{21}}{4},\frac{2+\sqrt{21}-\sqrt{46}}{4}\right) \end{aligned}$$

depicted in the right section of Figure 9. Given the coordinates of these points, since (x_2,y_2) is placed outside the area of triangle ABH, we conclude that the three grid disks placed at points: $(-1,0),(-1,2),(0,0)$, cover any disk centered inside the shaded area given by the intersection of the two disks depicted in the left section of Figure 9. It follows that intersecting the three shaded areas of Figure 8: ABC,$AC'H$,$AEC''D$, we cover triangle ABH completely. Hence, any disk centered inside triangle ABH is covered by three grid disks, placed in one of the three configurations depicted in Figure 8. By symmetry, the same holds for triangle OBH and we can tile the plane with triangles inside which disks can be centered and covered by three grid disks. □

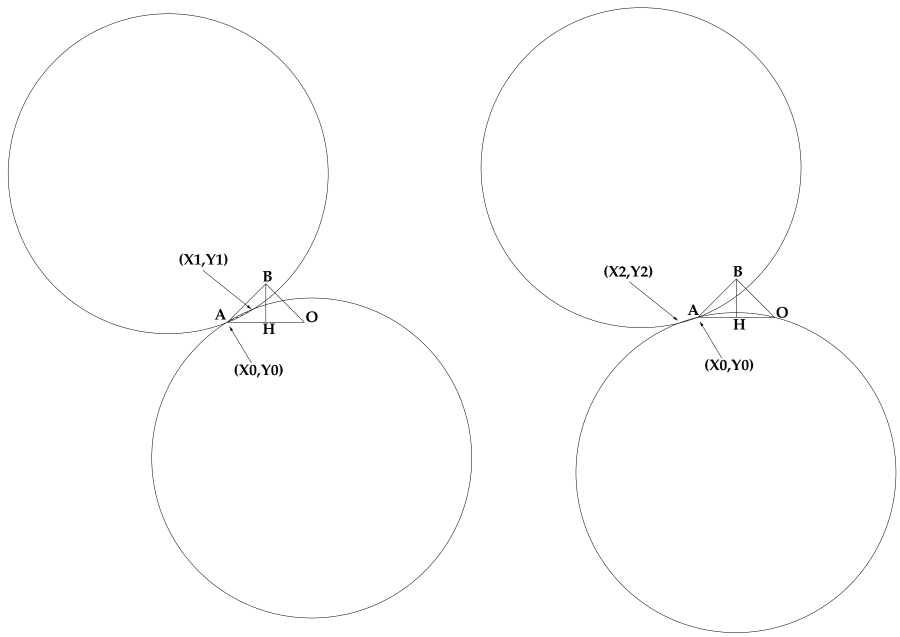

Figure 9: **CASE 5, sufficient condition.** *Point* (x_1, y_1) *is inside triangle* ABH*, point* (x_2, y_2) *is outside triangle* ABH.

B The Finite Dimension of the Lattice

Results of Section 2 refer to a finite lattice placed on top of a square of side length $l \times D$. In this Appendix we find the number of vertices p of this finite lattice for any given approximation. This number is given by the product of the number of lattice vertices on each of the two dimensions of the lattice boundary. The number of lattice vertices on each dimension is the sum of the side length of the square divided by the lattice distance and of an extra factor accounting for the finite dimension of the lattice.

Given the value of p, we obtain values of $K_{(l,p)} = l^2 \sum_{i=1}^{\lceil l\sqrt{2}\rceil^2-1} \binom{p}{i} i$ that are depicted in Figure 11.

B.1 Maintaining an Approximation Factor of 3

Consider the case $r = \frac{5\sqrt{2}}{4}$, depicted in Figure 10. The disks represented in this figure are centered at lattice vertices that are on the boundary of a finite lattice. By the Corollary 1, the approximation factor resulting from covering a set of n points in the plane, using disks of radius $r = \frac{5\sqrt{2}}{4}$ centered at lattice vertices, is three. We want to determine how large must a finite lattice be, in order to have the same approximation factor of three when covering a set of n points in the square of side length $l \times D$. The spacing between the boundary of the lattice and of the square must be at most $\overline{AC} = \sqrt{r^2 - 1} = \frac{\sqrt{34}}{4}$. That is because with a larger distance one could cover the four points placed at an infinitesimal distance ϵ to the right of B,C,D,E using a single (non-grid) disk (since $\overline{BE} = 3 < 2r$), while four grid disks

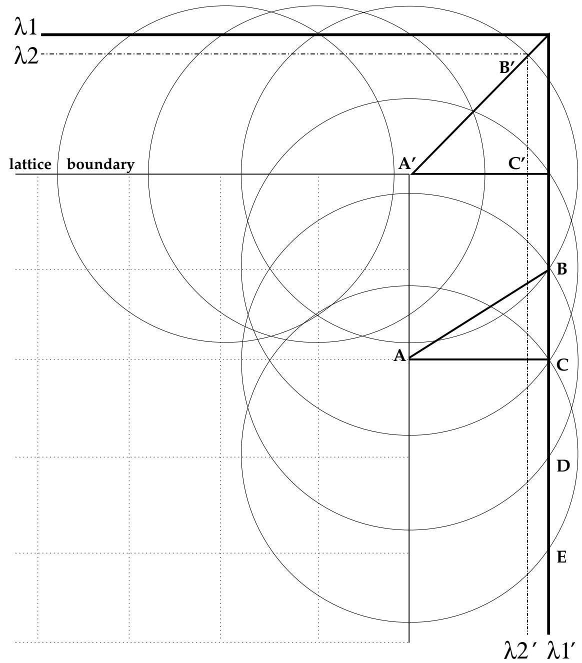

Figure 10: **Placement of the boundary.** *In order to obtain the value p, we determine how much do we need to extend the finite lattice, in order to cover all the points in a square of side length $l \times D$ with an approximation factor of three, using disks centered at lattice vertices.*

would be necessary to cover them. A different constraint is given by the four corners of the square. In order to cover them, the distance between the boundary of the finite lattice and the side of the square must be at most $\overline{A'C'} = \frac{r\sqrt{2}}{2} = \frac{5}{4}$; otherwise we need to add four more lattice vertices to be able to cover the four corners.

Lines λ_1, λ_2, λ'_1 and λ'_2, depicted in Figure 10, visualize the two constraints described above: if one side of the square is below line λ_2 and the side orthogonal to it on the left of line λ'_2, the approximation factor of three is met. However, if one side of the square is between lines λ_1 and λ_2 and the side orthogonal to it is between lines λ'_1 and λ'_2, we need to add an extra lattice point for each corner of the square to maintain the above approximation.

As stated above, to calculate the value of p we need to multiply the number of lattice vertices that lie on each of the two dimensions of the lattice boundary. This number is obtained by dividing the side length of the square by the lattice distance and by adding an extra factor that takes into account the fact that the lattice does not extend up to the side of the square, but there is a distance $d \leq \overline{AC}$ between the boundary of the lattice and the side of the square. Hence, dividing the side length of the square by the lattice distance we have:

$$\frac{2r\,l}{\frac{4r}{5\sqrt{2}}} = \frac{5\sqrt{2}\,l}{2}\;; \tag{15}$$

adding the extra factor to equation (15) and taking the product of the number of lattice vertices on each of the two dimensions of the lattice boundary, we obtain: $p = min\{p_1, p_2, p_3, p_4\}$, where:

$$p_1 = \left\lceil \frac{5\sqrt{2}l}{2} + 1 - \frac{5}{2} \right\rceil^2 , \tag{16}$$

$$p_2 = \left\lceil \frac{5\sqrt{2}l}{2} + 1 - \frac{5}{4} - \frac{\sqrt{34}}{4} \right\rceil \left\lceil \frac{5\sqrt{2}l}{2} + 1 - \frac{5}{2} \right\rceil + 2 , \tag{17}$$

$$p_3 = \left\lceil \frac{5\sqrt{2}l}{2} + 1 - \frac{5}{4} - \frac{\sqrt{34}}{4} \right\rceil^2 + 3 , \tag{18}$$

$$p_4 = \left\lceil \frac{5\sqrt{2}l}{2} + 1 - \frac{\sqrt{34}}{2} \right\rceil^2 + 4 . \tag{19}$$

Equation (16) expresses the number of lattice vertices if we extend the lattice boundary up to a distance $\overline{A'C'} = \frac{5}{4}$ from the boundary of the square. Equation (19) corresponds to extending the lattice boundary up to a distance $\overline{AC} = \frac{\sqrt{34}}{4}$ from the boundary of the square and to add four more lattice vertices to cover the four corners of the square. Equation (17) corresponds to extending three sides of the lattice boundary up to a distance $\overline{A'C'} = \frac{5}{4}$ from the boundary of the square, the fourth side of the lattice boundary up to a distance $\overline{AC} = \frac{\sqrt{34}}{4}$ from the boundary of the square and to add two more lattice vertices to cover two corners. Equation (18) corresponds to extending two sides of the lattice boundary up to a distance $\overline{A'C'} = \frac{5}{4}$ from the boundary of the square, the remaining two sides of the lattice boundary up to a distance $\overline{AC} = \frac{\sqrt{34}}{4}$ from the boundary of the square and to add three more lattice vertices to cover three corners.

B.2 Maintaining Approximation Factors of 4, 5 and 6

By similar reasonings, it is easy to find the smallest possible finite lattice and consequently the value of p in the remaining three cases defined by the Corollary 1. We have that: for $r = 1$, $p = 2l^2$; for $r = \frac{\sqrt{10}}{4}$, $p = min\{p_1, p_2, p_3, p_4\}$, where:

$$p_1 = \left\lceil \frac{\sqrt{10}\, l}{2} + 1 - \frac{\sqrt{5}}{2} \right\rceil^2 , \tag{20}$$

$$p_2 = \left\lceil \frac{\sqrt{10}\, l}{2} + 1 - \frac{\sqrt{5}}{4} - \frac{\sqrt{6}}{4} \right\rceil \left\lceil \frac{\sqrt{10}\, l}{2} + 1 - \frac{\sqrt{5}}{2} \right\rceil + 2 , \tag{21}$$

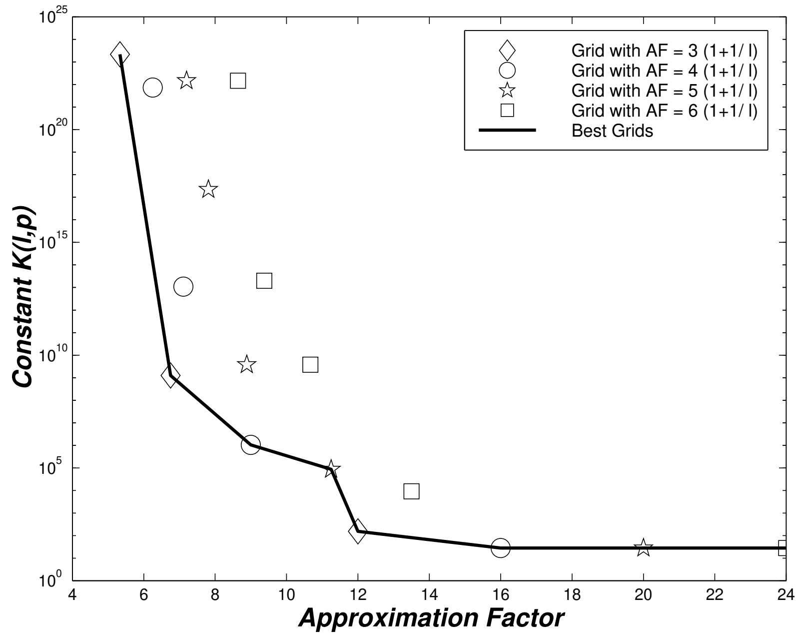

Figure 11: **Values of K for given approximation factors.** *By using algorithms with larger approximations, we can control the size of the constant K and thus achieve practical running times. The most convenient algorithms of our family are highlighted by connecting them with a solid line.*

$$p_3 = \left\lceil \frac{\sqrt{10}\, l}{2} + 1 - \frac{\sqrt{5}}{4} - \frac{\sqrt{6}}{4} \right\rceil^2 + 3\,, \tag{22}$$

$$p_4 = \left\lceil \frac{\sqrt{10}\, l}{2} + 1 - \frac{\sqrt{6}}{2} \right\rceil^2 + 4 \tag{23}$$

and for $r = \frac{\sqrt{2}}{2}$, $p = \lceil l\sqrt{2} \rceil^2$.

B.3 The Slope Factor $K_{(l,p)}$

In Section 2 we have shown that the running time of our family of algorithms is linear in the number of points to cover. The slope of the linear function is a constant K that depends from the approximation factor of a specific algorithm. Values of K for different approximation factors are depicted in figure 11. We note that by using larger approximation factors we can reduce the value of K and thus achieve reasonable running times.