CALIFORNIA INSTITUTE OF TECHNOLOGY

EARTHQUAKE ENGINEERING RESEARCH LABORATORY

# RAPID CALCULATION OF SELECTED FOURIER SPECTRUM ORDINATES

BY

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EERL 72-05

A REPORT ON RESEARCH CONDUCTED UNDER A GRANT FROM THE NATIONAL SCIENCE FOUNDATION

PASADENA, CALIFORNIA

NOVEMBER 1972

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CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91109

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## INTRODUCTION

The use of Fourier spectrum techniques in earthquake engineering has grown rapidly in recent years because of the economy of programs using the Fast Fourier Transform (FFT) and the widespread use of Fourier techniques in other fields of engineering and science. Typically, the standard FFT programs take $2^N$ equally spaced data points in the time domain as input and produce as output $2^{N-1}$ Fourier amplitude spectrum ordinates equally spaced in the frequency domain from 0 cps to the maximum frequency permitted by the digitization interval. By appropriate choice of filters, sampling interval and length of record, the FFT approach can be adapted to most purposes, but there is occasionally a need to calculate a few spectrum points in narrow frequency bands or to analyze, over selected frequency bands, records of longer duration than can be accommodated conveniently by standard FFT programs. The technique presented below permits such calculations to be made rapidly and accurately. In addition, the method helps in the interpretation of Fourier spectra used in earthquake engineering because it is developed from the point of view of elementary vibration theory.

The first part of the text reviews the relation between the Fourier amplitude and phase spectra of an accelerogram $\ddot{z}(t)$ and the

response of an undamped, single degree-of-freedom oscillator subjected to the same accelerogram. This review shows that the calculation of the Fourier amplitude and phase spectrum ordinates is equivalent to finding the potential and kinetic energies of an undamped oscillator at the end of the excitation. The analysis is then extended to include an associated free vibration problem useful in the interpretation of Fourier spectra.

The next portion of the study shows that these final response values can be calculated rapidly and accurately by reducing the accelerogram, regardless of length, to an equivalent excitation with a duration of one natural period, and by further reduction to two excitations - one for displacement and one for velocity - of only one-quarter period duration. The response of the oscillator to the shortened excitations can then be calculated by standard methods. The next section is devoted to the development of a subroutine for calculating ordinates of Fourier amplitude spectra by this approach, and to the presentation of examples of its use. The study concludes with a discussion of possible applications and extensions of the method.

## FOURIER SPECTRA AND EARTHQUAKE RESPONSE

The ordinates of the Fourier spectra of an accelerogram are closely related to the response of single degree-of-freedom oscillators, as is well-documented in the literature (Kawasumi, 1956; Rubin, 1961; Hudson, 1962; Jenschke, 1970). In what follows, this relation is reviewed and extended to aid the interpretation of Fourier spectra

and to guide the development of the subsequent calculation technique.

Consider an accelerogram, $\ddot{z}(t)$, of duration T and an undamped, single degree-of-freedom oscillator, initially at rest, subjected to this accelerogram (Figure 1). The Fourier transform of the accelerogram is the complex number defined by

$$F(\omega) = \int_{-\infty}^{\infty} \ddot{z}(t) e^{-i\omega t} dt \tag{1}$$

From equation (1) it is clear that the Fourier transform is a function of the frequency parameter $\omega$. To apply equation (1) it is necessary to define $\ddot{z}(t)$ in the intervals from $-\infty$ to 0 and from T to $+\infty$. Presuming $\ddot{z}(t)$ from 0 to T includes all the non-trivial parts of the accelerogram, the acceleration is assumed on physical grounds to be zero outside this interval. [It is noted in this regard that the FFT calculates the discrete Fourier transform which assumes the accelerogram is periodic with period T. (Bergland, 1969)].

Applying the assumed extension of the record by zeroes and introducing the variable of integration $\tau$, equation (1) becomes

$$F(\omega) = \int_{0}^{T} \ddot{z}(\tau) e^{-i\omega t} d\tau \tag{2}$$

Equation (2) can also be written as

$$F(\omega) = \int_{0}^{T} \ddot{z}(\tau) \cos \omega\tau \, d\tau \; - i \int_{0}^{T} \ddot{z}(\tau) \sin \omega\tau d\tau, \tag{3}$$

which conveniently separates the real and imaginary parts of $F(\omega)$.

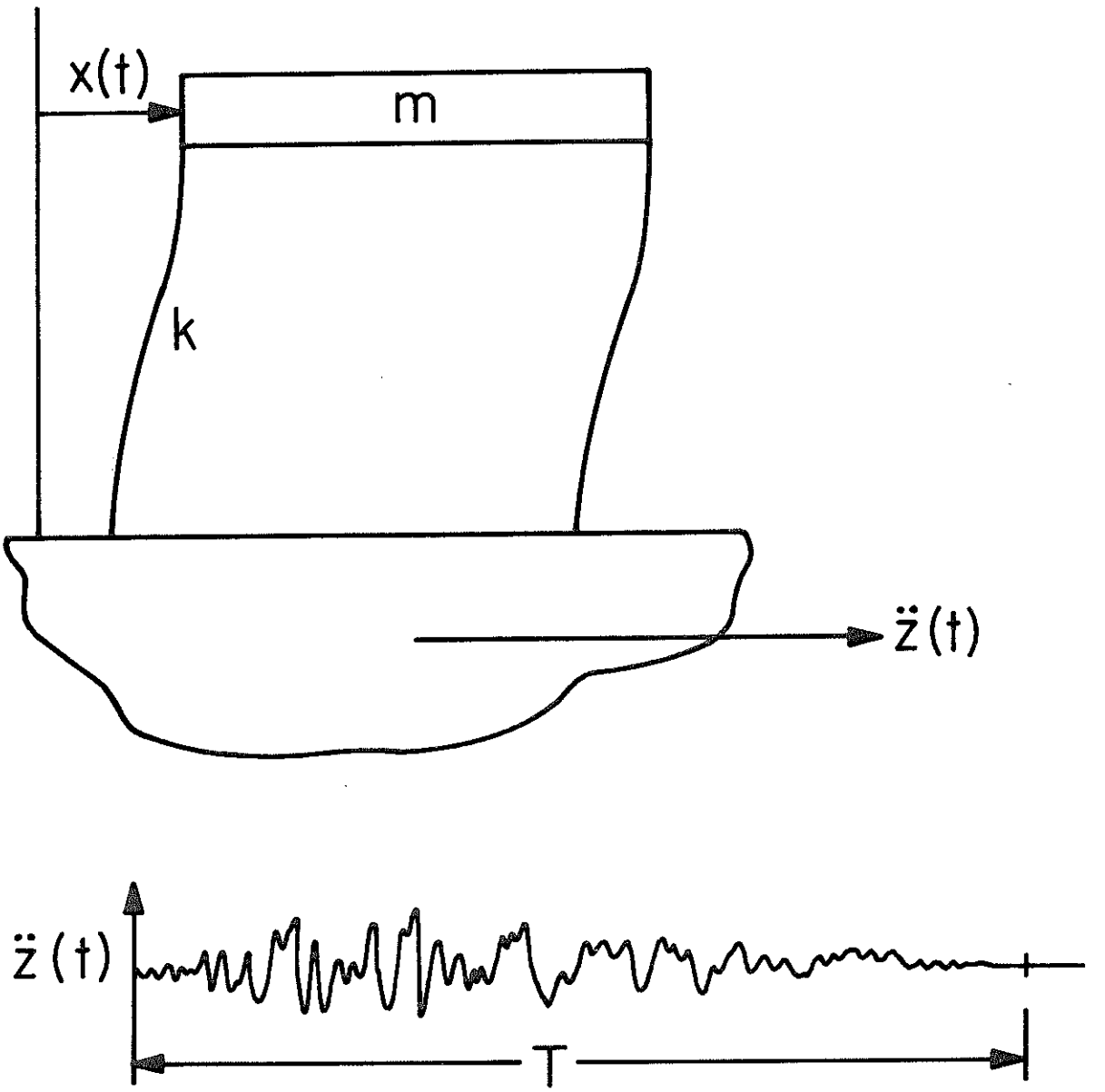


FIGURE 1

Undamped single degree of freedom oscillator subjected to ground acceleration, $\ddot{z}(t)$.

The modulus, or absolute value, of this complex number is

$$|F(\omega)| = \left\{ \left[ \int_0^T \ddot{z}(\tau) \cos \omega\tau \, d\tau \right]^2 + \left[ \int_0^T \ddot{z}(\tau) \sin \omega\tau \, d\tau \right]^2 \right\}^{\frac{1}{2}}, \qquad (4)$$

and the phase angle, $\Psi$, is defined by

$$\tan \Psi = \frac{-\int_0^T \ddot{z}(\tau) \sin \omega\tau \, d\tau}{\int_0^T \ddot{z}(\tau) \cos \omega\tau \, d\tau} \qquad (5)$$

In the polar form of $F(\omega)$, the real part is given by $|F(\omega)| \cos \Psi$ and the imaginary part by $|F(\omega)| \sin \Psi$. Both $|F(\omega)|$ and $\Psi$ are functions of $\omega$ and plots of their values over a frequency range define the Fourier amplitude and phase spectra respectively. In many applications, only the amplitude spectrum or its square, the power spectrum, is of interest. ($T^{-1} |F(\omega)|^2$ is an estimate of the power spectral density.)

For comparison, the displacement and velocity of an undamped single degree-of-freedom oscillator subjected to $\ddot{z}(t)$ (Figure 1) can be written in a related form by means of Duhamel's integral. For an oscillator initially at rest, with $\omega = \sqrt{k/m}$

$$x(t) = -\frac{1}{\omega} \int_0^t \ddot{z}(\tau) \sin \omega(t-\tau) \, d\tau \qquad (6a)$$

and

$$\dot{x}(t) = -\int_0^t \ddot{z}(\tau) \cos \omega(t-\tau) \, d\tau . \qquad (6b)$$

Introducing the temporary notation

$$C(t) = \int_0^t \ddot{z}(\tau) \cos \omega\tau \, d\tau , \tag{7a}$$

$$S(t) = \int_0^t \ddot{z}(\tau) \sin \omega\tau \, d\tau , \tag{7b}$$

and using trigonometric entities, equations (6) become

$$x(t) = -\frac{1}{\omega} \sin \omega t \, C(t) + \frac{1}{\omega} \cos \omega t \, S(t), \tag{8a}$$

$$\dot{x}(t) = -\cos \omega t \, C(t) - \sin \omega t \, S(t). \tag{8b}$$

Defining a phase angle $\Phi$ by

$$\cos \Phi = \frac{C(t)}{\sqrt{C^2(t) + S^2(t)}} \tag{9a}$$

$$\sin \Phi = \frac{-S(t)}{\sqrt{C^2(t) + S^2(t)}} , \tag{9b}$$

the displacement and velocity can be written as

$$x(t) = -\frac{1}{\omega} \sqrt{C^2(t) + S^2(t)} \sin (\omega t + \Phi) \tag{10a}$$

$$\dot{x}(t) = -\sqrt{C^2(t) + S^2(t)} \cos (\omega t + \Phi). \tag{10b}$$

The total energy of the relative motion of the oscillator is E(t), the sum of the kinetic energy of relative motion and the strain energy,

$$E(t) = \tfrac{1}{2} m\dot{x}^2(t) + \tfrac{1}{2}kx^2 \tag{11}$$

which can also be written

$$\frac{2E(t)}{m} = \dot{x}^2(t) + \omega^2 x^2(t). \tag{12}$$

Substituting from equation (10) into (12) leads to the result

$$\frac{2E(t)}{m} = C^2(t) + S^2(t). \tag{13}$$

Evaluating the energy at the end of excitation (t = T) and substituting from equations (7),

$$\frac{2E(T)}{m} = \left[\int_0^T \ddot{z}(\tau) \cos \omega \tau \, d\tau\right]^2 + \left[\int_0^T \ddot{z}(\tau) \sin \omega \tau \, d\tau\right]^2. \tag{14}$$

Comparing equations (14) and (4) shows that the square root of twice the energy per unit mass of the undamped oscillator at the end of the excitation is equal to the modulus of the Fourier transform, i. e.,

$$|F(\omega)| = [\dot{x}^2(T) + \omega^2 x^2(T)]^{\frac{1}{2}} = \left[\frac{2E(T)}{m}\right]^{\frac{1}{2}} \tag{15}$$

From equations (9) and (7),

$$\tan \Phi = \frac{-\int_0^t \ddot{z}(\tau) \sin \omega\tau \, d\tau}{\int_0^t \ddot{z}(\tau) \cos \omega\tau \, d\tau} \tag{16}$$

Equations (5) and (16) imply that

$$\Phi(T) = \Psi . \tag{17}$$

The phase angle $\Psi$ can be expressed in terms of x(T) and $\dot{x}(T)$ by means of equations (8), (9) and (17):

$$\tan \Psi = \frac{\omega x(T) \cos \omega T - \dot{x}(T) \sin \omega T}{\dot{x}(T) \cos \omega T + \omega x(T) \sin \omega T} . \tag{18}$$

A particular case of (18) occurs when the duration, T, is an integer multiple of the natural period, $2\pi/\omega$, of the oscillator. In this instance

$$\tan \Psi = \frac{\omega x(T)}{\dot{x}(T)} = \left[\frac{P.E.}{K.E.}\right]^{\frac{1}{2}}, \tag{19}$$

in which P.E. and K.E. denote potential and kinetic energy, respectively.

Because the addition of a segment of zero excitation to the beginning of the accelerogram does not change the mechanics of the vibration problem, equation (19) can be made to apply in all cases by addition of a segment of zeroes sufficient to make T an integer multiple of $T_o = 2\pi/\omega$. Thus, the Fourier phase spectrum ordinate can be thought of as a measure of the relative kinetic and potential energies of the oscillator at the end of the excitation.

It is of some help in interpreting Fourier spectra to form an associated free vibration problem. If, at time $t' = 0$, an undamped oscillator is released with initial velocity $\dot{x}(0)$ and initial displacement $x(0)$, then the subsequent vibrations are given by

$$x(t') = \frac{\dot{x}(0)}{\omega} \sin \omega t' + x(0) \cos \omega t'. \tag{20}$$

This motion can be treated as a response according to the above analysis. By applying equations (15) and (18), it is found that the amplitude and phase indicated by equation (20) for some duration of motion T are constants, independent of the duration.

$$|F(\omega)| = [\dot{x}^2(0) + \omega^2 x^2(0)]^{\frac{1}{2}} \tag{21}$$

$$\tan \Psi = \frac{\omega x(0)}{\dot{x}(0)} \tag{22}$$

Comparison of equations (21) and (22) with (15) and (19) suggests the corresponding vibration problems shown in Figure 2. Figure 2b

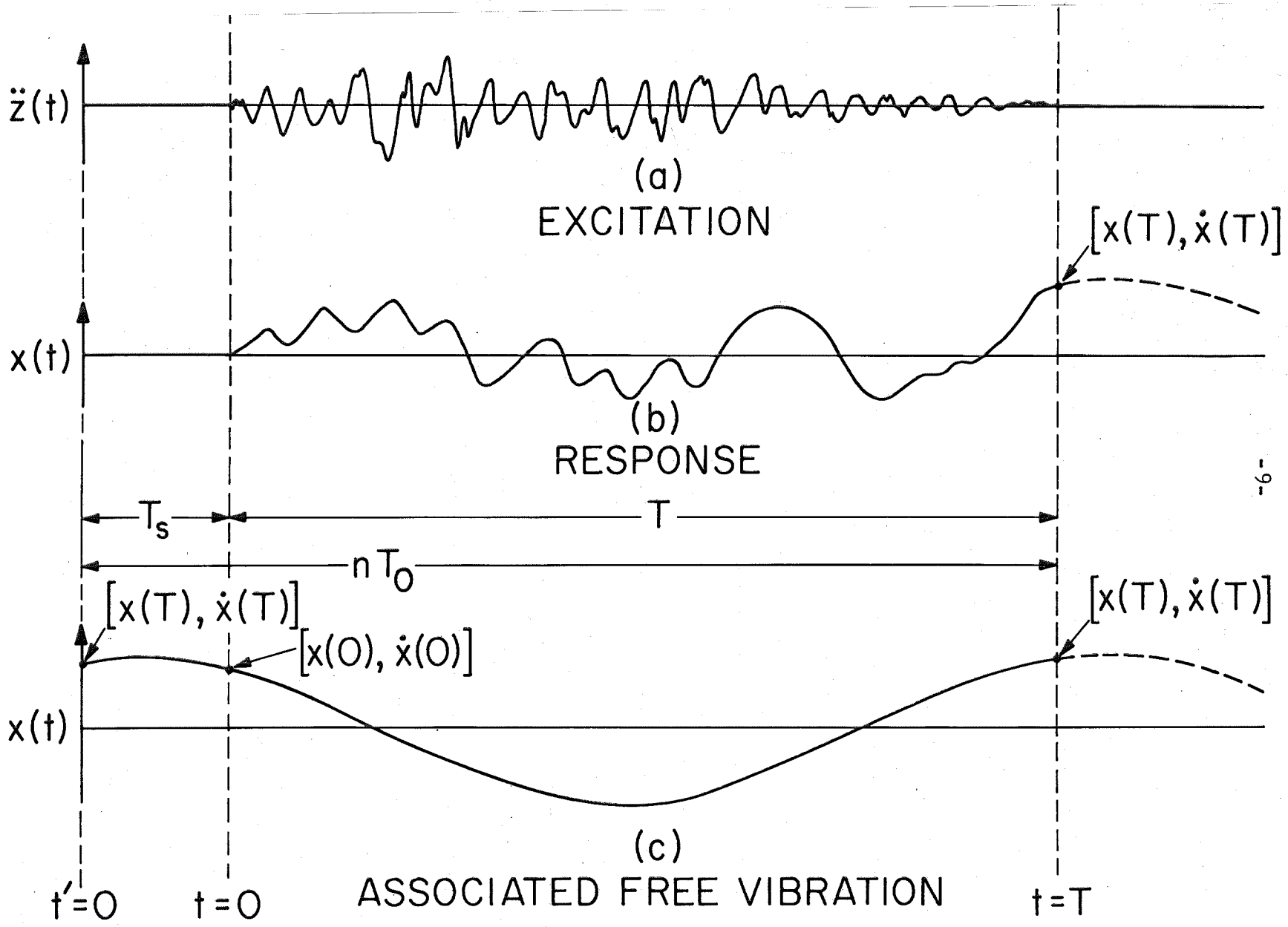


FIGURE 2

Relation between earthquake response and associated free vibration.
(a) Accelerogram $\ddot{z}(t)$; (b) Response of undamped oscillator to $\ddot{z}(t)$;
(c) The associated free vibration problem for the undamped oscillator.

is the earthquake response of an undamped oscillator initially at rest, subjected to $\ddot{z}(t)$ (Figure 2a). A segment of zero acceleration of duration $T_s$ has been added to the beginning of the excitation, and hence the reponse, to make the duration a multiple, n, of the period $T_o$. At the end of the excitationthe displacement and velocity are x(T) and $\dot{x}(T)$, respectively. Figure 2c shows the free vibration response of the same oscillator with initial displacement x(T) and initial velocity $\dot{x}(T)$ applied at $t' = 0$, as indicated in the figure. Because the total duration $T + T_s$ comprises a whole number of periods $T_o$, the periodic vibration with these initial conditions (Figure 2c) will have the same velocity and displacement at $t' = nT_o$ as it had at $t' = 0$. Thus, the associated free vibration problem has just the right initial conditions to match the earthquake response at $t = T$. Because the earthquake response for $t > T$ consists only of free vibrations, matching conditions at $t = T$ will ensure that the earthquake response and the associated free vibration are identical for $t \geq T$. Therefore, according to equations (15) and (18), the two motions will have the same Fourier amplitude and phase spectrum ordinates.

Alternatively, the associated free vibration can be viewed as an extension, backwards in time, of the free vibration which occurs after the end of the excitation.

The zero segment of duration $T_s$ need not actually be added to the curves in Figure 2, because it is clear from Figure 2c and equations 20 that the same free vibrations result for $t \geq 0$ if the initial conditions

$$\omega x(0) = \dot{x}(T) \sin \omega T_s + \omega x(T) \cos \omega T_s \tag{23}$$

$$\dot{x}(0) = \dot{x}(T) \cos \omega T_s - \omega x(T) \sin \omega T_s \tag{24}$$

are applied at $t = 0$ $(t' = T_s)$. For comparison, if the substitution $T = nT_o - T_s$ is made in equation (18), the result is

$$\tan \Psi = \frac{\dot{x}(T) \sin \omega T_s + \omega x(T) \cos \omega T_s}{\dot{x}(T) \cos \omega T_s - \omega x(T) \sin \omega T_s} , \tag{25}$$

and it is seen that $\tan \Psi$ is equal to the square root of the quotient of the initial potential and kinetic energies of the associated free vibration problem.

The above analysis shows the correspondence that exists among the Fourier transform of an accelerogram $\ddot{z}(t)$ of duration T, the response of an undamped oscillator to $\ddot{z}(t)$ and an associated free vibration problem. Using the associated free vibration problem, the amplitude and phase of the Fourier transform may be interpreted as specifying the initial conditions for the free vibration of an undamped oscillator which will have for $t \geq T$ the same displacement and velocity as the same oscillator subjected to $\ddot{z}(t)$. Fourier amplitude and phase spectra can therefore be interpreted as a spectrum of initial conditions for undamped free vibrations with the amplitude spectrum describing the total energy of the vibration, and the phase spectrum specifying how the energy is initially divided into kinetic and potential energy. It is thought that this physical interpretation of Fourier spectra will be useful in understanding Fourier spectra of accelerograms and other time histories encountered in earthquake engineering.

## RAPID CALCULATION OF RESPONSE AND FOURIER SPECTRA

In terms of vibration theory, the calculation of Fourier spectrum ordinates is seen from the preceding to be equivalent to calculating the displacement and velocity of an undamped oscillator at the end of an excitation of duration T. Equation (6a) is the integral representation of the response of such an oscillator to arbitrary excitation and is, in essence, the summation of responses at time t to impulses occurring at time $\tau$. This is illustrated by Figure 3, which shows two such impulses and the portions of the total displacement, x(t), contributed by the two impulses. The two impulses in Figure 3 are one period, $T_o$, apart and it can be seen from Figure 3 that as far as the response at t = T is concerned, the two pulses could have been superposed. Thus, for computing x(T) and $\dot{x}(T)$, the impulses at $T_o - \tau$, $2T_o - \tau$, etc. could first be added and the response to the sum computed, with the sum considered to be acting at $4T_o - \tau$ in the case shown in Figure 3. Extension of this argument implies that the excitation from 0 to $T_o$ can be added at corresponding points to that from $T_o$ to $2T_o$, etc., finally yielding an equivalent excitation of duration $T_o$. If the excitation is not an integer multiple of the period $T_o$, a zero acceleration segment of duration $T_s$ can be added at the beginning of the accelerogram. As discussed above, such an artifice does not change the response at t = T.

The derivation may be formalized by beginning with the integral form of the response to an excitation of duration $T = nT_o$,

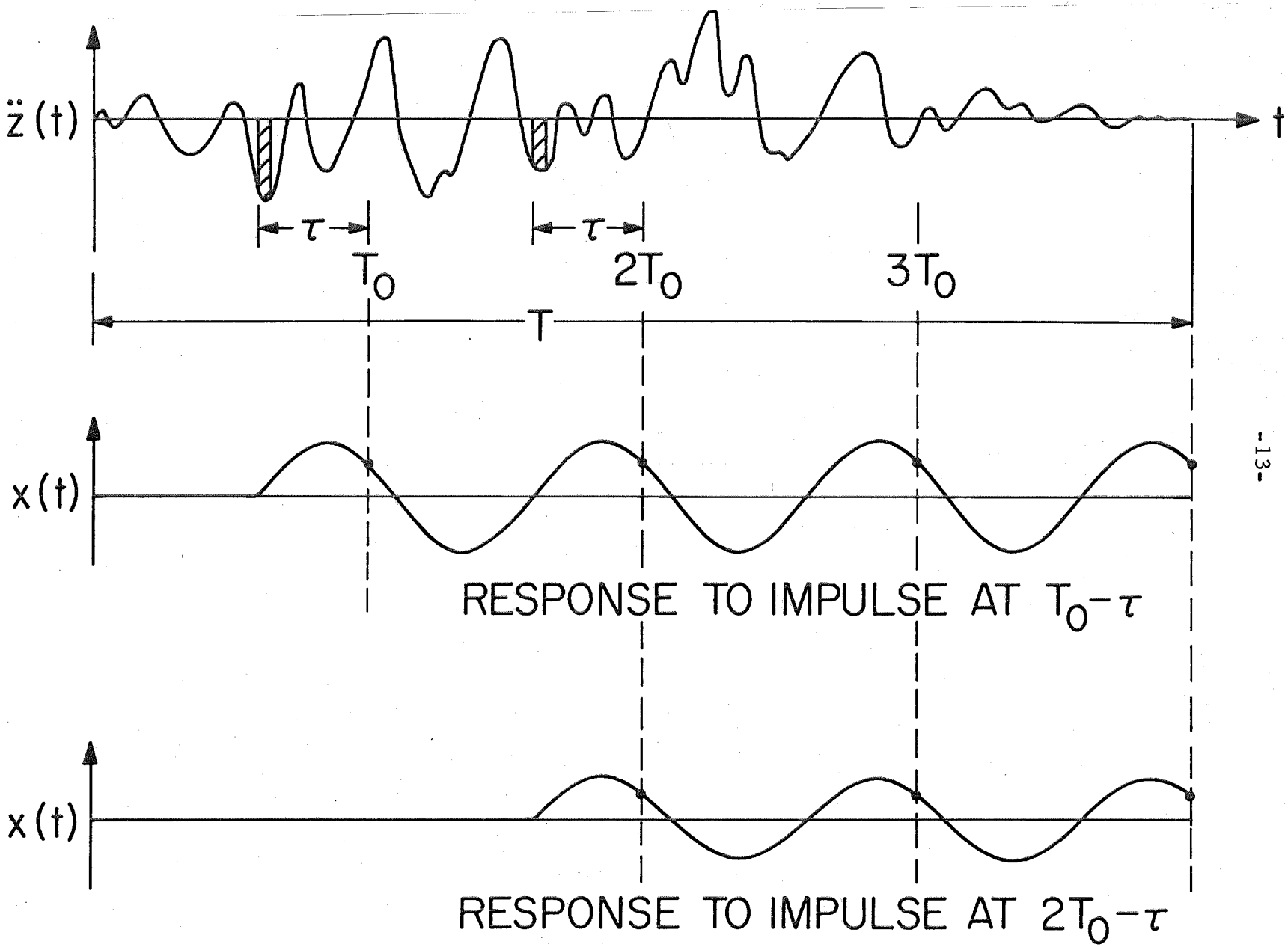


FIGURE 3

Portion of response of undamped oscillator contributed by impulses at $T_o - \tau$ and $2T_o - \tau$.

$$\omega x(T) = -\int_0^T \ddot{z}(\tau) \sin \frac{2\pi}{T_o} (T - \tau)\, d\tau. \tag{26}$$

Dividing the integration into segments of length $T_o$ gives

$$\omega x(T) = -\sum_{j=1}^{n} \int_{(j-1)T_o}^{jT_o} \ddot{z}(\tau) \sin \frac{2\pi}{T_o} (T - \tau)\, d\tau \ . \tag{27}$$

Letting $\tau_j = \tau - (j - 1)T_o$ in the general term of (27) produces

$$\int_{(j-1)T_o}^{jT_o} \ddot{z}(\tau) \sin\frac{2\pi}{T_o} (T-\tau) d\tau = \int_0^{T_o} \ddot{z}[\tau_j + (j-1)T_o] \sin\frac{2\pi}{T_o} [T-\tau_j-(j-1)T_o] d\tau_j, \quad (28$$

which reduces the sum to

$$\omega x(T) = -\sum_{j=1}^{n} \int_0^{T_o} \ddot{z}\, [\tau_j + (j-1)T_o] \sin\frac{2\pi}{T_o} (T_o - \tau_j)\, d\tau_j \ . \tag{29}$$

The series can be summed by interchanging the operations of integration and summation. Defining

$$\ddot{z}_o(\tau) = \ddot{z}(\tau) + \ddot{z}(\tau + T_o) + \cdot \ \cdot \ \cdot + \ddot{z}[\tau + (n-1)\, T_o], \ 0 \leq \tau \leq T_o, \tag{30}$$

which adds the one-period segments of the accelerogram, the displacement becomes

$$\omega x(T) = -\int_0^{T_o} \ddot{z}_o(\tau) \sin \frac{2\pi}{T_o} (T_o - \tau)\, d\tau \tag{31a}$$

and, by a similar analysis

$$\dot{x}(T) = -\int_0^{T_o} \ddot{z}_o(\tau) \cos \frac{2\pi}{T_o} (T_o - \tau)\, d\tau \ . \tag{31b}$$

Equations (31) state that $x(T)$ and $\dot{x}(T)$ are found from the res-

ponse of the oscillator to the excitation $\ddot{z}_o(\tau)$, of duration $T_o$. Any method can be used to find this response; it is not necessary to evaluate x(T) and $\dot{x}(T)$ by the convolution integrals of equation (31). For example, the differential equation corresponding to (31) could be solved by numerical integration.

A further reduction in computing time results if the equivalent excitation is shortened still further. By applying trigonometric identities to equation (31),

$$\omega x(T) = \int_0^{T_o} \ddot{z}_o(\tau) \sin\frac{2\pi\tau}{T_o} \, d\tau , \tag{32a}$$

$$\dot{x}(T) = -\int_0^{T_o} \ddot{z}_o(\tau) \cos\frac{2\pi\tau}{T_o} \, d\tau . \tag{32b}$$

The functions involved in equations (32) are shown in Figure 4. Because both trigonometric functions are repetitions of the basic quarter-cycle shape, introducing new time scales $\tau_1$, $\tau_2$, $\tau_3$, $\tau_4$, of quarter-cycle duration as shown in Figure 4, and defining the corresponding quarter-cycle accelerograms $\ddot{z}_1(\tau_1)$, · · · , $\ddot{z}_4(\tau_4)$ allows the computation of x(T) and $\dot{x}(T)$ to be reduced to integrations of duration $T_o/4$.

$$\omega x(T) = \int_0^{T_o/4} \ddot{z}_5(\tau) \sin\frac{2\pi\tau}{T_o} \, d\tau , \tag{33a}$$

$$\dot{x}(T) = -\int_0^{T_o/4} \ddot{z}_6(\tau) \cos\frac{2\pi\tau}{T_o} \, d\tau . \tag{33b}$$

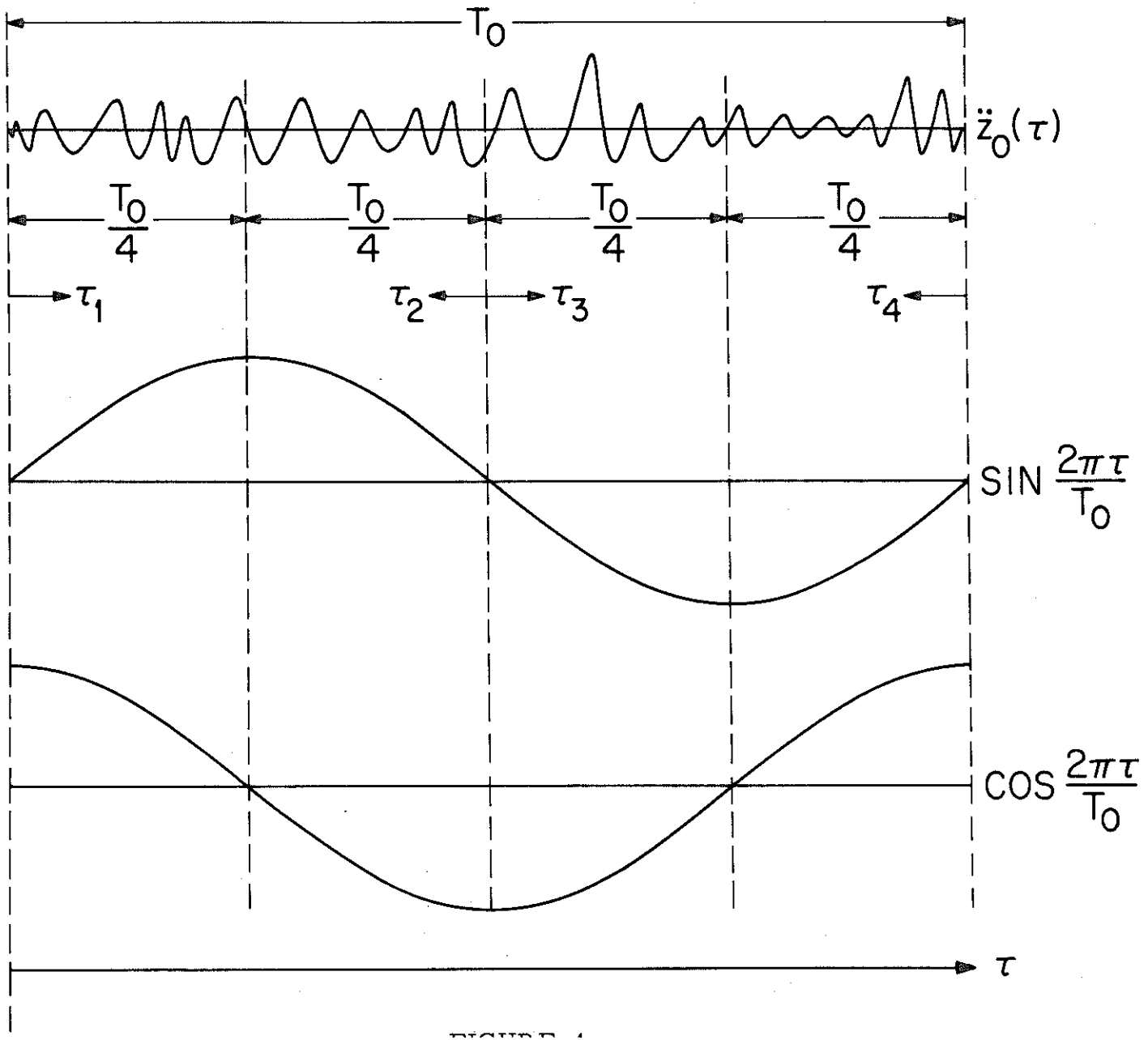
$T_0$
$\ddot{z}_0(\tau)$
$\frac{T_0}{4}$
$\frac{T_0}{4}$
$\frac{T_0}{4}$
$\frac{T_0}{4}$
$\tau_1$
$\tau_2$
$\tau_3$
$\tau_4$
$\text{SIN}\ \frac{2\pi\tau}{T_0}$
$\text{COS}\ \frac{2\pi\tau}{T_0}$
$\tau$

FIGURE 4

in which

$$\ddot{z}_5(\tau) = \ddot{z}_1(\tau) + \ddot{z}_2(\tau) - \ddot{z}_3(\tau) - \ddot{z}_4(\tau) , \tag{34a}$$

$$\ddot{z}_6(\tau) \quad z_1(\tau) - z_2(\tau) - z_3(\tau) + z_4(\tau) . \tag{34b}$$

Equations (33) can also be written as convolution integrals,

$$x(T) = \frac{1}{\omega} \int_0^{T_o/4} \ddot{z}_5(\tau) \cos\frac{2\pi}{T_o}\left(\frac{T_o}{4} - \tau\right) d\tau , \tag{35a}$$

$$\dot{x}(T) = - \int_0^{T_o/4} \ddot{z}_6(\tau) \sin\frac{2\pi}{T_o}\left(\frac{T_o}{4} - \tau\right) d\tau . \tag{35b}$$

Comparing equations (35) with (31) or (6), it is seen that the desired displacement and velocity are reversed from what would be found from a standard approach using $\ddot{z}_5$ or $\ddot{z}_6$ as input accelerograms. Letting $x^*(T_o/4, \ddot{z})$ and $\dot{x}^*(T_o/4, \ddot{z})$ be the displacement and velocity found by using a quarter-cycle accelerogram in the usual way, the desired quantities are

$$x(T) = - \frac{1}{\omega} \dot{x}^*(T_o/4, \ddot{z}_5) , \tag{36a}$$

$$\dot{x}(T) = \omega x^*(T_o/4, \ddot{z}_6) . \tag{36b}$$

$x(T)$ and $\dot{x}(T)$ can therefore be found by means of equations (36) from the velocity and displacement, respectively, of an oscillator subjected to the shortened excitations $\ddot{z}_5$ and $\ddot{z}_6$, respectively. The responses $\dot{x}^*$ and $x^*$ can be found by any convenient method.

Comparing the numerical calculations required to evaluate equations (31) and (36), it is seen that (31) requires the response of an oscillator subjected to an accelerogram of duration $T_o$, whereas equa-

tions (36) require the response of two oscillators to excitations of duration $T_o/4$. Thus the use of equation (36) requires more manipulation of the input, but the numerical integration is only half as long as that needed to evaluate equations (31).

## SUBROUTINE AND EXAMPLE CALCULATIONS

A subroutine has been developed to implement the analysis and to calculate Fourier amplitude spectra according to equation (15). The subroutine forms the equivalent accelerogram $\ddot{z}_o(\tau)$, equation (30), and the reduced accelerograms $\ddot{z}_5(\tau)$ and $\ddot{z}_6(\tau)$ according to equations (34). The response to the quarter-cycle accelerograms, equations (36), is then computed by adapting the computation method developed by Nigam and Jennings (1969) to the problem.

The calculation method assumes that the accelerogram $\ddot{z}(t)$ consists of straight line segments joining equally spaced data points at intervals $\Delta t$. Under this assumption, and the further condition that $T_o/4\Delta t$ is an integer, the equivalent excitations $\ddot{z}_o(\tau)$, $\ddot{z}_5(\tau)$ and $\ddot{z}_6(\tau)$ also consist of straight-line segments. The technique for computing the response is based on the exact solution to such piecewise linear excitation (Nigam and Jennings, 1969), so if the conditions are met, the calculation is exact in the sense that the only errors arise from round-off in the numerical operations.

The logic of the subroutine is shown in Figure 5. The basic steps are: 1) a check of whether or not $T/T_o$ and $T_o/4\Delta t$ are integers; if the first is not the case enough zeroes are considered to exist at the

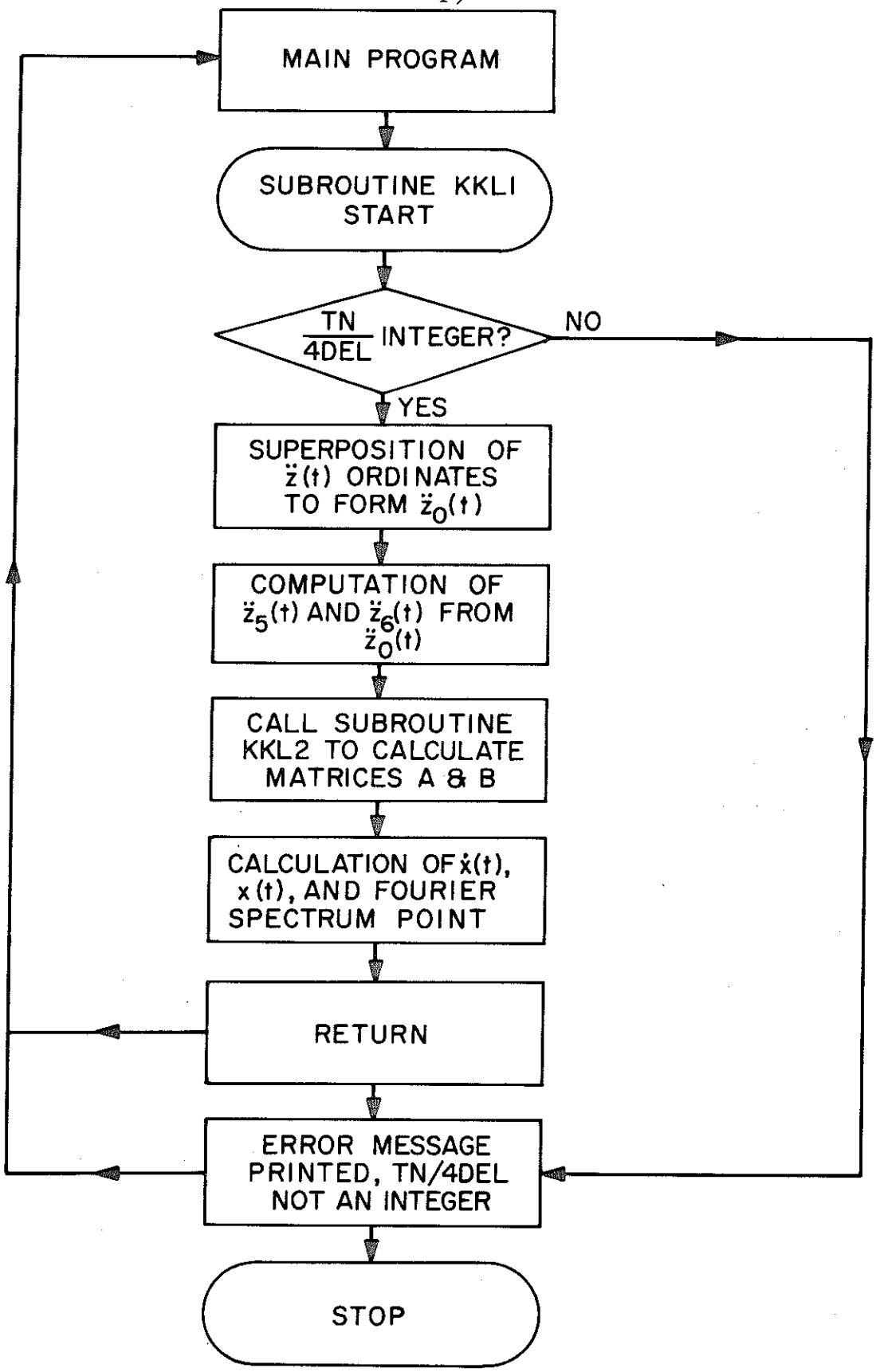


FIGURE 5

Flow chart of calculations.

beginning of the accelerogram to make it so; if the second condition is not satisfied, the calculation is not performed and an error message is printed; 2) the superposition of the accelerogram ordinates to form $\ddot{z}_o(\tau)$ and the quarter-cycle excitation $\ddot{z}_5(\tau)$ and $\ddot{z}_6(\tau)$; and 3) the calculation of x(T), $\dot{x}(T)$ and $|F(\omega)|$. The phase spectrum ordinate is not calculated, but could be evaluated from the output of the subroutine according to equation (18).

A listing of the subroutine is included in the Appendix.

An example of the use of the technique is illustrated by Figure 6 and Table I. The accelerogram $\ddot{z}(t)$ in Figure 6a is artificial earthquake C-1 (Jennings, Housner and Tsai, 1969), which consists of 12 seconds of motion defined by points spaced at intervals of 0.025 sec. Figure 6b shows the equivalent one-period acceleration $\ddot{z}_o(t)$ for a period, $T_o$, of one second, and Figures 6c and 6d are the quarter-cycle excitations $\ddot{z}_5(t)$ and $\ddot{z}_6(t)$, respectively. The accelerograms in Figure 6 are equivalent in the sense that a and b produce the same displacement and velocity at the end of the excitation in an oscillator of one second period and, with the application of equations (36), c and d produce the same terminal displacement and velocity, respectively.

The values of x(T), $\dot{x}(T)$ and the Fourier amplitude spectrum ordinate for $T_o = 1$ sec computed from $\ddot{z}(t)$, $\ddot{z}_o(t)$, and $\ddot{z}_5(t)$ and $\ddot{z}_6(t)$ are compared in Table I. The values using $\ddot{z}(t)$ and $\ddot{z}_o(t)$ were found by direct use of the program developed by Nigam and Jennings (1969), while the values of $\ddot{z}_5(t)$ and $\ddot{z}_6(t)$ were calculated using the subroutine in the Appendix. The values in Table I agree to within the accuracy of

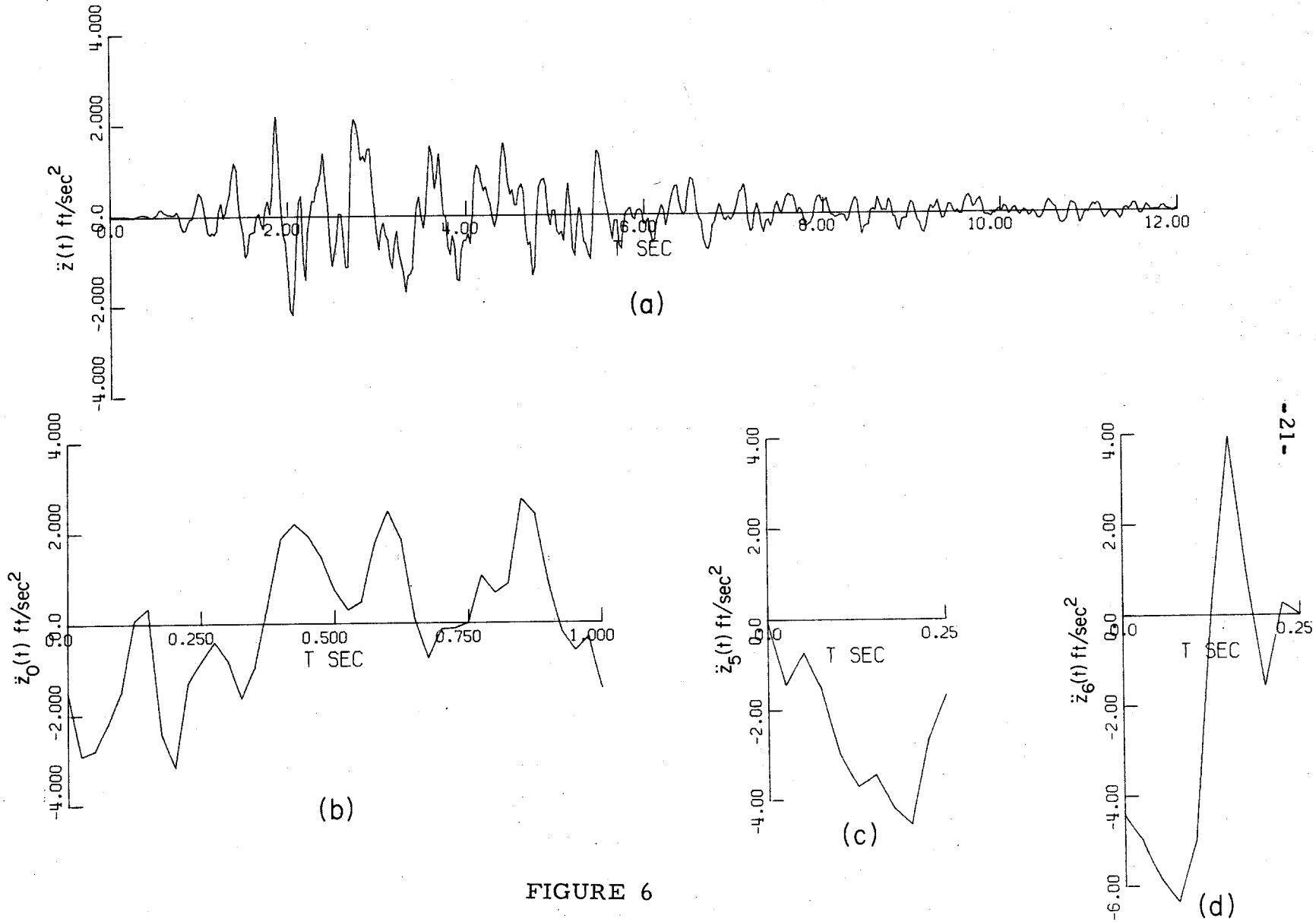


FIGURE 6

Application of the calculation method to artificial earthquake C - 1 for $T_o$ = 1 sec. (a) Original accelerogram; (b) Equivalent one period duration excitation; (c) and (d) Equivalent quarter-cycle excitations.

TABLE I

COMPARISON OF FOURIER AMPLITUDE SPECTRA AND TERMINAL VALUES C

DISPLACEMENT AND VELOCITY

$T_o = 1$ sec

| Accelerogram | Duration (sec) | Displacement, x(T) (in) | Velocity, $\dot{x}$(T) (in/sec) | Fourier spectrum (in/sec) |
|---|---|---|---|---|
| $\ddot{z}(t)$ = C-1 | 12 | -0.942364 | 5.96582 | 8.40801 |
| $\ddot{z}_o(t)$ | 1 | -0.942519 | 5.97092 | 8.40965 |
| $\ddot{z}_5(t)$ | 0.25 | -0.942533 | | 8.40980 |
| $\ddot{z}_6(t)$ | 0.25 | | 5.97104 | |

the computations; the differences are believed to be a result of the different ways round-off error accumulates in the calculations.

A more extreme example of the method is illustrated by Figure 7 and the values in Table II. This second case is like the first except that the period $T_o$ has been changed to 0.1 sec. Because the interval of definition of C-1 is 0.025 sec, $\ddot{z}_o(t)$ is defined by only five points, and the accelerograms $\ddot{z}_5(t)$ and $\ddot{z}_6(t)$ are straight lines defined by only two points. The numbers in Table II again agree within the accuracy of the computations.

The examples show some of the potential advantages of the method for calculating selected Fourier amplitude ordinates, particularly for long records. In the second example the effective reduction is a factor of 240.

The computing time required to calculate ordinates of the Fourier amplitude spectrum by the present method was not studied extensively, but some example calculations were made to determine approximately the time required. For an accelerogram consisting of 1025 points, the execution time for one application of the subroutine is approximately 22 milliseconds on the Caltech Computing Center's IBM 370/155 system. Depending on the value of $T_o$, about three-fourths of the time is spent forming the inputs $\ddot{z}_5(t)$ and $\ddot{z}_6(t)$, with the remainder used for numerical integration. The overall execution time appears to be roughly independent of $T_o$ because of offsetting trends in the forming of the inputs and the numerical integration.

For comparison, the Computing Center's FFT program pro-

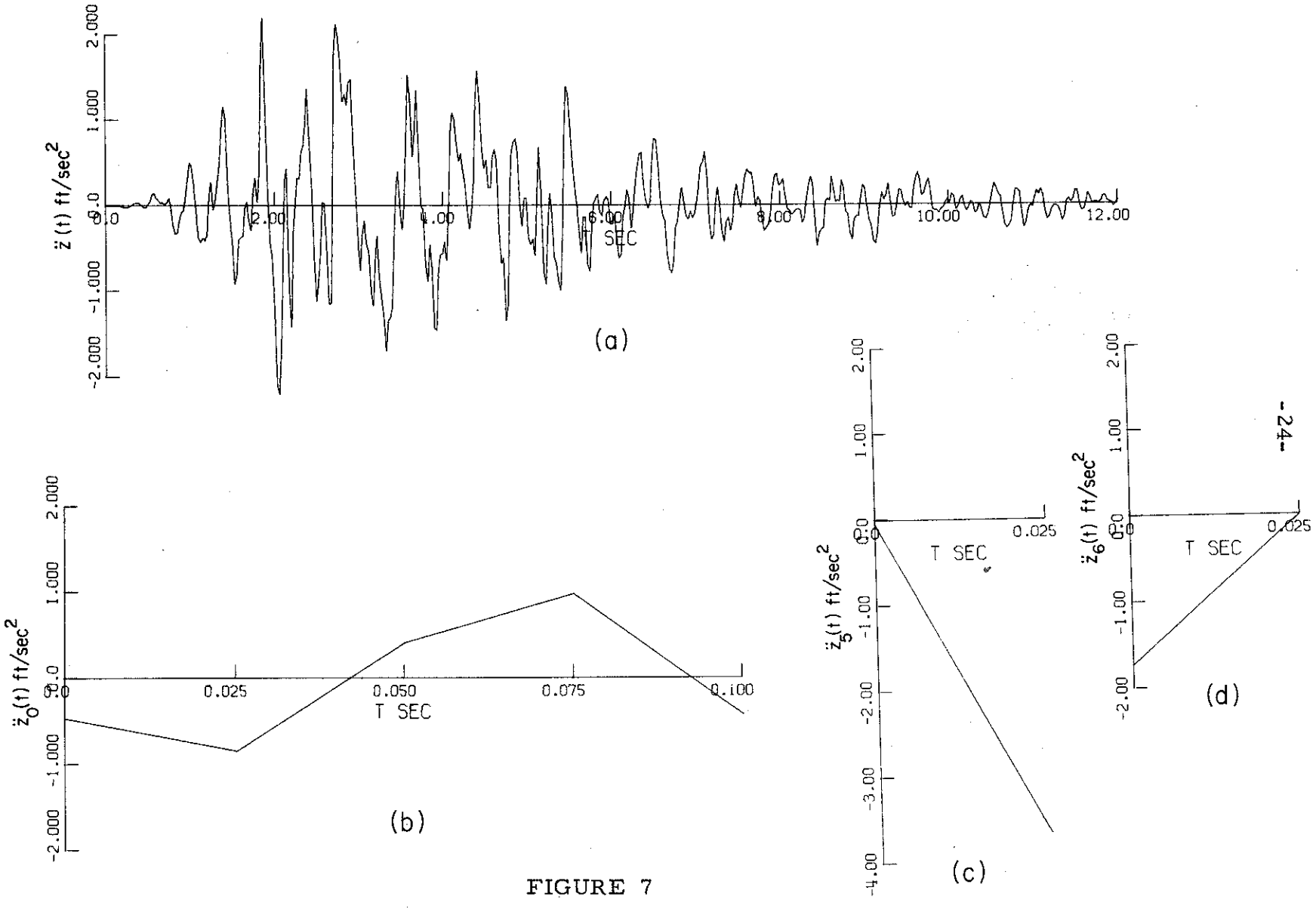
$\ddot{z}$ (t) ft/sec$^2$
2.000
1.000
0.0
-1.000
-2.000
0.0
2.00
4.00
6.00
8.00
10.00
12.00
T SEC
(a)
$\ddot{z}_0$(t) ft/sec$^2$
0.025
0.050
0.075
0.100
T SEC
(b)
$\ddot{z}_5$(t) ft/sec$^2$
1.00
-1.00
-2.00
-3.00
-4.00
(c)
$\ddot{z}_6$(t) ft/sec$^2$
(d)

FIGURE 7

TABLE II

MPARISON OF FOURIER AMPLITUDE SPECTRA AND TERMINAL VALUES OF DISPLACEMENT AND VELOCITY

$T_o = 0.1$ sec

| elerogram | Duration (sec) | Displacement, x(T) (in) | Velocity, $\dot{x}$(T) (in/sec) | Fourier Spectrum ordinate (in/sec) |
|---|---|---|---|---|
| (t) = C-1 | 12 | -0.00712361 | 0.209959 | 0.494927 |
| (t) | 0.10 | -0.00712990 | 0.210385 | 0.494388 |
| (t) | 0.025 | -0.00712995 | | 0.494930 |
| (t) | 0.025 | | 0.210388 | |

duced 512 Fourier amplitude spectrum ordinates from the same 1024 points [$\ddot{z}(1025) = \ddot{z}(0)$] in about 350 milliseconds. Thus, about 16 points at any periods could be found by the present method in the time it would take to calculate the 512 spectrum points at equally spaced frequencies by the FFT. The present method is not intended for such use, but if the entire spectrum were calculated the computing time would take 32 times longer than the FFT. For 1024 points Bergland (1969) states that the FFT is over 200 times faster than direct calculation of the discrete Fourier transforms. This comparison suggests the present subroutine has a computational advantage of about 6 over the direct approach considered by Bergland.

The accuracy of the method was verified by comparison with the well-known exact solution for a truncated sine-wave (e.g., Bendat, 1958, p. 49). In accordance with the way accelerograms are treated, the sine wave was considered to consist of straight line segments between calculated points, taking approximately 250 points per period to insure accurate definition. In all cases tried the results coincided with the analytic solution to better than four significant figures, even for the smallest permissible period, $T_o = 4\Delta t$, when the equivalent quarter-cycle excitations reduce to single straight line segments. This comparison suggests that if the original digitization of an accelerogram is detailed enough that the assumption of piecewise linearity is acceptable, then the proposed calculation method does not introduce significant additional error.

## DISCUSSION AND CONCLUSION

The above development was presented from the point of view of vibration theory, but it should be clear that in a mathematical sense the essential feature of the method arises from the fact that the integrand in the Fourier transform is periodic, which allows superposition of the function to be transformed. This fact is well-known in the field of numerical analysis and there exists a considerable body of literature on the subject of Fourier integrals (see, for example, de Balbine and Franklin, 1966). These mathematical studies have not yet found their way into earthquake engineering, however, and the contribution of the present study consists primarily in applying the advantages of this periodicity to the type of problems treated in earthquake engineering, and in preparing a subroutine to perform the calculations. The periodicity of the integrand is also used to advantage in the FFT (Bergland, 1969).

The calculation method is developed with the intent of calculating a single Fourier spectrum ordinate, but it is not difficult to see that the method could be extended to calculation of ordinates for several periods, if the periods were such that the equivalent inputs could be formed rapidly. For example, the inputs, $\ddot{z}_o(t)$, for $T_o/n$, $(n=2,3,\ldots)$ can be formed from that for $T_o$, without rehandling the basic data. This is illustrated in Figure 8. Beginning with a reduced record $\ddot{z}_o(t)$ of length $T_o$, application of the present method to $\ddot{z}_o(t)$ produces the Fourier amplitude ordinate at a frequency $f_1 = 1/T_o$ (cps). If $\ddot{z}_o(t)$ is then halved and superposed, the Fourier amplitude ordinate for $f_2=2/T_o$

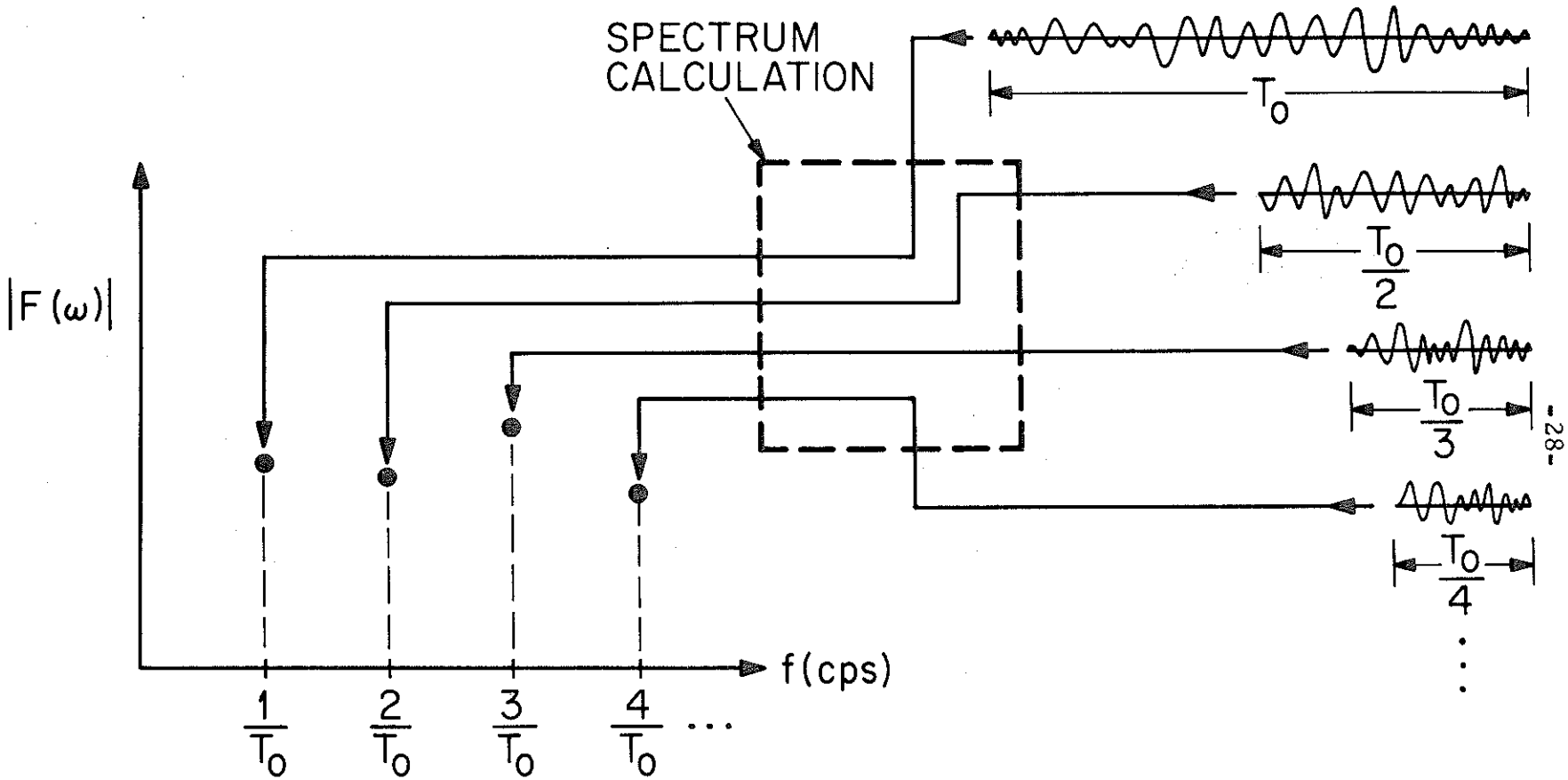


FIGURE 8

Diagram illustrating the use of the method to calculate the same Fourier spectrum ordinates produced by application of the Fast Fourier Transform (FFT)

can be found. Similarly, if $\ddot{z}_o(t)$ is divided into thirds and added, the Fourier ordinate for $f_3 = 3/T_o$ can be determined, etc. If the reduced accelerogram $\ddot{z}_o(t)$ is of duration T, i.e., one begins with the entire record, the results of such a procedure will be to produce Fourier amplitude spectrum ordinates at frequencies 1/T, 2/T, . . . , the same results produced by application of the FFT. Moreover, it is easy to see that this calculation could be made much more efficient than merely the repeated application of the subroutine in the Appendix. This relation between the present approach and the FFT can be used to guide the use of the FFT in problems in earthquake engineering. For instance, an overly long record could be superposed to form a record of convenient length, $T_o$, as indicated herein, and the FFT applied to the resulting reduced time series.

The calculation of Fourier spectrum ordinates is relatively common in earthquake engineering research and it is hoped that the method presented herein may have a number of uses. One possible application appears to be in the analysis of data from ambient vibration tests, where relatively narrow frequency bands may need to be studied, using relatively long records. In this case, the method might be used to investigate resonance peaks in detail, after the general nature of the Fourier spectra had been established by use of the FFT on a relatively short record. This application illustrates the two primary advantages of the computing method: the capability of selecting a small number of arbitrary frequencies for analysis, and the ability to handle long records with little difficulty.

1/50 of a second is a representative interval of definition for strong-motion accelerograms, depending on the characteristics of the motion and the recording instrument (Hudson, et al., 1971). The smallest period that can be used directly by the present method on such accelerograms would be $T_o = 4/50$ sec ($f = 12.5$ cps). To calculate Fourier spectra at higher frequencies with the present subroutine would require redefinition of the accelerogram at smaller intervals. If this is done by simple interpolation, inaccurate values of spectrum ordinates will be obtained for sufficiently high frequencies, except in the unusual case in which the accelerogram is exactly a piecewise-linear function between the original points of digitization.

## ACKNOWLEDGEMENTS

The subroutine in the Appendix was developed primarily by Mr. Kwak Kam Lo, a student at the California Institute of Technology. The author is grateful to Mr. Lo for his many thoughtful contributions to this study.

The study was supported in part by a grant from the National Science Foundation.

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# APPENDIX

The following two pages are a listing of a sample subroutine package which performs the calculations discussed in the text. The subroutine package consists of two parts: The main routine KKL1 performs the basic operations while the subsidiary program KKL2 evaluates the two 2 x 2 matrices needed for the numerical integration. The subroutines were written in Fortran IV for use on the California Institute of Technology's IBM 370/155 system. As assembled in Fortran IV G, the programs use about 45,000 units of core.

Although care has been taken to make the programs error-free and easy to follow, the sample subroutines are presented mainly to illustrate the type of programs that can be developed to implement the calculation method discussed in this report. They may have to be modified to meet specific needs of potential users.

```
C     ****************************************************************
C
C          SUBROUTINE FOR COMPUTATION OF END DISPLACEMENT,VELOCITY AND
C           MODULUS OF FOURIER SPECTRUM OF ACCELEROGRAMS DIGITIZED AT
C                              EQUAL INTERVALS
C                                   KKL1
C
C     ****************************************************************
C
C     *******DEFINITIONS*******
C
C     AC1 IS THE ARRAY OF ACCELERATION ORDINATES
C     N1 IS THE NUMBER OF ACCELERATION ORDINATES
C     T IS THE DURATION OF THE EARTHQUAKE
C     TN IS THE PERIOD
C     DEL IS THE INTERVAL OF DIGITIZATION
C     SF IS THE SCALE FACTOR TO PUT ACCELERATION ORDINATES INTO DESIRED UNITS
C     DISP IS THE END DISPLACEMENT,VEL END VELOCITY
C     FOS IS THE FOURIER SPECTRUM
C     N3 IS THE NO. OF POINTS IN A PERIOD,INCLUDING THE END POINTS
C     M3 IS THE NO. OF POINTS IN A FOURTH OF A PERIOD,INCLUDING END POINTS
C     ZO IS THE ARRAY FOR EQUIVALENT EXCITATION OF ONE SECOND DURATION
C     GA IS THE ARRAY FOR TWO,QUARTER-PERIOD EQUIVALENT EXCITATIONS
C     W IS THE FREQUENCY
C
C     *****BEGINNING OF PROGRAM*****
C
      SUBROUTINE KKL1(AC1,N1,T,TN,DEL,SF,DISP,VEL,FOS)
      DIMENSION AC1(1500),Y1(2),Y2(2),X(2),G(2),TY(2),GA(2,101),ZO(400),
     1A(2,2),B(2,2)
      S1=TN/(4.*DEL)
      IF((S1-INT(S1+.001)).GT.1.E-03) GO TO 99
      N3=4.*S1+1.1
      M3=(N3-1)/4.+1.1
      M=T/TN+0.001
C
C     *****COMPUTATION OF ZO"(T)****
C
      K=N3-1
      N2=N1-M*K
      L=N3-N2
      K1=N1-1
      L1=L+1
      IF(N2.EQ.1) GO TO 12
      DO 10 I=1,L
      ZO(I)=0.0
      KI=N2+I-1
      DO 10 J=KI,K1,K
      ZO(I)=ZO(I)+AC1(J)
   10 CONTINUE
      DO 11 I=L1,N3
      ZO(I)=0.0
      DO 11 J=1,N1,K
      ZO(I)=ZO(I)+AC1(J+I-L-1)
   11 CONTINUE
      GO TO 13
   12 DO 13 I=1,N3
      ZO(I)=0.0
      DO 13 J=1,K1,K
      ZO(I)=ZO(I)+AC1(J+I-1)
   13 CONTINUE
```

FIGURE A-1

Listing of Fortran IV subroutine package.

```
C       *****COMPUTATION OF Z5"(T) AND Z6"(T)*****
C
0031          DO 60 I=1,M3
C             *******Z5"(T)*******
0032          GA(1,I)=ZC(I)+ZO(2*M3-I)-ZO(2*M3-2+I)-ZO(4*M3-2-I)
C             *******Z6"(T)*******
0033          GA(2,I)=ZO(I)-ZO(2*M3-I)-ZO(2*M3-2+I)+ZC(4*M3-2-I)
0034       60 CONTINUE
0035          W=2.*3.1415927/TN
0036          W2=W**2
0037          CALL KKL2(W,DEL,A,B)
C
C       *****COMPUTATION OF RESPONSE*****
C
0038          DO 111 I2=1,2
0039          X(1)=0.0
0040          X(2)=0.0
0041          I=1
0042        7 G(1)=GA(I2,I)*SF
0043          G(2)=GA(I2,I+1)*SF
0044          TY(1)=A(1,1)*X(1)+A(1,2)*X(2)+B(1,1)*G(1)+B(1,2)*G(2)
0045          TY(2)=A(2,1)*X(1)+A(2,2)*X(2)+B(2,1)*G(1)+B(2,2)*G(2)
0046          X(1)=TY(1)
0047          X(2)=TY(2)
0048          I=I+1
0049          IF (I.EQ.M3) GO TO 18
0050          GO TO 7
0051       18 Y1(I2)=-X(2)/W
0052          Y2(I2)=W*X(1)
0053      111 CONTINUE
0054          FCS=SQRT(Y2(2)**2+W2*Y1(1)**2)
0055          DISP=Y1(1)
0056          VEL=Y2(2)
0057          RETURN
0058       99 WRITE(6,567)
0059      567 FORMAT(5X,66HQUARTER-PERIOD NOT AN INTEGER MULTIPLE OF INTERVAL OF
             1 DIGITIZATION)
0060          END

C       ****************************************************************
C
C          SUBROUTINE FOR COMPUTATION OF MATRICES A AND B
C                        FOR AN UNDAMPED OSCILLATOR
C                                  KKL2
C
C       ****************************************************************
C
C
0001          SUBROUTINE KKL2(W,DEL,A,B)
0002          DIMENSION A(2,2),B(2,2)
0003          A1=W*DEL
0004          A2=SIN(A1)
0005          A3=COS(A1)
0006          W2=W**2
0007          A4=-1./W2
0008          A5=1./A1
0009          A6=A4*A3
0010          A7=A4*A5*A2
0011          A8=(1./W)*A2
0012          A(1,1)=A3
0013          A(1,2)=A8
0014          A(2,1)=-W*A2
0015          A(2,2)=A3
0016          B(1,1)=A7-A6
0017          B(1,2)=-A7+A4
0018          B(2,1)=(A6-A4)/DEL-A8
0019          B(2,2)=(-A6+A4)/DEL
0020          RETURN
0021          END
```

FIGURE A-1 (contd.)