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# PIECEWISE LINEAR DYNAMIC SYSTEMS WITH TIME DELAYS

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## ABSTRACT

A new method of constructing periodic solutions for piecewise linear dynamic systems with time delays is investigated. Although the existence and the uniqueness of the periodic solution are guaranteed by well-known theorems, existing schemes for actually constructing the periodic solution are either purely formal or approximate.

The idea of constructing the exact solution is first pursued with the linear delay systems. The formal representation of the solution to the linear problem is viewed as a system of Fredholm integral equations of the second kind. Since the matrix kernel for this system of integral equations is separable, the integral equation can be reduced to a system of algebraic equations involving certain integral moments of the initial function. These observations lead to a transfer relationship between two state vectors in the form of a matrix equation. Then the problem can be posed as either an initial value problem (if one is seeking the transient solution), or a periodic solution problem (if one is seeking the unknown initial data).

This Fredholm Integral Equation Method is used effectively to construct periodic solutions to piecewise linear differential-difference equations. The periodic solutions are constructed from a cascaded product of matrix equations derived for each linear region. The stability of the periodic solution is determined by solving an associated eigenvalue problem. The periodic solution and its stability analysis are exact in the sense that the error induced by the truncation process

in the Fredholm Integral Equation Method can be made exponentially small as the size of the transfer matrix is increased.

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# Chapter I

# INTRODUCTION

The exact science of describing and predicting physical phenomena has always been a challenging area of research. Solutions to differential equations play a central role in describing phenomena in the present state, and also determining the subsequent behavior.

Investigation of dynamic systems with time delays is an attempt to describe physical phenomena more accurately, whenever the state of the system depends not only on the present, but also on the past. This constitutes a class of history dependent systems, and is known in the literature as 'delay systems', 'time lag systems', 'oscillations caused by retarded actions', or 'equations with deviating arguments'. The subject introduces a new field of mathematics, namely, functional-differential equation of the form

$$\frac{d\underline{x}(t)}{dt} = \underline{f}(t, \underline{x}(t), \underline{x}(t-\tau_1(t)), \underline{x}(t-\tau_2(t)), \cdots). \qquad (1.1)$$

The functional-differential equation becomes a differential-difference equation when the delays in the arguments, $\tau_1(t), \tau_2(t), \cdots$, are all constants. The differential equation (1.1) is categorized into three types. Equation (1.1) is called a retarded type, if the highest-order derivative of the unknown function appears for just one value of the argument, and this argument is not less than all other arguments. Equation (1.1) is called an advanced type if the highest-order derivative of the unknown function appears for just one value of the argument, and this argument is not larger than the remaining arguments. All other differential equations (1.1) with deviating arguments are called neutral types.

Historically, the eighteenth-century mathematicians, Euler, Laplace, and Bernoulli[1] first formulated the differential-difference equations. These equations arose when they tried to extend the knowledge of the discrete particle mechanics to the continuum mechanics, which later came to be studied in terms of partial differential equations. Practically nothing was done in the field throughout the nineteenth-century, and only after the First World War did the study of these equations gain momentum principally due to problems which arose in the field of automatic controls. It is observed that whenever a servomechanism is constructed in a closed loop form, the feedback signal is always delayed by a finite amount of time. This time lag is due to the transmission period, as well as the observation and guidance period of the signal.

In the last twenty years the area of application of the delay equations has greatly expanded, and now it encompasses not only many questions of physics and engineering, but also certain areas of econometrics and biological science. There exist numerous examples of time delay phenomena in practice, primarily in the area of feedback control systems. To name a few, the time lag in a control system was first formulated by Callender, Hartree and Porter[2] in 1936. Tsien and Crocco[3] considered the time delay occurring in the problem of rocket fuel combustion, and the associated stability problems. They found the system is governed by a first-order linear differential-difference equation of retarded type. Minorsky[4] described the problem of antirolling stabilization of ship motion by a second-order linear differential-difference equation. Continuously stirred tank reactors in chemical engineering are known to be describable in terms of coupled nonlinear differential-difference equations, and these were solved numerically by Seinfeld, Gavalas and Hwang[5].

The delays occurring in differential equations do not need to be limited to physical time lags. Corngold and Yan[6] considered the problem of inelastic scattering of neutrons in neutron slowing down theory, and found the governing equation to be a first-order linear partial differential-difference equation. The delays of the argument represent the finite energy levels associated with a nucleus. Also, there are many complicated physical problems described by partial differential equations posed as initial-boundary value problems. These equations can occasionally be transformed to much simpler coupled differential-difference equations posed as initial value problems.

In the past twenty years, mathematical theories have grown significantly in connection with the growth of physical applications. The first systematic investigation of differential-difference equations started in the early 1950's, resulting in the first series of comprehensive books by Pinney[7], Bellman and Cooke[8], El'sgol'ts[9] and Rubanik[10].

The fundamental theorems of existence and uniqueness for nonlinear differential-difference equations of retarded type were proved by extending analogous theorems from the theory of ordinary differential equations. Using topological methods, Hale[11], Halanay[12], Driver[13] and others extended these theorems to the more general class of hereditary dynamic systems, functional-differential equations. Jones[14] showed the existence of the periodic solution for a class of functional-differential equations using the fixed point theorems. Although most of the delay problems arising in practice are describable by functional-differential equations of retarded type, recent theoretical contributions for the neutral type are made by Cruz and Hale[15], Cooke[16] and others. Stability of solutions for linear differential-difference equation of retarded type was first discussed by Ansoff[17] in 1949 using the Nyquist diagrams, and later extended by Hsu and Bhatt[18]. Approximate solutions for quasi-linear dynamic systems are constructed by Rubanik[19], and other Russian authors, using the method of averaging and the general asymptotic method.

Among the classes of hereditary systems, we shall restrict our attention to nonlinear differential-difference equations of retarded type. There exist a number of basic theorems for the existence, uniqueness, and asymptotic behavior of the solutions. The existing schemes for constructing solutions are:

1) Step-by-step marching procedure, or a numerical integration process. This approach does not exhibit the qualitative aspects of the solution, and stability criteria cannot be obtained.
2) General asymptotic method, which yields the approximate steady state solutions, as well as

their stability. However, the validity of this method is crucially depended on the small parameters assumption.

Thus the present scope of this investigation is to develop a new method of constructing the exact solution to an important class of nonlinear differential-difference equations with constant parameters, namely, piecewise linear delay systems. The piecewise linear system is chosen not only for the mathematical simplicity, but also for the fact that many nonlinearities can be closely approximated by piecewise linear models. This study is designed to provide the bridge between mathematical theories and practical applications. The aim is to supply an analytical tool to construct solutions and establish stability criteria, such that a parameter study may be performed.

The principal difficulty of analyzing differential-difference equations arises in the linear problem itself. As will be shown in Chapter III, the characteristic equation is of a special transcendental character. The linear problem always leads to an infinite spectrum of frequencies with which a dynamic system can oscillate. The determination of this spectrum requires a corresponding determination of the roots of certain analytic functions. In order to find the exact solution, one must deal with an infinite series of residue contributions. A new method of constructing the exact solution is derived in this investigation using the knowledge of Fredholm integral equations.

Structure of the Thesis

Chapter II begins with the mathematical preliminaries for general retarded differential-difference equations. The basic issues of existence and unicity of solution, and the stability in the sense of Liapunov-Poincaré, are briefly discussed.

Starting with Chapter III, we restrict the attention to delay systems with constant coefficients and constant delays. Formulation of the problem is done in vector-matrix form such that higher-order systems and multidegree systems may be treated similarly. Chapter III contains a discussion of the roots of characteristic exponential polynomials, which determine the solution to homogeneous linear differential-difference equations. Asymptotic root distributions as well as exact locations of the roots are discussed, and the stability criteria are established for the linear system.

Knowing the nature of the root distributions, Chapter IV formulates the exact periodic solution for forced linear delay systems. A new method of constructing the solution is derived using a Fredholm integral equation theory. This method is subsequently called the Fredholm Integral Equation Method. The crucial step of truncating the kernel in this method is rigorously justified by an error bound analysis.

Knowledge of Chapter IV becomes the basic element of the discussions in Chapter V, when forced piecewise linear delay systems are considered. Periodic solutions of piecewise linear differential-difference equations are constructed by the Fredholm Integral Equation

Method, and the stability of the solutions is determined. For the case of a doubly bilinear delay system, triple solutions are found for a certain range of the frequency parameter, and their stability characteristics are obtained. It is observed that the results of this delay system closely resemble the solutions of a damped Duffing's oscillator. A comparison study between the solutions given by the Fredholm Integral Equation Method and the conventional approximate scheme (the method of slowly varying parameters) is presented.

# Chapter II

# GENERAL DISCUSSION OF RETARDED DIFFERENTIAL-DIFFERENCE EQUATIONS

This chapter is intended to serve as mathematical preliminaries of well-posed differential-difference equations of retarded type which will be discussed in the subsequent chapters. The basic issues of existence and uniqueness of solution as well as stability are briefly discussed here.

## 2.1 Existence and Uniqueness of Initial Value Problem Solutions

History dependent time delay dynamic systems are in general governed by retarded type differential-difference equation

$$\frac{d\underline{x}(t)}{dt} = \underline{f}(t, \underline{x}(t), \underline{x}(t-\tau)) \tag{2.1}$$

with initial function

$$\underline{x}(t) = \underline{g}(t) \quad , \quad \text{for } 0 \le t \le \tau$$

where the delay term $\tau$ is assumed to be positive constant. We have posed Equation (2.1) with single delay term, but the effect of multiple delays brings no essential difficulty.

The basic question of existence and uniqueness of solution for the nonlinear delay system (2.1) can be answered in a similar way as for a nonlinear ordinary differential equation. In fact, Equation (2.1) is reduced to an ordinary differential equation if one wishes to solve it by the method of steps (marching), that is

$$\underline{x}(t) = \underline{g}(t) \quad , \quad 0 \le t \le \tau$$

$$\dot{\underline{x}}(t) = \underline{f}(t, \underline{x}(t), \underline{g}(t)) \quad , \quad \tau \le t \le 2\tau$$

with initial condition

$$\underline{x}(\tau) = \underline{g}(\tau)$$

and this process can be repeated for every interval of $\tau$.

Theorem 2.1 (Existence and uniqueness proof)*

Suppose that $\underline{g}(t)$ is continuous and bounded for $0 \le t \le \tau$, $\|\underline{g}(t)\| \le m_g$, and $\underline{f}(t, \underline{x}(t), \underline{x}(t-\tau))$ satisfies a Lipschitz condition

$$\begin{gathered} \|\underline{f}(t, \underline{u}_1, \underline{v}_1) - \underline{f}(t, \underline{u}_2, \underline{v}_2)\| \le m(t)\,(\|\underline{u}_1 - \underline{u}_2\| + \|\underline{v}_1 - \underline{v}_2\|) \\ \underline{u} = \underline{x}(t) \quad , \quad \underline{v} = \underline{x}(t-\tau) \end{gathered} \tag{2.2}$$

for the solutions $(\underline{u}_1, \underline{v}_1)$, $(\underline{u}_2, \underline{v}_2)$ in a Region R

$$R: \ \|\underline{u}\| + \|\underline{v}\| \le c_1.$$

Let $c_2$ denote the maximum of the continuous function $\|\underline{f}(t, \underline{u}, \underline{v})\|$ for $(\underline{u}, \underline{v})$ in the Region R. Then if $2m_g < c_1$, there exists a unique solution $\underline{x}(t)$ of Equation (2.1) for $0 \le t \le \tau + c_3$, where $c_3 < (c_1 - 2m_g)/2c_2$.

Theorem 2.1 is in essence a local existence and uniqueness theorem. That is, it guarantees a unique solution only over a small interval in the neighborhood of the initial interval due to general nonlinearity assumption. However, for a special class of nonlinearity,

*Proof of this theorem is given in Bellman and Cooke[8], p. 341.

namely, $\underline{f}(t, \underline{x}(t), \underline{x}(t-\tau))$ is piecewise linear in $\underline{x}$, then the above Theorem 2.1 reduces to a global existence and uniqueness theorem since the Lipschitz condition (2.2) is satisfied everywhere in $(\underline{u}, \underline{v})$ space.

## 2.2 Stability of Solutions

In the construction of a differential equation which describes some physical phenomenon it is always necessary to simplify or idealize the phenomenon. In real process, the initial data are usually the result of direct measurements, and therefore, are unavoidably defined with some degree of inexactness.

Under these conditions, in order for the differential equation to describe even approximately the phenomenon under investigation, it is necessary that a small change in the initial function produce only a small change in the solution. Stability theory investigates the conditions under which small changes in the differential equation itself and small changes in the initial function lead to also small changes in the solution.

First we shall state the definitions of stability for the solutions of differential-difference equation, which are the exact analogy of ordinary differential equation cases.

**Definition 2.1** (Stability)

A solution $\underline{x}(t)$ of Equation (2.1) is said to be stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that the inequality $\|\underline{g}_1(t) - \underline{g}_2(t)\| < \delta(\epsilon)$ on the initial function implies that $\|\underline{x}_1(t) - \underline{x}_2(t)\| < \epsilon$ for $t \geq \tau$, where $\underline{x}_i(t)$ is the solution corresponding to $\underline{g}_i(t)$ for $i = 1, 2$.

Definition 2.2 (Asymptotic Stability)

A stable solution $\underline{x}(t)$ of Equation (2.1) is called asymptotically stable if $\lim_{t\to\infty} \|\underline{x}_1(t)-\underline{x}_2(t)\|=0$ for any continuous initial function $\underline{g}(t)$ satisfying $\|\underline{g}_1(t)-\underline{g}_2(t)\|<\delta$ for sufficiently small $\delta$.

Definition 2.3 (Exponential Asymptotic Stability)

A solution $\underline{x}(t)$ of Equation (2.1) is called exponentially asymptotically stable if there exist positive constants $\delta$, $\alpha$ and B such that the inequality $\|\underline{g}_1(t)-\underline{g}_2(t)\|<\delta$ implies $\|\underline{x}_1(t)-\underline{x}_2(t)\|<B\cdot\max_{0\le t\le\tau}\|\underline{g}_1(t)-\underline{g}_2(t)\|e^{-\alpha(t-\tau)}$ for $t>\tau$.

With above definitions, we shall construct the Liapunov-Poincaré stability theorem which extends to differential-difference equations.

Theorem 2.2 (Liapunov-Poincaré Stability Theorem)

Suppose that

(i) every continuous solution of linear differential-difference equation

$$\frac{d\underline{y}(t)}{dt}=A\,\underline{y}(t)+B\,\underline{y}(t-\tau)$$
$$\underline{y}(t)=\underline{g}(t) \quad \text{for} \quad 0\le t\le\tau \tag{2.3}$$

with A, B constant square matrices and $\tau$ positive constant, approaches zero as $t\to\infty$, i.e., all characteristic roots of $\det(zI-A-Be^{-\tau z})=0$ possess negative real parts;

(ii) $\underline{f}^*(t,\underline{y}(t),\underline{y}(t-\tau))$ is a continuous function of $\underline{y}(t)$ and $\underline{y}(t-\tau)$ in a neighborhood of the origin $\|\underline{y}(t)\|+\|\underline{y}(t-\tau)\|\le c_1$; and

(iii) $\underline{f}^*$ satisfies the nonlinearity condition, i.e.,

$$\lim_{\|\underline{y}(t)\| + \|\underline{y}(t-\tau)\| \to 0} \frac{\|\underline{f}^*(t, \underline{y}(t), \underline{y}(t-\tau))\|}{\|\underline{y}(t)\| + \|\underline{y}(t-\tau)\|} = 0. \tag{2.4}$$

Then any solution of the nonlinear differential-difference equation

$$\frac{d\underline{y}(t)}{dt} = A\,\underline{y}(t) + B\,\underline{y}(t-\tau) + \underline{f}^*(t, \underline{y}(t), \underline{y}(t-\tau)) \tag{2.5}$$

with initial function

$$\underline{y}(t) = \underline{g}(t) \quad , \quad 0 \le t \le \tau$$

is also asymptotically stable, i. e. $\lim_{t\to\infty} \|\underline{y}(t)\| = 0$, provided $m_g = \max_{0 \le t \le \tau} \|\underline{g}(t)\|$ is sufficiently small.

Proof

Let the solution of Equation (2.3) be denoted as $\underline{y}_0(t)$, then from the formal solution representation developed in Chapter IV,

$$\underline{y}_0(t) = K(t-\tau)\underline{g}(\tau) + \int_0^{\tau} K(t-\tau-\xi) B\, \underline{g}(\xi)\, d\xi \quad , \quad t > \tau$$

with

$$\|K(t)\| \le m_1 e^{\sigma_1 t} \quad , \quad \sigma_1 < 0 \quad , \quad 0 < m_1 < \infty$$

from the hypothesis (i) and

$$\|\underline{g}(t)\| \le m_g \quad , \quad 0 \le t \le \tau,$$

$$\|\underline{y}_0(t)\| \le m_2 e^{\sigma_1 t} \tag{2.6}$$

where

$$m_2 = m_1 m_g e^{-\sigma_1 \tau}\left[1+\|B\|\frac{1-e^{-\sigma_1\tau}}{\sigma_1}\right] < \infty$$

$$\therefore \qquad \|y_0(t-\tau)\| \le m_2 e^{\sigma_1(t-\tau)}. \tag{2.7}$$

And the solution to the full nonlinear equation (2.5) can be written as

$$\underline{y}(t) = \underline{y}_0(t) + \int_\tau^t K(t-\xi)\underline{f}^*(\xi, \underline{y}(\xi), \underline{y}(\xi-\tau))d\xi$$

$$\|\underline{y}(t)\| \le m_2 e^{\sigma_1 t} + m_1 e^{\sigma_1 t}\int_\tau^t e^{-\sigma_1\xi}\|\underline{f}^*(\xi, \underline{y}(\xi), \underline{y}(\xi-\tau))\| d\xi$$

but from the hypothesis (iii),

$$\|\underline{f}^*(t, \underline{y}(t), \underline{y}(t-\tau))\| \le m_3\{\|\underline{y}(t)\| + \|\underline{y}(t-\tau)\|\}\ , \quad m_3 > 0$$

$$\therefore \qquad \|\underline{y}(t)\| e^{-\sigma_1 t} \le m_2 + m_1 m_3 \int_\tau^t e^{-\sigma_1\xi}\{\|\underline{y}(\xi)\| + \|\underline{y}(\xi-\tau)\|\}\, d\xi$$

and

$$\|\underline{y}(t-\tau)\| e^{-\sigma_1 t} \le e^{-\sigma_1\tau}\left[m_2 + m_1 m_3 \int_\tau^t e^{-\sigma_1\xi}\{\|\underline{y}(\xi)\| + \|\underline{y}(\xi-\tau)\|\}d\xi\right].$$

Add the above two inequalities,

$$e^{-\sigma_1 t}\{\|\underline{y}(t)\| + \|\underline{y}(t-\tau)\|\} \le (1+e^{-\sigma_1\tau})\left[m_2 + m_1 m_3\int_\tau^t e^{-\sigma_1\xi}\{\|\underline{y}(\xi)\| + \|\underline{y}(\xi-\tau)\|\}d\xi\right] \tag{2.8}$$

and applying the Gronwall's lemma to the inequality (2.8),

$$
\begin{aligned}
&e^{-\sigma_1 t}\{\|\underline{y}(t)\|+\|\underline{y}(t-\tau)\|\} \le (1+e^{-\sigma_1\tau})m_2 \exp\{m_1 m_3(t-\tau)\} \\
\therefore \quad &\|\underline{y}(t)\|+\|\underline{y}(t-\tau)\| \le m_4 \exp\{(m_1 m_3+\sigma_1)t\}
\end{aligned}
\tag{2.9}
$$

with

$$m_4 = (1+e^{-\sigma_1\tau})m_2 e^{-m_1 m_3 \tau} < \infty.$$

Thus the solution to the nonlinear delay system (2.5) is asymptotically stable, i. e.,

$$\lim_{t\to\infty} \|\underline{y}(t)\|=0$$

provided

$$m_3 < -\frac{\sigma_1}{m_1} \;, \quad \sigma_1 < 0$$

and this is guaranteed for $m_g$ sufficiently small.

This stability theorem will be used extensively later in Chapter V when we discuss the stability of the periodic solutions of a nonlinear differential-difference equation (2.1): i. e., let

$$\underline{u} = \underline{x}(t) \quad , \quad \underline{v} = \underline{x}(t-\tau)$$

and consider the stability of the solution $\underline{x}$ in the neighborhood of the known periodic solution $\underline{x}^*$,

$$\underline{x}(t) = \underline{x}^*(t) + \underline{\eta}(t) \tag{2.10}$$

and substitute Equation (2.10) to (2.1), then $\underline{\eta}(t)$ satisfies

$$\frac{d\underline{\eta}(t)}{dt} = J_1\,\underline{\eta}(t) + J_2\underline{\eta}(t-\tau) + \underline{f}^*(t, \underline{\eta}(t), \underline{\eta}(t-\tau)) \tag{2.11}$$

where $J_1$, $J_2$ are the Jacobian matrices

$$J_1 = \frac{\partial \underline{f}}{\partial \underline{u}}\Bigg|_{\underline{u}=\underline{u}^*,\ \underline{v}=\underline{v}^*}$$

$$J_2 = \frac{\partial \underline{f}}{\partial \underline{v}}\Bigg|_{\underline{u}=\underline{u}^*,\ \underline{v}=\underline{v}^*}$$

and

$$\underline{f}^*(t, \underline{\eta}(t), \underline{\eta}(t-\tau)) = \begin{pmatrix} f_1^* \\ f_2^* \\ \vdots \\ f_n^* \end{pmatrix}$$

with

$$f_i^* = \frac{1}{2} \frac{\partial^2 f_i}{\partial u_j \partial u_k}\Bigg|_{\substack{\underline{u}=\underline{u}^* \\ \underline{v}=\underline{v}^*}} \cdot \eta_j(t)\eta_k(t) + \frac{\partial^2 f_i}{\partial u_j \partial v_k}\Bigg|_{\substack{\underline{u}=\underline{u}^* \\ \underline{v}=\underline{v}^*}} \cdot \eta_j(t)\eta_k(t-\tau)$$

$$+ \frac{1}{2} \frac{\partial^2 f_i}{\partial v_j \partial v_k}\Bigg|_{\substack{\underline{u}=\underline{u}^* \\ \underline{v}=\underline{v}^*}} \cdot \eta_j(t-\tau)\eta_k(t-\tau) + O(r^3) \quad , \quad i = 1, 2, \cdots, n$$

where the repeated index notation for higher order tensor is adopted here.

Then $\underline{f}^*$ is guaranteed to satisfy the nonlinearity condition (2.4), thus the stability of $\underline{x}^*(t)$ is completely determined by the linearized variational equation

$$\frac{d\underline{\eta}(t)}{dt} = J_1\underline{\eta}(t) + J_2\underline{\eta}(t-\tau). \tag{2.12}$$

Using the Theorem 2.2, if every solution of Equation (2.12) approaches zero as $t \to \infty$, then the periodic solution $\underline{x}^*(t)$ is asymptotically stable in the sense of Liapunov-Poincaré.

There exists a series of stability theorems which yield sufficient conditions for asymptotic stability of solutions of linear differential-difference equations (constant or variable coefficients), and when non-delayed term is governed by periodic coefficients, we have the following theorem.

Theorem 2.3 Given a linear differential-difference equation with periodic coefficients

$$\frac{d\underline{x}(t)}{dt} = P(t)\underline{x}(t) + B\,\underline{x}(t-\tau) \tag{2.13}$$

with initial function

$$\underline{x}(t) = \underline{g}(t) \quad \text{for} \quad 0 \le t \le \tau,$$

suppose (i) $P(t) = P(t+T)$, periodic matrix,

(ii) B constant matrix, $\tau$ positive constant,

(iii) $\underline{g}(t)$ is continuous and bounded, i.e., $\max\limits_{0 \le t \le \tau} \|\underline{g}(t)\| = m_g$

and (iv) all solutions of $\frac{dX(t)}{dt} = P(t)X(t)$, $t > \tau$ are asymptotically stable,

then also are the solutions of Equation (2.13), provided $\tau$ is sufficiently small.

Proof

First, convert Equation (2.13) into an integral equation

$$\underline{x}(t) = X(t)\underline{g}(\tau) + \int_{\tau}^{t} X(t)X^{-1}(\xi)B\,\underline{x}(\xi-\tau)\,d\xi \;,\quad t > \tau$$

and take norms of both sides,

$$\|\underline{x}(t)\| \le \|X(t)\|\cdot\|\underline{g}(\tau)\| + \int_{\tau}^{t} \|X(t)\|\cdot\|X^{-1}(\xi)\|\cdot\|B\|\cdot\|\underline{x}(\xi-\tau)\|\,d\xi \quad (2.14)$$

from hypothesis (iv),

$$\|X(t)\| \le m_1 e^{-\alpha t} \;,\quad m_1,\ \alpha \text{ positive constants}$$

and

$$|X(t)| = \exp\left\{\int_{\tau}^{t} \mathrm{Tr}\,P(\xi)d\xi\right\} \ne 0,$$

$\therefore$ $X(t)$ is non-singular, and $\|X^{-1}(\xi)\| \le m_2 e^{\alpha\xi}$ , $m_2 > 0$.

Then Equation (2.14) becomes

$$\|\underline{x}(t)\| \le m_1 m_g e^{-\alpha t} + m_3 e^{-\alpha t}\int_{\tau}^{t} e^{\alpha\xi}\|\underline{x}(\xi-\tau)\|\,d\xi$$

$$m_3 = m_1 m_2 \|B\|.$$

Let $\eta = \xi - \tau$, then

$$e^{\alpha t}\|\underline{x}(t)\| \le m_1 m_g + m_3 e^{\alpha\tau}\int_{0}^{t} e^{\alpha\eta}\|\underline{x}(\eta)\|\,d\eta. \quad (2.15)$$

Apply the Gronwall's lemma to the inequality (2.15),

$$\|\underline{x}(t)\| \le m_1 m_g \exp\{(m_3 e^{\alpha\tau} - \alpha)t\}. \quad (2.16)$$

Thus

$$\lim_{t\to\infty} \|\underline{x}(t)\| = 0$$

provided

$$m_3 e^{\alpha\tau} < \alpha$$

or

$$\tau < \frac{1}{\alpha} \ln\left(\frac{\alpha}{m_3}\right) \tag{2.17}$$

and since $\tau$ must be positive,

$$\alpha > m_3 = m_1 m_2 \|B\|$$

then the solutions of Equation (2.13) are asymptotically stable.

# Chapter III

# FREE LINEAR DYNAMIC SYSTEMS WITH TIME DELAYS

## 3.1 Problem Formulation

As an opening chapter to constructive discussions of dynamic systems with time delays, this chapter is devoted to a homogeneous retarded type linear differential-difference equation with constant coefficients and constant delays.

This system can in general be written as

$$\frac{d\underline{x}(t)}{dt} = \sum_{i=0}^{m} A_i \underline{x}(t-\tau_i) \qquad (3.1a)$$

$$0=\tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m \qquad (3.1b)$$

with initial function

$$\underline{x}(t) = \underline{g}(t) \quad \text{for} \quad 0 \le t \le \tau_m. \qquad (3.1c)$$

Equation (3.1) represents a system of first order differential-difference equations in vector-matrix form with the following assumptions:

Assumption 1 (A1): Square matrices $A_i$'s are real constant matrices.
Assumption 2 (A2): Delays $\tau_i$'s are real, distinct, positive constants.
Assumption 3 (A3): Initial function $\underline{g}(t)$ is continuous and bounded.

Then it immediately follows that the retarded type linear differential-difference equation (3.1) coupled with the Assumptions A1, A2, and A3 must possess unique solution, since it satisfies all the requirements for existence and uniqueness of solution as discussed in Chapter II.

The linear equation (3.1) admits solutions of the form

$$\underline{x}(t) = e^{zt}\underline{c} \tag{3.2}$$

where z is complex constant.

Substituting (3.2) to (3.1) immediately yields

$$\left[zI - \sum_{j=0}^{m} A_j e^{-\tau_j z}\right] e^{zt}\underline{c} = \underline{0} \tag{3.3}$$

and thus in order to have non-trivial solutions, we must have

$$G(z) = \det\left[zI - \sum_{j=0}^{m} A_j e^{-\tau_j z}\right] = 0. \tag{3.4}$$

This is n-th order exponential polynomial in z, where n corresponds to the size of matrix $A_j$.

Equation (3.4) possesses infinite number of roots $z_i$, and each root $z_i$ corresponds to a solution, $e^{z_i t}$. Thus general solution to the original differential-difference equation (3.1) will be

$$\underline{x}(t) = \sum_{i=1}^{\infty} e^{z_i t}\underline{c}_i \tag{3.5}$$

where the infinite set of constant vectors $\underline{c}_i$ can be chosen to satisfy the initial function $\underline{g}(t)$. This implies the general solution of any linear differential-difference equation with constant coefficients spans infinite dimensional vector space $E_\infty$, regardless of the order of the highest derivative term. This can be shown immediately since all base solutions

$$\underline{x}_1(t)=e^{z_1 t}\underline{c}_1 \; , \; \underline{x}_2(t)=e^{z_2 t}\underline{c}_2 \; , \cdots , \; \underline{x}_N(t)=e^{z_N t}\underline{c}_N , \cdots$$

are mutually linearly independent. The fact that solution spans infinite dimensional vector space is a unique characteristic of differential-difference equations while the solution of ordinary differential equations spans finite n-dimensional vector space where n corresponds to the size of the matrix $A_i$. This presents an inherent difficulty of obtaining an exact solution in practice, but it will be shown later how this can be overcome by knowing the distribution of the characteristic roots.

Since the existence and uniqueness of solution for Equation (3.1) is established, it is clear that the infinite series (3.5) must converge and term by term differentiation is possible.

## 3.2 Characteristic Exponential Polynomials

Although the infinite series formulation of (3.5) offers valid representation for the solution, it lacks information about qualitative behavior of the solution. The main part of this chapter is devoted to discussions of the characteristic exponential polynomial as shown in Equation (3.4), since distribution of the roots will determine the stability of the solution and also quantitative measure of error bounds by truncating the infinite number of roots. In fact, the roots of the characteristic polynomial completely characterize the solution of a homogeneous linear differential-difference equation with constant parameters.

Various authors have investigated the roots of characteristic exponential polynomials. Bellman and Cooke[20], El'sgol'ts[21], and

Pinney[22] each devotes a chapter in their books to the asymptotic location of the roots of the exponential polynomials. Chow[23] tabulated digital roots for specific cases of exponential polynomials. More recently, Hsu and Lee[24] used $\tau$-decomposition method to determine the stability criteria as a function of the delay term $\tau$.

A characteristic exponential polynomial G(z) in Equation (3.4) is an entire analytic function of z. From the analytic function theory, it is clear that G(z) must possess countably infinite number of roots whose unique accumulation point occurs at the infinity. Since the original differential-difference equation (3.1) is of retarded type, all the roots $z_i$ of G(z) must lie in a left half-plane, $\Re e\, z_i \leq \sigma^*$ for all i. This also implies the unique accumulation point must be at the negative infinity.

This chapter will first discuss general distribution of the roots for several specific cases of retarded dynamic systems, which will give exact location of the roots as well as allowed and forbidden zones for the roots. This is motivated as a checking process for the roots which are computed by the Newton's method. Once exact locations of the roots are tabulated, it is desirable to find the exact number of roots that are lying in the Region D which lies left next to $z = \sigma^*$.

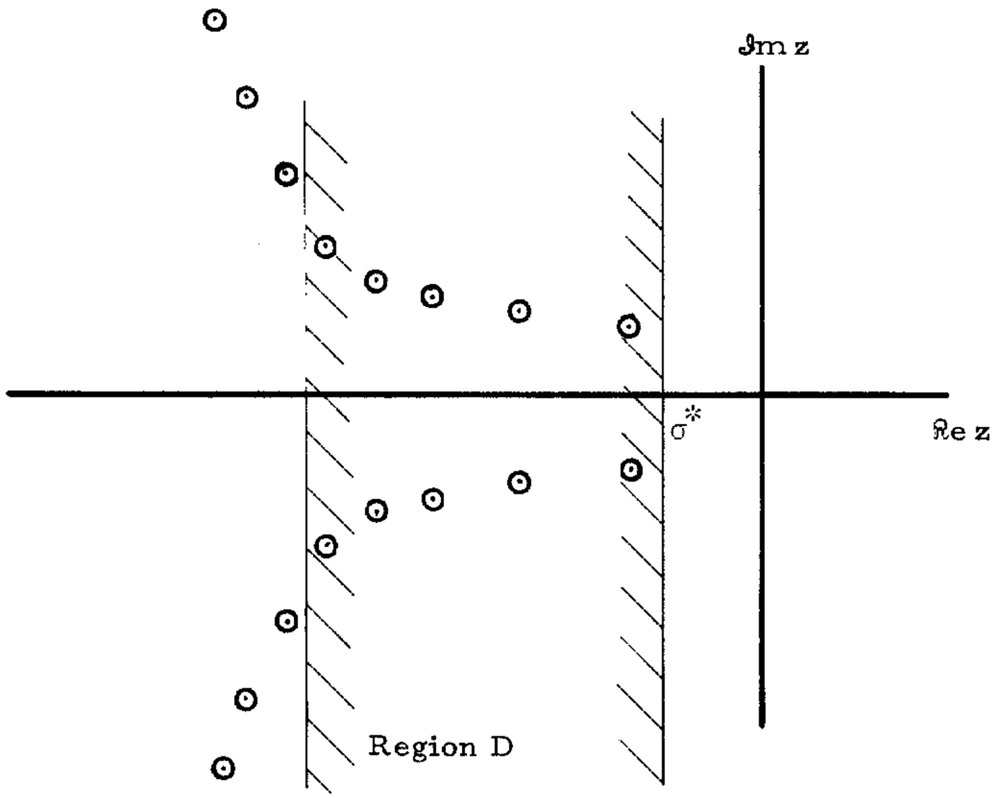


Figure 3.1 Typical Root Distribution of Retarded Type Characteristic Exponential Polynomial

Typical case of root distribution for a retarded type differential-difference equation is shown in Figure 3.1; the roots must occur as conjugate pairs for real coefficient systems. A great deal of attention is given to the N pair of rightmost roots lying in the Region D. For instance, in Figure 3.1, N=5. The size of the Region D, or the number of roots under consideration 2N is oriented towards particular solution methods for piecewise linear differential-difference systems, namely, the

Fredholm Integral Equation Method which will be discussed in detail in Chapters IV and V.

Thus, it is not the purpose of this chapter to consider the most general aspects of characteristic exponential polynomials, but to consider the exact distribution of the roots in a finite region, which will be used exclusively in the subsequent chapters. Several specific examples of linear dynamic systems with time delays are discussed here, along with the stability criteria which is governed by the negativity of the real part of the leading root $\sigma_1$.

### 3.2A Distribution of the Roots

As an illustration, we consider a typical linear dynamic system with time delays.

$$\ddot{x}(t) + c_1\dot{x}(t) + c_2\dot{x}(t-\tau_2) + k_1 x(t-\tau_1) + k_2 x(t) + k_3 x(t-\tau_3) = 0 \qquad (3.6)$$

with

$$0 < \tau_1 < \tau_2 < \tau_3.$$

Without loss of generality, Equation (3.6) represents a linear retarded type differential-difference equation with multiple delays.

The characteristic exponential polynomial for Equation (3.6) is

$$\hat{z}^2 + c_1\hat{z} + c_2\hat{z}\, e^{-\tau_2\hat{z}} + k_1\, e^{-\tau_1\hat{z}} + k_2 + k_3\, e^{-\tau_3\hat{z}} = 0. \qquad (3.7)$$

Now define dimensionless parameters

$$\zeta_1 = \frac{c_1}{2\sqrt{k_1}} \quad , \quad \zeta_2 = \frac{c_2}{2\sqrt{k_1}}$$

$$\gamma_1 = \sqrt{k_1}\tau_1 \ , \quad \gamma_2 = \sqrt{k_1}\tau_2 \ , \quad \gamma_3 = \sqrt{k_1}\tau_3 \tag{3.8}$$

$$\alpha = k_3/k_1 \ , \quad \beta = k_2/k_1$$

and let

$$z = \frac{\hat{z}}{\sqrt{k_1}} \ , \tag{3.9}$$

we get

$$z^2 + 2\zeta_1 z + 2\zeta_2 z\, e^{-\gamma_2 z} + e^{-\gamma_1 z} + \beta + \alpha\, e^{-\gamma_3 z} = 0. \tag{3.10}$$

Here $\gamma_1$, $\gamma_2$, and $\gamma_3$ represent non-dimensionalized delay parameters; $\zeta_1$, $\zeta_2$ represent damping factors, and $\alpha$, $\beta$ represent non-dimensionalized stiffness coefficients. In order-to investigate Equation (3.10) in further detail, we make the following case studies.

Case I ($\alpha = \zeta_2 = 0$)

This is the case of a single lag term occurring as a retarded spring force term. Suppressing the subscripts for simplicity, we have

$$z^2 + 2\zeta z + e^{-\gamma z} + \beta = 0 \ , \quad \gamma > 0. \tag{3.11}$$

Let

$$z = \sigma + i\omega \tag{3.12}$$

then Equation (3.11) yields

$$(\sigma^2 - \omega^2 + 2\zeta\sigma + \beta)e^{\gamma\sigma} = -\cos\gamma\omega \tag{3.13}$$

$$2(\sigma + \zeta)\omega\, e^{\gamma\sigma} = \sin\gamma\omega. \tag{3.14}$$

Combine (3.13) and (3.14),

$$\frac{2(\sigma+\zeta)\omega}{\omega^2-\sigma^2-2\zeta\sigma-\beta}=\tan\gamma\omega \tag{3.15}$$

$$\omega^2=-b+\sqrt{b^2-(\sigma^2+2\zeta\sigma+\beta)^2+e^{-2\gamma\sigma}} \tag{3.16}$$

with

$$b=\sigma^2+2\zeta\sigma+2\zeta^2-\beta. \tag{3.17}$$

Thus, the exact set of roots $z_i=\sigma_i+i\omega_i$ are such that they satisfy Equations (3.15) and (3.16) identically. Since the roots must occur as conjugate pairs, we shall only consider when $\omega\geq 0$. Actual calculation of the roots is carried out by using the Newton's method to solve Equations (3.15) and (3.16) simultaneously for two unknowns $\sigma$ and $\omega$.

In order to supplement actual tabulation of the roots for this case, it is possible to construct allowed zones and forbidden zones in complex z-plane where the roots may or may not occur. Consider Equation (3.14),

for

$$\sigma>-\zeta\ , \quad \sin\gamma\omega>0$$

$$\therefore \quad \frac{2n\pi}{\gamma}<\omega<\frac{(2n+1)\pi}{\gamma}\ , \quad n=0,1,2,\cdots \tag{3.18}$$

for

$$\sigma=-\zeta\ , \quad \sin\gamma\omega=0$$

$$\therefore \quad \omega=\frac{n\pi}{\gamma}\ , \quad n=0,1,2,\cdots \tag{3.19}$$

for

$$\sigma < -\zeta \ , \quad \sin \gamma\omega < 0$$

$$\therefore \quad \frac{(2n+1)\pi}{\gamma} < \omega < \frac{2(n+1)\pi}{\gamma} \ , \quad n = 0, 1, 2, \cdots \tag{3.20}$$

Now reconsider Equation (3.14) for $\omega > 0$

$$|2(\sigma+\zeta)\omega e^{\gamma\sigma}| \leq 1 \tag{3.21}$$

or

$$\omega \leq \frac{e^{-\gamma\sigma}}{2(\sigma+\zeta)} \ , \quad \sigma > -\zeta \tag{3.22}$$

and

$$\omega \leq -\frac{e^{-\gamma\sigma}}{2(\sigma+\zeta)} \ , \quad \sigma < -\zeta . \tag{3.23}$$

Thus summarizing the results of Equations (3.18), (3.19), (3.20), (3.22), and (3.23) it is clear that the roots must occur within the double cross-hatched area, thus constructing the allowed zones and the forbidden zones as in Figure 3.2.

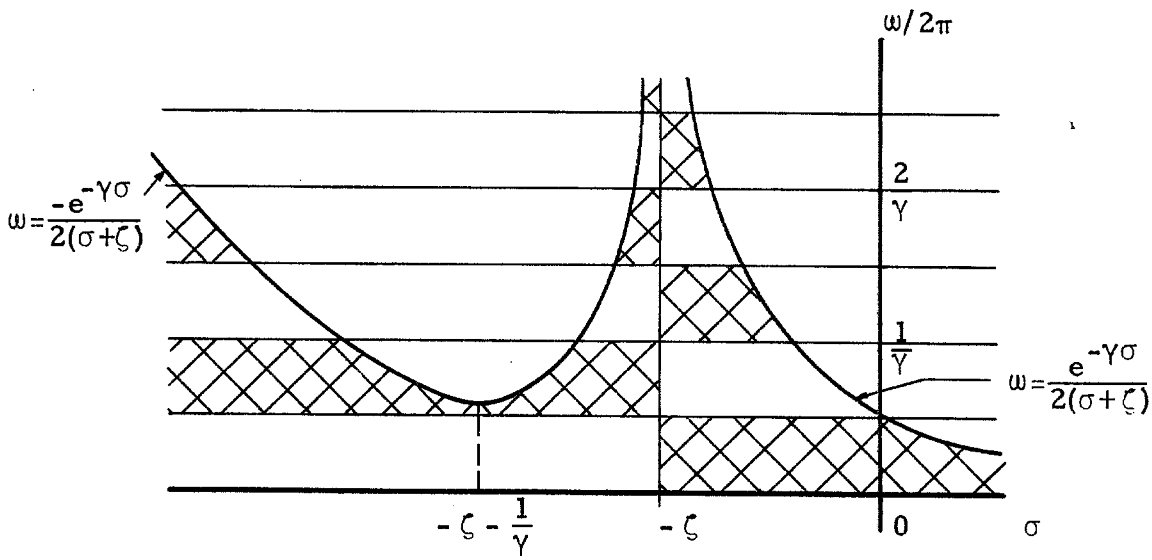


Figure 3.2 Allowed zones (double cross-hatched area) and forbidden zones (elsewhere) for $z^2 + 2\zeta z + e^{-\gamma z} + \beta = 0$.

In the following pages, Figures 3.3 and 3.4, numerical tabulation of the roots are shown as examples. We note all the roots do lie within the allowed zones as predicted.

Also, note that the roots form a single straight line chain formation on $\sigma$ vs. $\ell n\ \omega$ plot, and in fact this can be predicted by studying the asymptotic nature of the roots as $\sigma \to -\infty$. From Equation (3.16)

$$\lim_{\sigma \to -\infty} \omega \to e^{-\frac{\gamma}{2}\sigma} \quad \text{or} \quad \lim_{\sigma \to -\infty} \ell n\ \omega \to -\frac{\gamma}{2}\sigma. \tag{3.24}$$

<u>Case II ($\beta = \zeta_2 = 0$)</u>

If we let $\beta = \zeta_2 = 0$ from Equation (3.10), this is the case of two lag terms occurring as retarded spring force terms. This case will eventually lead to the study of multiple lag linear delay systems which will be discussed in the following Chapter IV. Equation (3.10) becomes

$$z^2 + 2\zeta_1 z + e^{-\gamma_1 z} + \alpha e^{-\gamma_3 z} = 0 \tag{3.25}$$

with

$$0 < \gamma_1 \neq \gamma_3 \quad , \quad \alpha \neq 0.$$

Let $z = \sigma + i\omega$, we have

$$\sigma^2 - \omega^2 + 2\zeta_1\sigma = -e^{-\gamma_1\sigma} \cos \gamma_1\omega - \alpha e^{-\gamma_3\sigma} \cos \gamma_3\omega \tag{3.26}$$

$$2(\sigma + \zeta_1)\omega = e^{-\gamma_1\sigma} \sin \gamma_1\omega + \alpha e^{-\gamma_3\sigma} \cos \gamma_3\omega. \tag{3.27}$$

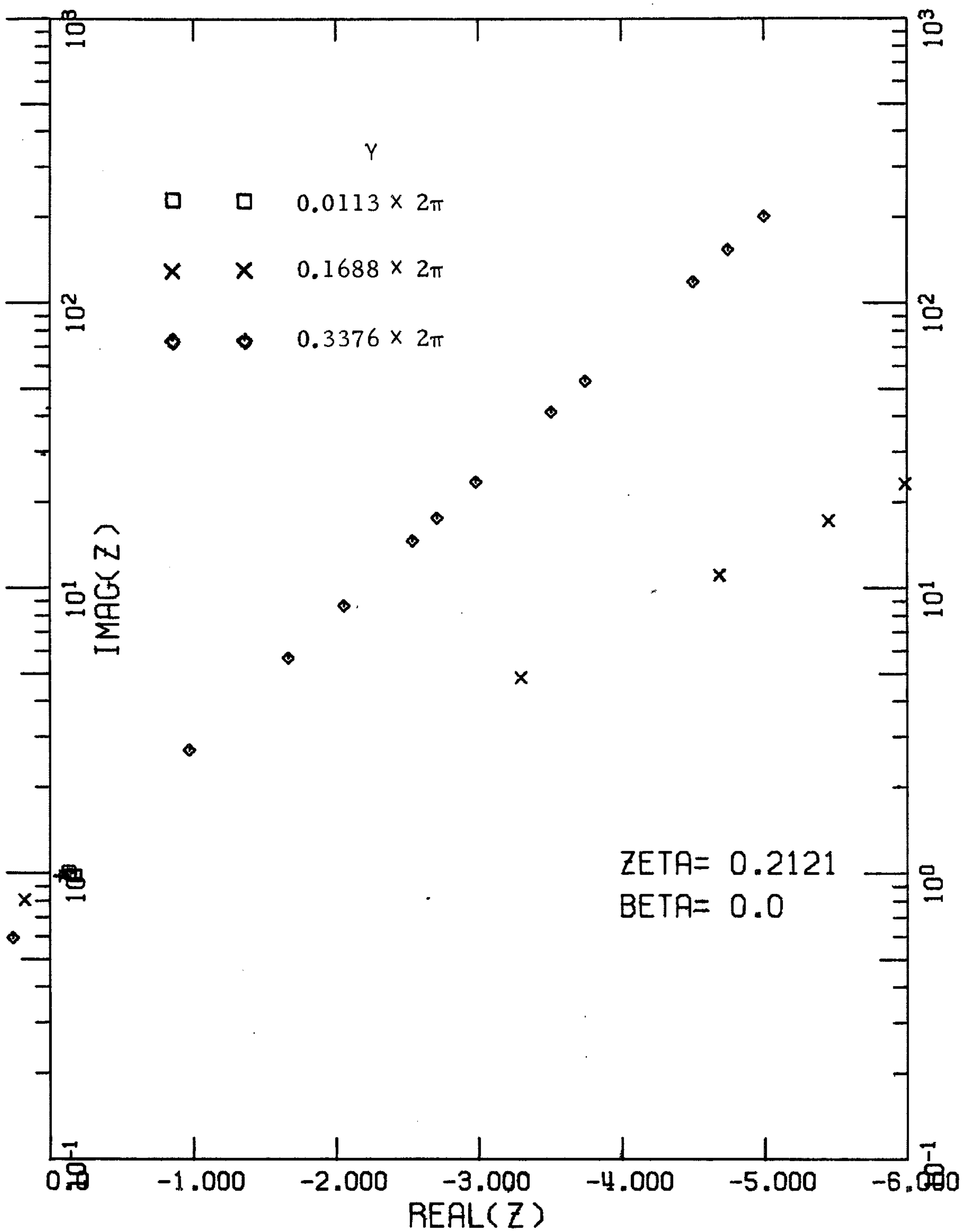


Figure 3.3 Root Locations of $z^2 + 2\zeta z + e^{-\gamma z} + \beta = 0$.

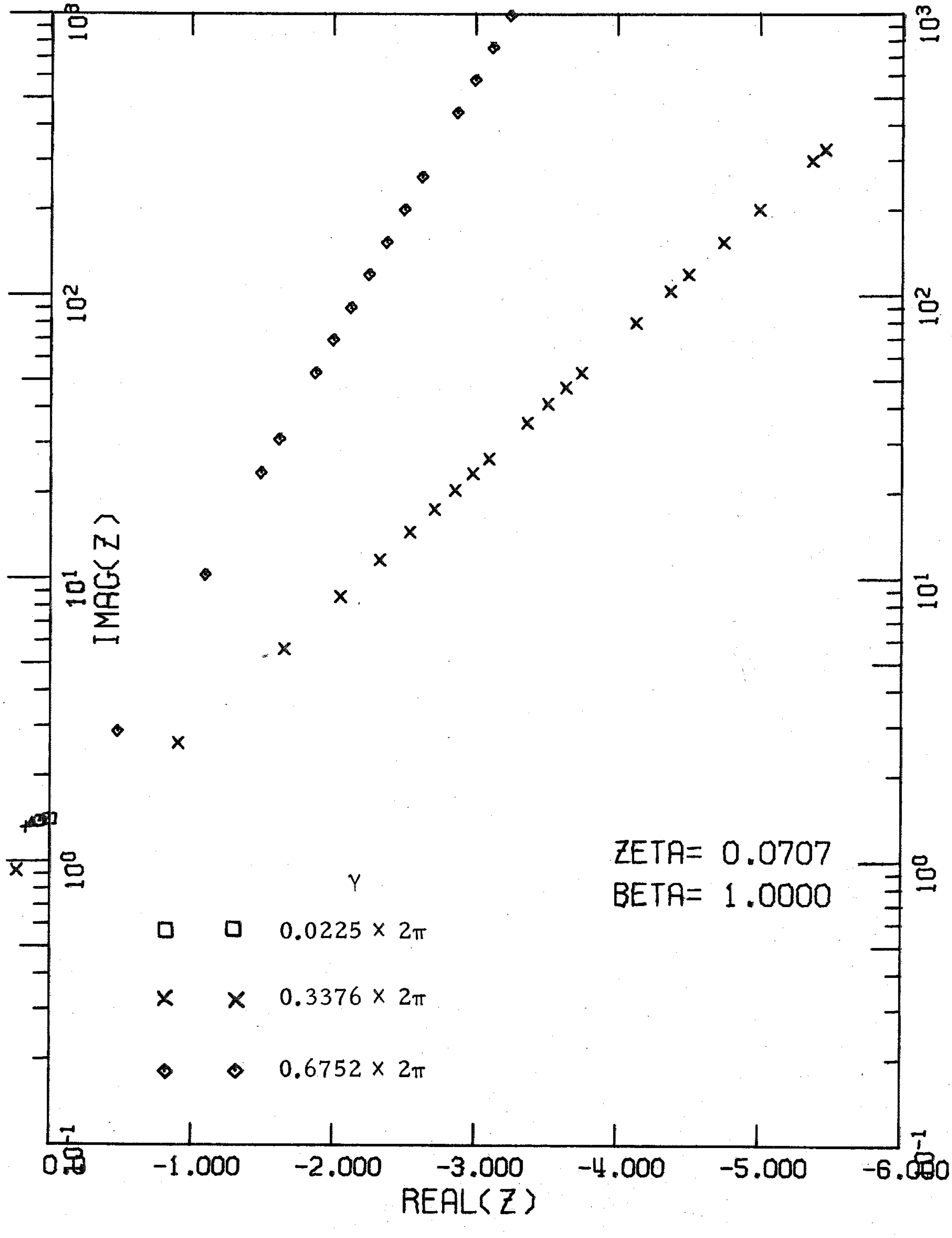


Figure 3.4 Root Locations of $z^2+2\zeta z+e^{-\gamma z}+\beta=0$.

Exact roots can be computed from Equations (3.26) and (3.27) numerically.

As $\sigma \to -\infty$, the last two terms of Equation (3.25) dominate, and

$$e^{-\gamma_1\sigma}\cos\gamma_1\omega + \alpha e^{-\gamma_3\sigma}\cos\gamma_3\omega \cong 0 \tag{3.28}$$

$$e^{-\gamma_1\sigma}\sin\gamma_1\omega + \alpha e^{-\gamma_3\sigma}\sin\gamma_3\omega \cong 0 \tag{3.29}$$

thus $\tan\gamma_1\omega = \tan\gamma_3\omega$, or

$$\omega = \frac{n\pi}{\gamma_1 - \gamma_3}\ ,\quad n = 0, \pm1, \pm2, \cdots \text{ as } \sigma \to -\infty. \tag{3.30}$$

Thus, asymptotically as $\sigma \to -\infty$, Equations (3.26) and (3.27) yields

$$\omega^2 = -b + \sqrt{b^2 - (\sigma^2 + 2\zeta_1\sigma)^2 + (e^{-\gamma_1\sigma} \pm \alpha e^{-\gamma_3\sigma})^2} \tag{3.31}$$

with

$$b = \sigma^2 + 2\zeta_1\sigma + 2\zeta_1^2$$

using

$$\omega = \frac{n\pi}{\gamma_1 - \gamma_3}$$

$$\therefore \qquad \lim_{\sigma \to -\infty} \omega^2 \to e^{-\gamma_1\sigma} \pm \alpha e^{-\gamma_3\sigma}. \tag{3.32}$$

If $\gamma_1 > \gamma_3 > 0$,

$$\lim_{\sigma \to -\infty} \ell n\, \omega = -\frac{\gamma_1}{2}\sigma + O(e^{(\gamma_1 - \gamma_3)\sigma}). \tag{3.33}$$

If $\gamma_3 > \gamma_1 > 0$,

$$\lim_{\sigma\to-\infty} \ell n\, \omega = \ell n \sqrt{\alpha} - \frac{\gamma_3}{2}\sigma + O(e^{(\gamma_3-\gamma_1)\sigma}). \tag{3.34}$$

Thus, slope of the root chain in $\sigma$ vs. $\ell n\, \omega$ plot is dominated by the largest lag term. In some cases of parameter combinations, it is observed that double trajectory of straight chains of roots occur. Figure 3.5 shows double trajectory and single trajectories for a case when two lag terms occur as restoring force terms.

Case III ($\alpha = \beta = 0$)

In this case, we let $\alpha = \beta = 0$ from the general equation (3.10), and we have

$$z^2 + 2\zeta_1 z + 2\zeta_2 z\, e^{-\gamma_2 z} + e^{-\gamma_1 z} = 0 \tag{3.35}$$

with

$$\gamma_1 > 0 \quad , \quad \gamma_2 > 0.$$

This case represents a delay linear differential-difference equation with one lag in the zeroth order term and the second lag in the first order term.

Let $z = \sigma + i\omega$, Equation (3.35) yields

$$\sigma^2 - \omega^2 + 2\zeta_1\sigma = -e^{-\gamma_1\sigma}\cos\gamma_1\omega - 2\zeta_2 e^{-\gamma_2\sigma}(\sigma\cos\gamma_2\omega + \omega\sin\gamma_2\omega) \tag{3.36}$$

$$2(\sigma+\zeta_1)\omega = e^{-\gamma_1\sigma}\sin\gamma_1\omega - 2\zeta_2 e^{-\gamma_2\sigma}(\omega\cos\gamma_2\omega - \sigma\sin\gamma_2\omega) \tag{3.37}$$

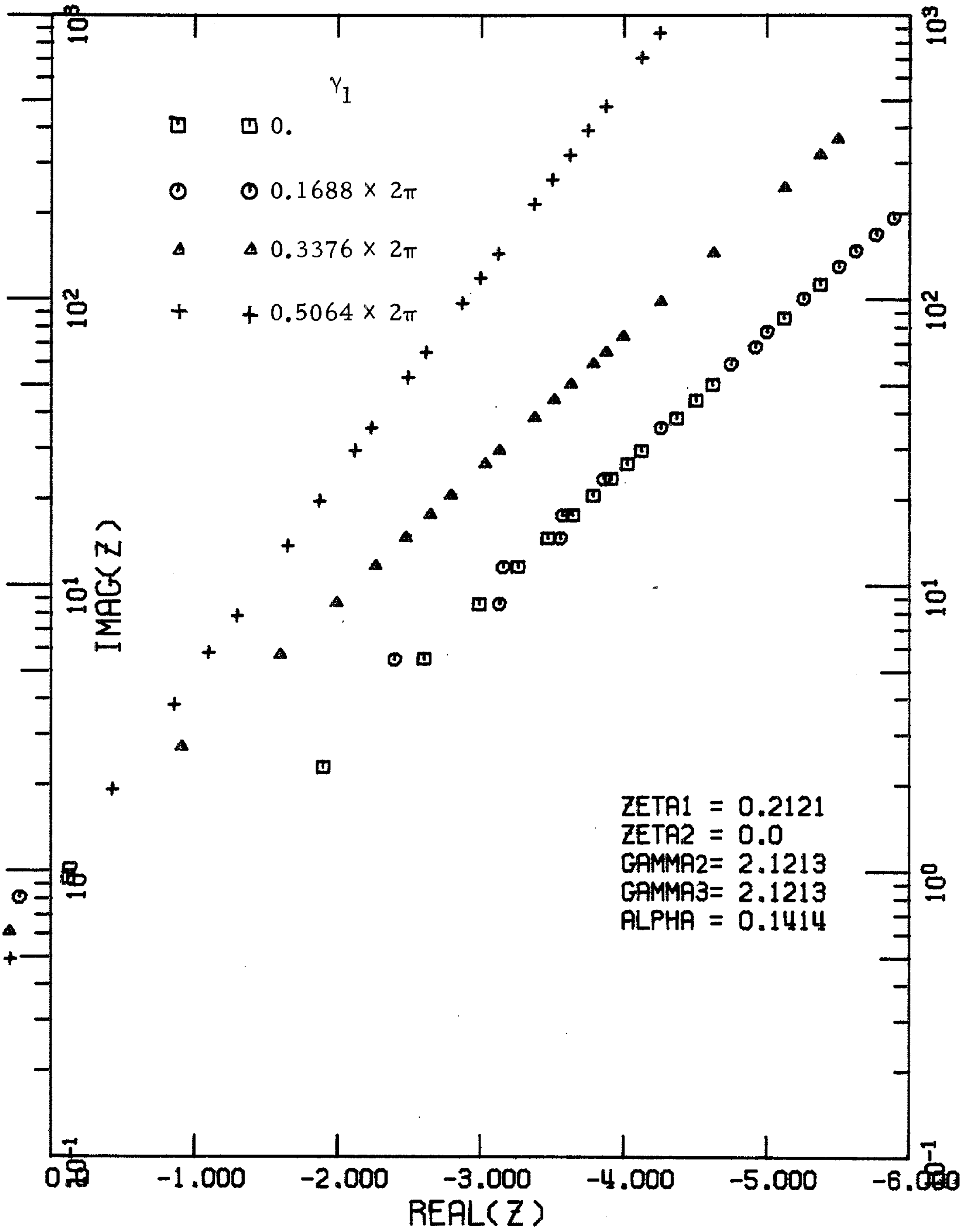


Figure 3.5 Root Locations of $z^2 + 2\zeta_1 z + e^{-\gamma_1 z} + \alpha e^{-\gamma_3 z} = 0$.

(Double Trajectory Occurs for $\gamma_1 = 0.1688 \times 2\pi$)

and the exact roots are tabulated from above.

If we consider the second lag $\gamma_2$ term as Taylor series,

$$e^{-\gamma_2 z} = 1 - \gamma_2 z + \frac{1}{2}\gamma_2^2 z^2 + \ldots \tag{3.38}$$

∴ Equation (3.35) becomes

$$(1-2\zeta_2\gamma_2)z^2 + 2(\zeta_1 + \zeta_2)z + e^{-\gamma_1 z} + O(\gamma_2^2) = 0. \tag{3.39}$$

Thus for $\gamma_2$ sufficiently small, effect of the second lag in the first derivative term does not alter the nature of the root distribution, and qualitative behaviors of the roots are extremely similar to the Case I when a single lag in the zeroth order term alone was considered.

### 3.2B Correct Number of Roots by Nyquist Diagram

Once the exact roots are tabulated by Newton's method, it is necessary to know that there are no missing roots within the finite Region D (refer to Figure 3.1). Since numerical procedure by Newton's method is based on the convergence to an exact root from an initial guess while there exist infinite number of roots, it is possible that the numerical procedure may have left out roots within D. Since the convergence of periodic solutions in forced linear systems with time delays by the Fredholm Integral Equation Method is entirely depended on the accuracy of the roots within D, it is extremely important that the tabulation is complete without a single missing root.

For this purpose it is most convenient to apply the Nyquist diagram in order to count the exact number of roots in the right half-plane. Construct the closed semi-circle countour as in Figure 3.6,

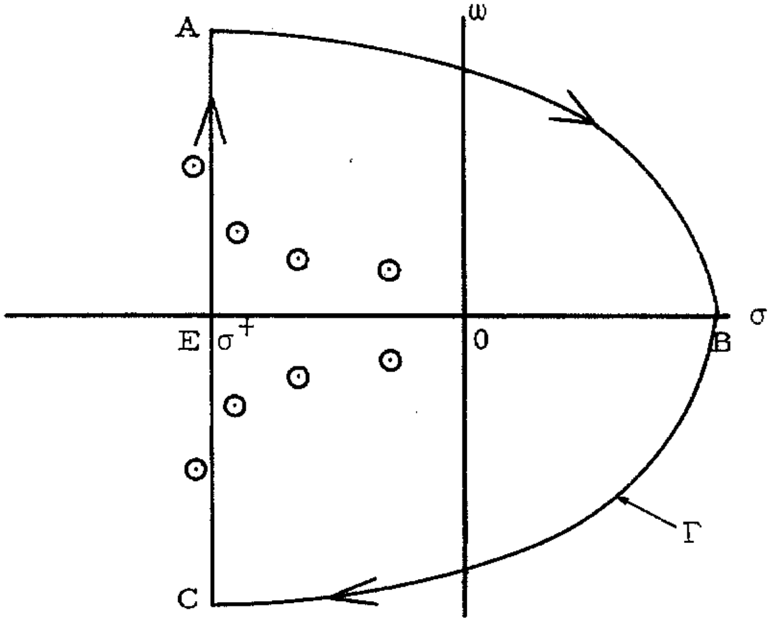


Figure 3.6 A Bromwich contour in z-plane

then the Cauchy's principle of argument states that, for a Bromwich contour $\Gamma$ followed by z (as $\overline{E\,ABC\,E}$ in Figure 3.6),

$$\text{Variation of argument } [G(z)] = 2\pi(Z-P) \tag{3.40}$$

where

Z = Number of zeroes within $\Gamma$

P = Number of poles within $\Gamma$.

Since the function G(z) under consideration is the characteristic exponential polynomial G(z),

$$G(z) = \det[zI - \sum_{j=0}^{m} A_j e^{-\tau_j z}] = 0 \tag{3.41}$$

it is clear G(z) has no poles, thus $P=0$.

As an illustrative purpose, we shall take a simple single lag case (this is considered in Section 3.2A, Case I with $\beta = 0$)

$$G(z) = z^2 + 2\zeta z + e^{-\gamma z} = 0 \tag{3.42}$$

with the tabulation of the roots as given in Table 3.1. Although the example of Equation (3.42) represents a special simple case of characteristic exponential polynomial, it will be clear that the following method will go through even though G(z) takes a most general form as in Equation (3.41).

Now construct the Nyquist Diagram in G(z)-plane as the contour $\Gamma$ in z-plane is mapped into G(z)-plane by the transformation (3.42).

As z traverses on $\overline{EA}$ of the contour $\Gamma$,

$$z = \sigma^+ + i\omega \quad , \quad 0 \le \omega < \infty \tag{3.43}$$

with $\sigma^+$ fixed constant.

Then

$$G(z) = x(\omega) + iy(\omega) \tag{3.44}$$

$$x(\omega) = \sigma^{+2} - \omega^2 + 2\zeta\sigma^+ + e^{-\gamma\sigma^+} \cos \gamma\omega \tag{3.45}$$

$$y(\omega) = 2(\zeta + \sigma^+)\omega - e^{-\gamma\sigma^+} \sin \gamma\omega \tag{3.46}$$

and as z traverses on $\overline{ABC}$ of the contour $\Gamma$,

$$z = Re^{i\theta} \quad , \quad R \to \infty \quad , \quad 0 \le \theta \le \pi \tag{3.47}$$

thus

$$G(z) = R^2 e^{2i\theta} + 2\zeta R\, e^{i\theta} + e^{-\gamma Re^{i\theta}} \tag{3.48}$$

and as $R \to \infty$, the first term dominates. Thus, the Nyquist contour is constructed by plotting Equations (3.44) and (3.48).

For a numerical example, roots of Equation (3.42) are tabulated for $\zeta = 0.0707$ and $\gamma = 2\pi \times 0.1688$.

Right-most five roots $z_j = \sigma_j + i\omega_j$ are:

| | $\sigma_j$ | $\omega_j$ | Number of roots to the right of $z = \sigma_j$ |
|---|---|---|---|
| $z_1$ | 0.2795 | 0.7858 | 2 |
| $z_2$ | -3.314 | 4.804 | 4 |
| $z_3$ | -4.690 | 11.10 | 6 |
| $z_4$ | -5.452 | 17.20 | 8 |
| $z_5$ | -5.990 | 23.22 | 10 |

Table 3.1 First five roots of $z^2 + 2\zeta z + e^{-\gamma z} = 0$ with $\zeta = 0.0707$, $\gamma = 2\pi \times 0.1688$.

Thus by choosing $z = \sigma^+$ appropriately, we can count the number of roots within $\Gamma$ by counting the number of net rotations of the angle $\varphi$ in the Nyquist diagram. The next four pages (Figures 3.7 - 3.10) show for each case of $\sigma^+$, and the number of net rotations do coincide with the number of roots within $\Gamma$ by the Newton's method.

This graphical method by Nyquist diagram will eliminate any possibilities of missing roots to the right of arbitrary $z = \sigma^+$, provided the number of net rotations coincides with the tabulation.

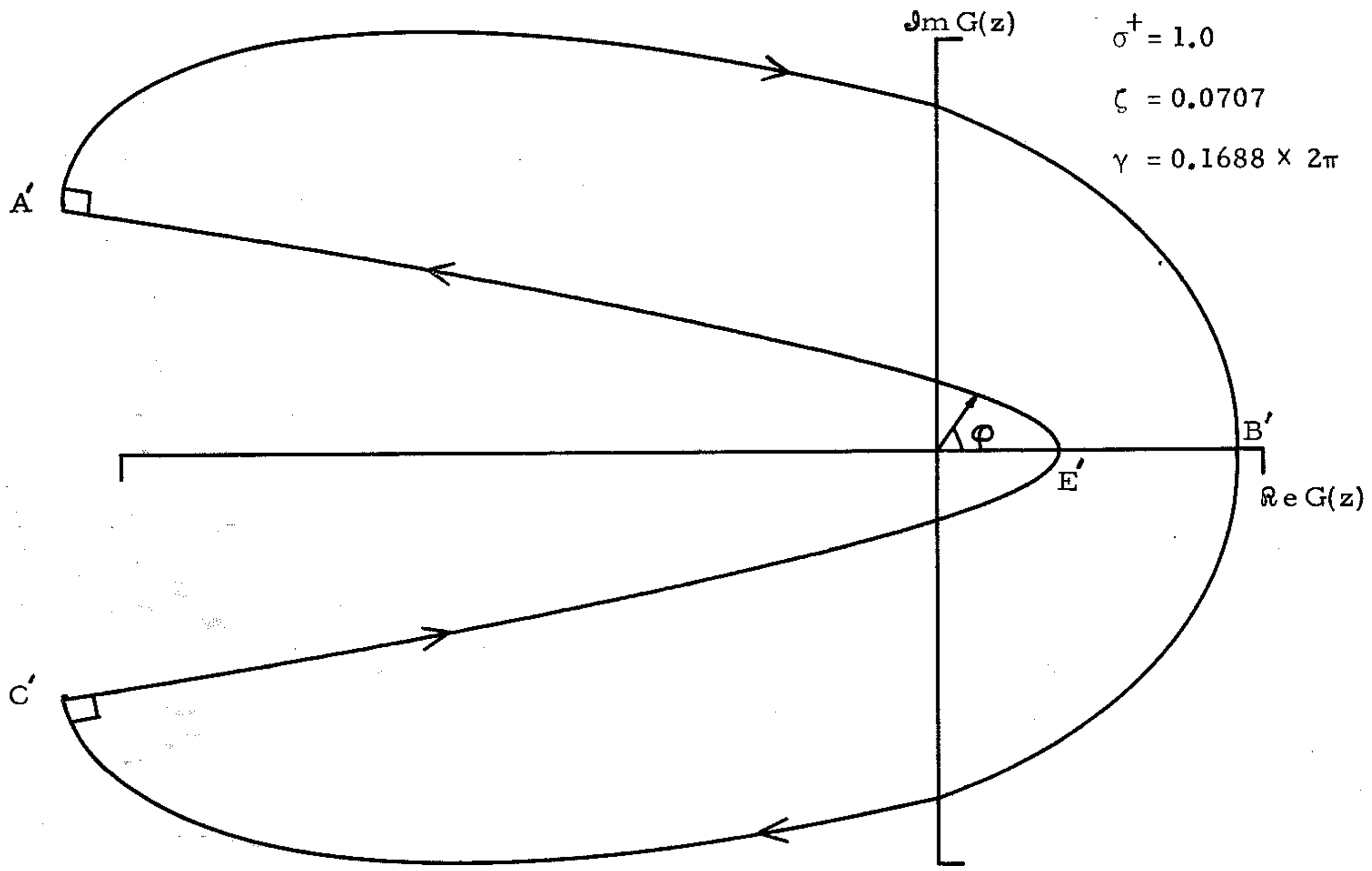


Figure 3.7 Nyquist Diagram of $G(z) = z^2 + 2\zeta z + e^{-\gamma z}$

(Zero Net Rotation of $\varphi \Rightarrow$ No Roots to the Right of $\sigma^+$)

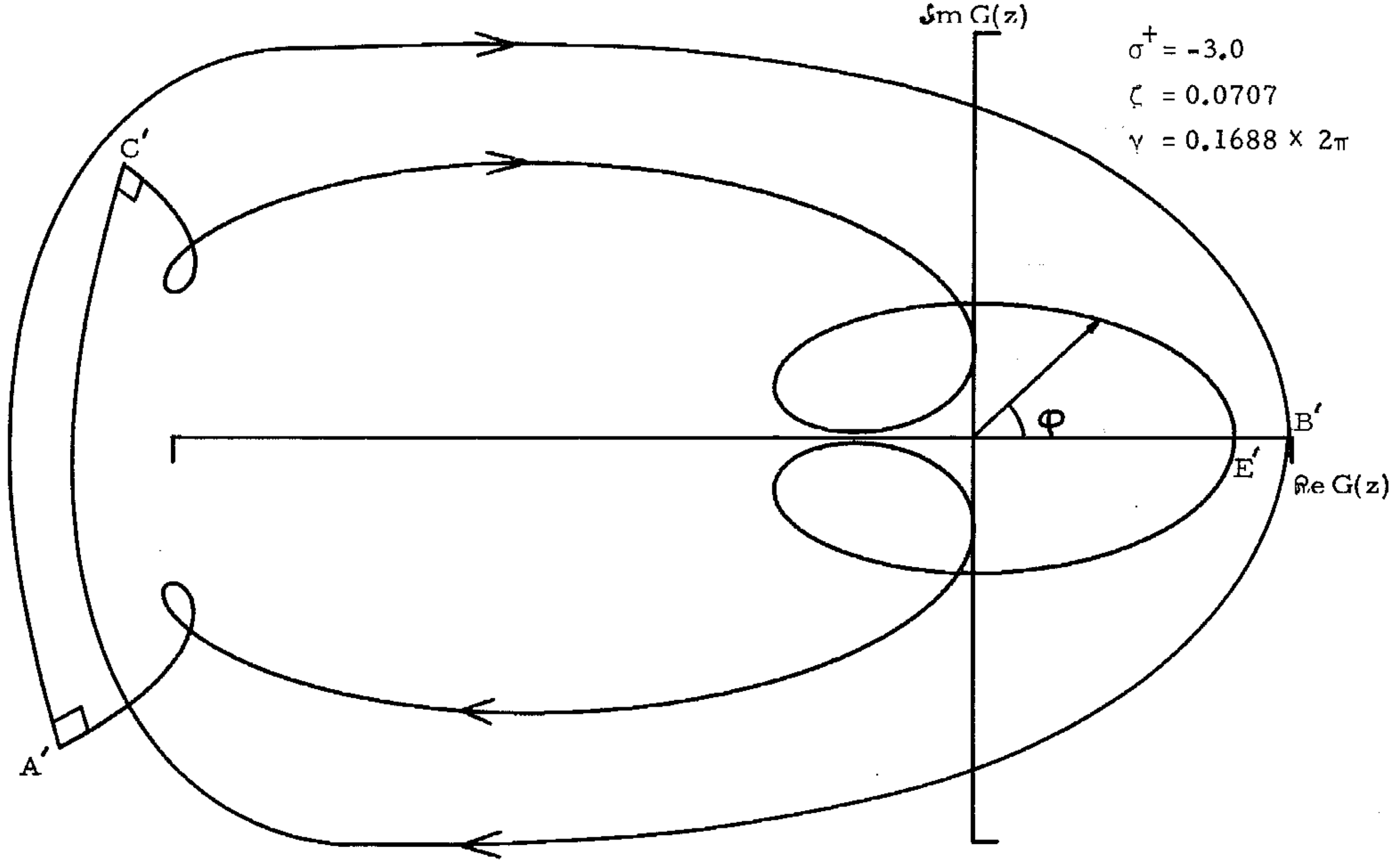


Figure 3.8 Nyquist Diagram of $G(z) = z^2 + 2\zeta z + e^{-\gamma z}$

(2 Net Rotations of $\varphi \Rightarrow 2$ Roots to the Right of $\sigma^+$)

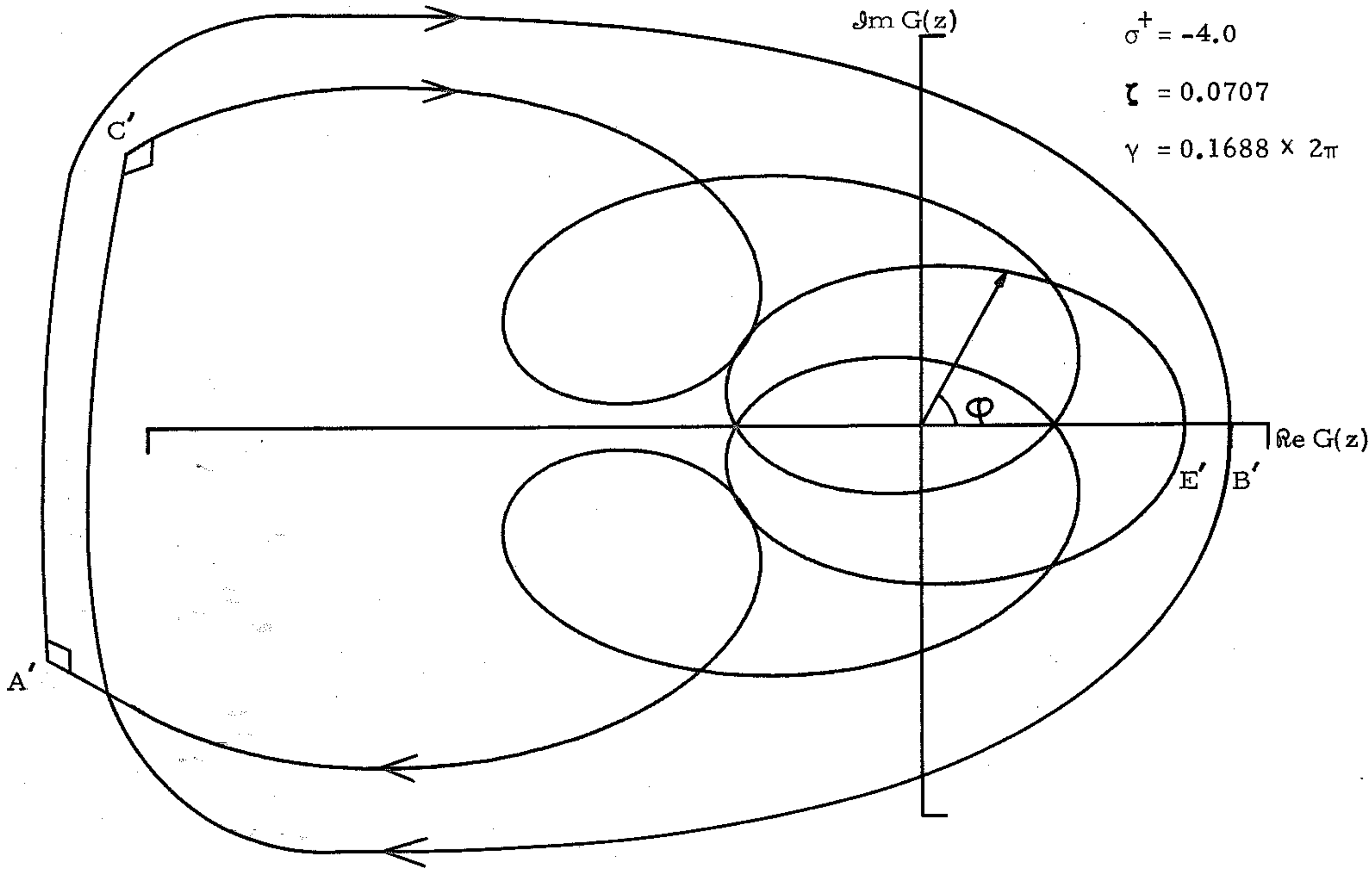


Figure 3.9 Nyquist Diagram of $G(z) = z^2 + 2\zeta z + e^{-\gamma z}$

(4 Net Rotations of $\varphi \Rightarrow$ 4 Roots to the Right of $\sigma^+$)

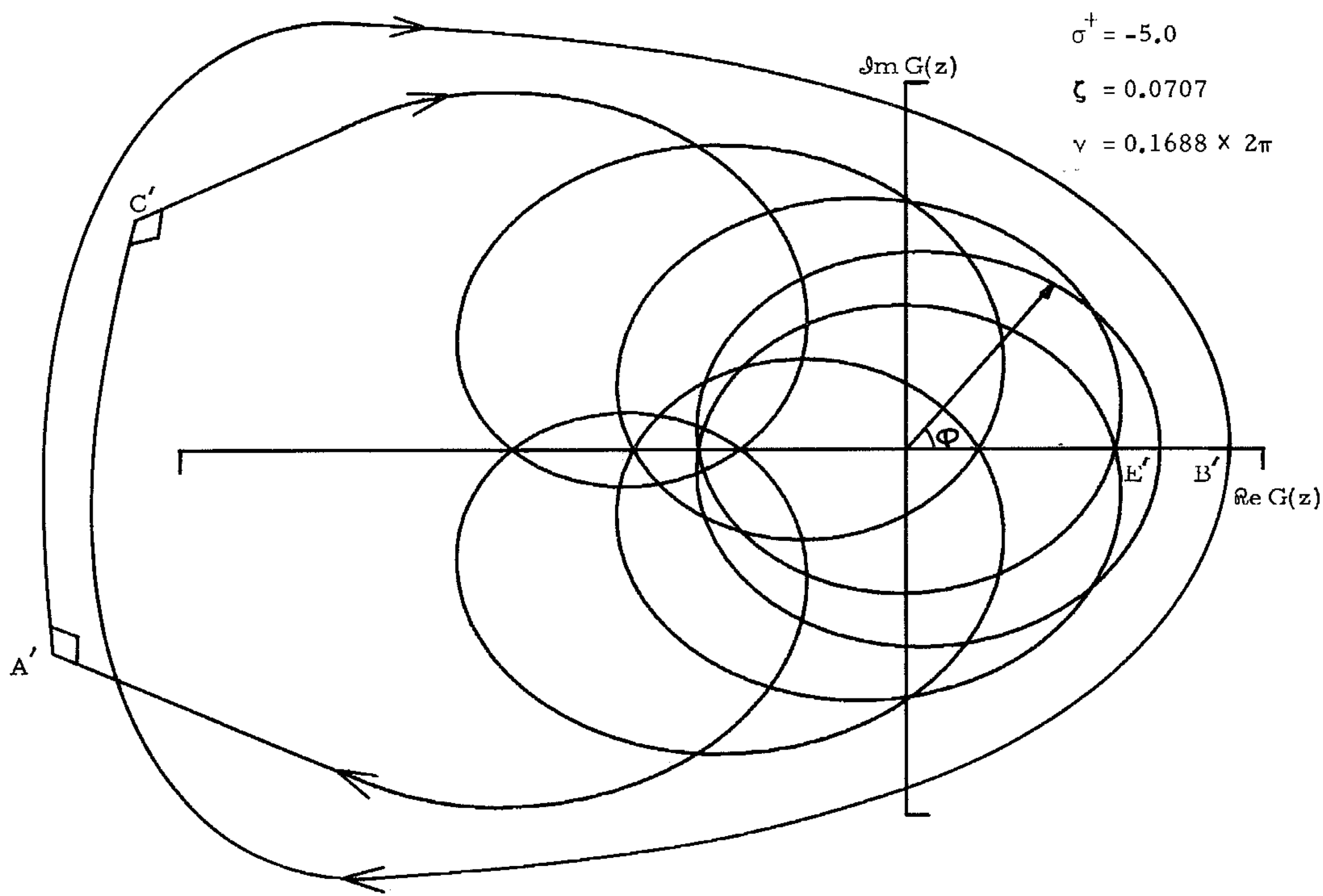


Figure 3.10 Nyquist Diagram of $G(z) = z^2 + 2\zeta z + e^{-\gamma z}$

(6 Net Rotations of $\varphi \Rightarrow$ 6 Roots to the Right of $\sigma^{+}$)

### 3.2C Stability Criteria by Satche Diagram

So far attention was restricted to the distribution of roots of the characteristic exponential polynomials. Most frequently, however, the question of stability of solution for a linear differential-difference equation is of prime importance, and this can be answered directly by analyzing the roots of the characteristic exponential polynomial.

It is a particularly simple matter to investigate the stability of solutions of linear equations with constant coefficients and constant delays of retarded type as in Equation (3.1). Any solution $\underline{x}(t)$ may be expanded in a uniformly and absolutely converging series of basic solutions

$$\underline{x}(t) = \sum_{i=1}^{\infty} e^{z_i t} \underline{c}_i(t) \tag{3.49}$$

where $\underline{c}_i(t)$ are polynomials of degree not less than $\alpha_i - 1$, where $\alpha_i$ is the multiplicity of the root $z_i$ of the characteristic exponential polynomial

$$G(z) = \det[zI - \sum_{j=0}^{m} A_j e^{-\tau_j z}] = 0 \tag{3.50}$$

and order the roots as

$$\Re e\, z_1 \geq \Re e\, z_2 \geq \cdots \geq \Re e\, z_N \geq \cdots \tag{3.51}$$

If all the roots $z_i$ have negative real parts, it is clear that the solution (3.49) must be asymptotically stable. Therefore, for $\Re e z_1 < 0$, all solutions of Equation (3.1) are exponentially asymptotically stable:

$$\|\underline{x}(t)\| \leq Me^{\Re e\, z_1 t} \tag{3.52}$$

where M is a constant.

Thus it is necessary to know the conditions between parameters such that all the roots of G(z) satisfy $\Re e\, z_i < 0$ for all i.

Although there exist many schemes to deduce stability criteria in parameter spaces, the idea of Satche diagram is found to be most useful for the class of equations considered here. The Satche diagram, in essence, is a modified version of the Nyquist diagram, where it is most ideally suited for a dynamic system with single delay. Let

$$G(z) = f(z) - g(z) \tag{3.53}$$

with

$$f(z) = \text{rational function in } z \tag{3.54}$$

$$g(z) = -e^{-\gamma z}. \tag{3.55}$$

Applying the Cauchy's principle of argument, we have

$$\text{Variation of argument } [f(z)-g(z)] = 2\pi(Z-P) \tag{3.56}$$

where

$$Z = \text{Number of zeroes within } \Gamma$$

$$P = \text{Number of poles within } \Gamma$$

with the Bromwich contour $\Gamma$ is chosen on z-plane, as in Figure 3.11.

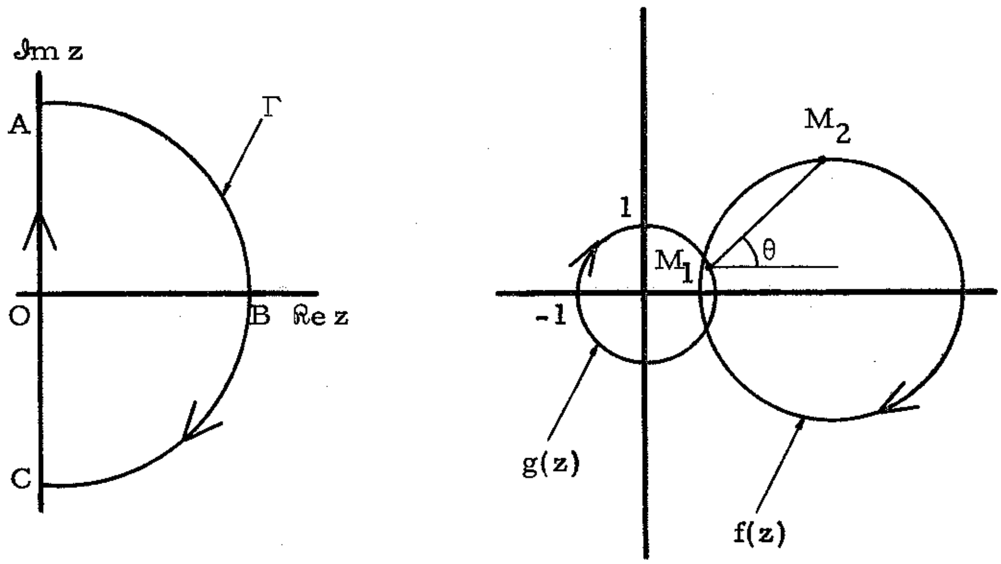


Figure 3.11 Bromwich contour and typical case of Satche diagram.

If we draw the diagrams of f(z), g(z) separately as z traces the Bromwich contour $\Gamma$, the argument of $[f(z)-g(z)]$ is the angle $\theta$ traced by the corresponding points $M_1$, $M_2$.

As $z = i\omega$, $-\infty < \omega < \infty$ on $\overline{COA}$, $g(i\omega)$ forms a unit circle centered at the origin, and $f(i\omega)$ may form a straight line, circle, or parabola depending on the function f(z). This double trajectory of f(z), g(z) is called Satche diagram[25], and, of course, the ordinary Nyquist diagram is a special case of Satche diagram when g(z) reduces to a simple point at the origin. We will show how this idea is used by means of an example.

Example

Consider a second order dynamic system with single delay term occurring in the zeroth order term, thus Equation (3.50) becomes

$$G(z) = z^2 + 2\zeta z + e^{-\gamma z} + \beta = 0 \tag{3.57}$$

with

$$\zeta > 0 \quad , \quad \gamma \geq 0 \quad , \quad -\infty < \beta < \infty .$$

In order to find conditions on the parameters $\zeta$, $\gamma$, and $\beta$ such that all the roots of Equation (3.57) possess negative real parts, we proceed to construct the Satche diagram. Let

$$G(z) = f(z) - g(z) \tag{3.58}$$

with

$$f(z) = z^2 + 2\zeta z + \beta \tag{3.59}$$

$$g(z) = -e^{-\gamma z} . \tag{3.60}$$

On $\overline{OA}$, $z = i\omega$, $0 \leq \omega < \infty$,

$$f(i\omega) = -\omega^2 + \beta + i\, 2\zeta\omega \tag{3.61}$$

$$g(i\omega) = -\cos\gamma\omega + i \sin\gamma\omega . \tag{3.62}$$

Thus for $|\beta| \leq 1$, the Satche diagram becomes Figure 3.12.

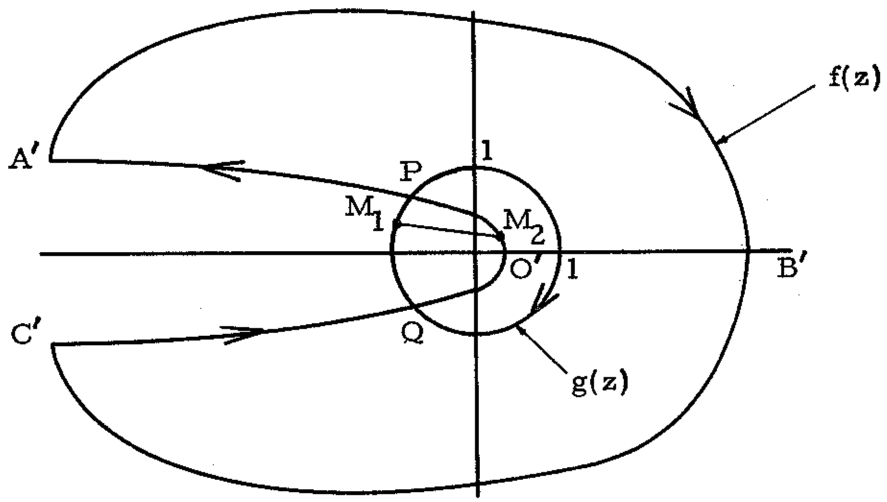


Figure 3.12 Satche diagram for $|\beta| \leq 1$.

In order to have zero net rotation of the argument $\theta$, we must have $M_1$ left of $M_2$ as $M_2$ travels between $O'$ and P. That is,

$$\Re e\, f(i\omega) > \Re e\, g(i\omega) \tag{3.63}$$

$$\therefore \qquad \cos\gamma\omega > \omega^2 - \beta \quad \text{for stability.} \tag{3.64}$$

Solve for the critical value of $\omega$ at P from Equations (3.61) and (3.62),

$$\omega^2 = -(2\zeta^2 - \beta) \pm \sqrt{(2\zeta^2-\beta)^2 + 1 - \beta^2}\,. \tag{3.65}$$

Choose the + sign,

$$\omega^2 = (\beta - 2\zeta^2) + \sqrt{(\beta-2\zeta^2)^2 + 1 - \beta^2} > 0 \quad \text{for } |\beta| \leq 1 \tag{3.66}$$

then

$$\omega^2 - \beta = -2\zeta^2 + \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1} < 1. \qquad (3.67)$$

Thus Equations (3.66), (3.67) and (3.64) give the stability criteria

$$\gamma \leq \frac{\cos^{-1}[-2\zeta^2 + \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}]}{\sqrt{(\beta - 2\zeta^2) + \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}}} \quad \text{for } |\beta| \leq 1 \qquad (3.68)$$

For $\beta > 1$, we can have two cases. First, $O'A'$ of f(z) intersects twice with the unit circle as shown in Figure 3.13.

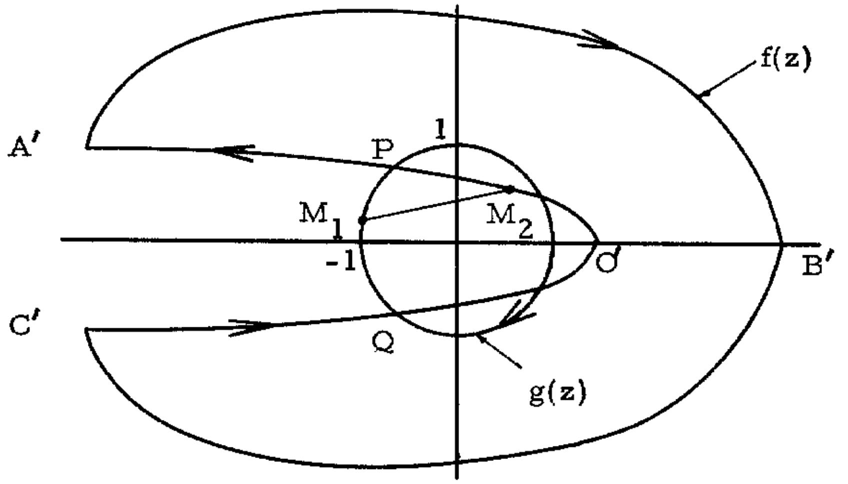


Figure 3.13 Satche diagram for $\beta > 1$.

Condition for stability is that $M_1$ must lie left of $M_2$ as $M_2$ travels between $O'$ and P. That is,

$$\cos \gamma\omega > \omega^2 - \beta \qquad (3.69)$$

with

$$\omega^2 = (\beta - 2\zeta^2) - \sqrt{(\beta - 2\zeta^2)^2 + 1 - \beta^2}$$

$$\omega^2 > 0 \quad \text{if } \beta > 2\zeta^2 \quad , \quad \beta > 1 \tag{3.70}$$

$$\omega^2 < 0 \quad \text{if } \beta < 2\zeta^2 \quad , \quad \beta > 1.$$

Thus, real $\omega$ exists only for $\beta > 2\zeta^2$, $\beta > 1$

$$\therefore \qquad \gamma \leq \frac{\cos^{-1}[-2\zeta^2 - \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}]}{\sqrt{(\beta - 2\zeta^2) - \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}}} \tag{3.71}$$

provided

$$4\zeta^4 - 4\beta\zeta^2 + 1 > 0. \tag{3.72}$$

If O′A′ has no intersections with the unit circle as shown in Figure 3.14,

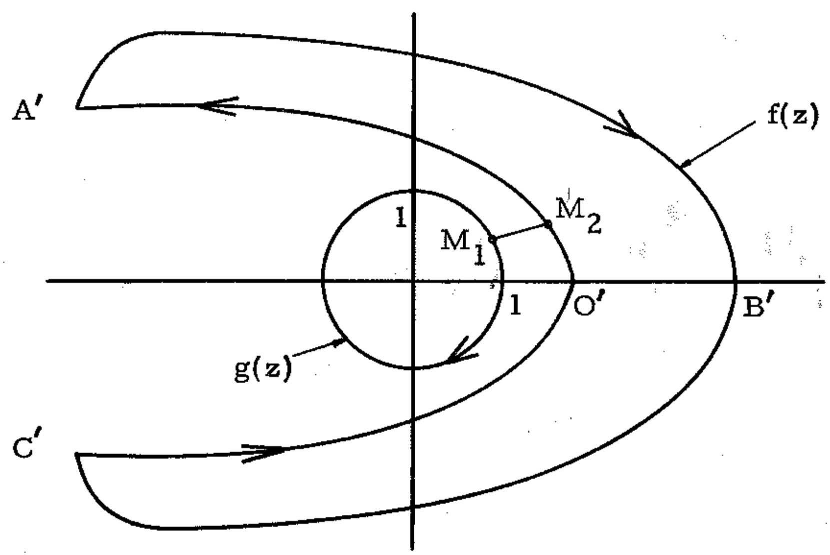


Figure 3.14 Satche diagram for $\beta > 1$.

it becomes clear that the net rotation of the argument $\theta$ is zero for all values of $\gamma$. This implies the unconditional stability on $\nu$ provided

$$4\zeta^4 - 4\beta\zeta^2 + 1 < 0 \tag{3.73}$$

or

$$\beta > \zeta^2 + \frac{1}{4\zeta^2} > 1.$$

For $\beta < -1$, we have Figure 3.15.

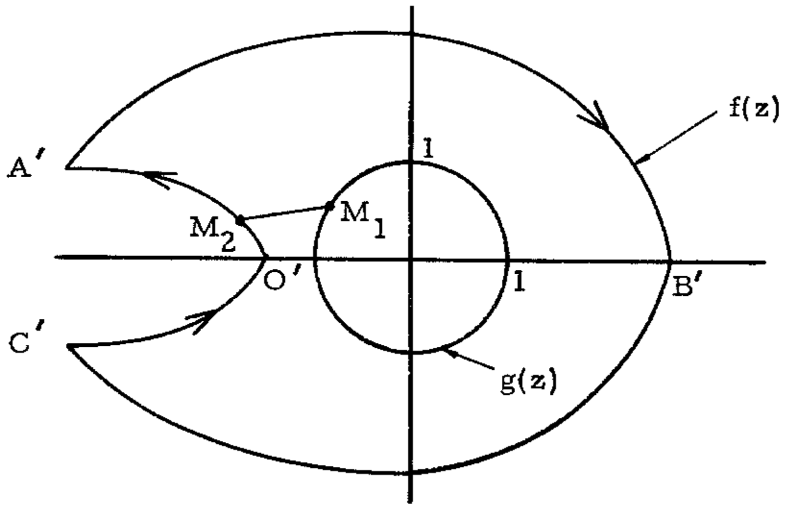


Figure 3.15 Satche diagram for $\beta < -1$.

Since $M_2$ is always to the left of $M_1$, it implies one net rotation of the argument $\theta$, thus we have unconditional instability for all values of $\zeta$, $\gamma$.

Summarizing the above results in $\zeta$, $\beta$ space, we have the stability regions of Figure 3.16.

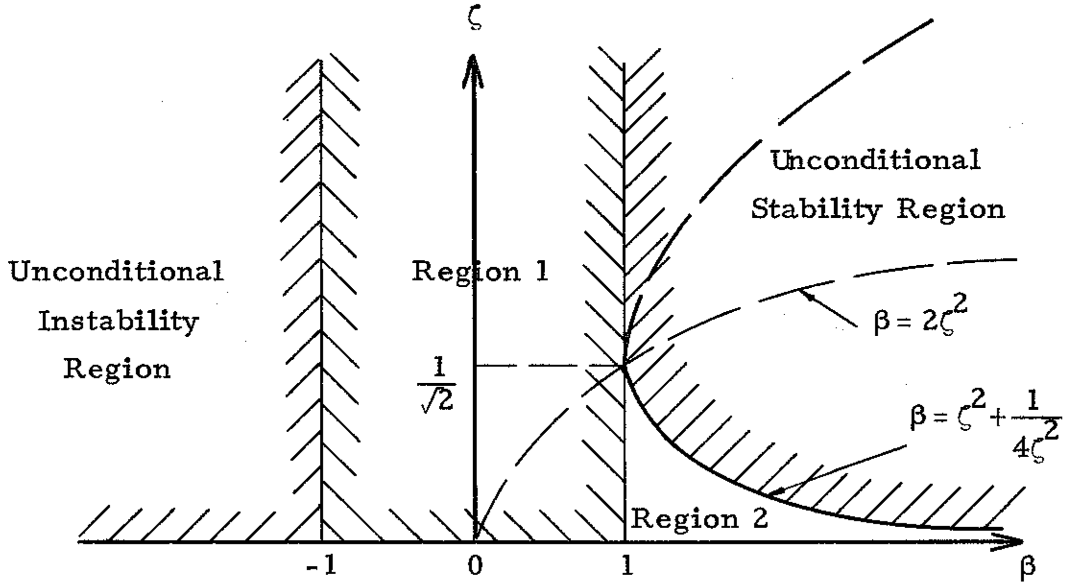


Figure 3.16 Region of stabilities for $G(z) = z^2 + 2\zeta z + e^{-\gamma z} + \beta$.

In Region 1, solution is stable if

$$\gamma \leq \frac{\cos^{-1}\left[-2\zeta^2 + \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}\right]}{\sqrt{(\beta - 2\zeta^2) + \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}}}. \tag{3.74}$$

In Region 2, solution is stable if

$$\gamma \leq \frac{\cos^{-1}\left[-2\zeta^2 - \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}\right]}{\sqrt{(\beta - 2\zeta^2) - \sqrt{4\zeta^4 - 4\beta\zeta^2 + 1}}}. \tag{3.75}$$

Plotting $\gamma$ vs. $\zeta$ parametric in $\beta$, from Equations (3.74) and (3.75), we have Figure 3.17.

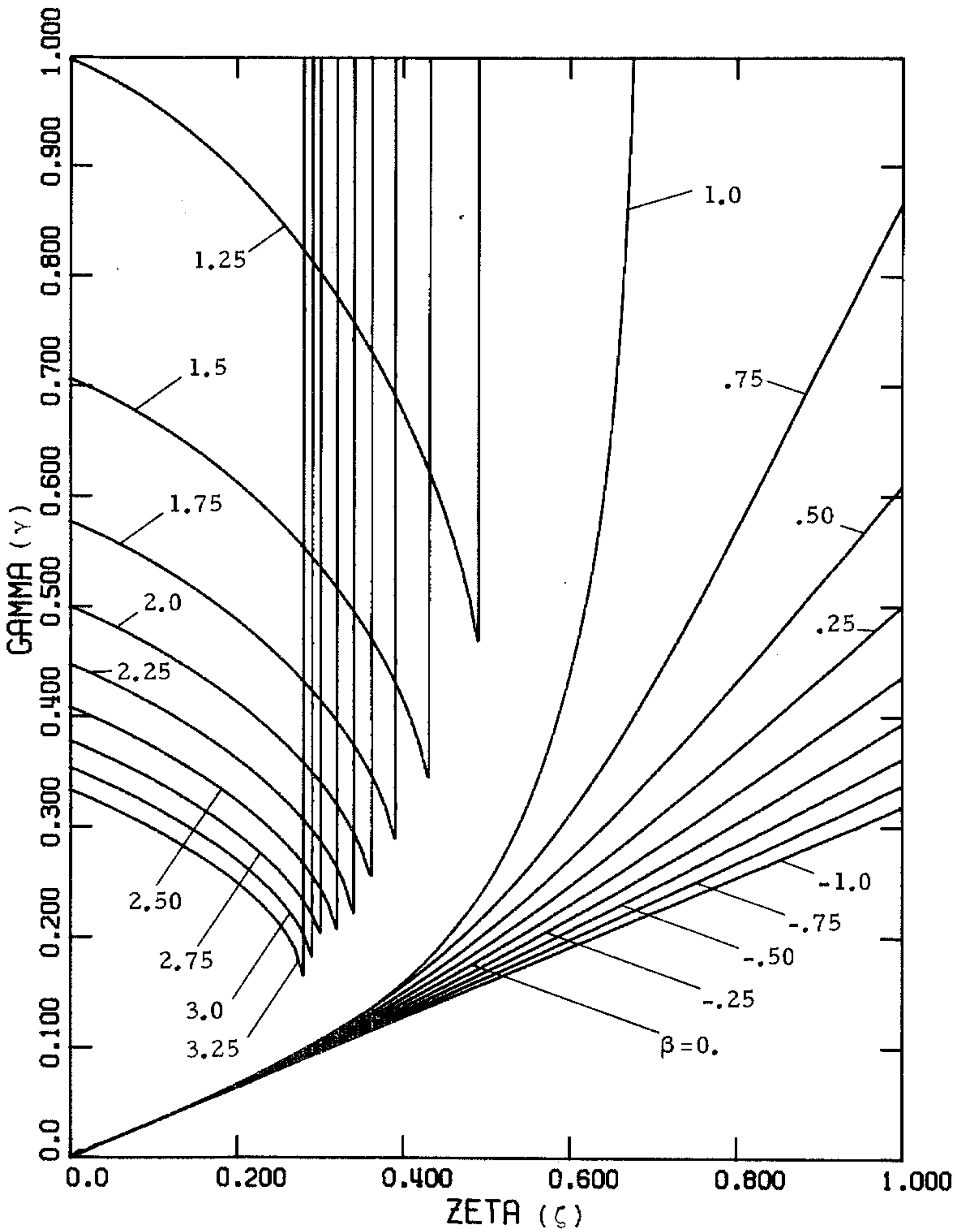


Figure 3.17 Stability Boundary Curves for $z^2 + 2\zeta z + e^{-\gamma z} + \beta = 0$, Parametric with $\beta$.

# Chapter IV

# FORCED LINEAR DYNAMIC SYSTEMS WITH TIME DELAYS

## 4.1 Problem Formulation

Knowing the mathematical preliminaries of free linear systems, it is logical to extend the analyses to forced linear systems with time delays. Again, knowledge of the forced linear system will be a preliminary foundation for the forced piecewise linear systems which will be discussed in Chapter V.

Consider

$$\frac{d\underline{x}(t)}{dt} = \sum_{i=0}^{m} A_i \underline{x}(t-\tau_i) + \underline{F}(t) \tag{4.1}$$

with initial function

$$\underline{x}(t) = \underline{g}(t) \quad \text{for} \quad 0 \le t \le \tau_m$$

and

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m.$$

Here Equation (4.1) carries the same Assumptions A1, A2, and A3 as in the free linear system. In addition to that,

Assumption 4 (A4): Characteristic exponential polynomial of the homogeneous part $\det\left[zI - \sum_{i=0}^{m} A_i e^{-\tau_i z}\right] = 0$ possesses roots with $\Re e\, z_i < 0$ for all i.

Assumption 5 (A5): The forcing function $\underline{F}(t)$ is bounded and continuous.

## 4.1A Formal Solution

Under the five Assumptions A1-A5, the existence and uniqueness of solution of Equation (4.1) is guaranteed, and the formal solution is constructed as follows:

Definition. Let K(t) be the unique matrix function which satisfies

$$\text{a) } K(t)=0 \quad \text{for} \quad t<0 \tag{4.2}$$

$$\text{b) } K(0)=I \tag{4.3}$$

$$\text{c) } L(K)=\frac{dK(t)}{dt}-\sum_{i=0}^{m} A_i K(t-\tau_i)=0 \tag{4.4}$$

for $t>0$.

Again the existence and uniqueness of K(t) is guaranteed, and actual construction of the matrix function K(t) will be discussed later in this Chapter.

Note the solutions of Equation (4.4) may possess jumps in the derivative, and denote S to be the set of all points of the form

$$t=\sum_{i=0}^{m} J_i \tau_i$$

where $J_i$ are integers. Let $S_+$ denote the intersection of S with the half-line $[0, \infty)$. Then, in general, the solution K(t) may possess jump discontinuities in the derivative at the points in $S_+$.

Theorem 4.1. If K(t) satisfies the conditions (4.2), (4.3) and (4.4), then the following is true.

$$K(t) = \int_{Br} e^{ts} H^{-1}(s)\, ds \quad , \quad t > 0 \tag{4.5}$$

$$\int_0^{\tau_m} K'(t-\xi)\underline{g}(\xi)d\xi = \int_{Br} e^{ts} H^{-1}(s)\, s \int_0^{\tau_m} e^{-s\xi}\underline{g}(\xi)d\xi\, ds \tag{4.6}$$

$$\int_0^{\tau_m} \sum_{i=0}^{m} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi = \int_{Br} e^{ts} H^{-1}(s) \sum_{i=0}^{m} A_i e^{-\tau_i s} \int_0^{\tau_m} e^{-s\xi}\underline{g}(\xi)d\xi\, ds \tag{4.7}$$

where

$$H(s) = sI - \sum_{i=0}^{m} A_i e^{-\tau_i s}. \tag{4.8}$$

Proof.

Let $\hat{K}(s)$ be the Laplace transform of K(t),

$$\hat{K}(s) = \int_0^{\infty} K(t) e^{-st} dt = \mathcal{L}[K(t)]$$

then

$$\mathcal{L}[K(t-\tau_i)] = \int_0^{\infty} K(t-\tau_i) e^{-st} dt = e^{-s\tau_i} \hat{K}(s) \tag{4.9}$$

$$\mathcal{L}[K'(t)] = \int_0^{\infty} K'(t) e^{-st} dt = s\hat{K}(s) - I. \tag{4.10}$$

Substitute (4.9) and (4.10) to Laplace transform of Equation (4.4),

$$\left[sI - \sum_{i=0}^{m} A_i e^{-\tau_i s}\right]\hat{K}(s) = I$$

thus

$$\hat{K}(s) = \int_0^{\infty} K(t)e^{-st}dt = \left[sI - \sum_{i=0}^{m} A_i e^{-\tau_i s}\right]^{-1}$$

$$= H^{-1}(s) \quad , \quad \text{for} \quad \Re e\, s > b$$

so K(t) and $H^{-1}(s)$ form a transform pair, thus K(t) can be recovered by inverse Laplace transform

$$K(t) = \int_{Br} e^{ts} H^{-1}(s) ds \quad , \quad t > 0$$

where the integration is taken along a suitable Bromwich contour. Thus,

$$K'(t) = \int_{Br} e^{ts} sH^{-1}(s) ds \quad , \quad t > 0. \tag{4.11}$$

Now using the convolution theorem in a vector-matrix sense,

$$\begin{aligned} K' * \underline{g} &= \int_0^t K'(t-\xi)\, \hbar(\tau_m - \xi)\underline{g}(\xi)d\xi \\ &= \int_0^{\tau_m} K'(t-\xi)\underline{g}(\xi)d\xi \end{aligned} \tag{4.12}$$

where $\hbar(t)$ denotes the Heaviside function,

$$\hbar(t) = 1 \quad \text{for} \quad t > 0$$
$$= 0 \quad \text{for} \quad t < 0$$

then

$$\mathcal{L}[K' * \underline{g}] = \mathcal{L}[K']\,\mathcal{L}[\underline{g}]$$
$$= sH^{-1}(s) \int_0^{\infty} e^{-s\xi} \hbar(\tau_m - \xi) \underline{g}(\xi) d\xi$$
$$= sH^{-1}(s) \int_0^{\tau_m} e^{-s\xi} \underline{g}(\xi) d\xi .$$

Thus $\int_0^{\tau_m} K'(t-\xi)\underline{g}(\xi)d\xi$ and $sH^{-1}(s)\int_0^{\tau_m} e^{-s\xi}\underline{g}(\xi)d\xi$ form a transform pair, i. e.,

$$\int_0^{\tau_m} K'(t-\xi)\underline{g}(\xi)d\xi = \mathcal{L}^{-1}\left[sH^{-1}(s)\int_0^{\tau_m} e^{-s\xi}\underline{g}(\xi)d\xi\right] \tag{4.13}$$

and similarly, it can be shown that

$$\int_0^{\tau_m} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi = \mathcal{L}^{-1}\left[H^{-1}(s)A_i e^{-\tau_i s}\int_0^{\tau_m} e^{-s\xi}\underline{g}(\xi)d\xi\right]. \tag{4.14}$$

Thus we complete the proof for Equations (4.6) and (4.7). Then the formal solution to Equation (4.1) becomes as follows:

<u>Theorem 4.2.</u> (Verification of the formal solution). Suppose Equation (4.1) satisfies the Assumptions A1, A2, A3 and A5. Let $\underline{x}(t)$ be the continuous solution of Equation (4.1) with the initial function

$\underline{x}(t) = \underline{g}(t)$ for $0 \le t \le \tau_m$. Then, for $t > \tau_m$,

$$\underline{x}(t) = K(t-\tau_m)\underline{g}(\tau_m) + \sum_{i=0}^{m} \int_{\tau_m-\tau_i}^{\tau_m} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi + \int_{\tau_m}^{t} K(t-\xi)\underline{F}(\xi)d\xi. \quad (4.15)$$

Proof. First we will see the solution (4.15) indeed satisfies the original differential equation (4.1) for $t > \tau_m$. Differentiate Equation (4.15) and use (4.3) and (4.4),

$$\frac{d\underline{x}(t)}{dt} = K'(t-\tau_m)\underline{g}(\tau_m) + \sum_{j=0}^{m} \int_{\tau_m-\tau_j}^{\tau_m} K'(t-\tau_j-\xi)A_j\underline{g}(\xi)d\xi + \int_{\tau_m}^{t} K'(t-\xi)\underline{F}(\xi)d\xi + K(0)\underline{F}(t)$$

$$= \sum_{i=0}^{m} A_i \Big[K(t-\tau_i-\tau_m)\underline{g}(\tau_m) + \sum_{j=0}^{m} \int_{\tau_m-\tau_j}^{\tau_m} K(t-\tau_i-\tau_j-\xi)A_j\underline{g}(\xi)d\xi \quad (4.16)$$

$$+ \int_{\tau_m}^{t} K(t-\tau_i-\xi)\underline{F}(\xi)d\xi\Big] + \underline{F}(t) = \sum_{i=0}^{m} A_i\underline{x}(t-\tau_i) + \underline{F}(t)$$

thus satisfies the original differential equation for $t > \tau_m$. However, the solution $\underline{x}(t)$ becomes zero for $0 \le t \le \tau_m$ (since K(t) obeys expression (4.2)), thus does not satisfy the initial function $\underline{g}(t)$. In fact, the solution (4.15) is only defined for $t > \tau_m$.

In order to see the solution satisfies the initial function $\underline{g}(t)$, we look for an alternate formulation of the solution which covers the interval $0 \le t \le \tau_m$. That is, for $t > 0$,

$$\underline{x}(t)=K(t)\underline{g}(0)+\int_0^{\tau_m} K(t-\xi)\underline{g}'(\xi)d\xi-\sum_{i=0}^{m}\int_0^{\tau_m-\tau_i} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi+\int_{\tau_m}^{t} K(t-\xi)\underline{F}(\xi)d\xi \tag{4.17}$$

provided

$$\underline{g}(t)\in C'[0,\tau_m].$$

Solution (4.17) can also be verified immediately as in Equation (4.16). Thus, solutions (4.15) and (4.17) are equivalent except for $0\le t\le \tau_m$. This can be also shown by using integration by parts on the second term of the right hand side of Equation (4.17). Equation (4.17) reduces identically to Equation (4.15).

For $0\le t\le \tau_m$, using the condition (4.2), Equation (4.17) becomes

$$\underline{x}(t)=K(t)\underline{g}(0)+\int_0^{\tau_m} K(t-\xi)\underline{g}'(\xi)d\xi-\sum_{i=0}^{m}\int_0^{\tau_m-\tau_i} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi \tag{4.18}$$

thus the solution for $0\le t\le \tau_m$ is independent of the forcing function $\underline{F}(t)$.

In order to see Equation (4.18) is identical to $\underline{g}(t)$, it is necessary to reformulate the solution by Laplace transform. Expand the second term of Equation (4.18) by integration by parts,

$$\int_0^{\tau_m} K(t-\xi)\underline{g}'(\xi)d\xi=K(t-\tau_m)\underline{g}(\tau_m)-K(t)\underline{g}(0)+\int_0^{\tau_m} K'(t-\xi)\underline{g}(\xi)d\xi \tag{4.19}$$

and for $0\le t\le \tau_m$,

$$\int_{\tau_m-\tau_i}^{\tau_m} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi=0$$

since

$$K(t)=0' \quad \text{for} \quad t<0.$$

Thus the last term of Equation (4.18) becomes

$$\sum_{i=0}^{m}\left[\int_0^{\tau_m-\tau_i} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi+\int_{\tau_m-\tau_i}^{\tau_m} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi\right]=\sum_{i=0}^{m}\int_0^{\tau_m} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi. \tag{4.20}$$

Thus substitute Equations (4.19), (4.20) to (4.18),

$$\underline{x}(t)=K(t-\tau_m)\underline{g}(\tau_m)+\int_0^{\tau_m}\left[K'(t-\xi)-\sum_{i=0}^{m}K(t-\tau_i-\xi)A_i\right]\underline{g}(\xi)d\xi$$

but the first term vanishes for $0\le t\le \tau_m$, and using the results of Theorem 4.1, Equations (4.6) and (4.7), we have

$$\underline{x}(t)=\int_{Br} e^{ts}H^{-1}(s)\left[sI-\sum_{i=0}^{m}A_ie^{-\tau_i s}\right]\int_0^{\tau_m} e^{-s\xi}\underline{g}(\xi)d\xi\, ds=\int_{Br} e^{ts}\int_0^{\tau_m} e^{-s\xi}\underline{g}(\xi)d\xi\, ds. \tag{4.21}$$

Interchange the order of integration,

$$\underline{x}(t)=\int_0^{\tau_m}\left[\int_{Br} e^{(t-\xi)s}ds\right]\underline{g}(\xi)d\xi=\int_0^{\tau_m}\delta(t-\xi)\underline{g}(\xi)d\xi \tag{4.22}$$

$$=\underline{g}(t) \qquad \text{for} \qquad 0\le t\le \tau_m,$$

since we define the Dirac delta function $\delta(t)$ as

$$\mathcal{L}[\delta(t-\xi)] = \int_0^\infty e^{-st}\delta(t-\xi)dt = e^{-s\xi}$$

$$\mathcal{L}^{-1}[e^{-s\xi}] = \int_{Br} e^{(t-\xi)s} ds = \delta(t-\xi).$$

This completes the proof of Theorem 4.2, and establishes the verification of the solution (4.15) for the differential-difference equation (4.1).

From now on, $K(t)$ is called a 'matrix kernel' $K(t)$ for the differential-difference equation. We note the matrix kernel $K(t)$ as it appears in the solution scheme of Theorem 4.2 is exactly analogous to the principal matrix solution $X(t)$ for an ordinary differential equation

$$\frac{d\underline{x}}{dt} = A\underline{x} + \underline{F}(t) \quad , \quad \underline{x}(0) = \underline{c}$$

then

$$\underline{x}(t) = X(t)\underline{c} + \int_0^t X(t-\xi)\underline{F}(\xi)\, d\xi \tag{4.23}$$

where $X(t)$ satisfies

$$\frac{dX}{dt} = AX \quad , \quad X(0) = I.$$

Note also the solution (4.15) is made of homogeneous solution and particular solution analogous to the ordinary differential equation solution (4.23). However, the basic difficulty in differential-difference equation is that the matrix kernel K(t) is made of the residue contributions from an infinite number of poles, where the principal matrix solution X(t) is made of the residue contributions from a finite number of poles. We shall see how the difficulty can be overcome by understanding the nature of the kernel K(t) later in this Chapter.

Boundedness of the Formal Solution

Theorem 4.3. Suppose the system of differential-difference equation (4.1) satisfies

(i) Initial function $\underline{g}(t)$ is continuous and bounded.

i. e., $\|\underline{g}(t)\| \leq m_1$ for $0 \leq t \leq \tau_m$. (4.24)

(ii) Forcing function $\underline{F}(t)$ is continuous and bounded.

i. e., $\|\underline{F}(t)\| \leq m_2$ for all $t > 0$. (4.25)

(iii) Characteristic exponential polynomial

$$\det\left[zI - \sum_{i=0}^{m} A_i e^{-\tau_i z}\right] = 0$$ possesses the roots

with $\Re e\, z_i < 0$ for all i.

i. e., $\|K(t)\| \leq m_3 e^{\Re e\, z_1 t}$ (4.26)

then the solution (4.15) is bounded.

Proof. Take norms both sides of solution (4.15),

$$\|\underline{x}(t)\| \le \|K(t-\tau_m)\|\cdot\|\underline{g}(\tau_m)\| + \sum_{i=0}^{m}\int_{\tau_m-\tau_i}^{\tau_m}\|K(t-\tau_i-\xi)\|\cdot\|A_i\|\cdot\|\underline{g}(\xi)\|d\xi$$

$$+\int_{\tau_m}^{t}\|K(t-\xi)\|\cdot\|\underline{F}(\xi)\|d\xi.$$

Using (4.24), (4.25) and (4.26), let

$$m_4 = \max_{0\le i\le m}\|A_i\| \quad , \quad \sigma_1 = \Re e\, z_1$$

$$\|\underline{x}(t)\| \le m_1 m_3 e^{\sigma_1(t-\tau_m)} + m_1 m_3 m_4 \sum_{i=0}^{m} e^{\sigma_1(t-\tau_i)} \frac{\left[e^{-\sigma_1(\tau_m-\tau_i)} - e^{-\sigma_1\tau_m}\right]}{\sigma_1}$$

$$+ m_2 m_3 e^{\sigma_1 t}\frac{\left[e^{-\sigma_1\tau_m} - e^{-\sigma_1 t}\right]}{\sigma_1}.$$

Knowing $\sigma_1 < 0$, $\|\underline{x}(t)\|$ is bounded for all time $t > 0$, and further,

$$\lim_{t\to\infty}\|\underline{x}(t)\| \le \frac{m_2 m_3}{|\sigma_1|} < \infty. \qquad (4.27)$$

This result is expected since for a stable system (i.e., $\Re e\, z_1 < 0$) all the transient solution will die out eventually and we are left only with the steady state solution which is bounded.

## 4.1B Structure of the Matrix Kernel K(t)

Earlier in Theorem 4.1, it was shown that the matrix kernel

K(t) can be formulated as

$$K(t) = \int_{Br} e^{ts} H^{-1}(s)\, ds$$

with

$$H(s) = sI - \sum_{i=0}^{m} A_i e^{-\tau_i s} .$$

At this point, it is convenient to show how K(t) can be computed for a simple class of dynamic equation with single lag.

Consider

$$\ddot{x}(t) + c\dot{x}(t) + kx(t-\tau) = 0 \tag{4.28}$$

with

$$x(t) = g(t) \quad \text{for} \quad 0 \le t \le \tau.$$

Write (4.28) in vector-matrix form,

$$\frac{d\underline{x}(t)}{dt} = A_0 \underline{x}(t) + A_1 \underline{x}(t-\tau) \tag{4.29}$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & -c \end{bmatrix} , \quad A_1 = \begin{bmatrix} 0 & 0 \\ -k & 0 \end{bmatrix} , \quad \underline{x} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

then

$$H(s) = sI - A_0 - A_1 e^{-\tau s} = \begin{bmatrix} s & -1 \\ ke^{-\tau s} & s+c \end{bmatrix} \tag{4.30}$$

$$\therefore \qquad H^{-1}(s) = \frac{1}{\Delta}\begin{bmatrix} s+c & 1 \\ -ke^{-\tau s} & s \end{bmatrix} \tag{4.31}$$

$$\Delta = s^2 + cs + ke^{-\tau s}.$$

Thus

$$K(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{ts} H^{-1}(s)\,ds$$

$$= \frac{1}{2\pi i}\begin{bmatrix} \displaystyle\int_{b-i\infty}^{b+i\infty} \frac{(s+c)e^{ts}}{\Delta}\,ds & \displaystyle\int_{b-i\infty}^{b+i\infty} \frac{e^{ts}}{\Delta}\,ds \\ \displaystyle\int_{b-i\infty}^{b+i\infty} \frac{-ke^{(t-\tau)s}}{\Delta}\,ds & \displaystyle\int_{b-i\infty}^{b+i\infty} \frac{se^{ts}}{\Delta}\,ds \end{bmatrix}. \tag{4.32}$$

Each element of the matrix (4.32) can be computed by summing the infinite number of residue contributions. From the investigation of Chapter III, we recognize $\Delta(z)$ as a characteristic exponential polynomial, thus possesses infinite chain of roots. For each simple zero $z_i$ of $\Delta(z) = z^2 + cz + ke^{-\tau z} = 0$,

$$\int_{b-i\infty}^{b+i\infty} \frac{(s+c)e^{ts}}{\Delta}\,ds = 2\pi i \sum_{i=1}^{\infty} \text{Residue } (z_i)$$

$$= 2\pi i \sum_{i=1}^{\infty} \left[\lim_{s\to z_i} \frac{(s-z_i)(s+c)e^{ts}}{\Delta}\right].$$

Using the l'Hopital's rule to evaluate the residue,

$$\int_{b-i\infty}^{b+i\infty} \frac{(s+c)e^{ts}}{\Delta} ds = 2\pi i \sum_{i=1}^{\infty} \frac{(z_i+c)e^{tz_i}}{\Delta'(z_i)} \tag{4.33}$$

where

$$\Delta'(z_i) = 2z_i + c - k\tau e^{-\tau z_i}.$$

Similarly,

$$\int_{b-i\infty}^{b+i\infty} \frac{e^{ts}}{\Delta} ds = 2\pi i \sum_{i=1}^{\infty} \frac{e^{tz_i}}{\Delta'(z_i)} \tag{4.34}$$

$$\int_{b-i\infty}^{b+i\infty} \frac{-ke^{(t-\tau)s}}{\Delta} ds = 2\pi i \sum_{i=1}^{\infty} \frac{-ke^{(t-\tau)z_i}}{\Delta'(z_i)} \tag{4.35}$$

$$\int_{b-i\infty}^{b+i\infty} \frac{s\,e^{ts}}{\Delta} ds = 2\pi i \sum_{i=1}^{\infty} \frac{z_i e^{tz_i}}{\Delta'(z_i)} \tag{4.36}$$

Thus the matrix kernel K(t) can be computed from Equations (4.33), (4.34), (4.35) and (4.36),

$$K(t) = \sum_{i=1}^{\infty} e^{z_i t} R_i \quad \text{for} \quad t > 0 \tag{4.37}$$

where $R_i$ is the $2 \times 2$ matrix

$$R_i = \frac{1}{\Delta'(z_i)} \begin{bmatrix} z_i + c & 1 \\ -ke^{-\tau z_i} & z_i \end{bmatrix}.$$

At this point, the infinite series representation of the kernel (4.37) is partitioned into two parts for pragmatic computational reasons. Let

$$K(t) = K_N(t) + K_\varepsilon(t) = \sum_{i=1}^{2N} e^{z_i t} R_i + \sum_{i=2N+1}^{\infty} e^{z_i t} R_i \tag{4.38}$$

where $K_N(t)$ represents the finite-summed kernel which is the residue contributions of N-pair of rightmost roots, and $K_\varepsilon(t)$ is the error kernel with the remaining residue contributions.

In the process of developing the periodic solution scheme, we shall use the truncated N-kernel $K_N(t)$ exclusively, and the error induced by neglecting the error kernel $K_\varepsilon(t)$ will be rigorously analyzed in Section 4.3 of Error Analyses. It will be shown that the error is not only bounded, but also can be made arbitrarily small by increasing N for the N-kernel. Also, due to the exponential nature of the kernel, we recognize $K_N(t-\xi)$ to be of a separable kernel type, i.e.,

$$K_N(t-\xi) = \sum_{i=1}^{2N} e^{z_i t} \cdot e^{-z_i \xi} R_i . \tag{4.39}$$

## 4.2 Construction of Exact Periodic Solution by Fredholm Integral Equation Method

With the knowledge of the formal solution (4.15) and the properties of the kernel as in Equations (4.38) and (4.39), we proceed to construct the exact periodic solution of a forced linear differential-difference equation. The word "exact" is used here in a sense that the error caused by using the truncated N-kernel $K_N(t)$ can be made arbitrarily small by taking N sufficiently large.

The new method, which will be called Fredholm Integral Equation Method, is derived here to provide a constructive algorithm for general solutions of linear differential-difference equations. Examples will be given for the most interesting type of solutions, namely, periodic solutions; however, transient solutions can also be obtained by the same method.

In order to illustrate how the method is precisely derived, it is appropriate to introduce a basic theory of Fredholm integral equations at this point.

## 4.2A A Theory of Fredholm Integral Equations

Consider a Fredholm integral equation of second type in vector-matrix form of dimension n,

$$\underline{\varphi}(x) = \underline{f}(x) + K_N(x, 0)\underline{\varphi}(a) + \lambda \int_0^a K_N(x, y)\underline{\varphi}(y)dy \quad , \quad a > 0 \quad (4.40)$$

where $\underline{\varphi}(x)$ is the unknown vector function and $K_N(x, y)$ is the given matrix kernel. Then from the classical theory of integral equations, the solution of Equation (4.40) is given as infinite Neumann series in general whose convergence is only guaranteed for $\lambda$ sufficiently small. However, when the matrix kernel $K_N(x, y)$ is of separable type, i.e.,

$$K_N(x, y) = \sum_{i=1}^{N} X_i(x)Y_i(y) \quad (4.41)$$

then the Fredholm integral equation (4.40) reduces identically to (nN+n) system of algebraic equations. This separable class of kernel is called a Pincherle-Goursat kernel or, briefly, a PG-kernel. It is evident that the matrix kernel in the formal solution (4.15) is always a PG-kernel as in Equation (4.39).

Since the process of reducing a Fredholm integral equation with PG-kernel to a system of algebraic equations is the most important step of the entire solution scheme, we shall treat it in some detail here.

First, substitute Equation (4.41) to (4.40),

$$\underline{\varphi}(x) = \underline{f}(x) + \sum_{i=1}^{N} X_i(x) Y_i(0) \underline{\varphi}(a) + \lambda \sum_{i=1}^{N} X_i(x) \int_0^a Y_i(y) \underline{\varphi}(y) dy \qquad (4.42)$$

and let us seek the solution $\underline{\varphi}(x)$ over the fixed interval $x_0 \leq x \leq x_0 + a$, $x_0 > a$, and let

$$\begin{aligned} \underline{\varphi}_1(x) &= \underline{\varphi}(x) \qquad \text{for} \quad 0 \leq x \leq a \\ \underline{\varphi}_2(x) &= \underline{\varphi}(x + x_0) \qquad \text{for} \quad 0 \leq x \leq a. \end{aligned} \qquad (4.43)$$

Then at $x = a + x_0$, Equation (4.42) becomes

$$\underline{\varphi}_2(a) = \underline{f}(a + x_0) + \sum_{i=1}^{N} X_i(a + x_0) Y_i(0) \underline{\varphi}_1(a) + \lambda \sum_{i=1}^{N} X_i(a + x_0) \underline{m}_{1i} \qquad (4.44)$$

if we define the integral moments of the unknowns $\underline{\varphi}_1(x)$, $\underline{\varphi}_2(x)$ as

$$\begin{aligned} \underline{m}_{1i} &= \int_0^a Y_i(y) \underline{\varphi}_1(y) dy \\ & \qquad\qquad\qquad i, j = 1, 2, \cdots, N. \\ \underline{m}_{2j} &= \int_0^a Y_j(x) \underline{\varphi}_2(x) dx \ . \end{aligned} \qquad (4.45)$$

Also from Equations (4.42) and (4.43),

$$\underline{\varphi}_2(x) = \underline{f}(x+x_0) + \sum_{i=1}^{N} X_i(x+x_0)Y_i(0)\underline{\varphi}_1(a) + \lambda \sum_{i=1}^{N} X_i(x+x_0)\underline{m}_{1i} \qquad (4.46)$$

and premultiply both sides of Equation (4.46) by $Y_j(x)$ and integrate from 0 to a, we have

$$\underline{m}_{2j} = \underline{h}_j + \sum_{i=1}^{N} Q_{ji} Y_i(0)\underline{\varphi}_1(a) + \lambda \sum_{i=1}^{N} Q_{ji}\underline{m}_{1i} \quad , \quad j = 1, 2, \cdots, N \qquad (4.47)$$

with

$$\underline{h}_j = \int_0^a Y_j(x)\underline{f}(x+x_0)dx \quad , \quad \text{n-size vector}$$

$$Q_{ji} = \int_0^a Y_j(x)X_i(x+x_0)dx \quad , \quad n \times n \text{ matrix.}$$

Now combine Equations (4.44) and (4.47) and construct the $(nN+n)$ system of algebraic equations

$$\{\underline{\varphi}_2\} = [M]\{\underline{\varphi}_1\} + \{\underline{h}\} \qquad (4.48)$$

with $(nN+n)$ size vectors

$$\{\underline{\varphi}_1\} = \begin{Bmatrix} \underline{\varphi}_1(a) \\ \underline{m}_{11} \\ \underline{m}_{12} \\ \vdots \\ \underline{m}_{1N} \end{Bmatrix} , \quad \{\underline{\varphi}_2\} = \begin{Bmatrix} \underline{\varphi}_2(a) \\ \underline{m}_{21} \\ \underline{m}_{22} \\ \vdots \\ \underline{m}_{2N} \end{Bmatrix} , \quad \{\underline{h}\} = \begin{Bmatrix} \underline{f}(a+x_0) \\ \underline{h}_1 \\ \underline{h}_2 \\ \vdots \\ \underline{h}_N \end{Bmatrix} \qquad (4.49)$$

and $(nN+n) \times (nN+n)$ matrix

$$[M] = \begin{bmatrix} \sum_{j=1}^{N} X_j(a+x_0)Y_j(0) & \lambda X_1(a+x_0) & \cdots \lambda X_N(a+x_0) \\ \sum_{j=1}^{N} Q_{1j}Y_j(0) & \lambda Q_{11} & \cdots \lambda Q_{1N} \\ \vdots & \vdots & \vdots \\ \sum_{j=1}^{N} Q_{Nj}Y_j(0) & \lambda Q_{N1} & \cdots \lambda Q_{NN} \end{bmatrix} . \quad (4.50)$$

Thus we have established the transfer relationship between two state vectors $\{\underline{\varphi}_1\}$ and $\{\underline{\varphi}_2\}$ which are exactly $x_0$ apart, and the matrix [M] is called the transfer matrix.

The matrix equation (4.48) can be posed in two ways. First, given the initial vector $\{\underline{\varphi}_1\}$ at one point, then the state vector at any other point $\{\underline{\varphi}_2\}$ can be computed by Equation (4.48) since [M] and $\{\underline{h}\}$ are solely depended on $x_0$. Second, when a periodic solution is sought with known period T, then the periodicity requirement is

$$\{\underline{\varphi}_2\} = [M(T)]\{\underline{\varphi}_1\} + \{\underline{h}(T)\} = \{\underline{\varphi}_1\}$$

$$\therefore \quad \{\underline{\varphi}_1\} = [I - M(T)]^{-1}\{\underline{h}(T)\}. \quad (4.51)$$

The first case implies the transient solution of the integral equation (4.40) can be constructed by Equation (4.48), and the second case is for the steady state periodic solutions. This will eventually lead to the construction of periodic solutions for linear and piecewise linear differential-difference equations.

It is to be emphasized that the equivalence between the Fredholm integral equation (4.40) and the matrix equation (4.48) is mathematically exact, thus yielding the exact solution in terms of the integral moments. Once the integral moments are known, the solution $\underline{\varphi}(x)$ itself is immediate from the relationship (4.42).

We note the formal solution (4.15) of a linear differential-difference equation can be transformed into a Fredholm integral equation (4.40) as one seeks the periodic solution such that $\underline{x}(t+T) = \underline{x}(t) = \underline{g}(t)$ for $0 \le t \le \tau_m$. Then the periodic solution is obtained as Equation (4.51), and this is how the name Fredholm Integral Equation Method for the periodic solution is derived. We will see later in Chapter V that the method is ideally suited for piecewise linear systems with time delays where the closed form periodic solution is not known otherwise.

## 4.2B Single Delay Case

In order to illustrate the constructive solution scheme by the Fredholm Integral Equation Method, a simple case of dynamic system with single delay is chosen. We note the method is well suited for a more general class of Equation (4.1).

Consider the following problem

$$\ddot{x}(t) + c\dot{x}(t) + kx(t-\tau) = p \cos(\omega t + \psi) \tag{4.52}$$

with initial function

$$x(t) = g(t) \quad \text{for} \quad 0 \le t \le \tau.$$

Assumptions are:

A1) $\underline{g}(t)$ is continuous and bounded, but unknown.

A2) Homogeneous solution of Equation (4.52) is stable, i. e., roots of $z^2 + cz + ke^{-\tau z} = 0$ have $\Re e\, z_i < 0$ for all i.

A3) System parameters, c, k, $\tau$, p, $\omega$ are given.

A4) The phase angle $\psi$ between forcing function and the solution is unknown.

Then find the steady state periodic solution subject to the symmetry condition as shown in Figure 4.1.

$$x\left(\frac{T}{2}+\tau\right)=0 \tag{4.53}$$

where $T=\frac{\pi}{\omega}$ , half period of the forcing function.

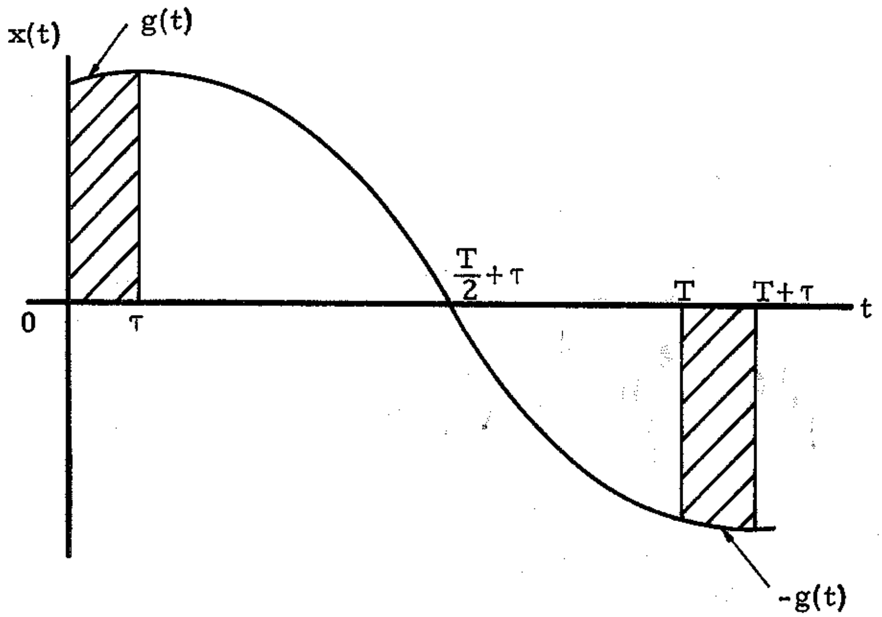


Figure 4.1 Formulation of the problem (4.52)

Since the homogeneous solution is assumed to be stable, the steady state periodic solution must possess the same period $2T = \frac{2\pi}{\omega}$ as the forcing function.

We note the periodic solution of the forced linear differential-difference equation (4.52) can be obtained immediately by the method of harmonic balance, and this is given in Section 4.4. Although the harmonic balance method is simple and exact for the linear delay systems, it can only be extended for piecewise linear delay systems to yield approximate solutions. Since the Fredholm Integral Equation Method is ultimately for the piecewise linear delay systems to give exact periodic solutions, we shall see how the method is applied to a linear delay system.

Rewrite Equation (4.52) in vector-matrix form,

$$\frac{d\underline{x}(t)}{dt} = A\underline{x}(t) + B\underline{x}(t-\tau) + \underline{F}(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -c \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -k & 0 \end{bmatrix}, \quad \underline{F}(t) = \begin{pmatrix} 0 \\ p\cos(\omega t + \psi) \end{pmatrix} \tag{4.54}$$

with initial function

$$\underline{x}(t) = \underline{g}_1(t) = \begin{pmatrix} g(t) \\ g'(t) \end{pmatrix} \quad \text{for} \quad 0 \le t \le \tau.$$

Then the formal solution can be written as in Equation (4.15),

$$\underline{x}(t) = K(t-\tau)\underline{g}_1(\tau) + \int_0^\tau K(t-\tau-\xi)B\underline{g}_1(\xi)d\xi + \int_\tau^t K(t-\xi)\underline{F}(\xi)d\xi \quad \text{for} \quad t > \tau. \tag{4.55}$$

At this point we shall replace the matrix kernel K(t) with the finite N-kernel $K_N(t)$, and denote the corresponding unknown initial function and solution as $\underline{g}_{1N}(t)$, $\underline{x}_N(t)$ respectively. Then Equation (4.55) becomes

$$\underline{x}_N(t) = K_N(t-\tau)\underline{g}_{1N}(\tau) + \int_0^\tau K_N(t-\tau-\xi)B\underline{g}_{1N}(\xi)d\xi + \int_\tau^t K_N(t-\xi)\underline{F}(\xi)d\xi \tag{4.56}$$

where the matrix kernel is given as in Equation (4.38)

$$K_N(t) = \sum_{i=1}^{2N} e^{z_i t} R_i \quad \text{for} \quad t > 0$$

$$K_N(t) = I \quad \text{for} \quad t = 0 \tag{4.57}$$

$$= 0 \quad \text{for} \quad t < 0$$

with $z_i$ being the roots of

$$z^2 + cz + ke^{-\tau z} = 0$$

and

$$R_i = \frac{1}{\Delta_i}\begin{bmatrix} z_i + c & 1 \\ -ke^{-\tau z_i} & z_i \end{bmatrix}$$

$$\Delta_i = 2z_i + c - \tau k e^{-\tau z_i}$$

as in Equation (4.37). Substitute Equation (4.57) to (4.56), for $t > \tau$

$$\underline{x}_N(t) = \sum_{i=1}^{2N} e^{z_i(t-2\tau)} R_i \left[ e^{z_i \tau} \underline{g}_{1N}(\tau) + \int_0^{\tau} e^{z_i(\tau-\xi)} B \underline{g}_{1N}(\xi) d\xi \right] + \sum_{i=1}^{2N} R_i \underline{f}_i(t) \qquad (4.58)$$

with the integrated forcing term $\underline{f}_i(t)$ being

$$\underline{f}_i(t) = p \left\{ \begin{matrix} 0 \\ \\ \{C_i(t,t) - C_i(t,\tau)\} \cos\psi - \{S_i(t,t) - S_i(t,\tau)\} \sin\psi \end{matrix} \right\}$$

where the new symbols $C_i$ and $S_i$ are defined as

$$C_i(t_1, t_2) = (\omega \sin \omega t_2 - z_i \cos \omega t_2) e^{z_i(t_1 - t_2)} / (\omega^2 + z_i^2)$$

$$S_i(t_1, t_2) = -(\omega \cos \omega t_2 + z_i \sin \omega t_2) e^{z_i(t_1 - t_2)} / (\omega^2 + z_i^2).$$

Now we define the integral moments of the initial functions

$$\underline{m}_{1i} = \int_0^{\tau} e^{z_i(\tau-\xi)} B \underline{g}_{1N}(\xi) d\xi \qquad (4.59a)$$

$$i = 1, 2, \cdots, N$$

$$\underline{m}_{2i} = \int_0^{\tau} e^{z_i(\tau-\xi)} B \underline{g}_{2N}(\xi) d\xi \qquad (4.59b)$$

where

$$\underline{g}_{2N}(t) = \underline{x}_N(t + T) \quad , \quad 0 \le t \le \tau. \qquad (4.60)$$

Thus $\underline{g}_{2N}(t)$ has become the new initial function for the second half period of the solution.

Examining Equation (4.58), the solution is composed of the integral moments of the initial function and the forcing terms. Since the solution is completely determined in terms of the integral moments of the initial function, which are constant vectors dependent only on the root $z_i$, then it becomes logical to treat the integral moments as if they are initial condition vectors in ordinary differential equations.

Thus we can treat the solution in a finite (2N + 2) dimensional vector space where the solution is completely determined by (2N + 2) size initial vectors

$$\{\underline{g}_1\} = \begin{Bmatrix} \underline{g}_{1N}(\tau) \\ \underline{m}_{11} \\ \underline{m}_{12} \\ \vdots \\ \underline{m}_{1N} \end{Bmatrix} , \quad \{\underline{g}_2\} = \begin{Bmatrix} \underline{g}_{2N}(\tau) \\ \underline{m}_{21} \\ \underline{m}_{22} \\ \vdots \\ \underline{m}_{2N} \end{Bmatrix} . \tag{4.61}$$

Then we proceed to construct $\{\underline{g}_2\}$ in terms of $\{\underline{g}_1\}$ as they are related by Equations (4.60) and (4.58).

$$\underline{g}_{2N}(\tau) = \underline{x}_N(\tau + T) = \sum_{i=1}^{2N} e^{z_i(T-\tau)} R_i \left[ e^{z_i \tau} \underline{g}_{1N}(\tau) + \underline{m}_{1i} \right] + \sum_{i=1}^{2N} R_i \underline{f}_i(\tau + T) \tag{4.62}$$

$$\begin{aligned} \underline{m}_{2i} &= \int_0^{\tau} e^{z_i(\tau-\xi)} B\, \underline{x}_N(\xi + T) d\xi \quad \text{for} \quad i = 1, 2, \cdots, N \\ &= \sum_{j=1}^{2N} e^{z_j(T-2\tau)} q_{ji} B R_j \left[ e^{z_j \tau} \underline{g}_{1N}(\tau) + \underline{m}_{1j} \right] + \underline{h}_i \end{aligned} \tag{4.63}$$

with

$$q_{ji} = \frac{e^{z_j \tau} - e^{z_i \tau}}{z_j - z_i} \quad , \quad i \neq j$$

$$= \tau e^{z_i \tau} \quad , \quad i = j$$

and

$$\underline{h}_i = \sum_{j=1}^{2N} \int_0^{\tau} e^{z_i(\tau-\xi)} B R_j \underline{f}_j(\xi+T) d\xi = \begin{pmatrix} 0 \\ h_i \end{pmatrix} . \qquad (4.64)$$

Integral moment of the forcing term $h_i$ is integrated out to be

$$h_i = k p \cos\psi \sum_{j=1}^{2N} \frac{1}{\Delta_j} \Big[ S_j(T, T) \{S_i(\tau, \tau) - S_i(\tau, 0)\}$$

$$- C_j(T, T) \{C_i(\tau, \tau) - C_i(\tau, 0)\} + C_j(T, \tau) q_{ji} \Big]$$

$$+ k p \sin\psi \sum_{j=1}^{2N} \frac{1}{\Delta_j} \Big[ S_j(T, T) \{C_i(\tau, \tau) - C_i(\tau, 0)\}$$

$$+ C_j(T, T) \{S_i(\tau, \tau) - S_i(\tau, 0)\} - S_j(T, \tau) q_{ji} \Big] .$$

<u>The Transfer Matrix [M]</u>

Combine the results (4.62) and (4.63) in the form of (4.48) and the resulting vector-matrix equation becomes

$$\{\underline{g}_2\} = [M] \{\underline{g}_1\} + \{\underline{h}\} \qquad (4.65a)$$

where

$$[M] = 2\mathcal{R}e \begin{bmatrix} \sum_{j=1}^{N} e^{z_j T} R_j & e^{z_1(T-\tau)} R_1 & \cdots & e^{z_N(T-\tau)} R_N \\ \sum_{j=1}^{N} e^{z_j(T-\tau)} q_{j1} B R_j & e^{z_1(T-2\tau)} q_{11} B R_1 & \cdots & e^{z_N(T-2\tau)} q_{N1} B R_N \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{N} e^{z_j(T-\tau)} q_{jN} B R_j & e^{z_1(T-2\tau)} q_{1N} B R_1 & \cdots & e^{z_N(T-2\tau)} q_{NN} B R_N \end{bmatrix} \tag{4.65b}$$

$(2N+2) \times (2N+2)$ matrix

and

$$\{\underline{h}\} = \begin{Bmatrix} \sum_{j=1}^{2N} R_j \underline{f}_j(\tau + T) \\ \underline{h}_1 \\ \underline{h}_2 \\ \vdots \\ \underline{h}_N \end{Bmatrix} . \qquad (2N+2) \text{ size vector} \tag{4.65c}$$

Reduction of the problem to Equation (4.65) form is an important step for the solution method of Fredholm Integral Equation. The initial vector at one point $t = t_1$ can be related to the initial vector at another point $t = t_2$, $t_2 > t_1 \geq \tau$, by the linear relationship (4.65), and this is exactly analogous to Equation (4.48) of the integral equation theory. From this relationship the matrix is called the "transfer matrix" which is the sole function of the transfer time $T = t_2 - t_1$ and the homogeneous part of the original differential equation (4.52).

If we let $\tau = 0$, Equations (4.59a), (4.59b) and (4.63) yield

$$\underline{m}_{1i} = \underline{m}_{2i} = 0 \quad , \quad q_{ji} = 0 \quad \text{for} \quad i, j = 1, 2, \cdots, N$$

thus the $(2N+2) \times (2N+2)$ matrix equation, (4.65a), reduces to a simple $2 \times 2$ relationship,

$$g_{2N}(0) = \sum_{i=1}^{2} e^{z_i T} R_i g_{1N}(0) + \sum_{i=1}^{2} R_i \underline{f}_i(T) \tag{4.66}$$

And this is indeed the case of a second order ordinary differential equation with forcing term, since Equation (4.66) is equivalent to

$$\underline{x}(T) = X(T)\,\underline{x}(0) + \int_0^T X(T-\xi)\underline{F}(\xi)d\xi$$

where X(t) is the principal matrix solution for the homogeneous ordinary differential equation.

As it was seen in the general discussion of Chapter II, a general solution of a constant coefficient differential-difference equation spans infinite dimensional space $E_\infty$. However, the constructive solution method presented here is to reduce the solution into (2N + 2) finite dimensional space $E_{2N+2}$, which is not only manageable from the viewpoint of practical computations, but also mathematically rigorous about being able to bound the error induced by truncating the kernel.

Closure Condition (Periodicity Requirement)

In order to have a periodic solution, it is required that the identical initial vector is reproduced after every full period 2T, or that the initial vectors are skew-symmetric at the half period T interval. That is,

$$\{\underline{g}_2\} = -\{\underline{g}_1\} = [M]\{\underline{g}_1\} + \{\underline{h}\}. \tag{4.67}$$

This is in essence the closure condition which states the periodicity requirement for the solution. Combine Equations (4.65) and (4.67) and we can solve for $\{g_1\}$

$$\{\underline{g}_1\} = -[M+I]^{-1} \cdot \{\underline{h}\} \tag{4.68}$$

where [I] being (2N + 2) × (2N + 2) identity matrix. Construction of the periodic solution from the formal solution kernel-integral form of Equation (4.56) to the matrix equation (4.68), is conceived by recognizing Equation (4.56) as a Fredholm integral equation under the closure condition

$$\underline{g}_{2N}(t) = \underline{x}_N(t+T) = -\underline{g}_{1N}(t) \quad \text{for} \quad 0 \le t \le \tau. \tag{4.69}$$

Then the formal solution (4.56) becomes

$$-\underline{g}_{1N}(t)=K_N(t+T-\tau)\underline{g}_{1N}(\tau)+\int_0^{\tau}K_N(t+T-\tau-\xi)B\,\underline{g}_{1N}(\xi)d\xi+\int_{\tau}^{T+t}K_N(t+T-\xi)\underline{F}(\xi)d\xi. \tag{4.70}$$

We recognize Equation (4.70) as a Fredholm integral equation of the second type in vector-matrix equation (4.40) with the close analogy between Equations (4.40) and (4.70),

$$\begin{aligned}
\underline{\varphi}(x) &\Rightarrow \underline{g}_{1N}(t)\\
\underline{f}(x) &\Rightarrow -\int_{\tau}^{T+t}K_N(t+T-\xi)\underline{F}(\xi)d\xi\\
K_N(x,y) &\Rightarrow -K_N(t+T-\tau-\xi)B \qquad (4.71)\\
\lambda &\Rightarrow -I\\
a &\Rightarrow \tau.
\end{aligned}$$

Then the solution process follows precisely as described in Section 4.2A.

<u>Calculation of the Unknown Phase $\psi$</u>

Construction of the periodic solution by Equation (4.68) is not yet complete since the integral moments of forcing term $\{\underline{h}\}$ possess the unknown phase angle $\psi$ between the forcing function and the solution. $\{\underline{h}\}$ can be written as

$$\{\underline{h}\}=\{\underline{h}_c\}\cos\psi+\{\underline{h}_s\}\sin\psi. \tag{4.72}$$

Then the solution (4.68) becomes

$$\{\underline{g}_1\} = -[M+I]^{-1}(\{\underline{h}_c\}\cos\psi + \{\underline{h}_s\}\sin\psi).$$

Thus the initial vector $\{\underline{g}_1\}$ is given in terms of two unknowns $\cos\psi$ and $\sin\psi$. Knowing $\{\underline{g}_1\}$, the time domain solution $\underline{x}_N(t)$ immediately follows from Equation (4.58) in terms of $\cos\psi$ and $\sin\psi$. Now apply the symmetry condition (4.53) which yields a scalar equation

$$c_1 \cos\psi + c_2 \sin\psi = 0 \tag{4.73}$$

$c_1$, $c_2$ are known constants.

Solve Equation (4.73) simultaneously with $\cos^2\psi + \sin^2\psi = 1$,

$$\cos\psi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \quad , \quad \sin\psi = \frac{-c_1}{\sqrt{c_1^2 + c_2^2}} \; . \tag{4.74}$$

Now the periodic solution to the differential-difference equation is complete, which satisfies the governing equation as well as the symmetry condition.

## 4.2C Multiple Delay Case

It is to be pointed out that the solution scheme of the Fredholm Integral Equation Method is completely adaptable to the general multiple delay differential-difference equation (4.1) with a slight increase in algebra in the process of forming the transfer matrix [M]. To illustrate this, consider a forced linear dynamic system with two delay terms,

$$\frac{d\underline{x}(t)}{dt} = A\,\underline{x}(t) + B\,\underline{x}(t-\tau_1) + C\,\underline{x}(t-\tau_2) + \underline{F}(t)$$

$$0 < \tau_1 < \tau_2 \qquad (4.75)$$

with initial function

$$\underline{x}(t) = \underline{g}_1(t) \quad \text{for} \quad 0 \le t \le \tau_2.$$

Assumptions are

A1) $\underline{g}_1(t)$ is continuous and bounded, but unknown.

A2) $\underline{F}(t)$ is continuous, bounded and periodic.

$\underline{F}(t+2T) = \underline{F}(t)$

A3) Homogeneous solution of Equation (4.75) is stable, i. e., roots of $\det\left[zI - A - Be^{-\tau_1 z} - Ce^{-\tau_2 z}\right] = 0$ possesses $\mathrm{Re}\, z_i < 0$ for all i.

Then the problem is to find the steady state periodic solution which possesses the same period as the forcing function.

The formal solution can be written from Equation (4.15) with the truncated N-kernel,

$$\underline{x}_N(t) = K_N(t-\tau_2)\underline{g}_{1N}(\tau_2) + \int_0^{\tau_2}\left[K_N(t-\tau_1-\xi)\,\hbar(\xi-\tau_2+\tau_1)B + K_N(t-\tau_2-\xi)C\right]\underline{g}_{1N}(\xi)d\xi$$

$$+ \int_{\tau_2}^{t} K_N(t-\xi)F(\xi)d\xi \quad , \qquad \text{for} \qquad t > \tau_2. \qquad (4.76)$$

Here $\hbar(t)$ implies the Heaviside function, and the matrix kernel $K_N(t)$ is given as

$$K_N(t) = \sum_{i=1}^{2N} e^{z_i t} R_i \qquad \text{for} \quad t > 0$$

$$= I \qquad \text{for} \quad t = 0$$

$$= 0 \qquad \text{for} \quad t < 0.$$

Define the integral moments of initial functions

$$\underline{m}_{1i} = \int_0^{\tau_2} e^{z_i(\tau_2 - \xi)} \left[ e^{z_i(\tau_2 - \tau_1)} \hbar(\xi - \tau_2 + \tau_1) B + C \right] \underline{g}_{1N}(\xi) d\xi \tag{4.77a}$$

$$\underline{m}_{2i} = \int_0^{\tau_2} e^{z_i(\tau_2 - \xi)} \left[ e^{z_i(\tau_2 - \tau_1)} \hbar(\xi - \tau_2 + \tau_1) B + C \right] \underline{g}_{2N}(\xi) d\xi \tag{4.77b}$$

where

$$\underline{g}_{2N}(t) = \underline{x}_N(t + T) \quad , \quad 0 \le t \le \tau_2.$$

Construct $(2N+2)$ size initial vectors

$$\{\underline{g}_1\} = \begin{Bmatrix} \underline{g}_{1N}(\tau_2) \\ \underline{m}_{11} \\ \underline{m}_{12} \\ \vdots \\ \underline{m}_{1N} \end{Bmatrix} \quad , \quad \{\underline{g}_2\} = \begin{Bmatrix} \underline{g}_{2N}(\tau_2) \\ \underline{m}_{21} \\ \underline{m}_{22} \\ \vdots \\ \underline{m}_{2N} \end{Bmatrix} \tag{4.78}$$

then following the identical steps of formulating the transfer matrix [M] as in the single delay case, we get

$$\{\underline{g}_2\} = [M] \{\underline{g}_1\} + \{\underline{h}\} \tag{4.79a}$$

with the $(2N+2) \times (2N+2)$ transfer matrix

$$[M] = 2\,\mathcal{R}e \begin{bmatrix} \sum_{j=1}^{N} e^{z_j T} R_j & e^{z_1(T-\tau_2)} R_1 & \cdots & e^{z_N(T-\tau_2)} R_N \\ \sum_{j=1}^{N} e^{z_j(T-\tau_2)} Q_{j1} R_j & e^{z_1(T-2\tau_2)} Q_{11} R_1 & \cdots & e^{z_N(T-2\tau_2)} Q_{N1} R_N \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{N} e^{z_j(T-\tau_2)} Q_{jN} R_j & e^{z_1(T-2\tau_2)} Q_{1N} R_1 & \cdots & e^{z_N(T-2\tau_2)} Q_{NN} R_N \end{bmatrix} \tag{4.79b}$$

with $(2 \times 2)$ matrix $Q_{ji}$ given as

$$Q_{ji} = \frac{1}{z_j - z_i}\left[\left\{e^{z_j\tau_2 + z_i(\tau_2-\tau_1)} - e^{z_i\tau_2 + z_j(\tau_2-\tau_1)}\right\}B + \left\{e^{z_j\tau_2} - e^{z_i\tau_2}\right\}C\right] \quad \text{for} \quad i \neq j$$

$$= e^{z_i\tau_2}\left[\tau_1 e^{z_i(\tau_2-\tau_1)} B + \tau_2 C\right] \quad \text{for} \quad i = j$$

$$i, j = 1, 2, \cdots, N$$

and the integral moments of forcing vector $\{\underline{h}\}$ is

$$\{\underline{h}\} = \left\{\begin{array}{l} \sum_{i=1}^{2N} R_i \underline{f}_i(\tau_2 + T) \\ \underline{h}_1 \\ \underline{h}_2 \\ \vdots \\ \underline{h}_N \end{array}\right\} \qquad (2N+2) \text{ size vector} \qquad (4.79c)$$

with

$$\underline{h}_i = \sum_{j=1}^{2N} \int_0^{\tau_2} e^{z_i(\tau_2 - \xi)} \left[ e^{z_i(\tau_2 - \tau_1)} \hbar(\xi - \tau_2 + \tau_1) B + C \right] R_j \underline{f}_j(\xi + T) d\xi$$

$$i = 1, 2, \ldots, N$$

while $\underline{f}_j(t)$ is obtained similarly as in the single delay case.

Thus it is shown that the formal solution for the multiple delay case, Equation (4.76) is reduced to $(2N+2)$ system of algebraic equations (4.79) by the Fredholm Integral Equation Method. From here on, the periodic solution is found precisely the same way as the single delay case.

## 4.3 Error Analyses of the Periodic Solution

It is mentioned several times earlier that the entire justification of the Fredholm Integral Equation Method is based on the fact that we can bound the error caused by truncating the matrix kernel K(t). In fact, the arbitrary smallness of the error enables us to call the periodic solution "exact". With this in mind, we shall show the error bound for the periodic solution of linear differential-difference equations.

Reconsider Equation (4.1)

$$\frac{d\underline{x}(t)}{dt} = \sum_{i=0}^{m} A_i \underline{x}(t-\tau_i) + \underline{F}(t)$$

with initial function

$$\underline{x}(t) = \underline{g}(t) \quad \text{for} \quad 0 \le t \le \tau_m$$

with the same assumptions A1 ~ A5. Recall the formal solution to (4.1) is given as an integral form of Equation (4.15)

$$\underline{x}(t) = K(t-\tau_m)\underline{g}(\tau_m) + \sum_{i=0}^{m} \int_{\tau_m-\tau_i}^{\tau_m} K(t-\tau_i-\xi)A_i\underline{g}(\xi)d\xi$$

$$+ \int_{\tau_m}^{t} K(t-\xi)\underline{F}(\xi)d\xi \quad , \quad \text{for} \quad t > \tau_m$$

where the matrix kernel K(t) is given as an infinite series

$$K(t) = \sum_{i=1}^{\infty} e^{z_i t} R_i .$$

Suppose the characteristic roots $z_i$ are ordered as follows:

$$0 > \Re e\, z_1 > \Re e\, z_2 > \cdots > \Re e\, z_N > \Re e\, z_{N+1} > \cdots$$

and define

$$\sigma_1 = \Re e\, z_1 \quad , \quad \sigma_\epsilon = \Re e\, z_{N+1} .$$

Rewrite the kernel of (4.37) in two parts,

$$K(t) = K_N(t) + K_\varepsilon(t) = \sum_{i=1}^{2N} e^{z_i t} R_i + \sum_{i=2N+1}^{\infty} e^{z_i t} R_i. \tag{4.80}$$

Here $K_N(t)$ represents the finite summed N-kernel used in the Fredholm Integral Equation Method and $K_\varepsilon(t)$ represents the error kernel which is truncated.

Since the kernels are composed of exponential terms, we recognize the kernels as uniformly converging series, and the norms of the matrix kernels can be bounded as

$$\begin{aligned} \|K(t)\| &\le m\, e^{\sigma_1 t} \\ \|K_\varepsilon(t)\| &\le m_\varepsilon e^{\sigma_\varepsilon t} \end{aligned} \quad , \quad t > 0 \tag{4.81}$$

for $m, m_\varepsilon$ positive constants.

<u>Error Bound Formulation</u>

Let us denote

$\underline{x}(t)$ ... exact solution

$\underline{x}_N(t)$ ... solution calculated by Fredholm Integral Equation Method with N-pair of terms in $K_N(t)$

$\underline{x}_\varepsilon(t)$ ... error in the solution by truncating the matrix kernel.

Then

$$\underline{x}(t) = \underline{x}_N(t) + \underline{x}_\varepsilon(t), \tag{4.82}$$

$$\underline{g}(t) = \underline{g}_N(t) + \underline{g}_\epsilon(t) \quad , \quad 0 \leq t \leq \tau_m . \tag{4.83}$$

Adopting the supremum norm notation, let

$$G = \sup_{0 \leq t \leq \tau_m} \|\underline{g}(t)\|$$

$$G \leq G_N + G_\epsilon$$

where $G_N$ is bounded.

Substitute Equations (4.80), (4.82) and (4.83) into the formal solution (4.15)

$$\begin{aligned} \underline{x}(t) = \underline{x}_N(t) + \underline{x}_\epsilon(t) &= \left[K_N(t-\tau_m) + K_\epsilon(t-\tau_m)\right]\left\{\underline{g}_N(\tau_m) + \underline{g}_\epsilon(\tau_m)\right\} \\ &+ \sum_{i=0}^{m} \int_{\tau_m - \tau_i}^{\tau_m} \left[K_N(t-\tau_i-\xi) + K_\epsilon(t-\tau_i-\xi)\right] A_i \left\{\underline{g}_N(\xi) + \underline{g}_\epsilon(\xi)\right\} d\xi \\ &+ \int_{\tau_m}^{t} \left[K_N(t-\xi) + K_\epsilon(t-\xi)\right] \underline{F}(\xi) d\xi \end{aligned} \tag{4.84}$$

and the solution by the Fredholm Integral Equation Method satisfies

$$\begin{aligned} \underline{x}_N(t) = K_N(t-\tau_m)\underline{g}_N(\tau_m) &+ \sum_{i=0}^{m} \int_{\tau_m-\tau_i}^{\tau_m} K_N(t-\tau_i-\xi) A_i \underline{g}_N(\xi) d\xi \\ &+ \int_{\tau_m}^{t} K_N(t-\xi)\underline{F}(\xi) d\xi . \end{aligned} \tag{4.85}$$

Subtract Equation (4.85) from (4.84),

$$\underline{x}_\epsilon(t) = K_\epsilon(t-\tau_m)\underline{g}_N(\tau_m) + \sum_{i=0}^{m} \int_{\tau_m-\tau_i}^{\tau_m} K_\epsilon(t-\tau_i-\xi)A_i\underline{g}_N(\xi)d\xi$$

$$+\int_{\tau_m}^{t} K_\epsilon(t-\xi)\underline{F}(\xi)d\xi + K(t-\tau_m)\underline{g}_\epsilon(\tau_m) + \sum_{i=0}^{m} \int_{\tau_m-\tau_i}^{\tau_m} K(t-\tau_i-\xi)A_i\underline{g}_\epsilon(\xi)d\xi. \qquad (4.86)$$

Since the kernel $K_\epsilon(t)$ and the initial function $\underline{g}_N(t)$ by the Fredholm Integral Equation Method is known,

$$\underline{g}^*(t) = K_\epsilon(t-\tau_m)\underline{g}_N(\tau_m) + \sum_{i=0}^{m} \int_{\tau_m-\tau_i}^{\tau_m} K_\epsilon(t-\tau_i-\xi)A_i\underline{g}_N(\xi)d\xi$$

$$+\int_{\tau_m}^{t} K_\epsilon(t-\xi)\underline{F}(\xi)d\xi \qquad (4.87)$$

is known.

Now we observe the closure condition

$$\underline{x}(t+T) = \underline{x}_N(t+T) + \underline{x}_\epsilon(t+T)$$

$$= -\underline{x}(t) = -[\underline{x}_N(t) + \underline{x}_\epsilon(t)]$$

for T being half-period of the forcing function $\underline{F}(t)$. Then for $0 \le t \le \tau_m$,

$$\underline{x}_\epsilon(t+T) = -\underline{x}_\epsilon(t) = -\underline{g}_\epsilon(t). \qquad (4.88)$$

Thus let $t = \eta + T$, $0 \le \eta \le \tau_m$ in Equation (4.86) and apply the closure condition (4.88),

$$-\underline{g}_\epsilon(\eta) = \underline{g}^*(\eta+T) + K(\eta+T-\tau_m)\underline{g}_\epsilon(\tau_m) + \sum_{i=0}^{m} \int_{\tau_m-\tau_i}^{\tau_m} K(\eta+T-\tau_i-\xi)A_i\underline{g}_\epsilon(\xi)d\xi$$

$$\text{for} \quad 0 \le \eta \le \tau_m. \qquad (4.89)$$

This represents the error in the initial function $g_\epsilon(\eta)$ in an integral equation form, and the upper bound for $\|g_\epsilon(\tau)\|$ is to be pursued here since it represents the error bound for the periodic solution computed by the Fredholm Integral Equation Method.

First, let $\eta = \tau_m$ in Equation (4.89),

$$g_\epsilon(\tau_m) = -[I+K(T)]^{-1}\left\{g^*(\tau_m+T)+\sum_{i=0}^{m}\int_{\tau_m-\tau_i}^{\tau_m} K(\tau_m+T-\tau_i-\xi)A_i g_\epsilon(\xi)d\xi\right\} \quad (4.90)$$

Define an $n \times n$ matrix

$$W(\eta) = K(\eta+T-\tau_m)[I+K(T)]^{-1}, \quad (4.91)$$

then Equations (4.90), (4.91) and (4.89) yield

$$g_\epsilon(\eta) = \hat{h}(\eta) + \int_0^{\tau_m} \hat{K}(\eta,\xi)g_\epsilon(\xi)d\xi \quad , \quad 0 \le \eta \le \tau_m \quad (4.92a)$$

with

$$\hat{h}(\eta) = -g^*(\eta+T) + W(\eta)g^*(\tau_m+T) \quad (4.92b)$$

$$\hat{K}(\eta,\xi) = \sum_{i=0}^{m}[W(\eta)K(\tau_m+T-\tau_i-\xi) - K(\eta+T-\tau_i-\xi)]\, \hbar(\xi-\tau_m+\tau_i)A_i. \quad (4.92c)$$

Thus Equation (4.92) represents the error in the initial function $g_\epsilon(\eta)$ in the form of a Fredholm integral equation of the second type. In order to seek the solution, we search for a converging Neumann's sequence suggested by the recurrence relationship

$$\underline{g}_\epsilon^k(\eta) = \underline{\hat{h}}(\eta) + \int_0^{\tau_m} \hat{K}(\eta,\xi)\underline{g}_\epsilon^{k-1}(\xi)d\xi \tag{4.93}$$

with

$$\underline{g}_\epsilon^0(\eta) = \underline{\hat{h}}(\eta),$$

then

$$\|\underline{g}_\epsilon^1(\eta) - \underline{g}_\epsilon^0(\eta)\| \leq \int_0^{\tau_m} \|\hat{K}(\eta,\xi)\| \cdot \|\underline{\hat{h}}(\xi)\| d\xi.$$

Now define

$$\lambda = \tau_m \sup \|\hat{K}(\eta,\xi)\| \tag{4.94a}$$

$$\mu = \sup \|\underline{\hat{h}}(\xi)\| < \infty \tag{4.94b}$$

and the supremum is taken over the variables $0 \leq \eta \leq \tau_m$, $0 \leq \xi \leq \tau_m$, unless otherwise specified, then

$$\sup \|\underline{g}_\epsilon^1(\eta) - \underline{g}_\epsilon^0(\eta)\| \leq \lambda\,\mu.$$

Similarly,

$$\|\underline{g}_\epsilon^2(\eta) - \underline{g}_\epsilon^1(\eta)\| \leq \int_0^{\tau_m}\int_0^{\tau_m} \|\hat{K}(\eta,\xi_2)\| \cdot \|\hat{K}(\xi_2,\xi_1)\| \cdot \|\underline{\hat{h}}(\xi_1)\| d\xi_1 d\xi_2$$

$$\therefore \qquad \sup \|\underline{g}_\epsilon^2(\eta) - \underline{g}_\epsilon^1(\eta)\| \leq \lambda^2 \mu$$

$$\cdots\cdots\cdots$$

$$\sup \|\underline{g}_\epsilon^k(\eta) - \underline{g}_\epsilon^{k-1}(\eta)\| \leq \lambda^k \mu.$$

Constructing a telescopic series

$$\|g_\epsilon^k(\eta)\| \le \|g_\epsilon^k - g_\epsilon^{k-1}\| + \|g_\epsilon^{k-1} - g_\epsilon^{k-2}\| + \cdots$$

$$\cdots + \|g_\epsilon^2 - g_\epsilon^1\| + \|g_\epsilon^1 - g_\epsilon^0\| + \|g_\epsilon^0\|$$

$$\therefore \qquad \sup\|g_\epsilon^k(\eta)\| \le (\lambda^k + \lambda^{k-1} + \cdots + \lambda^2 + \lambda + 1)\mu$$

$$\sup\|g_\epsilon(\eta)\| \le \left(\lim_{k\to\infty} \sum_{i=0}^{k} \lambda^i\right)\mu \tag{4.95}$$

and knowing $\mu < \infty$, the series can be summed as

$$\mathcal{G}_\epsilon = \sup\|g_\epsilon(\eta)\| \le \frac{\mu}{1-\lambda} \tag{4.96}$$

provided $|\lambda| < 1$. Thus it becomes crucial to show $|\lambda| < 1$, since the series (4.95) converges if and only if $|\lambda| < 1$ holds true.

So far it is shown that the error in the solution is bounded as in Equation (4.96) provided certain conditions are met. At this point, it is appropriate to show the following asymptotic properties of the error bound.

## Asymptotic Properties of the Error Bound $\mathcal{G}_\epsilon$

Consider Equation (4.94a) with (4.92c) and (4.91), and let $\tau_m$ approach zero,

$$\lim_{\tau_m\to 0} \lambda = \lim_{\tau_m\to 0} \tau_m \cdot \sup\|\hat{K}(\eta, \xi)\|$$

and

$$\lim_{\tau_m\to 0} \sup\|\hat{K}(\eta, \xi)\| \le \sum_{i=0}^{m} \left\{\|K(T)\| \cdot \|[I+K(T)]^{-1}\| \cdot \|K(T)\| + \|K(T)\|\right\} \cdot \|A_i\| \tag{4.97}$$

Since the matrix kernel K(t) is continuous and bounded for finite $t = T$, and $[I + K(T)]$ is non-singular for $T > 0$, the expression (4.97) is clearly bounded, say $\le k_1$. Therefore,

$$\lim_{\tau_m \to 0} \lambda = \lim_{\tau_m \to 0} \tau_m k_1 = 0 = \lambda_0 . \tag{4.98}$$

Similarly, consider Equations (4.94b) with (4.92b) and (4.87),

$$\lim_{\tau_m \to 0} \mu = \lim_{\tau_m \to 0} \sup \|\underline{\hat{h}}(\xi)\|$$

$$\le \{\|I\| + \|K(T)\| \cdot \|[I+K(T)]^{-1}\|\} \cdot \lim_{\tau_m \to 0} \|\underline{g}^*(T)\|$$

but from the nature of the characteristic root distribution,

$$\lim_{\tau_m \to 0} \|K_\epsilon(t)\| = 0 \quad \Rightarrow \quad \lim_{\tau_m \to 0} \|\underline{g}^*(T)\| = 0$$

$$\therefore \qquad \lim_{\tau_m \to 0} \mu = 0 = \mu_0 . \tag{4.99}$$

Thus Equations (4.98) and (4.99) yield

$$\lim_{\tau_m \to 0} \sigma_\epsilon \le \lim_{\tau_m \to 0} \frac{\mu}{1-\lambda} = 0. \tag{4.100}$$

This implies that as the delay terms approach zero, in other words as the given linear differential-difference equation system approaches ordinary differential equation status, the solution error due to truncation of the kernel also approaches asymptotically to zero. This result is quite expected, since the distribution of the characteristic roots from Chapter III shows that infinite chain of roots vanishes to the negative infinity except the leading n roots where n is the order of the differential equation. This implies $\|K_\varepsilon(t)\| \to 0$ since $N > n$, thus $\mu \to 0$, and eventually the error bound $\mathcal{G}_\varepsilon$ approaches asymptotically to zero.

Now consider the case when $\tau_m > 0$, sufficiently small but bounded away from zero. The error bound for this case is established by invoking the local implicit function theorem. It is necessary to show that in order to have $0 < \lambda < 1$, there must exist some value of $\tau_m = \tau^* > 0$ so that the error bound (4.96) be valid. First consider an implicit function of $\tau$

$$f(\lambda;\tau) = \mathcal{G}_\varepsilon - \frac{\mu}{1-\lambda} + r$$

with

$$r \geq 0.$$

At $\tau = 0$,

$$f(\lambda_0;\tau_0) = \mathcal{G}_\varepsilon - \frac{\mu_0}{1-\lambda_0} = 0 \quad , \quad r = 0$$

since $\mu_0 = \lambda_0 = \mathcal{G}_\varepsilon = 0$ at $\tau = \tau_0 = 0$. At this point it is appropriate to quote the general implicit function theorem proved for n-dimensional vector space.

<u>Theorem 4.4</u> (Implicit Function Theorem)*

Let $\underline{f} = (f_1, f_2, \ldots, f_n)$ be a vector-valued function defined on an open set S in $E_{n+k}$, (n+k) dimensional Euclidean space, with values in

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*See Apostol[26], p. 147 for the complete proof.

$E_n$. Suppose $\underline{f} \in C'$ on S. Let $(\underline{\lambda}_0; \underline{\tau}_0)$ be a point in S for which $\underline{f}(\underline{\lambda}_0; \underline{\tau}_0) = 0$ and for which the $n \times n$ determinant Jacobian $\det[D_j f_i(\underline{\lambda}_0; \underline{\tau}_0)] \neq 0$. Then there exists a k-dimensional vector space neighborhood $T_0$ of $\underline{\tau}_0$ and a unique vector-valued function $\underline{\lambda}$ defined on $T_0$ and having values in $E_n$, $\ni$

i) $\underline{\lambda} \in C'$ on $T_0$

ii) $\underline{\lambda}(\underline{\tau}_0) = \underline{\lambda}_0$

iii) $\underline{f}(\underline{\lambda}(\underline{\tau}); \underline{\tau}) = 0$ for all $\underline{\tau}$ in $T_0$

Applying the theorem to our case, $n = k = 1$, there must exist $\tau_m = \tau^*$ in the neighborhood of $\tau_m = \tau_0 = 0$, $\ni \lambda(\tau_0) = \lambda_0 = 0$, and $\lambda(\tau_m) \in C'$, thus there must exist $\lambda \ni 0 < \lambda < 1$, and thus convergence of the error bound is established.

Furthermore, we shall show that the error bound $G_\epsilon$ can be made arbitrarily small by taking more terms in the truncated kernel $K_N(t)$. This is the case when $\tau_m > 0$ but $\tau_m \in T_0$ and N approaches infinity.

Reconsider Equation (4.94a) and by inspection $\lambda$ is independent of the number of terms in the truncated kernel N, and from the implicit function theorem, we know $0 < \lambda < 1$ for $\tau_m = \tau^* > 0$. Reconsider Equation (4.94b),

$$\mu = \sup \| W(\eta)\underline{g}^*(\tau_m + T) - \underline{g}^*(\eta + T) \|.$$

But from Equation (4.87),

$$\lim_{N \to \infty} \|\underline{g}^*(\eta)\| = 0 \quad \text{since} \quad \lim_{N \to \infty} \|K_\epsilon(t)\| = 0$$

and knowing $W(\eta)$ is bounded,

$$\lim_{N\to\infty} \mu = 0$$

and thus

$$\lim_{N\to\infty} \alpha_\epsilon \le \lim_{N\to\infty} \frac{\mu}{1-\lambda} = 0. \tag{4.101}$$

This implies the error bound decreases exponentially as N increases, and approaches asymptotically to zero as $N\to\infty$.

Computation of the Error Bound

Since the qualitative error bound has been proved with its existence and asymptotic nature, it is desired to investigate the quantitative aspects of the error bound. Without loss of generality, we consider a simple case of second order dynamic system with single delay occurring in the zeroth-order terms,

$$\ddot{x}(t) + 2\zeta\dot{x}(t) + x(t-\gamma) = p\cos(\omega t+\psi) \tag{4.102}$$

with initial function

$$x(t) = g(t) \quad \text{for} \quad 0 \le t \le \gamma.$$

Rewrite in vector-matrix form,

$$\frac{d\underline{x}(t)}{dt} = A\,\underline{x}(t) + B\,\underline{x}(t-\gamma) + \underline{F}(t) \tag{4.103}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2\zeta \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad F = \begin{pmatrix} 0 \\ p\cos(\omega t+\psi) \end{pmatrix}$$

and it is desired to investigate the error bound in the periodic solution by the Fredholm Integral Equation Method, namely, $\mathcal{G}_\epsilon \leq \frac{\mu}{1-\lambda}$.

Recall from Equation (4.94) that $\mu$, $\lambda$ are functions of the supremum norms of the matrix kernels $K_\epsilon(t)$, and $K(t)$. It becomes crucial to estimate $\|K_\epsilon(t)\|$ as well as $\|K(t)\|$.

Consider the error kernel of the system (4.103)

$$K_\epsilon(t) = \sum_{j=2N+1}^{\infty} e^{z_j t} R_j$$

with $2 \times 2$ matrix as given in Equation (4.37),

$$R_j = \frac{1}{2z_j + 2\zeta - \gamma e^{-\gamma z_j}} \begin{bmatrix} z_j + 2\zeta & 1 \\ -e^{-\gamma z_j} & z_j \end{bmatrix} .$$

Since the characteristic roots $z_j$ must occur as conjugate pairs,

$$K_\epsilon(t) = 2 \sum_{j=N+1}^{\infty} \Re e\,[e^{z_j t} R_j]$$

or

$$\|K_\epsilon(t)\| \leq \sum_{j=N+1}^{\infty} m_j(t) \tag{4.104}$$

with $m_j(t) = 2\|\Re e\,[e^{z_j t} R_j]\|$.

If we choose the norm of a matrix to be

$$\|A\| = \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}| ,$$

then

$$m_j(t, \sigma_j) \le \frac{2e^{\sigma_j t}}{\sqrt{\alpha_j^2 + \beta_j^2}} \left[1 + 2|\sigma_j + \zeta + \omega_j| + \sqrt{2}e^{-\gamma\sigma_j}\right] \tag{4.105}$$

with

$$z_j = \sigma_j + i\,\omega_j$$

$$\alpha_j = 2\sigma_j + 2\zeta - \gamma e^{-\gamma\sigma_j} \cos \gamma\omega_j$$

$$\beta_j = 2\omega_j + \gamma e^{-\gamma\sigma_j} \sin \gamma\omega_j$$

and

$$\omega_j = e^{-(\gamma/2)\sigma_j} \quad \text{as } \sigma \to -\infty.$$

Inspecting Equation (4.105), $m_j(t, \sigma_j)$ is a monotone decreasing function of $\sigma_j$ since $\sigma_j < 0$ for all j, and graphically this is shown in Figure 4.2.

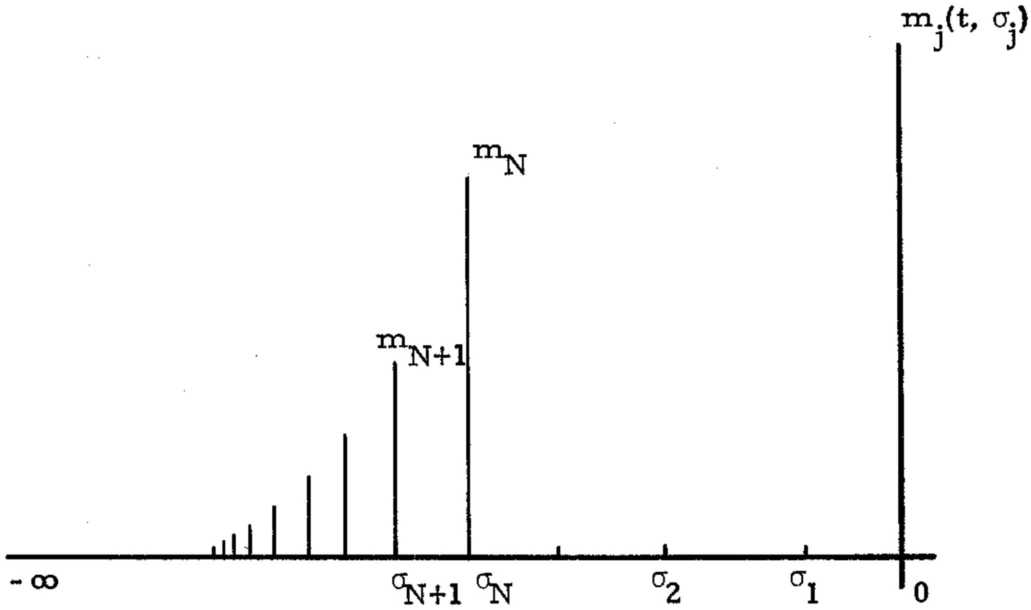


Figure 4.2 Monotone Decreasing Function of $m_j(t, \sigma_j)$

Now the idea is to convert the infinite summation of Equation (4.104) into a Riemann integration process such that the $\sigma$ dependence is integrated out.

Rewrite Equation (4.104) with the knowledge of Figure 4.2,

$$\begin{aligned} \|K_\varepsilon(t)\| &\le \sum_{j=N+1}^{\infty} m_j(t, \sigma_j) \\ &\le \sum_{j=N+1}^{\infty} \frac{1}{\sigma_{j-1} - \sigma_j} \int_{\sigma_j}^{\sigma_{j-1}} m(t, \sigma) d\sigma . \end{aligned} \tag{4.106}$$

If it is possible to find the expression $u(\sigma)$ such that

$$\sigma_{j-1} - \sigma_j \ge u(\sigma) \quad \text{for all } j = N+1, \cdots \tag{4.107}$$

then the infinite summation can be written as

$$\|K_\epsilon(t)\| \le \int_{-\infty}^{\sigma_N} \frac{m(t,\sigma)}{u(\sigma)}\, d\sigma. \tag{4.108}$$

In order to find a suitable expression for $u(\sigma)$, it is necessary to know the characteristic root distribution study of Chapter III. In particular, combine Equations (3.15) and (3.24),

$$\tan\left(\gamma e^{-\frac{\gamma}{2}\sigma}\right) = \frac{2(\sigma+\zeta)e^{-\frac{\gamma}{2}\sigma}}{e^{-\gamma\sigma} - \sigma^2 - 2\zeta\sigma} = 0 \quad \text{as} \quad \sigma \to -\infty$$

$$\therefore \qquad \gamma e^{-\frac{\gamma}{2}\sigma_j} = j\pi \quad , \quad j = 0, \pm 1, \pm 2, \cdots \tag{4.109}$$

and

$$\sigma_{j-1} - \sigma_j = \ln\left(\frac{j}{j-1}\right)^{2/\gamma} = -\frac{2}{\gamma}\ln\left(1 - \frac{1}{j}\right)$$

but $j \ge N+1$, $0 < \frac{1}{j} < 1$, thus expand $\ln(1-\frac{1}{j})$ in power series,

$$\sigma_{j-1} - \sigma_j = \frac{2}{\gamma}\left(\frac{1}{j} + \frac{1}{2j^2} + \frac{1}{3j^3} + \cdots\right) > \frac{2}{\gamma j}. \tag{4.110}$$

A suitable choice of $u(\sigma)$ is made by combining Equations (4.110) and (4.109),

$$u(\sigma) = \frac{2}{\gamma j} = \frac{2\pi}{\gamma^2} e^{\frac{\gamma}{2}\sigma} \quad \text{as} \quad \sigma \to -\infty \tag{4.111}$$

and finally the norm of $K_\epsilon(t)$ is integrated out to be, from Equation (4.108),

$$\|K_\epsilon(t)\| \le \frac{2\gamma}{\pi}\left[\frac{1}{t} + \frac{\sqrt{2}}{2t-\gamma}\, e^{-\frac{\gamma}{2}\sigma_N}\right] e^{\sigma_N t} \quad \text{for} \quad t > \gamma. \tag{4.112}$$

Also, the asymptotic behavior of the norm becomes clear by inspection,

$$\lim_{\gamma \to 0} \|K_\epsilon(t)\| = 0 \qquad \text{since } \sigma_N \to -\infty$$

$$\lim_{N \to \infty} \|K_\epsilon(t)\| = 0 \qquad \text{since } \sigma_N \to -\infty.$$

Once the error kernel norm $\|K_\epsilon(t)\|$ is specified, the whole kernel norm $\|K(t)\|$ is given immediately,

$$\|K(t)\| \le \|K_N(t)\| + \|K_\epsilon(t)\| \tag{4.113}$$

with

$$\|K_N(t)\| \le \sum_{j=1}^{N} m_j(t)$$

where $m_j(t)$ is given in Equation (4.105).

Knowing the upper bounds of the kernels for this particular case of single delay, then the error bound for the periodic solution $\mathfrak{a}_\epsilon$ can be computed. From Equations (4.94b), (4.92b), (4.87) and simplify

$$\begin{aligned} \mu &= \sup \|\underline{\hat{h}}(\xi)\| \\ &\le (1 - \mathfrak{a}_w)\,\mathfrak{a}_g \end{aligned} \tag{4.114}$$

with

$$\mathfrak{a}_w = \inf_{0 \le \eta \le T} \|W(\eta)\| = \frac{\|K(T)\|}{\|I\| + \|K(T)\|} < 1$$

$$\mathfrak{a}_g = \sup \|\underline{g}^*(\eta + T)\|$$

$$\leq [(1+\gamma)G_N + (T-\gamma)p] \cdot \|K_\epsilon(T-\gamma)\|$$

and

$$G_N = \sup \|g_N(\xi)\| < \infty.$$

Similarly from Equations (4.94a), (4.92c), (4.91) and simplify

$$\begin{aligned} \lambda &= \gamma \sup \|\hat{K}(\eta, \xi)\| \\ &\leq \gamma (1-G_w) \|K(T-\gamma)\| \end{aligned} \tag{4.115}$$

and for $v$ sufficiently small, $0 < \lambda < 1$. Combine the results of Equations (4.114) and (4.115),

$$\begin{aligned} G_\epsilon &\leq \frac{\mu}{1-\lambda} \\ &\leq \frac{(1-G_w)\, G_g}{1-\gamma(1-G_w)\|K(T-\gamma)\|} \end{aligned}$$

or

$$C_\epsilon \leq \frac{[(1+\gamma)G_N + (T-\gamma)p] \cdot \|K_\epsilon(T-\gamma)\|}{1 + \frac{1}{2}\|K(T)\| - \gamma \|K(T-\gamma)\|}. \tag{4.116}$$

This represents a constructive algorithm for computing the error bound of the periodic solution of Equation (4.102) by the Fredholm Integral Equation Method. The representation also verifies the asymptotic properties of the error bound, namely Equations (4.100) and (4.101) immediately.

## 4.4 Conventional Solution Method by Harmonic Balance

As the last part of this chapter, the conventional method of harmonic balance is studied to show the exact correlation with the solution by the Fredholm Integral Equation Method. Note the harmonic balance solution is exact only for the linear differential-difference system.

Consider a single delay dynamic equation

$$\ddot{x}(t) + c\dot{x} + kx(t-\tau) = p\cos\omega t \tag{4.117}$$

with initial function

$$x(t) = g(t) \quad \text{for} \quad 0 \le t \le \tau.$$

Assuming the homogeneous solution is stable, the linear system (4.117) must possess a unique periodic solution with the same period as the forcing function.

Assume a solution of the form

$$\begin{aligned} x(t) &= c_1\cos\omega t + c_2\sin\omega t \\ x(t-\tau) &= (c_1\cos\omega\tau - c_2\sin\omega\tau)\cos\omega t + (c_1\sin\omega\tau + c_2\cos\omega\tau)\sin\omega t. \end{aligned} \tag{4.118}$$

Substitute (4.118) to (4.117), we have

$$\begin{aligned} (-\omega^2 + k\cos\omega\tau)c_1 + (c\omega - k\sin\omega\tau)c_2 &= p \\ (-c\omega + k\sin\omega\tau)c_1 + (-\omega^2 + k\cos\omega\tau)c_2 &= 0. \end{aligned}$$

And $c_1$, $c_2$ are solved to give amplitude and phase of the solution

$$X = \sqrt{c_1^2 + c_2^2} = \frac{p/k}{\sqrt{\left(\cos\omega\tau - \frac{\omega^2}{k}\right)^2 + \left(\frac{c\omega}{k} - \sin\omega\tau\right)^2}}$$

$$\psi = \tan^{-1}\frac{c_2}{c_1} = \frac{c\omega - k\sin\omega\tau}{-\omega^2 + k\cos\omega\tau}.$$

Define the non-dimensional parameters

$$\gamma = \sqrt{k}\,\tau \quad , \quad \zeta = \frac{c}{2\sqrt{k}} \quad , \quad \Omega = \frac{\omega}{\omega_n} \quad \text{(frequency ratio)}$$

with

$$\omega_n = \sqrt{k} \quad , \quad X_0 = \frac{p}{k} \, ,$$

then the magnification factor $X/X_0$ is

$$\frac{X}{X_0} = \frac{1}{\sqrt{[\cos(\gamma\Omega) - \Omega^2]^2 + [2\zeta\Omega - \sin(\gamma\Omega)]^2}} \tag{4.119}$$

and the phase angle

$$\psi = \tan^{-1}\left[\frac{2\zeta\Omega - \sin(\gamma\Omega)}{\cos(\gamma\Omega) - \Omega^2}\right]. \tag{4.120}$$

Plotting the Equations (4.119) and (4.120) for the frequency response curves of amplitude and phase angle, we get Figures 4.3 and 4.4. Note the difference from the ordinary differential equation case, that is the presence of an unstable solution as the case of $\zeta=0$ in the figures. This is clear from the phase angle plot since $\frac{d\psi}{d\Omega} < 0$ for $\zeta=0$, and this is predicted from the stability criteria by the Satche diagram of Figure 3.17.

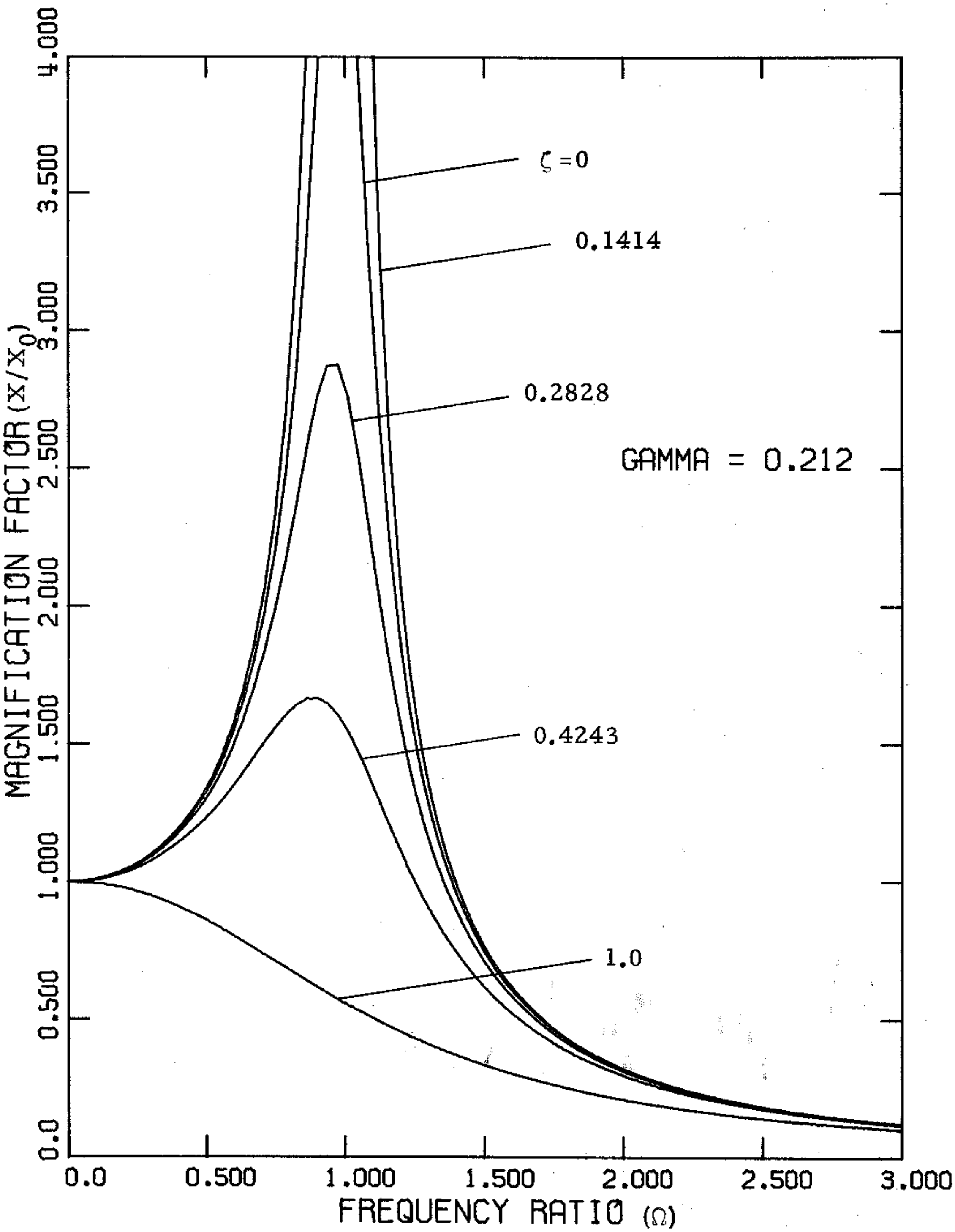


Figure 4.3 Amplitude Response Curves for $\ddot{x}(t)+c\dot{x}(t)+kx(t-\tau)=p\cos\omega t$.

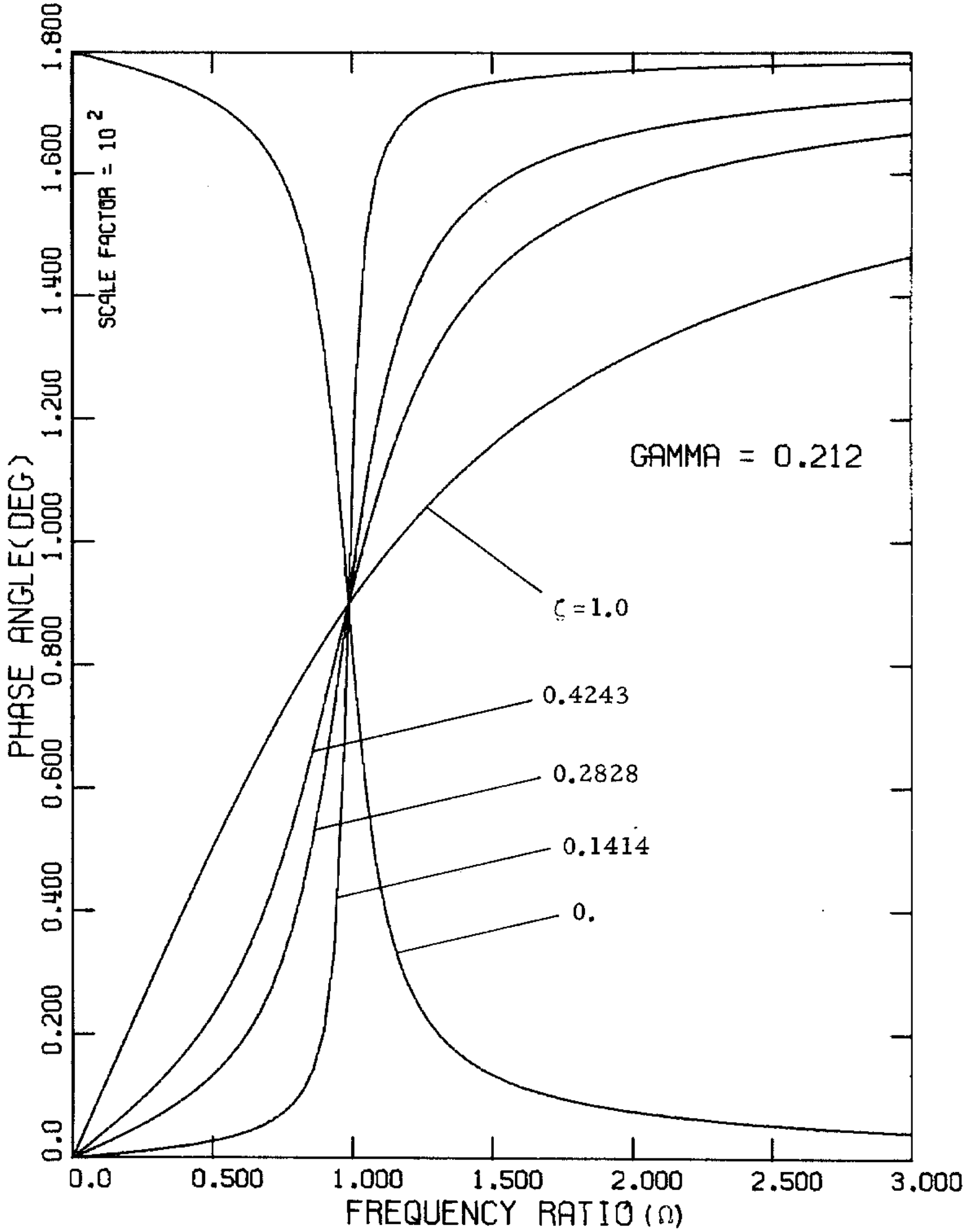


Figure 4.4 Phase Angle Response Curves for $\ddot{x}(t)+c\dot{x}(t)+kx(t-\tau)=p\cos\omega t$.

# Chapter V

# FORCED PIECEWISE LINEAR DYNAMIC SYSTEMS WITH TIME DELAYS

## 5.1 Problem Formulation

With the knowledge of forced linear delay systems discussed in Chapter IV, we will see how the same idea can be logically extended to a class of nonlinear delay systems, namely, piecewise linear delay systems. It is to be noted that for the linear delay system, the exact solution was readily obtained by the harmonic balance method, thus the real merit of the Fredholm Integral Equation Method is that the solution scheme for the linear delay system will serve as a building block for the piecewise linear delay system, since the harmonic balance method fails to provide an exact solution for this class of nonlinear delay systems.

Analyzing a nonlinear system in general, one is forced to consider the problem in a restricted sense: i.e., often times the existence and unicity of the solution is guaranteed for only local regions, and the closed form solution is in general not obtainable, thus forcing analysis by approximate methods which may fail to give validity of solutions in some interesting cases since the approximation error cannot be bounded.

For the sake of algebraic simplicity, we shall assume the problem to possess a single delay term for a typical piecewise linear dynamic system,

$$\ddot{x}(t) + c\dot{x}(t) + f(x(t-\tau)) = F(t) \tag{5.1}$$

with initial function

$$x(t) = g(t) \quad \text{for} \quad 0 \le t \le \tau$$

where the nonlinear restoring force term f(x) is piecewise linear in x and further we assume

$$\frac{df}{dx} < \infty \quad \text{for all x.} \tag{5.2}$$

## 5.2 Existence of a Unique Periodic Solution

Theorem 5.1 Suppose in Equation (5.1), the forcing function F(t) is piecewise continuous and bounded, and the initial function g(t) is continuous and bounded, then the problem (5.1) possesses a unique solution.

Proof

Rewrite Equation (5.1) in vector-matrix form,

$$\frac{d\underline{x}(t)}{dt} = \begin{pmatrix} \dot{x}(t) \\ \\ -c\dot{x}(t) - f(x(t-\tau)) + F(t) \end{pmatrix} = \underline{h}(t, \underline{x}(t), \underline{x}(t-\tau)) \tag{5.3}$$

with

$$\underline{x}(t) = \begin{pmatrix} x(t) \\ \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} g(t) \\ \\ \dot{g}(t) \end{pmatrix} \quad \text{for} \quad 0 \le t \le \tau .$$

Define $\underline{u} = \underline{x}(t)$, $\underline{v} = \underline{x}(t-\tau)$, then since F(t) is bounded,

$$\|\underline{h}(t, \underline{u}, \underline{v})\| \le m_1(t) \quad \text{for} \quad \|\underline{u}\| + \|\underline{v}\| \le c_1 . \tag{5.4}$$

Moreover, any pair of vectors $\underline{u}_1, \underline{u}_2$; $\underline{v}_1, \underline{v}_2$ lying in the phase plane satisfies

$$\|\underline{h}(t, \underline{u}_1, \underline{v}_1) - \underline{h}(t, \underline{u}_2, \underline{v}_2)\| \leq m_2(t)\left(\|\underline{u}_1 - \underline{u}_2\| + \|\underline{v}_1 - \underline{v}_2\|\right) \qquad (5.5)$$

since the first derivative of the nonlinear function f(x) is bounded. Thus, the conditions of the Cauchy-Lipschitz theorem for a non-autonomous delay system (see Bellman and Cooke[8], page 341) are satisfied, and thus a unique solution to Equation (5.1) must exist. It is to be noted that for the class of piecewise linear delay systems where Equation (5.2) is satisfied, the Lipschitz condition is satisfied everywhere in the phase plane, thus the unique solution exists globally. Also, it follows immediately from the Lipschitz condition that the solution is continuously depended on the initial function g(t).

Theorem 5.2 Suppose the Equation (5.1) satisfies the same hypotheses as Theorem 5.1. In addition to that, assume

A1) $F(t) = F(t+T)$,

A2) All characteristic roots of each linear region of f(x) possess negative real parts; i. e., solution of each linear region is stable,

then there exists at least one periodic solution of Equation (5.1), with the period T.

Proof

It was shown in Theorem 5.1 that the system (5.1) must possess a unique solution which may be written as

$$\underline{x}(t) = \underline{g}(t) \qquad \text{for} \qquad 0 \le t \le \tau$$

$$= \underline{g}(\tau) + \int_{\tau}^{t} \underline{h}(\xi, \underline{x}(\xi), \underline{x}(\xi-\tau))d\xi \qquad \text{for} \qquad t > \tau. \tag{5.6}$$

Hence there exists a continuous mapping M(T) which maps the solution $\underline{x}(t)$ in a finite set $\Omega$ into $\underline{x}(t+T)$, which is also in $\Omega$. This follows from the fact that the solution is ultimately bounded in $\Omega$ by the Assumption A2. Then, by the Brouwer's fixed point theorem (see Saaty[27], page 42), there must exist at least one fixed point $\underline{x}_0$ in $\Omega$ such that

$$\underline{x}_0(t+T) = M(T)\,\underline{x}_0(t) = \underline{x}_0(t). \tag{5.7}$$

Extending this result,

$$\underline{x}_0(t+2T) = M(T)\,\underline{x}_0(t+T) = M(T)\,\underline{x}_0(t) = \underline{x}_0(t)$$

- - - - - - - - - - - - - - - - - - -

$$\underline{x}_0(t+nT) = M(T)\,\underline{x}_0(t+(n-1)T) = \cdots = \underline{x}_0(t)$$

and this holds true for $n \to \infty$, thus establishing the existence of at least one periodic solution of (5.1) with the period T.

## 5.3 Construction of Exact Periodic Solution by Fredholm Integral Equation Method

For a class of nonlinear delay systems, namely piecewise linear systems, it was shown that there must exist a unique periodic solution for the problem as posed in Equation (5.1). As mentioned earlier, the process of Fredholm Integral Equation Method developed for the linear delay system will be used here to construct a periodic solution for a piecewise linear delay system.

Without loss of generality, we shall consider a trilinear delay system with single lag, with the implication that the same process will apply to a general n-piecewise linear delay system. In order to further simplify the algebra, trilinear delay terms are assumed to be doubly bilinear which is symmetric about the origin.

Graphical representation of the trilinear system is shown on Figure 5.1 along with typical representation of a periodic solution (shown for the half period).

Consider

$$\ddot{x}(t) + c\dot{x}(t) + f(x(t-\tau)) = p\cos(\omega t + \psi) \tag{5.8}$$

with

$$x(t) = g(t) \quad \text{for} \quad 0 \le t \le \tau$$

and

$$\begin{aligned} f(x) &= x-1+k && \text{for} \quad x > 1 \\ &= kx && \text{for} \quad |x| \le 1 \\ &= x+1-k && \text{for} \quad x < -1 \end{aligned}$$

where the restoring force term f(x) is doubly bilinear, and the system parameters, c, k, p, $\omega$, $\tau$ are given.

Then the problem is posed in the following manner. Find the periodic solution(s) to Equation (5.8) which possesses the same period as the forcing function and the $|x(t)|$ crosses +1 only once in the quarter period as shown in Figure 5. 1. Unknowns of the problem are:

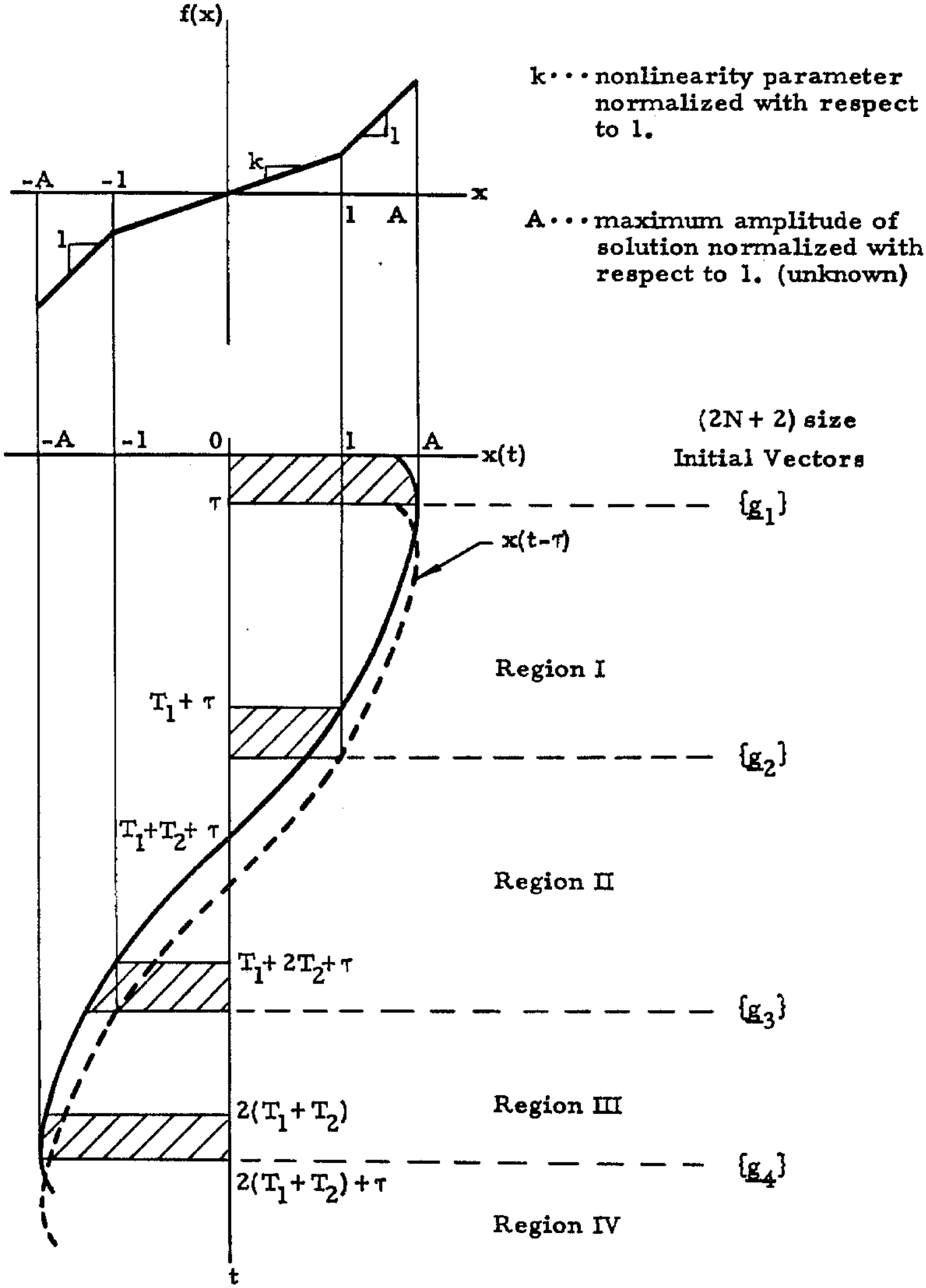


Figure 5.1 Trilinear Restoring Force f(x) and Corresponding Periodic Solution with the Constraints (5.9).

1) the initial function $g(t)$ for $0 \leq t \leq \tau$,
2) the phase angle $\psi$ between the forcing function and the solution $x(t)$,
3) the switch-over time $T_1$ between two linear regions I and II.

And thus the problem is to find the correct initial function $g(t)$ such that the periodic solution as specified is obtained, as well as the unknowns $\psi$ and $T_1$ which are found using the following constraints,

$$1)\quad x(T_1+\tau)=1 \tag{5.9a}$$

$$2)\quad x(T_1+T_2+\tau)=0 \tag{5.9b}$$

with

$$T=\frac{2\pi}{\omega}=4(T_1+T_2). \tag{5.10}$$

Steady State Solution

In order to construct a periodic solution to Equation (5.8), we shall consider the problem as three separate regions where each region is governed by a linear differential-difference equation. Thus, for each region the solution process of the Fredholm Integral Equation Method applies by constructing the linear relationship of transfer matrix as described in Chapter IV.

1) Region I $(\tau \leq t \leq T_1 + 2\tau)$

Equation (5.8) becomes a linear differential-difference equation

$$\ddot{x}_1(t)+c\dot{x}_1(t)+x_1(t-\tau)=p\cos(\omega t+\psi)+1-k \tag{5.11}$$

with

$$x_1(t) = g_1(t) \quad \text{unknown for} \quad 0 \le t \le \tau.$$

Rewrite in vector-matrix form,

$$\frac{d\underline{x}_1(t)}{dt} = A_1\underline{x}_1(t) + B_1\underline{x}_1(t-\tau) + \underline{F}_1(t) + \underline{D}_1 \tag{5.12}$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -c \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \underline{F}_1(t) = \begin{Bmatrix} 0 \\ p\cos(\omega t + \psi) \end{Bmatrix}, \quad \underline{D}_1 = \begin{Bmatrix} 0 \\ 1-k \end{Bmatrix}$$

with initial function

$$\underline{x}_1(t) = \underline{g}_1(t) = \begin{pmatrix} g_1(t) \\ \dot{g}_1(t) \end{pmatrix} \qquad \text{for} \qquad 0 \le t \le \tau.$$

And the formal solution is given as Equation (4.15) for this region, in particular,

$$\begin{aligned} \underline{x}_1(t) = K_1(t-\tau)\underline{g}_1(\tau) + \int_0^{\tau} K_1(t-\tau-\xi)B_1\underline{g}_1(\xi)d\xi \\ + \int_{\tau}^{t} K_1(t-\xi)\left\{\underline{F}_1(\xi) + \underline{D}_1\right\} d\xi. \end{aligned} \tag{5.13a}$$

At this point we recognize the matrix kernel $K_1(t)$ is in the form of an infinite series, and the kernel is partitioned into two parts,

$$K_1(t) = K_{1N}(t) + K_{1\varepsilon}(t) = \sum_{i=1}^{2N} e^{z_{1i}t}R_{1i} + \sum_{i=2N+1}^{\infty} e^{z_{1i}t}R_{1i} \tag{5.13b}$$

and the entire solution method is based on the finite N-kernel $K_{1N}(t)$ with corresponding solution and initial function $\underline{x}_{1N}(t)$, $\underline{g}_{1N}(t)$ respectively. This process will be rigorously justified in the error bound analysis where the error induced by truncating the error kernel $K_{1\epsilon}(t)$ is closely examined.

Similar truncation process is used for the Region II and Region III, where the kernels, solutions, and initial functions are denoted by $K_{2N}(t)$, $\underline{x}_{2N}(t)$, $\underline{g}_{2N}(t)$; $K_{3N}(t) = K_{1N}(t)$, $\underline{x}_{3N}(t)$, $\underline{g}_{3N}(t)$ respectively.

Rewrite Equation (5.13a) with the finite N-kernel, for $t = T_1' + \eta$,

$$\underline{x}_{1N}(\eta+T_1') = K_{1N}(\eta+T_1'-\tau)\underline{g}_{1N}(\tau) + \int_0^{\tau} K_{1N}(\eta+T_1'-\tau-\xi)B_1\underline{g}_{1N}(\xi)d\xi$$

$$+ \int_{\tau}^{\eta+T_1'} K_{1N}(\eta+T_1'-\xi)\left\{\underline{F}_1(\xi)+\underline{D}_1\right\}d\xi \qquad \begin{array}{l} 0\le\eta\le\tau \\ T_1' = T_1+\tau \end{array} \tag{5.14a}$$

where the matrix kernel $K_{1N}(t)$ is given as Equation (4.57),

$$K_{1N}(t) = \sum_{i=1}^{2N} e^{z_{1i}t} R_{1i} \tag{5.14b}$$

$$R_{1i} = \frac{1}{\Delta_{1i}}\begin{bmatrix} z_{1i}+c & 1 \\ -e^{-\tau z_{1i}} & z_{1i} \end{bmatrix}$$

$$\Delta_{1i} = 2z_{1i}+c-\tau e^{-\tau z_{1i}}$$

and $z_{1i}$ are the characteristic roots of

$$z_1^2 + c z_1 + e^{-\tau z_1} = 0.$$

Substitute Equation (5.14b) to (5.14a) and simplify the forcing terms,

$$\underline{x}_{1N}(\eta+T_1') = \sum_{i=1}^{2N} e^{z_{1i}(\eta+T_1'-2\tau)} R_{1i}\left\{e^{z_{1i}\tau}\underline{g}_{1N}(\tau)+\underline{m}_{1i}\right\}$$
$$+\sum_{i=1}^{2N} R_{1i}\left\{\underline{f}_{1i}(\eta+T_1')+\underline{d}_{1i}(\eta+T_1')\right\} \tag{5.15}$$

where the integral moment of the unknown initial function is defined as

$$\underline{m}_{1i} = \int_0^\tau e^{z_{1i}(\tau-\xi)} B_1\underline{g}_{1N}(\xi)d\xi \quad , \quad i=1,2,\cdots,N \tag{5.16}$$

and the integral moments of the forcing terms are

$$\underline{f}_{1i}(t) = p\begin{pmatrix} 0 \\ \{C_{1i}(t,t)-C_{1i}(t,\tau)\}\cos\psi - \{S_{1i}(t,t)-S_{1i}(t,\tau)\}\sin\psi \end{pmatrix} \tag{5.17a}$$

$$C_{1i}(T,t) = (\omega\sin\omega t - z_{1i}\cos\omega t)e^{z_{1i}(T-t)}/(\omega^2+z_{1i}^2) \tag{5.17b}$$
$$S_{1i}(T,t) = -(\omega\cos\omega t + z_{1i}\sin\omega t)e^{z_{1i}(T-t)}/(\omega^2+z_{1i}^2)$$

and

$$\underline{d}_{1i}(t) = \frac{1-k}{z_{1i}}\begin{pmatrix} 0 \\ e^{z_{1i}(t-\tau)}-1 \end{pmatrix} .$$

From the continuity of the solution, the initial function of the Region II is the solution (5.15), i.e.,

$$\underline{g}_{2N}(t) = \underline{x}_{1N}(t+T_1') \quad , \quad 0 \le t \le \tau. \tag{5.18}$$

Now we construct the finite (2N + 2) dimensional initial vectors

$$\{g_1\} = \begin{bmatrix} \underline{g}_{1N}(\tau) \\ \underline{m}_{11} \\ \underline{m}_{12} \\ \vdots \\ \underline{m}_{1N} \end{bmatrix} \quad , \quad \{g_2\} = \begin{bmatrix} \underline{g}_{2N}(\tau) \\ \underline{m}_{21} \\ \underline{m}_{22} \\ \vdots \\ \underline{m}_{2N} \end{bmatrix} \tag{5.19}$$

with

$$\underline{m}_{2i} = \int_0^\tau e^{z_{2i}(\tau-\xi)} B_2 \underline{g}_{2N}(\xi) d\xi \quad , \quad i = 1, 2, \cdots, N$$

integral moment of the initial function of Region II.

Then from Equations (5.18) and (5.15)

$$\underline{g}_{2N}(\tau) = \sum_{i=1}^{2N} e^{z_{1i}(T_1'-\tau)} R_{1i}\left\{e^{z_{1i}\tau} \underline{g}_{1N}(\tau) + \underline{m}_{1i}\right\} + \sum_{i=1}^{2N} R_{1i}\left\{\underline{f}_{1i}(T_1'+\tau) + \underline{d}_{1i}(T_1'+\tau)\right\} \tag{5.20a}$$

and

$$\underline{m}_{2i} = \sum_{j=1}^{2N} e^{z_{1j}(T_1'-2\tau)} q_{ji} B_2 R_{1j}\left\{e^{z_{1j}\tau} \underline{g}_{1N}(\tau) + \underline{m}_{1j}\right\} + \underline{h}_{1i} \tag{5.20b}$$

with

$$q_{ji} = \frac{e^{z_{1j}\tau} - e^{z_{2i}\tau}}{z_{1j} - z_{2i}}$$

$$\underline{h}_{1i} = \sum_{j=1}^{2N} \int_0^\tau e^{z_{2i}(\tau-\xi)} B_2 R_{1j}\left\{\underline{f}_{1j}(\xi+T_1') + \underline{d}_{1j}(\xi+T_1')\right\} d\xi = \begin{pmatrix} 0 \\ h_{1i} \end{pmatrix}, \quad i, j = 1, 2, \cdots, N$$

$$h_{1i} = \sum_{j=1}^{2N} \frac{k(k-1)}{\Delta_{1j} z_{1j}} \left\{ e^{z_{1j}(T_1' - \tau)} q_{ji} + \frac{1}{z_{2i}} (1 - e^{z_{2i}\tau}) \right\} \tag{5.21}$$

$$+ kp \cos\psi \sum_{j=1}^{2N} \frac{1}{\Delta_{1j}} \Big[ S_{1j}(T_1', T_1') \left\{ S_{2i}(\tau, \tau) - S_{2i}(\tau, 0) \right\}$$

$$- C_{1j}(T_1', T_1') \left\{ C_{2i}(\tau, \tau) - C_{2i}(\tau, 0) \right\} + C_{1j}(T_1', \tau) q_{ji} \Big]$$

$$+ kp \sin\psi \sum_{j=1}^{2N} \frac{1}{\Delta_{1j}} \Big[ S_{1j}(T_1', T_1') \left\{ C_{2i}(\tau, \tau) - C_{2i}(\tau, 0) \right\}$$

$$+ C_{1j}(T_1', T_1') \left\{ S_{2i}(\tau, \tau) - S_{2i}(\tau, 0) \right\} - S_{1j}(T_1', \tau) q_{ji} \Big]$$

with

$$C_{2i}(T, t) = (\omega \sin \omega t - z_{2i} \cos \omega t) e^{z_{2i}(T-t)} / (\omega^2 + z_{2i}^2)$$

$$S_{2i}(T, t) = -(\omega \cos \omega t + z_{2i} \sin \omega t) e^{z_{2i}(T-t)} / (\omega^2 + z_{2i}^2)$$

and $C_{1i}$, $S_{1i}$ are given in Equation (5.17b).

Combine the expressions (5.20a) and (5.20b), we obtain the desired transfer matrix relationship between the Region I and Region II initial vectors,

$$\{g_2\} = [M_1] \{g_1\} + \{\underline{h}_1\} \tag{5.22}$$

where the $(2N+2) \times (2N+2)$ transfer matrix is formed as

$$[M_1] = 2\Re e$$

$$\times \begin{bmatrix} \sum_{j=1}^{N} e^{z_{1j}T_1'} R_{1j} & e^{z_{11}(T_1'-\tau)} R_{11} & \cdots & e^{z_{1N}(T_1'-\tau)} R_{1N} \\ \sum_{j=1}^{N} e^{z_{1j}(T_1'-\tau)} q_{j1} B_2 R_{1j} & e^{z_{11}(T_1'-2\tau)} q_{11} B_2 R_{11} & \cdots & e^{z_{1N}(T_1'-2\tau)} q_{N1} B_2 R_{1N} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{N} e^{z_{1j}(T_1'-\tau)} q_{jN} B_2 R_{1j} & e^{z_{11}(T_1'-2\tau)} q_{1N} B_2 R_{11} & \cdots & e^{z_{1N}(T_1'-2\tau)} q_{NN} B_2 R_{1N} \end{bmatrix} \tag{5.23}$$

and

$$\{\underline{h}_1\} = \begin{Bmatrix} \sum_{j=1}^{2N} R_{1j} \left\{ \underline{f}_{1j}(T_1'+\tau) + \underline{d}_{1j}(T_1'+\tau) \right\} \\ \underline{h}_{11} \\ \vdots \\ \underline{h}_{1N} \end{Bmatrix} \quad (2N+2) \text{ size vector.} \tag{5.24}$$

2) <u>Region II</u> $(T_1 + 2\tau \le t \le T_1 + 2T_2 + 2\tau)$

Governing differential-difference equation for this region is, from Equation (5.8),

$$\ddot{x}_2(t) + c\dot{x}_2(t) + kx_2(t-\tau) = p\cos(\omega t + \psi) \tag{5.25}$$

and

$$T_2 = \frac{\pi}{2\omega} - T_1.$$

Rewrite Equation (5.25) in vector-matrix form,

$$\frac{d\underline{x}_2(t)}{dt} = A_2\underline{x}_2(t) + B_2\underline{x}_2(t-\tau) + \underline{F}_2(t)$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -c \end{bmatrix} \quad , \quad B_2 = \begin{bmatrix} 0 & 0 \\ -k & 0 \end{bmatrix} \quad , \quad \underline{F}_2(t) = \begin{pmatrix} 0 \\ p\cos(\omega t + \psi) \end{pmatrix} \tag{5.26}$$

initial function

$$\underline{x}_2(t) = \underline{g}_2(t) = \underline{x}_1(t + T_1') \quad , \quad 0 \le t \le \tau.$$

Formal solution to Equation (5.26) is given with the finite N-kernel,

$$\underline{x}_{2N}(t) = K_{2N}(t-\tau)\underline{g}_{2N}(\tau) + \int_0^\tau K_{2N}(t-\tau-\xi)B_2\underline{g}_{2N}(\xi)d\xi + \int_\tau^t K_{2N}(t-\xi)\underline{F}_2(\xi+T_1')d\xi \tag{5.27}$$

with the matrix kernel

$$K_{2N}(t) = \sum_{i=1}^{2N} e^{z_{2i}t} R_{2i}$$

$$R_{2i} = \frac{1}{\Delta_{2i}} \begin{bmatrix} z_{2i}+c & 1 \\ -ke^{-\tau z_{2i}} & z_{2i} \end{bmatrix} \tag{5.28}$$

$$\Delta_{2i} = 2z_{2i} + c - \tau k e^{-\tau z_{2i}}$$

and $z_{2i}$ are the characteristic roots of

$$z_2^2 + cz_2 + ke^{-\tau z_2} = 0.$$

Now we proceed to construct the initial vector of Region III in terms of the initial vector of Region II, with the continuity condition

$$\underline{g}_{3N}(t) = \underline{x}_{2N}(t + 2T_2) \quad , \quad 0 \le t \le \tau \tag{5.29}$$

and

$$\{g_2\} = \begin{Bmatrix} \underline{g}_{2N}(\tau) \\ \underline{m}_{21} \\ \underline{m}_{22} \\ \vdots \\ \underline{m}_{2N} \end{Bmatrix} \quad , \quad \{\underline{g}_3\} = \begin{Bmatrix} \underline{g}_{3N}(\tau) \\ \underline{m}_{31} \\ \underline{m}_{32} \\ \vdots \\ \underline{m}_{3N} \end{Bmatrix} \tag{5.30}$$

where the integral moments of the initial function of the Region III are

$$\underline{m}_{3i} = \int_0^\tau e^{z_{1i}(\tau-\xi)} B_1 \underline{g}_{3N}(\xi) d\xi \quad , \quad i = 1, 2, \cdots, N.$$

Then following the identical procedures of the Region I, we obtain the transfer matrix relationship for the Region II and III,

$$\{\underline{g}_3\} = [M_2] \{\underline{g}_2\} + \{\underline{h}_2\} \tag{5.31}$$

with

$[M_2] = 2\Re e$

$$\times \begin{bmatrix} \sum_{j=1}^{N} e^{2z_{2j}T_2} R_{2j} & e^{z_{21}(2T_2-\tau)} R_{21} & \cdots e^{z_{2N}(2T_2-\tau)} R_{2N} \\ \sum_{j=1}^{N} e^{z_{2j}(2T_2-\tau)} q'_{j1} B_1 R_{2j} & e^{z_{21}(2T_2-2\tau)} q'_{11} B_1 R_{21} & \cdots e^{z_{2N}(2T_2-2\tau)} q'_{N1} B_1 R_{2N} \\ \vdots & \vdots & \vdots \\ \sum_{j=1}^{N} e^{z_{2j}(2T_2-\tau)} q'_{jN} B_1 R_{2j} & e^{z_{21}(2T_2-2\tau)} q'_{1N} B_1 R_{21} & \cdots e^{z_{2N}(2T_2-2\tau)} q'_{NN} B_1 R_{2N} \end{bmatrix}$$

(5.32)

$$q'_{ji} = \frac{e^{z_{2j}\tau} - e^{z_{1i}\tau}}{z_{2j} - z_{1i}}$$

and

$$\{\underline{h}_2\} = \begin{Bmatrix} \sum_{j=1}^{2N} R_{2j}\underline{f}_{2j}(\tau + 2T_2) \\ \underline{h}_{21} \\ \vdots \\ \underline{h}_{2N} \end{Bmatrix} \tag{5.33}$$

where

$$\underline{f}_{2j}(t) = p\begin{pmatrix} 0 \\ \left\{C_{2j}(t', t') - C_{2j}(t', \tau')\right\}\cos\psi - \left\{S_{2j}(t', t') - S_{2j}(t', \tau')\right\}\sin\psi \end{pmatrix}$$

$$t' = t + T_1' \quad , \quad \tau' = \tau + T_1'$$

$$\underline{h}_{2i} = \sum_{j=1}^{2N} \int_0^{\tau} e^{z_{1i}(\tau - \xi)} B_1 R_{2j}\underline{f}_{2j}(\xi + 2T_2)d\xi = \begin{pmatrix} 0 \\ h_{2i} \end{pmatrix}$$

$$h_{2i} = p\cos\psi \sum_{j=1}^{2N} \frac{1}{\Delta_{2j}}\Big[S_{2j}(T_3, T_3)\left\{S_{1i}(\tau, \tau) - S_{1i}(\tau, 0)\right\}$$

$$- C_{2j}(T_3, T_3)\left\{C_{1i}(\tau, \tau) - C_{1i}(\tau, 0)\right\} + C_{2j}(T_3, T_4)q'_{ji}\Big]$$

$$+ p\sin\psi \sum_{j=1}^{2N} \frac{1}{\Delta_{2j}}\Big[S_{2j}(T_3, T_3)\left\{C_{1i}(\tau, \tau) - C_{1i}(\tau, 0)\right\}$$

$$+ C_{2j}(T_3, T_3)\left\{S_{1i}(\tau, \tau) - S_{1i}(\tau, 0)\right\} - S_{2j}(T_3, T_4)q'_{ji}\Big]$$

$$T_3 = 2T_2 + T_1' \quad , \quad T_4 = T_1' + \tau.$$

3) Region III $(T_1+2T_2+2\tau \le t \le 2(T_1+T_2)+\tau)$

Governing differential-difference equation for this linear region is, from Equation (5.8),

$$\ddot{x}_3(t)+c\dot{x}_3(t)+x_3(t-\tau)=p\cos(\omega t+\psi)+k-1 \tag{5.34}$$

or in vector-matrix form,

$$\frac{d\underline{x}_3(t)}{dt}=A_1\underline{x}_3(t)+B_1\underline{x}_3(t-\tau)+\underline{F}_1(t)-\underline{D}_1 \tag{5.35}$$

initial function

$$\underline{x}_3(t)=\underline{g}_3(t)=\underline{x}_2(t+2T_2) \quad , \quad 0\le t\le \tau$$

with $A_1, B_1, \underline{F}_1, \underline{D}_1$ given in Equation (5.12).

Construct the initial vectors of Region III and Region IV with the continuity condition

$$\underline{g}_{4N}(t)=\underline{x}_{3N}(t+T_1'') \quad , \quad 0\le t\le \tau \tag{5.36}$$

and

$$T_1''=T_1-\tau$$

$$\{g_3\}=\begin{Bmatrix}\underline{g}_{3N}(\tau)\\ \underline{m}_{31}\\ \underline{m}_{32}\\ \vdots\\ \underline{m}_{3N}\end{Bmatrix} \quad , \quad \{\underline{g}_4\}=\begin{Bmatrix}\underline{g}_{4N}(\tau)\\ \underline{m}_{41}\\ \underline{m}_{42}\\ \vdots\\ \underline{m}_{4N}\end{Bmatrix} \tag{5.37}$$

where the integral moments of the initial function of Region IV are

$$\underline{m}_{4i}=\int_0^\tau e^{z_{1i}(\tau-\xi)}B_1\underline{g}_{4N}(\xi)d\xi \quad , \quad i=1,2,\cdots,N.$$

Then the transfer matrix relationship is established as before,

$$\{\underline{g}_4\} = [M_3]\,\{\underline{g}_3\} + \{\underline{h}_3\} \tag{5.38}$$

with

$[M_3] = 2\Re e$

$$\times \begin{bmatrix} \sum_{j=1}^{N} e^{z_{1j}T_1''} R_{1j} & e^{z_{11}(T_1''-\tau)} R_{11} & \cdots e^{z_{1N}(T_1''-\tau)} R_{1N} \\ \sum_{j=1}^{N} e^{z_{1j}(T_1''-\tau)} q_{j1}'' B_1 R_{1j} & e^{z_{11}(T_1''-2\tau)} q_{11}'' B_1 R_{11} & \cdots e^{z_{1N}(T_1''-2\tau)} q_{N1}'' B_1 R_{1N} \\ \vdots & \vdots & \vdots \\ \sum_{j=1}^{N} e^{z_{1j}(T_1''-\tau)} q_{jN}'' B_1 R_{1j} & e^{z_{11}(T_1''-2\tau)} q_{1N}'' B_1 R_{11} & \cdots e^{z_{1N}(T_1''-2\tau)} q_{NN}'' B_1 R_{1N} \end{bmatrix} \tag{5.39}$$

$$q_{ji}'' = \frac{e^{z_{1j}\tau} - e^{z_{1i}\tau}}{z_{1j} - z_{1i}} \quad , \quad i \neq j$$

$$= \tau e^{z_{1i}\tau} \quad , \quad i = j$$

and

$$\{\underline{h}_3\} = \begin{Bmatrix} \sum_{j=1}^{2N} R_{1j} \left\{\underline{f}_{3j}(\tau + T_1'') + \underline{d}_{3j}(\tau + T_1'')\right\} \\ \underline{h}_{31} \\ \vdots \\ \underline{h}_{3N} \end{Bmatrix} \tag{5.40}$$

$$\underline{f}_{3j}(t) = p\begin{pmatrix} 0 \\ \left\{C_{1j}(t'', t'') - C_{1j}(t'', \tau'')\right\}\cos\psi - \left\{S_{1j}(t'', t'') - S_{1j}(t'', \tau'')\right\}\sin\psi \end{pmatrix}$$

$$t'' = T_1' + 2T_2 + t \quad , \quad \tau'' = T_1' + 2T_2 + \tau$$

$$\underline{d}_{3j}(t) = \frac{k-1}{z_{1j}}\begin{pmatrix} 0 \\ e^{z_{1j}(t-\tau)} - 1 \end{pmatrix} \quad , \quad \underline{h}_{3i} = \begin{pmatrix} 0 \\ h_{3i} \end{pmatrix}$$

$$h_{3i} = \sum_{j=1}^{2N} \frac{1-k}{\Delta_{1j} z_{1j}} \left\{ e^{z_{1j}(T_1'' - \tau)} q_{ji}'' + \frac{1 - e^{z_{1i}\tau}}{z_{1i}} \right\}$$

$$+ p\cos\psi \sum_{j=1}^{2N} \frac{1}{\Delta_{1j}} \Big[ S_{1j}(T_5, T_5)\left\{S_{1i}(\tau, \tau) - S_{1i}(\tau, 0)\right\}$$

$$- C_{1j}(T_5, T_5)\left\{C_{1i}(\tau, \tau) - C_{1i}(\tau, 0)\right\} + C_{1j}(T_5, T_6) q_{ji}'' \Big]$$

$$+ p\sin\psi \sum_{j=1}^{2N} \frac{1}{\Delta_{1j}} \Big[ S_{1j}(T_5, T_5)\left\{C_{1i}(\tau, \tau) - C_{1i}(\tau, 0)\right\}$$

$$+ C_{1j}(T_5, T_5)\left\{S_{1i}(\tau, \tau) - S_{1i}(\tau, 0)\right\} - S_{1j}(T_5, T_6) q_{ji}'' \Big]$$

$$T_5 = 2(T_1 + T_2) \quad , \quad T_6 = T_1' + 2T_2 + \tau.$$

Thus, construction of the transfer matrix relationship for any n-piece-wise linear delay system in general can be formulated as

$$\{\underline{g}_{i+1}\} = [M_i]\{\underline{g}_i\} + \{\underline{h}_i\} \quad , \quad i = 1, 2, \cdots, n. \tag{5.41}$$

4) <u>Closure (Periodicity) Condition</u>

In order to satisfy the requirements of periodic solution with same period as the forcing function, we require

$$\{\underline{g}_4\} = -\{\underline{g}_1\}. \tag{5.42}$$

Combine Equations (5.22), (5.31) and (5.38), and eliminate the intermediate initial vectors $\{\underline{g}_2\}$ and $\{\underline{g}_3\}$,

$$\{\underline{g}_4\} = [M]\{\underline{g}_1\} + \{\underline{h}\} \tag{5.43}$$

where

$$[M] = [M_3][M_2][M_1]$$

$$\{\underline{h}\} = [M_3][M_2]\{\underline{h}_1\} + [M_3]\{\underline{h}_2\} + \{\underline{h}_3\}$$

and combine Equations (5.42) and (5.43) and solve for $\{\underline{g}_1\}$,

$$\{\underline{g}_1\} = -\left[ M + I \right]^{-1} \{\underline{h}\}. \tag{5.44}$$

We recognize the periodic solution $\{\underline{g}_1\}$ is identical in form to the linear delay system case of Equation (4.68). We have thus extended the Fredholm Integral Equation Method solution scheme to piecewise linear delay systems.

As it was in the linear delay system of Chapter IV, the force vector $\{\underline{h}\}$ possesses two unknowns $\sin\psi$ and $\cos\psi$, and they are solved by the constraint condition (5.9b) and $\sin^2\psi + \cos^2\psi = 1$ as in the linear case.

There exists an additional difficulty, that of computing the switch-over time $T_1$, which did not exist in the linear case. Since the transfer matrices $[M_1]$, $[M_2]$, $[M_3]$ are implicit functions of $T_1$, the method of solution is to assume values of $T_1$, calculate $[M_1]$, $[M_2]$ and $[M_3]$, solve for $\{g_1\}$ from Equation (5.44) and hence obtain $x(T_1+\tau)$, then the correct value of $T_1$ is obtained when the condition (5.9a) is satisfied.

Thus we have obtained the periodic solution to a special example of trilinear dynamic system with delay as in Equation (5.8), constructing all the unknowns $g(t)$, $\psi$, and $T_1$ by requiring the periodicity condition and the constraints to the solution as in Equation (5.9).

It is to be noted here that the solution scheme of the Fredholm Integral Equation Method which was developed originally for the linear delay system applies logically to piecewise linear delay systems to produce exact solutions, while all other existing solution schemes (for instance, slowly varying parameters method) fail to give the exact solutions. We note also the accuracy of the periodic solution by the Fredholm Integral Equation Method depends only on the number of characteristic roots N and is completely free from the <u>small parameters</u> requirements which are strictly necessary for the other existing approximate schemes.

## 5.4 Stability of the Periodic Solution

Once the periodic solutions are obtained by the Fredholm Integral Equation Method (F. I. E. M.), the question of stability of the solution can be answered immediately. As it was defined in Chapter II, we shall consider the stability in the sense of Liapunov-Poincaré.

First, perturb the solution of Equation (5.8),

$$\begin{aligned} x(t) &= x^*(t) + \eta(t) \\ x(t-\tau) &= x^*(t-\tau) + \eta(t-\tau) \end{aligned} \tag{5.45}$$

where $x^*(t)$---exact periodic solution by F. I. E. M.

$\eta(t)$ ---perturbation variable in the neighborhood of $x^*(t)$

then we obtain the first variational equation of Equation (5.8),

$$\ddot{\eta}(t) + c\dot{\eta}(t) + \left.\frac{df}{dx}\right|_{x=x^*} \eta(t-\tau) = 0$$

or

$$\ddot{\eta}(t) + c\dot{\eta}(t) + \{\alpha + \beta\, q(t-\tau)\}\, \eta(t-\tau) = 0 \tag{5.46}$$

where

$$\alpha = \frac{1+k}{2} \quad , \quad \beta = \frac{1-k}{2}$$

and q(t) is the rectangular pulse as shown in Figure 5.2.

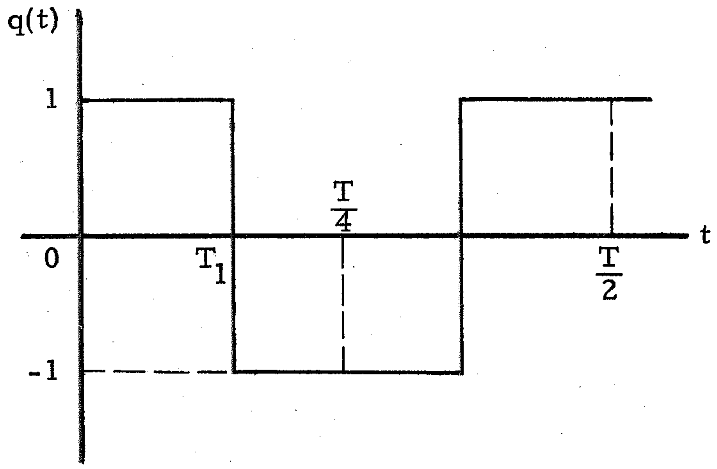


Figure 5.2 Rectangular Pulse q(t) for Equation (5.46).

Thus the original nonlinear differential-difference equation (5.8) has produced its variational equation (5.46) in the form of a linear differential-difference equation with variable coefficients. We recognize the Equation (5.46) as the Hill-Meisner equation with time delay, and stability criteria is established by posing an eigenvalue problem as it is done in the case of Hill-Meisner equation with no delay.

Solution of Equation (5.46) is obtained by constructing the transfer matrix relationship between the initial vectors in the perturbed $\eta$ variable, that is, perturb the initial vectors

$$\{\underline{g}_i\}=\{\underline{g}_i\}^*+\{\underline{\eta}_i\} \quad , \quad i=1, 2, 3, 4 \tag{5.47}$$

where $\{\underline{g}_i\}^*$ is the exact initial vector by F. I. E. M., then substitute Equation (5.47) to Equations (5.22), (5.31), (5.38) and cancel out

$$\{\underline{g}_{i+1}\}^*=[M_i]\{\underline{g}_i\}^*+\{\underline{h}_i\} \quad , \quad i=1, 2, 3$$

we have

$$\begin{aligned} \{\underline{\eta}_2\}&=[M_1]\{\underline{\eta}_1\} && \text{for Region I} \\ \{\underline{\eta}_3\}&=[M_2]\{\underline{\eta}_2\} && \text{for Region II} \\ \{\underline{\eta}_4\}&=[M_3]\{\underline{\eta}_3\} && \text{for Region III.} \end{aligned} \tag{5.48}$$

Combine Equation (5.48),

$$\{\underline{\eta}_4\}=[M_3][M_2][M_1]\{\underline{\eta}_1\}=[M]\{\underline{\eta}_1\}. \tag{5.49}$$

Now we pose an eigenvalue problem for stability,

$$\{\underline{\eta}_4\}=\lambda\{\underline{\eta}_1\}=[M]\{\underline{\eta}_1\} \tag{5.50}$$

then the necessary and sufficient condition for the periodic solution $\{\underline{g}_1\}^*$ to be stable is that all the eigenvalues of the cascaded transfer matrix [M] must lie within the unit circle; i.e.,

$$\{\underline{g}_1\}^* \text{ is stable if and only if } |\lambda_i| \leq 1 \text{ for all } i = 1, 2, \cdots (2N+2),$$

and

$$\{\underline{g}_1\}^* \text{ is unstable if } |\lambda_i| > 1 \text{ for any } i = 1, 2, \ldots, (2N+2).$$

Thus far we have shown the algorithm for constructing the periodic solutions by Equation (5.44) and determining their stability by Equation (5.50), using the same transfer matrix [M]. So the stability and the periodic solution of a nonlinear delay system are answered entirely by the system of algebraic equations which can be readily solved, and this task is particularly attractive with the aid of digital computers.

At the end of this chapter, a series of numerical examples are shown. The maximum amplitude of the periodic solution $\{\underline{g}_1\}^*$ is plotted versus the forcing frequency $\omega$ by taking a finite number of characteristic roots 2N for various sets of parameters. Stability is determined for each solution and comparison is made with an approximate method.

## 5.5 Error Analyses of the Periodic Solution

It was mentioned earlier that the "exactness" of the periodic solution by the Fredholm Integral Equation Method is crucially dependent on the fact that the error induced by truncating the matrix kernel K(t) is made exponentially smaller by increasing the number of characteristic root-pairs N. Thus, in essence, the solution can be made as accurate as one wishes by increasing the size of the transfer matrix, and this was shown for the linear delay systems in Chapter IV.

We shall see in this section that the idea of bounding the error of solutions for piecewise linear delay system follows closely to that of linear delay system case.

As an example, consider the trilinear delay system (5.8) and its periodic solution $\{\underline{g}_1\}$ of Equation (5.44). The formal solution for each region (refer to Figure 5.1) is given as

$$\underline{x}_i(t) = K_i(t-\tau)\underline{g}_i(\tau) + \int_0^{\tau} K_i(t-\tau-\xi)B_i\underline{g}_i(\xi)d\xi + \int_{\tau}^{t} K_i(t-\xi)\underline{F}_i(\xi)d\xi \qquad i=1,2,3 \tag{5.51}$$

and for this particular case of doubly bilinear system,

$$K_3(t) = K_1(t)$$

and the kernels are in infinite series form

$$K_i(t) = \sum_{j=1}^{\infty} e^{z_{ij}t} R_{ij} \quad , \quad i=1,2 \tag{5.52}$$

where the first 2N terms are used in the Fredholm Integral Equation Method. At this point, we consider the entire kernel as in Equation (5.52) and see the effect of the truncated terms.

Let

$$K_i(t) = K_{iN}(t) + K_{i\epsilon}(t) = \sum_{j=1}^{2N} e^{z_{ij}t} R_{ij} + \sum_{j=2N+1}^{\infty} e^{z_{ij}t} R_{ij} \quad , \quad i = 1, 2 \quad (5.53)$$

and accordingly the initial functions and the solutions are split into two parts,

$$\underline{x}_i(t) = \underline{x}_{iN}(t) + \underline{x}_{i\epsilon}(t) \quad (5.54)$$

$$i = 1, 2, 3.$$

$$\underline{g}_i(t) = \underline{g}_{iN}(t) + \underline{g}_{i\epsilon}(t) \quad (5.55)$$

<u>Properties of the Kernels</u>

Let the infinite chain of characteristic roots be ordered as

$$0 > \Re e\, z_{i1} > \Re e\, z_{i2} > \cdots > \Re e\, z_{iN} > \Re e\, z_{i(N+1)} > \cdots \quad , \quad i = 1, 2$$

and define

$$\sigma_{i1} = \Re e\, z_{i1} \quad , \quad \sigma_{iN} = \Re e\, z_{iN} \quad , \quad \sigma_{i\epsilon} = \Re e\, z_{i(N+1)} \quad , \quad i = 1, 2$$

then the kernels form uniformly converging series, and

$$\begin{aligned} \|K_i(t)\| &\le m_i e^{\sigma_{i1} t} \\ \|K_{i\epsilon}(t)\| &\le m_{i\epsilon} e^{\sigma_{i\epsilon} t} \end{aligned} \quad , \quad i = 1, 2 \quad (5.56)$$

with $m_i$, $m_{i\epsilon}$ being positive constants.

Also, asymptotic properties of the kernels are

$$\begin{aligned} &\lim_{\tau \to 0} \|K_i(t)\| \le k_i < \infty \\ &\lim_{\tau \to 0} \|K_{i\epsilon}(t)\| = 0 \qquad \text{provided } N > 1 \end{aligned} \quad (5.57)$$

$$\lim_{\substack{N\to\infty \\ \tau>0}} \|K_{i\epsilon}(t)\| = 0 \qquad \text{for } i=1,2.$$

Error Bound Formulation

The error in the solution is formulated in a similar way to the periodic solution itself by considering each region where it is governed by a linear differential-difference equation.

Substitute Equations (5.53), (5.54) and (5.55) to (5.51) for the Region I and cancel out $\underline{x}_{1N}(t)$ terms,

$$\underline{x}_{1\epsilon}(t) = K_{1\epsilon}(t-\tau)\underline{g}_{1N}(\tau) + \int_0^\tau K_{1\epsilon}(t-\tau-\xi)B_1\underline{g}_{1N}(\xi)d\xi + \int_\tau^t K_{1\epsilon}(t-\xi)\underline{F}_1(\xi)d\xi$$
$$+ K_1(t-\tau)\underline{g}_{1\epsilon}(\tau) + \int_0^\tau K_1(t-\tau-\xi)B_1\underline{g}_{1\epsilon}(\xi)d\xi. \tag{5.58}$$

From the continuity condition, error of the Region II initial function is given as, for $0 \le \eta \le \tau$

$$\underline{g}_{2\epsilon}(\eta) = \underline{x}_{1\epsilon}(\eta+T_1) = \underline{g}_1^*(\eta+T_1) + K_1(\eta+T_1-\tau)\underline{g}_{1\epsilon}(\tau) + \int_0^\tau K_1(\eta+T_1-\tau-\xi)B_1\underline{g}_{1\epsilon}(\xi)d\xi. \tag{5.59}$$

Similarly for Region III, $0 \le \eta \le \tau$

$$\underline{g}_{3\epsilon}(\eta) = \underline{x}_{2\epsilon}(\eta+2T_2) = \underline{g}_2^*(\eta+2T_2) + K_2(\eta+2T_2-\tau)\underline{g}_{2\epsilon}(\tau)$$
$$+ \int_0^\tau K_2(\eta+2T_2-\tau-\xi)B_2\underline{g}_{2\epsilon}(\xi)d\xi \tag{5.60}$$

and for Region IV, $0 \le \eta \le \tau$

$$\underline{g}_{4\epsilon}(\eta) = \underline{x}_{3\epsilon}(\eta+T_1) = \underline{g}_3^*(\eta+T_1) + K_1(\eta+T_1-\tau)\underline{g}_{3\epsilon}(\tau) + \int_0^\tau K_1(\eta+T_1-\tau-\xi)B_1\underline{g}_{3\epsilon}(\xi)d\xi \tag{5.61}$$

while the known terms are

$$\begin{aligned}
\underline{g}_1^*(\eta) &= K_{1\epsilon}(\eta-\tau)\underline{g}_{1N}(\tau) + \int_0^\tau K_{1\epsilon}(\eta-\tau-\xi)B_1\underline{g}_{1N}(\xi)d\xi + \int_\tau^\eta K_{1\epsilon}(\eta-\xi)\underline{F}_1(\xi)d\xi \\
\underline{g}_2^*(\eta) &= K_{2\epsilon}(\eta-\tau)\underline{g}_{2N}(\tau) + \int_0^\tau K_{2\epsilon}(\eta-\tau-\xi)B_2\underline{g}_{2N}(\xi)d\xi + \int_\tau^\eta K_{2\epsilon}(\eta-\xi)\underline{F}_2(\xi)d\xi \\
\underline{g}_3^*(\eta) &= K_{1\epsilon}(\eta-\tau)\underline{g}_{3N}(\tau) + \int_0^\tau K_{1\epsilon}(\eta-\tau-\xi)B_1\underline{g}_{3N}(\xi)d\xi + \int_\tau^\eta K_{1\epsilon}(\eta-\xi)\underline{F}_3(\xi)d\xi
\end{aligned} \tag{5.62}$$

since the $\underline{g}_{iN}(\xi)$, $i = 1, 2, 3$ are known from the Fredholm Integral Equation Method.

Substituting Equation (5.59) to (5.60), and then (5.60) to (5.61), thus eliminating the intermediate initial functions $\underline{g}_{2\epsilon}(\eta)$ and $\underline{g}_{3\epsilon}(\eta)$, and applying the closure (periodicity) condition on the error of the initial function,

$$\underline{g}_{4\epsilon}(\eta) = -\underline{g}_{1\epsilon}(\eta) \quad \text{for} \quad 0 \le \eta \le \tau \tag{5.63}$$

then finally we obtain the error in a Fredholm integral equation form,

$$\underline{g}_{1\epsilon}(\eta) = \hat{\underline{h}}(\eta) + \int_0^\tau \hat{K}(\eta,\rho)\underline{g}_{1\epsilon}(\rho)d\rho \quad , \quad 0 \le \eta \le \tau \tag{5.64}$$

with

$$\hat{\underline{h}}(\eta) = -\underline{h}^{**}(\eta) + W(\eta)\underline{h}^{**}(\tau) \tag{5.65a}$$

$$\hat{K}(\eta, \rho) = \int_0^T \int_0^T [W(\eta)K^{**}(\tau, \xi, \gamma, \rho) - K^{**}(\eta, \xi, \gamma, \rho)] B_1 d\gamma d\xi \tag{5.65b}$$

where

$$W(\eta) = \int_0^T \int_0^T K^{**}(\eta, \xi, \gamma, 0) d\gamma \, d\xi \left[I + \int_0^T \int_0^T K^{**}(\tau, \xi, \gamma, 0) d\gamma \, d\xi\right]^{-1}$$

$$K^{**}(\eta, \xi, \gamma, \rho) = \frac{1}{\tau} K_1(\eta + T_1 - \tau) K^*(\tau, \gamma, \rho) + K_1(\eta + T_1 - \tau - \xi) B_1 K^*(\xi, \gamma, \rho)$$

$$\underline{h}^{**}(\eta) = \underline{g}_3^*(\eta + T_1) + K_1(\eta + T_1 - \tau)\underline{h}^*(\tau) + \int_0^T K_1(\eta + T_1 - \tau - \xi) B_1 \underline{h}^*(\xi) d\xi$$

$$K^*(\xi, \gamma, \rho) = \frac{1}{\tau} K_2(\xi + 2T_2 - \tau) K_1(T_1 - \rho) + K_2(\xi + 2T_2 - \tau - \gamma) B_2 K_1(\gamma + T_1 - \tau - \rho)$$

$$\underline{h}^*(\xi) = \underline{g}_2^*(\xi + 2T_2) + K_2(\xi + 2T_2 - \tau)\underline{g}_1^*(\tau + T_1) + \int_0^T K_2(\xi + 2T_2 - \tau - \gamma) B_2 \underline{g}_1^*(\gamma + T_1) d\gamma.$$

It will be noted that Equation (5.64) which determines the error in the initial function for the trilinear case has exactly the same form as that for the error in the initial function for the linear problem of Equation (4.92).

It is important to note that although the algebra of constructing the error for the trilinear delay system has increased many fold, the basic structure of the error remains unchanged from the linear delay system. Furthermore, one can generalize that even for any n-piecewise linear delay system, the error will have the form of a Fredholm integral equation of second type like Equation (5.64).

Thus the solution of the integral equation (5.64) is given in the form of a Neumann series, and the error is bounded if the series is convergent. Construct a recurrence relationship,

$$\underline{g}_{1\epsilon}^{k}(\eta) = \hat{\underline{h}}(\eta) + \int_0^{T} \hat{K}(\eta, \rho)\underline{g}_{1\epsilon}^{k-1}(\rho)d\rho \qquad k = 1, 2, 3, \cdots \tag{5.66}$$

with

$$\underline{g}_{1\epsilon}^{0}(\eta) = \hat{\underline{h}}(\eta)$$

then following the same process of linear delay system case of Equation (4.93), we get

$$G_{1\epsilon} = \sup_{0 \leq \eta \leq \tau} \|\underline{g}_{1\epsilon}(\eta)\| \leq \frac{\mu}{1-\lambda} \tag{5.67}$$

with

$$\lambda = \tau \cdot \sup_{0 \leq \eta,\, \rho \leq T/4} \|\hat{K}(\eta, \rho)\| \tag{5.68a}$$

$$\mu = \sup_{0 \leq \eta \leq T/4} \|\hat{\underline{h}}(\eta)\| \tag{5.68b}$$

provided $0 < \lambda < 1$.

Note the supremum norms of $\hat{K}(\eta, \rho)$ and $\hat{\underline{h}}(\eta)$ are taken over the larger interval $0 \leq \eta \leq T/4$, $0 \leq \rho \leq T/4$, rather than the linear delay system case of $0 \leq \eta \leq \tau$. Here $T/4$, a quarter of the period, is assumed to be larger than the delay time $\tau$. The reason for taking the supremums over a larger interval is that the original nonlinear delay problem (5.8) possesses the unknown switch-over time $T_1$ and this was computed by the Fredholm Integral Equation Method of using only the finite kernels

$K_{iN}(t)$. Thus, true switch-over time $T_1$ may be of the form $T_1 + \Delta T$, where the amount of error $\Delta T$ is not known. However, from the physics of the trilinear problem, we must have

$$0 < T_1 + \Delta T < T/4,$$

thus the supremum norms are taken over the interval 0 to T/4 in order to give more conservative estimates for $\mu$ and $\lambda$, and eliminate the error effect of $\Delta T$.

It can be shown that the asymptotic properties of $\lambda$ and $\mu$ are;

$$\lim_{\tau\to 0} \lambda = \lim_{\tau\to 0} \mu = 0 \tag{5.69a}$$

$$\lim_{\substack{N\to\infty \\ \tau>0}} \mu = 0 \quad , \quad \lim_{\substack{N\to\infty \\ \tau>0}} \lambda > 0 \tag{5.69b}$$

since $\lambda$ depends only on the norm of entire kernels $\|K_i(t)\|$, $i = 1, 2$, and $\mu$ ultimately depends on the norm of error kernels $\|K_{i\epsilon}(t)\|$, $i = 1, 2$, and thus Equation (5.57) implies Equation (5.69). Therefore, the norm of error in the periodic solution $\mathfrak{a}_{1\epsilon} \leq \frac{\mu}{1-\lambda}$ becomes asymptotically

$$\lim_{\tau\to 0} \mathfrak{a}_{1\epsilon} = 0 \quad , \quad \lim_{\substack{N\to\infty \\ \tau>0}} \mathfrak{a}_{1\epsilon} = 0. \tag{5.70}$$

It is pointed out again that since Equation (5.69a) holds true, there must exist $\lambda$ such that $0 < \lambda < 1$ for $\tau = \tau^* > 0$ by invoking the local implicit function theorem, thus the convergence of the Neumann series is guaranteed to give Equation (5.67).

## Computation of the Error Bound

In order to compute the error bound of the periodic solution $\mathfrak{a}_{1\epsilon}$, it is necessary to establish the upper bounds for the norm of matrix kernels, $\|K_i(t)\|$ and $\|K_{i\epsilon}(t)\|$, $i=1,2$. This is done in Chapter IV, Equation (4.112),

$$\|K_{i\epsilon}(t)\| \le \frac{2\tau}{\pi}\left[\frac{1}{t}+\frac{\sqrt{2}}{2t-\tau}e^{-\frac{\tau}{2}\sigma_{iN}}\right]e^{\mathfrak{a}_{iN}t} \tag{5.71}$$

$$i=1,2$$

and

$$\|K_i(t)\| \le \|K_{iN}(t)\| + \|K_{i\epsilon}(t)\| \tag{5.72}$$

where $\|K_{iN}(t)\|$ is known for $i=1,2$. From Equations (5.68b) and (5.65a),

$$\mu = \sup_{0\le\eta\le T/4} \|\hat{\underline{h}}(\eta)\| = (1-\mathfrak{a}_w)\ \mathfrak{a}_g \tag{5.73}$$

where

$$\begin{aligned}\mathfrak{a}_w &= \inf_{0\le\eta\le T/4} \|W(\eta)\| \\ &= \frac{\tau^2\alpha_1\alpha_2}{2+\tau^2\alpha_1\alpha_2} < 1\end{aligned} \tag{5.74}$$

with

$$\alpha_1 = \frac{1}{\tau}\|K_1(T_1-\tau)\| + \|K_1(T_1-2\tau)\|\cdot\|B_1\|$$

$$\alpha_2 = \frac{1}{\tau}\|K_2(2T_2)\|\cdot\|K_1(T_1)\| + \|K_2(2T_2-\tau)\|\cdot\|B_2\|\cdot\|K_1(T_1)\|$$

and

$$\mathfrak{a}_g = \sup_{0\le\eta\le T/4} \|\underline{h}^{**}(\eta)\| \tag{5.75}$$

$$= a_3^* + \left\{\|K_1(T_1-\tau)\| + \tau\|K_1(T_1-2\tau)\|\cdot\|B_1\|\right\} \times \left\{a_2^* + \|K_2(2T_2)\|\ a_1^* + \tau\|K_2(2T_2-\tau)\|\cdot\|B_2\|\ a_1^*\right\} \tag{5.75 cont.}$$

with

$$a_1^* = \|K_{1\epsilon}(T_1-\tau)\|G_{1N} + \tau\|K_{1\epsilon}(T_1-2\tau)\|\cdot\|B_1\|\ G_{1N} + (T_1-\tau)\|K_{1\epsilon}(\tau)\|\cdot\|\underline{F}_1\|$$

$$a_2^* = \|K_{2\epsilon}(2T_2-\tau)\|G_{2N} + \tau\|K_{2\epsilon}(2T_2-2\tau)\|\cdot\|B_2\|\ G_{2N} + (2T_2-\tau)\|K_{2\epsilon}(\tau)\|\cdot\|\underline{F}_2\|$$

$$a_3^* = \|K_{1\epsilon}(T_1-\tau)\|G_{3N} + \tau\|K_{1\epsilon}(T_1-2\tau)\|\cdot\|B_1\|\ G_{3N} + (T_1-\tau)\|K_{1\epsilon}(\tau)\|\cdot\|\underline{F}_3\|$$

where

$$G_{iN} = \sup_{0\le t\le\tau}\|\underline{g}_{iN}(t)\| \quad , \quad \text{known from F. I. E. M. for } i=1, 2, 3.$$

From Equations (5.68a) and (5.65b),

$$\lambda = \tau\cdot\sup_{0\le\eta,\,\rho\le T/4}\|\hat{K}(\eta,\rho)\| = \tau^3(1-G_w)\,G_k \tag{5.76}$$

where

$$G_k = \sup\|K^{**}(\eta,\xi,\gamma,\rho)\| \quad , \quad 0\le\eta,\xi,\gamma,\rho\le T/4$$

$$= \left\{\frac{1}{\tau}\|K_1(T_1-\tau)\| + \|K_1(T_1-2\tau)\|\cdot\|B_1\|\right\}\cdot\left\{\frac{1}{\tau}\|K_2(2T_2)\|\cdot\|K_1(T_1-\tau)\| + \|K_2(2T_2-\tau)\|\cdot\|B_2\|\cdot\|K_1(T_1-\tau)\|\right\}.$$

Thus finally combine Equations (5.73) and (5.76),

$$G_{1\epsilon} \le \frac{\mu}{1-\lambda} = \frac{(1-G_w)\,G_g}{1-\tau^3(1-G_w)G_k} \tag{5.77}$$

and the error in the periodic solution of the trilinear delay system of Equation (5.8) is bounded by the expression (5.77). From this expression, the asymptotic nature of the error as shown in Equation (5.70) is verified immediately since $\mathfrak{a}_w$ and $\mathfrak{a}_k$ are bounded away from zero for $\tau > 0$ and independent of N, and from Equation (5.75),

$$\lim_{\tau \to 0} \mathfrak{a}_g = 0 \quad , \quad \lim_{\substack{N \to \infty \\ \tau > 0}} \mathfrak{a}_g = 0.$$

Furthermore, due to the nature of the error kernels, $\mathfrak{a}_g$ decreases exponentially as the number of terms N increases.

Therefore, the periodic solution obtained by the Fredholm Integral Equation Method is indeed exact in the sense that the error induced by truncating the kernel is made arbitrarily small by increasing N.

## 5.6 Approximate Periodic Solution by the Method of Slowly Varying Parameters

For the sake of comparison study, we shall consider an approximate solution method on the given nonlinear delay system Equation (5.8). Particular method chosen is based on the work of Krylov and Bogoliubov* and this is also known as the method of slowly varying parameters.

Although this method is applicable to more general class of nonlinear differential equations, the system must satisfy the small parameters requirement, namely, small nonlinearity, small forcing

*Good discussion of this subject is in Nonlinear Mechanics, N. Minorsky, p. 186.

coefficient, or small damping in order for the slowly varying parameters assumption to be valid.

Reposing the problem (5.8),

$$\ddot{x}(t) + c\dot{x}(t) + f(x(t-\tau)) = p \cos \omega t \tag{5.78}$$

with

$$\begin{aligned} f(x) &= x-1+k && \text{for } x > 1 \\ &= kx && \text{for } |x| \le 1 \\ &= x+1-k && \text{for } x < -1 \end{aligned}$$

and the harmonic response of Equation (5.78) is sought, i. e.,

$$|\omega-\omega_0| << 1 \ , \quad \omega_0 = \sqrt{k} .$$

Assume the main harmonic solution to be of the form

$$\begin{aligned} x(t) &= A(t) \cos (\omega t - \varphi(t)) = A(t) \cos \theta \\ \theta &= \omega t - \varphi(t) \end{aligned} \tag{5.79}$$

where $A(t)$ and $\varphi(t)$ are assumed to be slowly varying, therefore,

$$A(t-\tau) \simeq A(t) \quad , \quad \varphi(t-\tau) \simeq \varphi(t) \qquad \text{for} \qquad \tau << 1.$$

Then

$$x(t-\tau) = A(t) \{\cos \omega\tau \cdot \cos \theta + \sin \omega\tau \cdot \sin \theta\}$$

$$\dot{x}(t) = -A(t)\, \omega \sin \theta$$

and by setting

$$\dot{A}(t) \cos \theta + A\dot{\varphi}(t) \sin \theta = 0 \tag{5.80}$$

$$\therefore \qquad \ddot{x}(t) = -\dot{A}\, \omega \sin \theta - A\omega^2 \cos \theta + A\omega\, \dot{\varphi} \cos \theta .$$

Substitute the above in Equation (5.78)

$$-\dot{A}\omega \sin\theta - A\omega^2\cos\theta + A\omega\dot{\varphi}\cos\theta - cA\omega\sin\theta + f(A\cos(\theta-\omega\tau)) = p\cos\omega t. \tag{5.81}$$

Combine Equations (5.80) and (5.81), and integrate over a cycle, we obtain the first variational equations

$$2A\omega\dot{\varphi} - A\omega^2 + C(A, \omega) = p\cos\varphi \tag{5.82a}$$

$$2\dot{A}\omega + cA\omega - S(A, \omega) = p\sin\varphi \tag{5.82b}$$

where

$$C(A, \omega) = \frac{1}{\pi}\int_0^{2\pi} f(A\cos(\theta-\omega\tau))\cos\theta\, d\theta \tag{5.83a}$$

$$S(A, \omega) = \frac{1}{\pi}\int_0^{2\pi} f(A\cos(\theta-\omega\tau))\sin\theta\, d\theta. \tag{5.83b}$$

If we let $\theta^* = \cos^{-1}\frac{1}{A}$, $\theta^{**} = \pi - \theta^*$, then evaluation of Equation (5.83) becomes

$$C(A,\omega) = \frac{2}{\pi}\left[\int_0^{\theta^*}\{A\cos(\theta-\omega\tau) - 1 + k\}\cos\theta\, d\theta + \int_{\theta^*}^{\theta^{**}} kA\cos(\theta-\omega\tau)\cos\theta\, d\theta\right.$$

$$\left. + \int_{\theta^{**}}^{\pi}\{A\cos(\theta-\omega\tau) + 1 - k\}\cos\theta\, d\theta\right].$$

Simplifying the result,

$$C(A,\omega) = \cos\omega\tau\left\{kA + \frac{2}{\pi}(1-k)\left(A\cos^{-1}\frac{1}{A} + \frac{\sqrt{A^2-1}}{A}\right)\right\} - \frac{4(1-k)}{\pi A}\sqrt{A^2-1} \tag{5.84a}$$

for $A > 1$

$$= kA\cos\omega\tau$$

for $A \le 1$

and similarly

$$S(A,\omega) = \sin\omega\tau \left\{kA + \frac{2}{\pi}(1-k)\left(A\cos^{-1}\frac{1}{A} - \frac{\sqrt{A^2-1}}{A}\right)\right\} \qquad (5.84b)$$

$$\text{for } A > 1$$

$$= kA\sin\omega\tau$$

$$\text{for } A \le 1.$$

## Steady State Solution

From Equation (5.82), we let $\dot{A} = \dot{\varphi} = 0$,

$$\begin{aligned} -A\omega^2 + C(A,\omega) &= p\cos\varphi \\ cA\omega - S(A,\omega) &= p\sin\varphi \end{aligned} \qquad (5.85)$$

and eliminating $\varphi$ terms,

$$A^2\omega^4 - (2AC(A,\omega) - c^2A^2)\omega^2 - 2cAS(A,\omega)\quad \omega + C^2(A,\omega) + S^2(A,\omega) - p^2 = 0 \qquad (5.86)$$

and the steady state amplitude A versus the forcing frequency $\omega$ relationship is established.

Steady state phase angles are given by

$$\varphi = \tan^{-1}\left(\frac{cA\omega - S(A,\omega)}{-A\omega^2 + C(A,\omega)}\right). \qquad (5.87)$$

## Stability

Stability of the approximate solution is derived by perturbing the solution about its known amplitude $A^*$ and phase $\varphi^*$, let

$$\begin{aligned} A &= A^* + \xi \\ \varphi &= \varphi^* + \eta. \end{aligned} \qquad (5.88)$$

Substitute Equation (5.88) to the first variational equation (5.82), and neglecting the higher order terms of $\eta$ and $\xi$, we obtain two coupled linear first order ordinary differential equations in $\xi$ and $\eta$,

$$
\begin{aligned}
&2\omega A^{*}\dot{\eta}+(c\omega A^{*}-S(A^{*},\omega))\eta+\left(\left.\frac{\partial C}{\partial A}\right|_{A=A^{*}}-\omega^{2}\right)\xi=0 \\
&2\omega\dot{\xi}+\left(c\omega-\left.\frac{\partial S}{\partial A}\right|_{A=A^{*}}\right)\xi+(A^{*}\omega^{2}-C(A^{*},\omega))\eta=0
\end{aligned}
\tag{5.89}
$$

and we let

$$\eta=\hat{\eta}e^{\lambda t} \quad , \quad \xi=\hat{\xi}e^{\lambda t}$$

thus for non-trivial solution of $\hat{\eta}$, $\hat{\xi}$, the determinant must vanish, i. e.,

$$(2\omega\lambda)^{2}+b_{1}(2\omega\lambda)+b_{2}=0 \tag{5.90}$$

where

$$b_{1}=2c\omega-\frac{S(A^{*},\omega)}{A^{*}}-\left.\frac{\partial S}{\partial A}\right|_{A=A^{*}}$$

$$b_{2}=\left(c\omega-\frac{S(A^{*},\omega)}{A^{*}}\right)\left(c\omega-\left.\frac{\partial S}{\partial A}\right|_{A=A^{*}}\right)-\left(\omega^{2}-\frac{C(A^{*},\omega)}{A^{*}}\right)\left(\left.\frac{\partial C}{\partial A}\right|_{A=A^{*}}-\omega^{2}\right).$$

For stability, the complex roots $\lambda_{1,2}$ of Equation (5.90) must possess negative real parts, and this holds if and only if

$$\text{i)} \quad b_{1}>0 \tag{5.91}$$

$$\text{ii)} \quad b_{2}>0 \tag{5.92}$$

and the stability boundaries are given by setting $b_1=0, b_2=0$; and using Equation (5.84),

$$b_1 = 0 = c\omega - \sin\omega\tau\left\{k + \frac{2}{\pi}(1-k)\cos^{-1}\frac{1}{A^*}\right\} \quad , \quad A^* > 1$$
$$= c\omega - k\sin\omega\tau \qquad , \qquad A^* \le 1 \tag{5.93}$$

and

$$b_2 = 0 = \left(c\omega - \frac{S(A^*,\omega)}{A^*}\right)\left(c\omega - \frac{\partial S}{\partial A}\bigg|_{A=A^*}\right) + \left(\omega^2 - \frac{C(A^*,\omega)}{A^*}\right)\left(\omega^2 - \frac{\partial C}{\partial A}\bigg|_{A=A^*}\right)$$
$$A^* > 1 \tag{5.94}$$
$$= (c\omega - k\sin\omega\tau)^2 + (\omega^2 - k\cos\omega\tau)^2, \quad A^* \le 1$$

Also, Equation (5.94) coincides with the locus of vertical tangency obtained by taking partials of Equation (5.86) and setting $\frac{\partial\omega}{\partial A} = 0$.

Thus the approximate steady state frequency response A versus $\omega$ can be plotted from Equation (5.86) and their stability determined by Equations (5.93) and (5.94).

## 5.7 Numerical Examples

Numerical examples of the periodic solution (maximum amplitude A versus forcing frequency $\omega$) for the trilinear delay system (5.8) are given in the following Figures 5.3 through 5.6. First the exact solution by the Fredholm Integral Equation Method (F. I. E. M.) is given by solving the matrix equation (5.44), and its stability is determined by the eigenvalue study of Equation (5.50), and shown on the Figures as circles.

For reference, characteristic roots for the Region I ($z_1 = \sigma_1 + i\omega_1$) and Region II ($z_2 = \sigma_2 + i\omega_2$) are tabulated. The approximate periodic solution by the method of slowly varying parameters are also plotted using Equation (5.86) and its stability determined by Equations (5.93) and (5.94) for the same set of parameters.

Figure 5.3 shows the case when the system possesses very small nonlinearity ($k = 0.98$), small damping ($c = 0.02$), and small delay ($\tau$ is about one hundredth of the linear natural period). This is the case when the slowly varying parameters method is expected to yield good approximations, and this is verified here.

Figure 5.4 shows the case with large nonlinearity and large damping term present. Delay term ($\tau = 0.1$) is increased, but each linear region is kept stable.

Figure 5.5 is the case when delay term exceeds the damping coefficient ($\tau = 0.3$, $c = 0.2$), thus the entire nonlinear solution ($A > 1$) is unstable.

Figure 5.6 is the ordinary differential equation case ($\tau = 0$) with still large nonlinearity ($k = 0.5$).

Although general shape of a hardening system solution is similar between the two methods as shown in Figures 5.4 through 5.6, the amplitudes obtained by slowly varying parameters show large deviations from the exact solution whenever the large nonlinearity terms are present, while the stability criteria matched precisely in all cases.

Also we note in actual process of the Fredholm Integral Equation Method, it was sufficient to take the first three pairs of roots ($N = 3$) in both regions for the accuracy desired. This resulted in the transfer matrices of at the most $8 \times 8$ in size and further enlargement of the matrices did not improve the accuracy of the solution significantly.

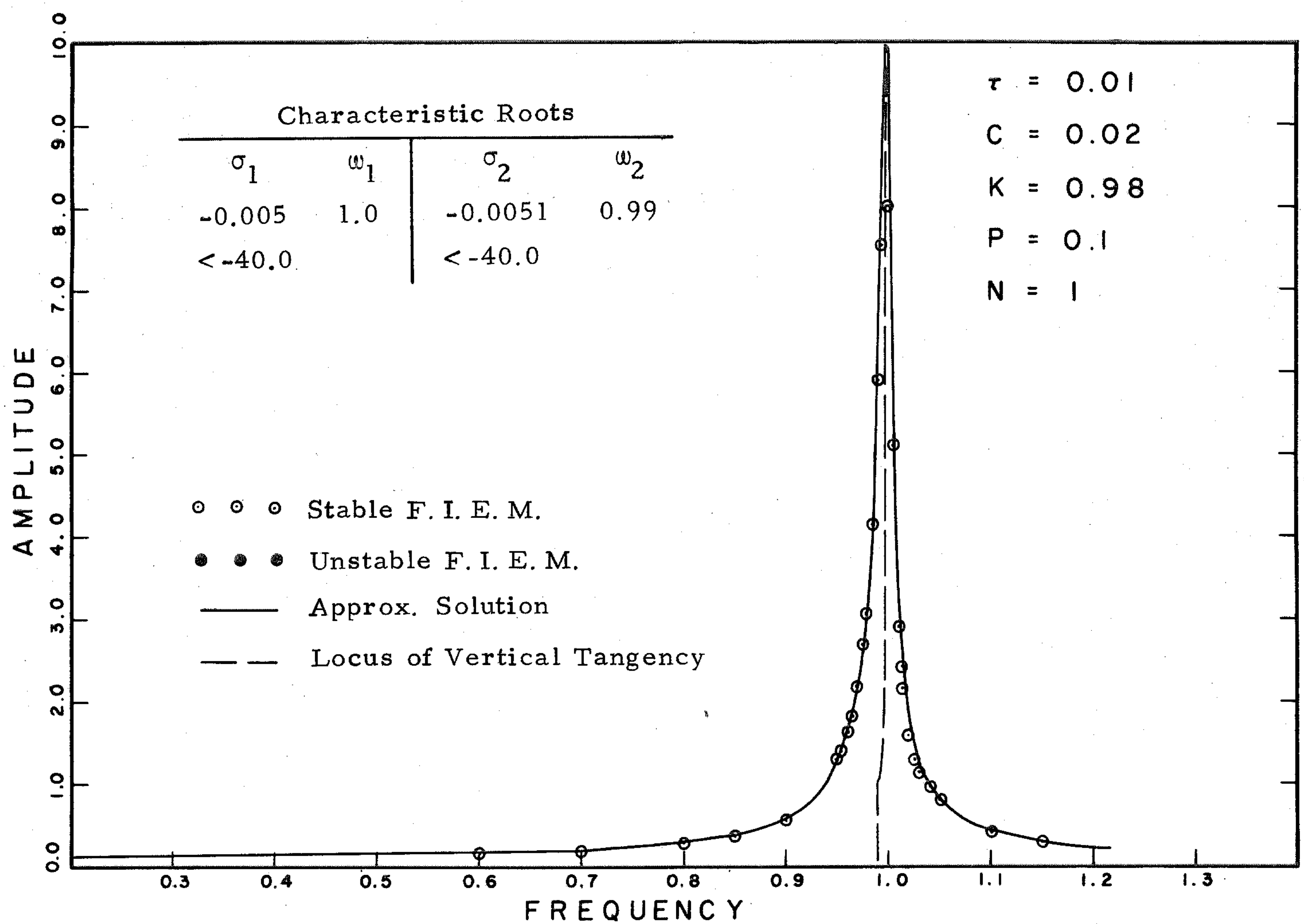


Fig. 5.3 Frequency Response for a Weakly Bilinear Delay System

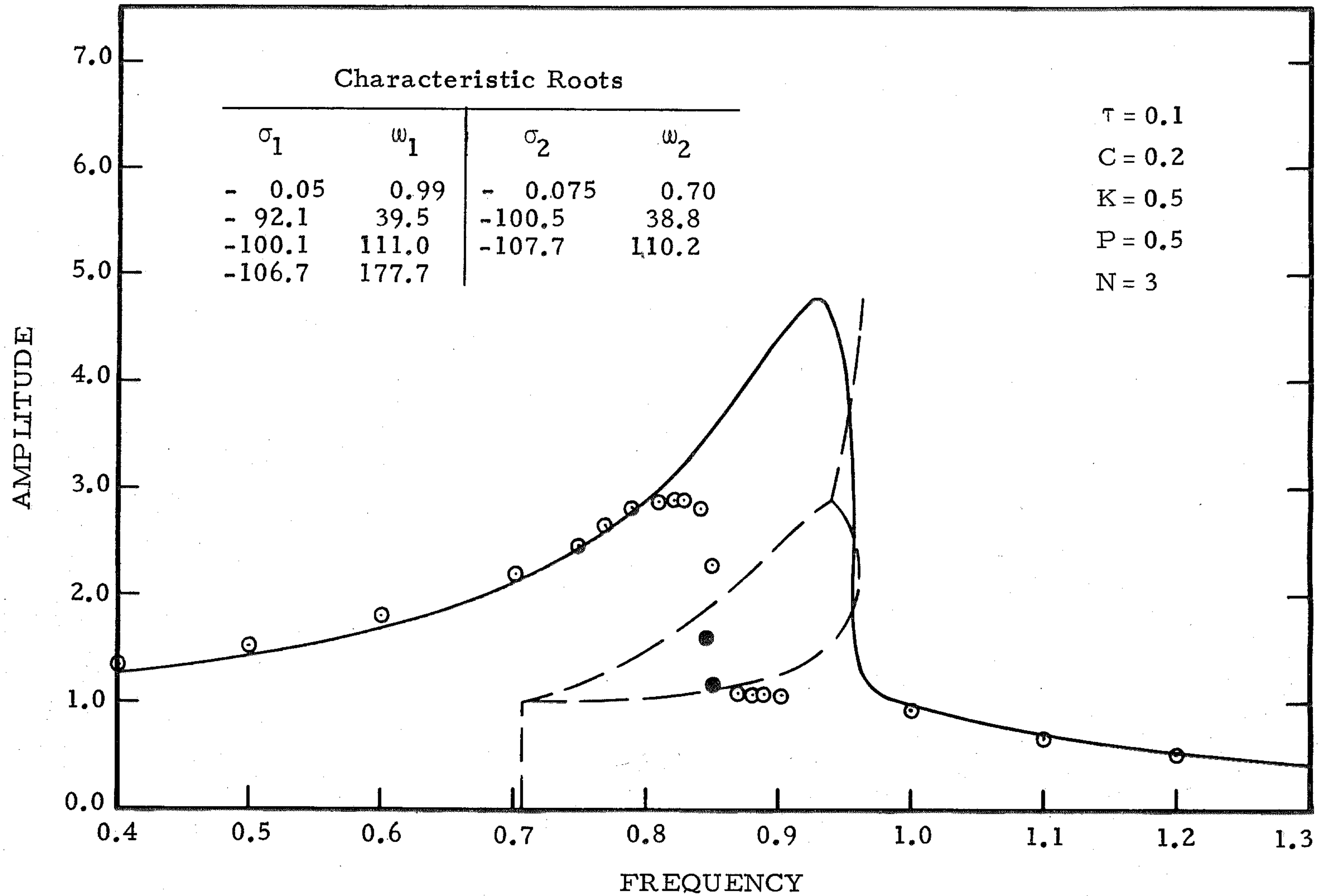


Fig. 5.4 Frequency Response for a Bilinear Delay System

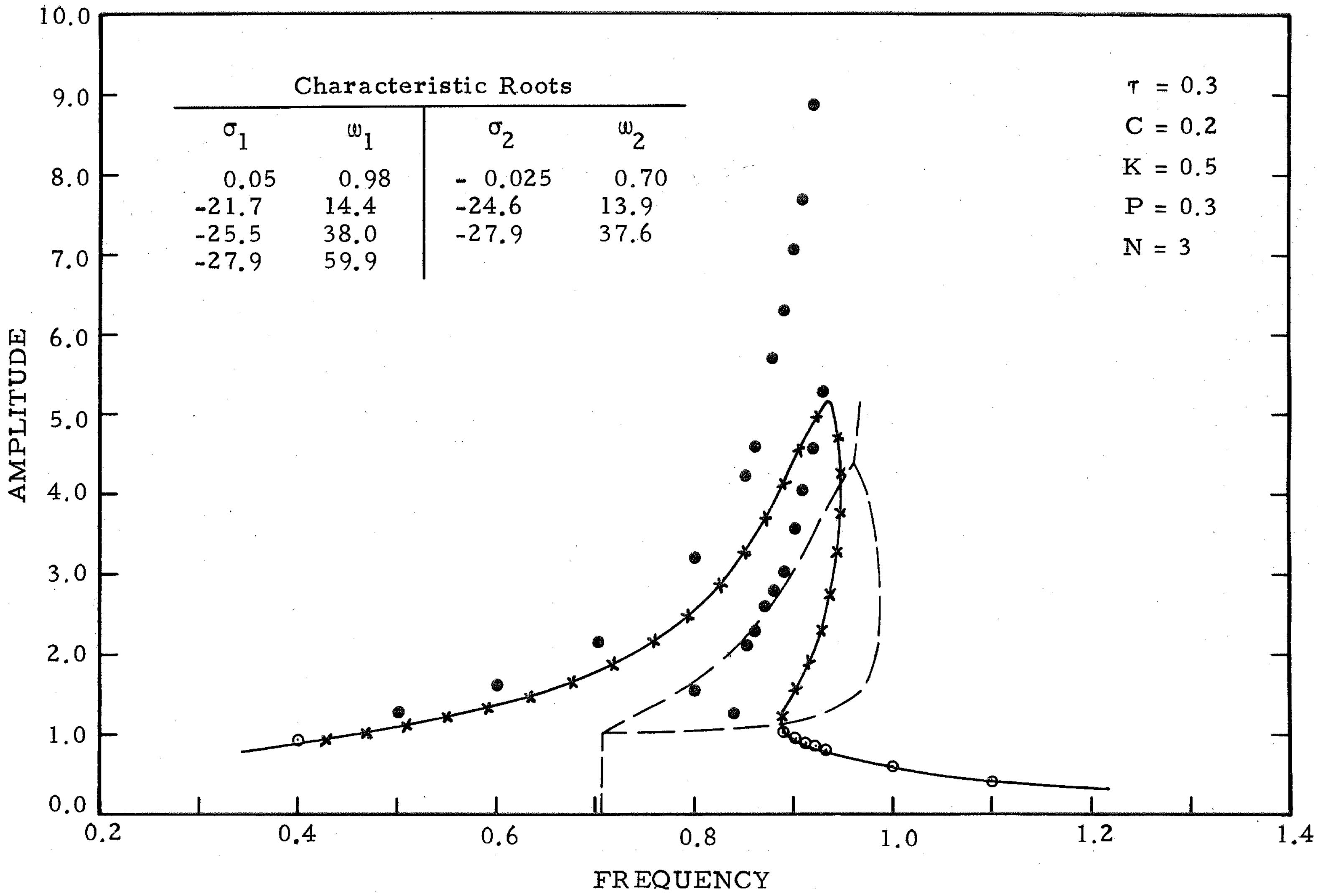


Fig. 5.5 Frequency Response for a Bilinear Delay System

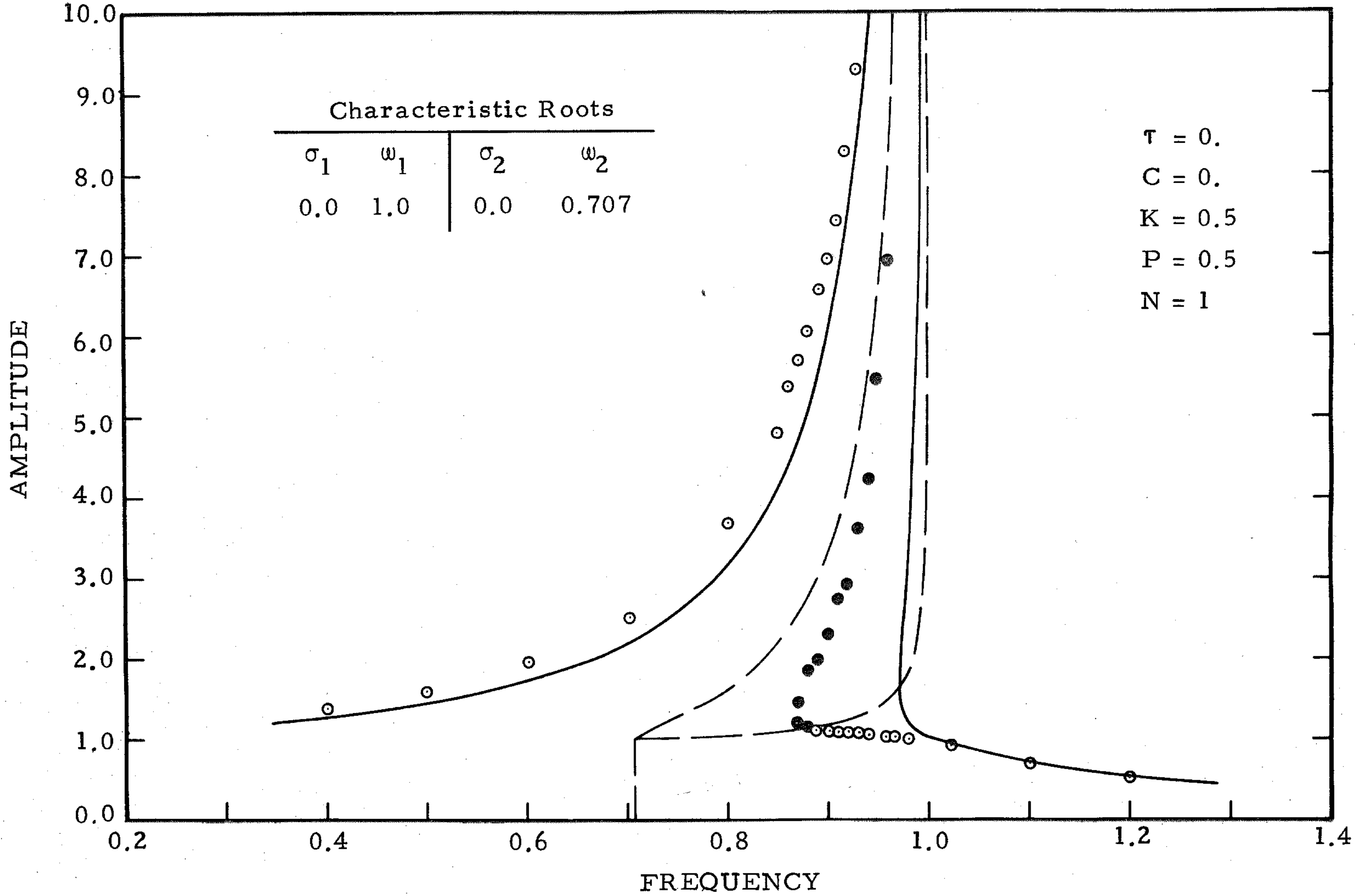


Fig. 5.6 Frequency Response for a Bilinear System

# Chapter VI

# CONCLUSIONS

## Summary of the Thesis

The main portion of this thesis is devoted to a new method of constructing periodic solutions for piecewise linear delay dynamic systems using the Fredholm Integral Equation Method. The method is developed for a linear delay system, and its most important use is made for piecewise linear delay systems. The key element during the process of developing the method is that as the formal solution is posed under the periodicity requirements, the solution takes the form of a Fredholm integral equation of the second kind, whose solution in general is not obtainable in a closed form. However, for the differential-difference equations arising from dynamic systems, it is observed that the matrix kernel $K(t)$ is always in the exponential form; thus $K(t-\xi)$ is guaranteed to be a separable kernel (Pincherle-Goursat kernel). Then from the well-established theory of integral equations, the Fredholm integral equation with separable kernel is reduced to a system of linear algebraic equations, thus the desired solution is obtained by a simple matrix equation. In other words, each linear region is represented by a transfer matrix $[M_i]$, such that the entire piecewise linear system is represented by a single transfer matrix $[M]$ which is a cascaded product of individual transfer matrices. Furthermore, the question of stability of the periodic solutions is answered immediately by calculating the eigenvalues of the transfer matrix [M]. A trilinear delay system with doubly bilinear case is examined as a numerical example to demonstrate the Fredholm Integral Equation Method, and the result is compared with an existing conventional approximate scheme.

One advantage of this method is that it is entirely free from the small parameter requirements which are strictly necessary for the conventional approximate schemes on nonlinear differential equations. Furthermore, the solution obtained by the Fredholm Integral Equation Method can be made as accurate as one wishes by enlarging the size of the transfer matrix [M], thus the solution is claimed to be exact.

The main limitation of this method is that it is applicable to only piecewise linear delay systems, and not applicable to a more general class of nonlinearity. However one may argue that there exist many classes of nonlinearities which can be closely approximated by an n-piecewise linear system, and thus the solution can be obtained by the same method. Thus whenever a physical nonlinearity can be mathematically modeled as an n-piecewise linear system, we have the exact solution method for the model.

## Suggestions for Further Study

The Fredholm Integral Equation Method developed for a piecewise linear delay system of single degree of freedom case may be extended beyond the present scope of this thesis. First, one may look for the harmonic response of a multidegree of freedom delay system, i. e., we have

$$M\ddot{\underline{x}}(t) + C\dot{\underline{x}}(t) + K\underline{x}(t-\tau) = \underline{F}(t) \qquad (6.1)$$

where M, C, and K are $n \times n$ matrices. If the system (6. 1) possesses classical normal modes such that M, C and K are simultaneously diagonalizable, then Equation (6. 1) reduces to n scalar delay equations, thus a multidegree piecewise linear delay system can be analyzed in a similar manner to the single degree case presented in this thesis. Another area of future interest is the case when a delay system is subjected to a random excitation. For example,

$$\ddot{x}(t) + c\dot{x}(t) + kx(t-\tau) = N(t) \qquad (6.2)$$

where N(t) is a Gaussian distributed white noise. In addition, the sub-harmonic and ultra-harmonic responses of a nonlinear delay system, which may in itself possess interesting features, are left untouched in this thesis. Finally, delay mechanisms occurring in continuous media, or mathematically in the area of partial differential-difference equations, brings entirely new area of research about which very little is known at the present time except for the very simple cases.

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