CALIFORNIA INSTITUTE OF TECHNOLOGY

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STABILITY OF PARAMETRICALLY EXCITED DIFFERENTIAL EQUATIONS

by

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ABSTRACT

Sufficient stability criteria for classes of parametrically excited differential equations are developed and applied to example problems of a dynamical nature.

Stability requirements are presented in terms of 1) the modulus of the amplitude of the parametric terms, 2) the modulus of the integral of the parametric terms and 3) the modulus of the derivative of the parametric terms.

The methods employed to show stability are Liapunov's Direct Method and the Gronwall Lemma. The type of stability is generally referred to as asymptotic stability in the sense of Liapunov.

The results indicate that if the equation of the system with the parametric terms set equal to zero exhibits stability and possesses bounded operators, then the system will be stable under sufficiently small modulus of the parametric terms or sufficiently small modulus of the integral of the parametric terms (high frequency). On the other hand, if the equation of the system exhibits individual stability for all values that the parameter assumes in the time interval, then the actual system will be stable under sufficiently small modulus of the derivative of the parametric terms (slowly varying)

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INTRODUCTION

Systems of explicit ordinary differential equations have been studied by pure and applied mathematicians since the beginning of the century. Cesari's work (1) is an excellent reference in this field. Sufficient existence and uniqueness theorems have been credited to Osgood, Kamke, Cauchy, Peano and Lipschitz. Most have been based on a bounded Lipschitz constant or an integrable Lipschitz constant over the period of applicability. Fortunately, the mathematical representations of physical problems encountered by the engineer and scientist usually fall into the class of equations which can be shown to exhibit solutions which are unique.

For application, the properties of the solution are desired. Ideally one would like the exact solution wherein the value of the solution vector can be readily obtained for any time t. However, with the exception of the linear equation with constant coefficients, this is not feasible. Some beneficial statements nevertheless can be directed to such solutions. The continuity with respect to initial conditions generally follows from the existence proof. Continuity of higher derivatives of the solution was studied by Coddington and Levinson. Investigation of equations often reduces to finding bounds on the solution and thus restricting the solution to some region of the phase space. For a fixed time, the bounds are considered improved if the region can be reduced in size. Two distinct methods have been utilized to demonstrate such bounds. The application of the Gronwall Lemma directly to the integral equation has been effective.

Many of the existence proofs are based on this lemma. The other approach is the Liapunov Direct Method (2) which considers an equation of the stability boundary rather than the trajectory equation directly.

The stability of the class of linear equations with parametric coefficients has received considerable attention in the literature. Liapunov, Hukahara and others investigated the case where the coefficients approach constant values with time. Conditions on the time dependent characteristic roots were analyzed by Cesari, Wintner and Markus. The Floquet Theory considers coefficients which are periodic in time. More recently Kozin (11) and Caughey (6) consider the cases where the coefficients are stochastic. Wang (9) extended Kozin's method to linear partial differential equations.

The concept of utilizing particular properties of the coefficient matrix has been the motivation of research into the stability of such equations. The introduction of the derivative and integral of the parameters (frequency content) is the aim of this thesis.

CHAPTER I

SECOND ORDER ORDINARY LINEAR SYSTEM

INTRODUCTION

In this chapter a Liapunov type theorem is proved (Theorem A) and then applied to demonstrate the asymptotic stability of the origin of $\ddot{x} + c\dot{x} + [\omega^2 + \ddot{s}(t)]x = 0$. Theorem B demonstrates the stability of the equation under sufficiently small bounds on the derivative of $\ddot{s}(t)$ and for a narrow banded function in the frequency domain this is equivalent to "low frequency bounds." Theorem C demonstrates the stability of the equation under sufficiently small bounds on the integral of $\ddot{s}(t)$ and for a narrow banded function this can be referred to as "high frequency bounds." Theorem D considers the stability under only the condition of a small modulus of $\ddot{s}(t)$ without any regard to its frequency content and hence is referred to as "universal stability bounds."

PRELIMINARIES

The following theorem is an extension, for time dependent functions, of Theorem VI of LaSalle and Lefschetz (8). It is similar to Liapunov's Direct Method but develops asymptotic stability for the case when $\frac{dV}{dt}$ is not negative definite.

Theorem A

Given the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{f}(\bar{x},t) \qquad t \geq 0 \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \qquad (1.1)$$

where $\bar{f}(\bar{x},t)$ together with its first partial derivatives is continuous in $\bar{x}$ and t, such that the existence and uniqueness of the initial value problem associated with 1.1 is assured for $\bar{x} \epsilon S_o$.

Then if there exist functions $V(\bar{x},t)$, $V_2(\bar{x})$, $V_3(\bar{x})$ and compact sets $S_t(\bar{x}_o)$, $S_1(\bar{x}_o)$ with the following properties:

i) $V(\bar{x},t)$ together with its first partial derivatives is continuous in $\bar{x}$ and t

ii) $V(\bar{0},t) \equiv 0 \qquad \forall t \geq 0$

iii) $V(\bar{x},t) \geq V_2(\bar{x}) > 0 \quad \forall t \qquad \forall \|\bar{x}\| \neq 0$

iv) $\dot{V}(\bar{x},t) \leq V_3(\bar{x}) \leq 0 \quad \forall t$, along any trajectory $\bar{x}(t)$ of 1.1

v) $V_2(\bar{x})$, $V_3(\bar{x})$ are continuous in $\bar{x}$ and $V_2(\bar{0}) = V_3(\bar{0}) = 0$

vi) $|\ddot{V}(\bar{x},t)| < \infty$ along any trajectory of 1.1

vii) R is the set R : $\{\bar{x} | V_3(\bar{x}) = 0\}$

viii) $S_t(\bar{x}_o)$: $\{\bar{x} | V(\bar{x},t) \leq V(\bar{x}_o,0)\} \subset S_1(\bar{x}_o) \subset S_o \qquad \forall t \geq 0$

then: 1) solutions of 1.1 are contained in $S_1(\bar{x}_o)$.

2) solutions of 1.1 tend to N as t tends to infinity, where N is the largest invarient set in $S_1(\bar{x}_o) \cap R$.

Proof:

Part 1. This part establishes the boundedness of the solutions. Along any trajectory of 1.1 $V(\bar{x}(t),t)$ is a positive, non-increasing function (by iii and iv)

$$\therefore\ V(\bar{x}(t),t) \leq V(\bar{x}_o,0) \tag{1.2}$$

Thus by viii) $$\bar{x}(t) \in S_t(\bar{x}_o) \tag{1.3}$$

$\therefore\ \bar{x}(t)$ is contained in $S_1(\bar{x}_o)$ $\forall t \geq 0$.

Part 2. This part establishes the asymptotic stability of the solutions. Before proving Part 2 of Theorem A, it is necessary to establish the following lemma. This lemma develops a sufficient condition for $\lim_{t\to\infty} \dot{V} = 0$ if $\lim_{t\to\infty} V = \ell$.

Lemma 1

If 1) $V(t)$ is twice differentiable for $t \in (t_o,\infty)$.

2) $V(t)$ tends to a limit $\underline{\ell}$ as t tends to infinity.

3) $\ddot{V}(t)$ is bounded for $t \in (t_o,\infty)$.

Then $\lim_{t\to\infty} \dot{V}(t) = 0$.

Proof:

By Taylor's Theorem:

$$V(t+h) = V(t) + h\dot{V}(t) + \frac{h^2}{2}\ddot{V}(\xi) \tag{1.4}$$

$\forall h \ni t$ and $t+h \in (t_o,\infty)$ and $t < \xi < t+h$.

thus $$\dot{V}(t) = \frac{1}{h}\,[(V(t+h)-\ell) - (V(t)-\ell)] - \frac{h}{2}\,\ddot{V}(\xi) \tag{1.5}$$

$$\therefore \max_{t_o \le t} |\dot{V}(t)| \le \frac{2}{h} \max_{t_o \le t} |V(t)-\ell| + \frac{h}{2} \max_{t_o \le t} |\ddot{V}(t)| \tag{1.6}$$

The minimum of the right hand side of 1.6 occurs for

$$\frac{h}{2} = \sqrt{\max_{t_o \le t} |V(t)-\ell| \Big/ \max_{t_o \le t} |\ddot{V}(t)|} \tag{1.7}$$

$$\therefore \max_{t_o \le t} |\dot{V}(t)| \le 2\sqrt{\max_{t_o \le t} |V(t)-\ell| \max_{t_o \le t} |\ddot{V}(t)|} \tag{1.8}$$

Let t_o , and hence t , tend to infinity

$$\therefore \lim_{t_o \to \infty} \max_{t_o \le t} |\dot{V}(t)| \le 2\sqrt{\{\lim_{t_o \to \infty} \max_{t_o \le t} |V(t)-\ell| \lim_{t_o \to \infty} \max_{t_o \le t} |\ddot{V}(t)|\}} \tag{1.9}$$

But $\lim_{t \to \infty} V(t) = \ell$ and $|\ddot{V}(t)| < \infty$ $t \in (t_o,\infty)$.

$$\therefore \lim_{t_o \to \infty} \max_{t_o \le t} |\dot{V}(t)| \le 0 \tag{1.10}$$

$$\therefore \lim_{t \to \infty} \dot{V}(t) \equiv 0 \tag{1.11}$$

Returning to the proof of Part 2 of Theorem A, $V(\bar{x}(t),t)$ is a positive, non-increasing function, bounded below; hence $V(\bar{x}(t),t) = V(t)$ must tend to a limit $\underline{\ell}$ as t tends to infinity.

Using vi) and Lemma 1, this implies that the $\lim_{t\to\infty} \dot{V}(\bar{x}(t),) = 0$, which in turn implies that $\lim_{t\to\infty} V_3(\bar{x}(t)) = 0$. Thus $\Gamma^+ \epsilon R$, where Γ^+ is the positive limit set for equation 1.1. Since $\bar{x}(t)$ is bounded in $S_1(\bar{x}_o)$, its positive limit set Γ^+ is a non-empty invariant set. Hence all solutions of equation 1.1 must tend to N as t tends to infinity, where N is the largest invariant set in the intersection of R and $S_1(\bar{x}_o)$, i.e., in $R \cap S_1(\bar{x}_o)$.

Main Problem

Given the differential equation:

$$\left.\begin{array}{ll} & \ddot{x} + c\dot{x} + [\omega^2 + \ddot{s}(t)]x = 0 \\ & x(0) = x_o , \qquad \dot{x}(0) = \dot{x}_o \\ \text{where} & \text{I)} \quad c > 0 , \qquad \omega^2 > 0 \\ & \text{II)} \quad |d^i/dt^i \, [s(t)]| < \infty \qquad i = 0, 1, 2, 3, 4 \end{array}\right\} \quad (1.12)$$

The space S_o of existence and uniqueness of this initial value problem is the two dimensional Euclidean space E_2 .

We have the following theorems:

Theorem B "Low Frequency Stability Bound"

If in equation 1.12 there exists B_1, ϵ_1' and ϵ greater than zero such that:

$$\left.\begin{array}{ll} \text{i)} \quad \infty > B_1 \geq \omega^2 - |\ddot{s}(t)| \geq \epsilon_1' > 0 & t \geq 0 \\ \text{ii)} \quad [2c - |\dddot{s}(t)| \; [\omega^2 - |\ddot{s}(t)|]^{-1}] \geq \epsilon > 0 & t \geq 0 \end{array}\right\} \quad (1.13)$$

Then equation 1.12 is asymptotically stable at the origin, and further:

$$\left.\begin{array}{l} |x| \leq \sqrt{x_o^2 + \dot{x}_o^2/\epsilon_1'} \\ |\dot{x}| \leq \sqrt{(x_o^2 + \dot{x}_o^2/\epsilon_1')B_1} \end{array}\right\} \quad (1.14)$$

<u>Proof</u>:

Let $$V(x,\dot{x},t) = x^2 + \dot{x}^2[\omega^2 + \ddot{s}(t)]^{-1} \quad (1.15)$$

Then $$\dot{V}(x(t), \dot{x}(t),t) = -\dot{x}^2[\omega^2 + \ddot{s}(t)]^{-1} \{2c + \dddot{s}(t) \; [\omega^2 + \ddot{s}(t)]^{-1}\} \quad (1.16)$$

Let $$V_2(x,\dot{x}) = x^2 + \dot{x}^2/B_1 > 0 \quad (1.17)$$

$$V_3(x,\dot{x}) = -\dot{x}^2\epsilon/B_1 \quad (1.18)$$

R is then the set $R: \{\bar{x} | \dot{x} = 0\}$

Let $S_1(\bar{x}_o)$ be the set $S_1(\bar{x}_o): \{\bar{x} | x^2 + \dot{x}^2 B_1^{-1} \leq x_o^2 + \dot{x}_o^2(\epsilon_1')^{-1}\}$ (1.19)

Let $S_t(\bar{x}_o)$ be the set $S_t(\bar{x}_o): \{\bar{x} | x^2 + \dot{x}^2[\omega^2 + \ddot{s}(t)]^{-1} \leq x_o^2 + \dot{x}_o^2[\omega^2 + \ddot{s}(0)]^{-1}\}$ (1.20)

Then $$S_t(\bar{x}_o) \subset S_1(\bar{x}_o) \subset S_o \quad (1.21)$$

All the conditions of Theorem A, except vi), are satisfied. Hence

1) All solutions of 1.12 are contained in $S_1(\bar{x}_o)$. Hence

$$\left.\begin{aligned} |x| &\leq \sqrt{x_o^2 + \dot{x}_o^2(\epsilon_1')^{-1}} \\ |\dot{x}| &\leq \sqrt{(x_o^2 + \dot{x}_o^2(\epsilon_1')^{-1})B_1} \end{aligned}\right\} \qquad (1.22)$$

Using 1.12, 1.13, 1.16 and 1.22 it is easily seen that $\ddot{V}(x(t),\dot{x}(t),t)$ is bounded, hence:

2) All solutions of 1.12 tend to N , the largest invariant set within $S_1(\bar{x}_o) \cap R$, i.e., within $\{\bar{x} \mid \dot{x} = 0,\ |x| \leq \sqrt{x_o^2+\dot{x}_o^2(\epsilon_1')^{-1}}\}$.

Now $$d\dot{x}/dx = [-c\dot{x} - [\omega^2 + \ddot{s}(t)]x]\ [\dot{x}]^{-1} \qquad (1.23)$$

$$\therefore\ |d\dot{x}/dx|_{\dot{x}=0} = \infty \text{ except at } x \equiv 0$$

Thus R contains no arc of a trajectory of 1.12 except at $x \equiv 0$. Hence N is the origin itself. Therefore all solutions are bounded and tend to zero as t tends to infinity, the origin is therefore asymptotically stable.

Theorem C "High Frequency Stability Bound"

If in equation 1.12 there exists B_2, ϵ_2', α and α' greater than zero such that:

$$\left.\begin{array}{lll} \text{i)} & \infty > B_2 \geq [\omega^2 + \dot{s}^2 - c|\dot{s}|] \geq \epsilon_2' > 0 & t \geq 0 \\ \text{ii)} & c - 2|\dot{s}| \geq \alpha > 0 & t \geq 0 \\ \text{iii)} & 2\omega^2 + 2\dot{s}^2 - 2c|\dot{s}| - |\ddot{s}| \geq \alpha' > 0 & t \geq 0 \end{array}\right\} \quad (1.24)$$

Then equation 1.12 is asymptotically stable at the origin, and further

$$|x| \leq \sqrt{x_o^2 + [\dot{x}_o + x_o\,\dot{s}(0)]^2\,[\epsilon_2']^{-1}}\,\exp[2\,\mathrm{Max}\,|s|]$$

$$|\dot{x}| \leq \sqrt{x_o^2 + [\dot{x}_o + x_o\,\dot{s}(0)]^2\,[\epsilon_2']^{-1}}\,[\sqrt{B_2'} + \mathrm{Max}|\dot{s}|]\,\exp[2\,\mathrm{Max}|s|] \quad (1.25)$$

<u>Proof:</u>

Let $\quad x = \exp(-s)y \quad (1.26)$

Equation 1.12 becomes

$$\ddot{y} + [c-2\dot{s}]\dot{y} + [\omega^2 + \dot{s}^2 - c\dot{s}]y = 0 \quad (1.27)$$

Let $\quad V(y,\dot{y},t) = y^2 + \dot{y}^2[\omega^2 + \dot{s}^2 - c\dot{s}]^{-1} \quad (1.28)$

$$\therefore\ \dot{V}(y(t),\dot{y}(t),t) = -\dot{y}^2[c-2\dot{s}]\,[\omega^2+\dot{s}^2-c\dot{s}]^{-1}\,[2-\ddot{s}[\omega^2+\dot{s}^2-c\dot{s}]^{-1}] \quad (1.29)$$

Let $\quad V_2(\bar{y}) = y^2 + \dot{y}^2/B_2 \quad (1.30)$

$$V_3(\bar{y}) = -\dot{y}^2\,\alpha\alpha'\,B_2^{-2} \quad (1.31)$$

The set R is given by $R:\{\bar{y}|\dot{y} = 0\}$

Let $S_1(\bar{y}_o)$ be the set $S_1(\bar{y}_o):\{\bar{y}|y^2+\dot{y}^2B_2^{-1} \leq y_o^2+\dot{y}_o^2(\epsilon_2')^{-1}\}$. $\quad (1.32)$

Let $S_t(\bar{y}_o)$ be the set:

$$S_t(\bar{y}_o):\{\bar{y}|y^2+\dot{y}^2[\omega^2+\dot{s}^2(t)-c\dot{s}(t)]^{-1} \leq y_o^2+\dot{y}_o^2[\omega^2+\dot{s}^2(0)-c\dot{s}(0)]^{-1}\} \quad (1.33)$$

Then $\quad S_t(\bar{y}_o) \subset S_1(\bar{y}_o) \subset S_o \qquad (1.34)$

All conditions of Theorem A, except vi), are satisfied, hence

1) All solutions of 1.27 are contained in $S_1(\bar{y}_o)$, hence

$$\left.\begin{aligned} |y| &\le \sqrt{y_o^2 + \dot{y}_o^2(\epsilon_2')^{-1}} \\ |\dot{y}| &\le \sqrt{(y_o^2 + \dot{y}_o^2(\epsilon_2')^{-1})B_2} \end{aligned}\right\} \qquad (1.35)$$

Using 1.26 and 1.35 it is easily shown that:

$$\begin{aligned} |x| &\le \sqrt{x_o^2 + [\dot{x}_o + x_o\,\dot{s}(0)]^2\,[\epsilon_2']^{-1}}\,\exp[2\,\mathrm{Max}|s|] \\ |\dot{x}| &\le \sqrt{x_o^2 + [\dot{x}_o + x_o\,\dot{s}(0)]^2\,[\epsilon_2']^{-1}}\,[\sqrt{B_2} + \mathrm{Max}|\dot{s}|]\,\exp[2\,\mathrm{Max}|s|] \end{aligned} \qquad (1.36)$$

Using 1.27, 1.29 and 1.35 it is easily seen that $\ddot{V}(y(t),\dot{y}(t),t)$ is bounded, hence:

2) All solutions of 1.27 tend to N, the largest invariant set within:

$$S_1(\bar{y}_o) \cap R, \text{ i.e., within } \{\bar{y} \mid \dot{y} = 0,\ |y| \le \sqrt{y_o^2 + \dot{y}_o^2[\epsilon_2']^{-1}}\}$$

Now: $\quad d\dot{y}/dy = [-[c-2\dot{s}]\dot{y} - [\omega^2 + \dot{s}^2 - c\dot{s}]y]\,[\dot{y}]^{-1} \qquad (1.37)$

$$\therefore\ |d\dot{y}/dy|_{\dot{y}=0} = \infty$$

Thus R contains no arc of a trajectory of 1.27 except at $y \equiv 0$. Hence N is the origin itself. Therefore all solutions of 1.27 are bounded and tend to zero as t tends to infinity; the origin is therefore asymptotically stable. Using 1.36, all solutions of 1.12 are bounded, and since s is bounded, all solutions of 1.12 tend to zero as t tends to infinity; therefore equation 1.12 is asymptotically stable at the origin.

<u>Theorem D</u> <u>"Universal Stability Bound"</u>

If in equation 1.12 there exists a δ greater than zero such that:

$$\text{i)} \quad 1-|\ddot{s}(t)|\left[\frac{1}{\omega^2}+\frac{1}{\omega c}\right] \geq \delta > 0 \qquad t > 0 \tag{1.38}$$

Then equation 1.12 is asymptotically stable at the origin, and further:

$$\left.\begin{aligned} |x| &\leq \sqrt{\left\{\left(\frac{c}{2}+\frac{\omega^2}{c}\right)x_o^2 + x_o\,\dot{x}_o + \frac{\dot{x}_o^2}{c}\right\}\frac{c}{\omega^2}} \\ |\dot{x}| &\leq \sqrt{\left\{\left(\frac{c}{2}+\frac{\omega^2}{2}\right)x_o^2 + x_o\,\dot{x}_o + \frac{\dot{x}_o^2}{c}\right\} 2c} \end{aligned}\right\} \tag{1.39}$$

<u>Proof</u>:

Let $$V(x,\dot{x},t) = V_2(\bar{x}) = \left(\frac{c}{2}+\frac{\omega^2}{c}\right)x^2 + x\dot{x} + \frac{\dot{x}^2}{c} \tag{1.40}$$

$$\therefore\ \dot{V}(x,\dot{x},t) = -[\dot{x}^2 + \omega^2 x^2] - \ddot{s}\left[x^2 + \frac{2}{c}\,x\dot{x}\right] \tag{1.41}$$

Let $$V_3(\bar{x}) = -[\dot{x}^2 + \omega^2 x^2]\,\delta \tag{1.42}$$

The set R is given by $R: \{\bar{x} \mid x = \dot{x} = 0\}$.

Let $S_1(\bar{x})$ be the set $S_1(\bar{x}_o): \{\bar{x} \mid V_2(\bar{x}) \leq V_2(\bar{x}_o)\}$.

All conditions of Theorem A, except vi are satisfied, hence:

1) All solutions of 1.12 are contained in $S_1(\bar{x}_o)$, therefore from 1.40:

$$|x| \leq \sqrt{\{(\frac{c}{2} + \frac{\omega^2}{c})x_o^2 + x_o\dot{x}_o + \frac{\dot{x}_o^2}{c}\} c/\omega^2}$$

$$|\dot{x}| \leq \sqrt{\{(\frac{c}{2} + \frac{\omega^2}{c})x_o^2 + x_o\dot{x}_o + \frac{\dot{x}_o^2}{c}\} 2c} \qquad (1.43)$$

In this case the set $S_t(\bar{x}_o)$ coincides with $S_1(\bar{x}_o)$. Using 1.12, 1.13, 1.41 and 1.43 it is easily shown that $\ddot{V}(\bar{x}(t),t)$ is bounded, hence:

2) All solutions of 1.12 tend to N , the largest invariant set in R $S_1(\bar{x}_o)$ which in this case is simply the origin. Therefore all solutions of 1.12 are bounded and tend to zero as t tends to infinity; therefore system 1.12 is asymptotically stable at the origin.

Examples

For illustration, the Liapunov stability domains will be applied to a restricted class of the Hill equation 1.12 where $\ddot{s}(t)$ is narrowly banded in the frequency domain. This includes the special cases:

i) $\ddot{s}(t) = M \sin \eta t$ (Mathieu equation) and

ii) $\ddot{s}(t) = \sum_i M_1 \sin(\eta_i t + \varphi_i)$ (generalized Mathieu equation)

where the set $\{\eta_i\}$ is narrowly banded.

The application of the theorems to the Mathieu equation allows comparison with the known stability boundary. The effect of broadening the frequency

band is to broaden the minimum amplitude region about $\omega/\eta = 1/2$, as would be expected. The sufficient stability regions of two cases of multiple degree of freedom systems which possess classical modes are developed and illustrated.

The interest in this class of Hill equation arises, for example, in the stability analysis of a base excited pendulum and also in the equation of first variation about a periodic solution of a non-linear equation (the forced Duffing equation).

Example 1.1

Consider the equation of the linear oscillator with a parametric stiffness term:

$$\left.\begin{aligned} &\ddot{x} + c\dot{x} + [\omega^2 + \ddot{s}(t)]x = 0 \\ &x(0) = x_o,\ \dot{x}(0) = \dot{x}_o \qquad t \geq 0 \end{aligned}\right\} \tag{1.44}$$

where

$$\left.\begin{aligned} &s(t) = \int_{\eta(1-\theta)}^{\eta(1+\theta)} A(\omega)\omega^{-2} \sin[\omega t + \varphi(\omega)]d\omega \\ &\eta > 0;\ \theta > 0;\ \theta \ll 1;\ 0 < \int_{\eta(1-\theta)}^{\eta(1+\theta)} |A(\omega)|d\omega = M < \infty \end{aligned}\right|$$

We note that:

$$\left.\begin{aligned}
&\text{i)}\quad \dot{s}(t) = \int_{\eta(1-\theta)}^{\eta(1+\theta)} A(\omega)\omega^{-1}\cos[\omega t + \varphi(\omega)]d\omega \\
&\qquad |\dot{s}(t)| \le M/\eta(1-\theta) \\
&\text{ii)}\quad \ddot{s}(t) = -\int_{\eta(1-\theta)}^{\eta(1+\theta)} A(\omega)\sin[\omega t + \varphi(\omega)]d\omega \\
&\qquad |\ddot{s}(t)| \le M \\
&\text{iii)}\quad \dddot{s}(t) = -\int_{\eta(1-\theta)}^{\eta(1+\theta)} \omega A(\omega)\cos[\omega t + \varphi(\omega)]d\omega \\
&\qquad |\dddot{s}(t)| \le \eta(1+\theta)M \\
&\text{iv)}\quad \ddddot{s}(t) = \int_{\eta(1-\theta)}^{\eta(1+\theta)} \omega^2 A(\omega)\cos[\omega t + \varphi(\omega)]d\omega \\
&\qquad |\ddddot{s}(t)| \le \eta^2(1+\theta)^2 M
\end{aligned}\right\} \quad (1.45)$$

The function $s(t)$ satisfies condition 12 II. Thus the criteria for asymptotic stability for this example reduce to the following:

"Low Frequency Stability Boundary"

From 1.13:

$$\left.\begin{aligned} &\text{i)} \quad M < \omega^2 \\ &\text{ii)} \quad \eta(1+\theta)M < 2c(\omega^2 - M) \end{aligned}\right\} \qquad (1.46)$$

"High Frequency Stability Boundary"

From 1.24:

$$\left.\begin{aligned} &\text{i)} \quad \omega^2 - cM/\eta(1-\theta) > 0 \\ &\text{ii)} \quad c - 2M/\eta(1-\theta) > 0 \\ &\qquad \text{[Note: ii) implies i) if } c^2 < 2\omega^2\text{].} \\ &\text{iii)} \quad 2\omega^2 - 2cM/\eta(1-\theta) - M > 0 \end{aligned}\right\} \qquad (1.47)$$

"Universal Stability Boundary"

From 1.38:

$$\text{i)} \quad M < \omega c/(1 + \frac{c}{\omega}) \qquad (1.48)$$

Figure 1 shows the resulting stability boundaries for this example. It will be noted that M/ω^2 reaches its minimum value in the vicinity of η/ω equal to two. Figure 2 shows the resulting stability boundaries for the case θ equal to zero. Superimposed on the graph are the known stability boundaries for the Mathieu equation (7).

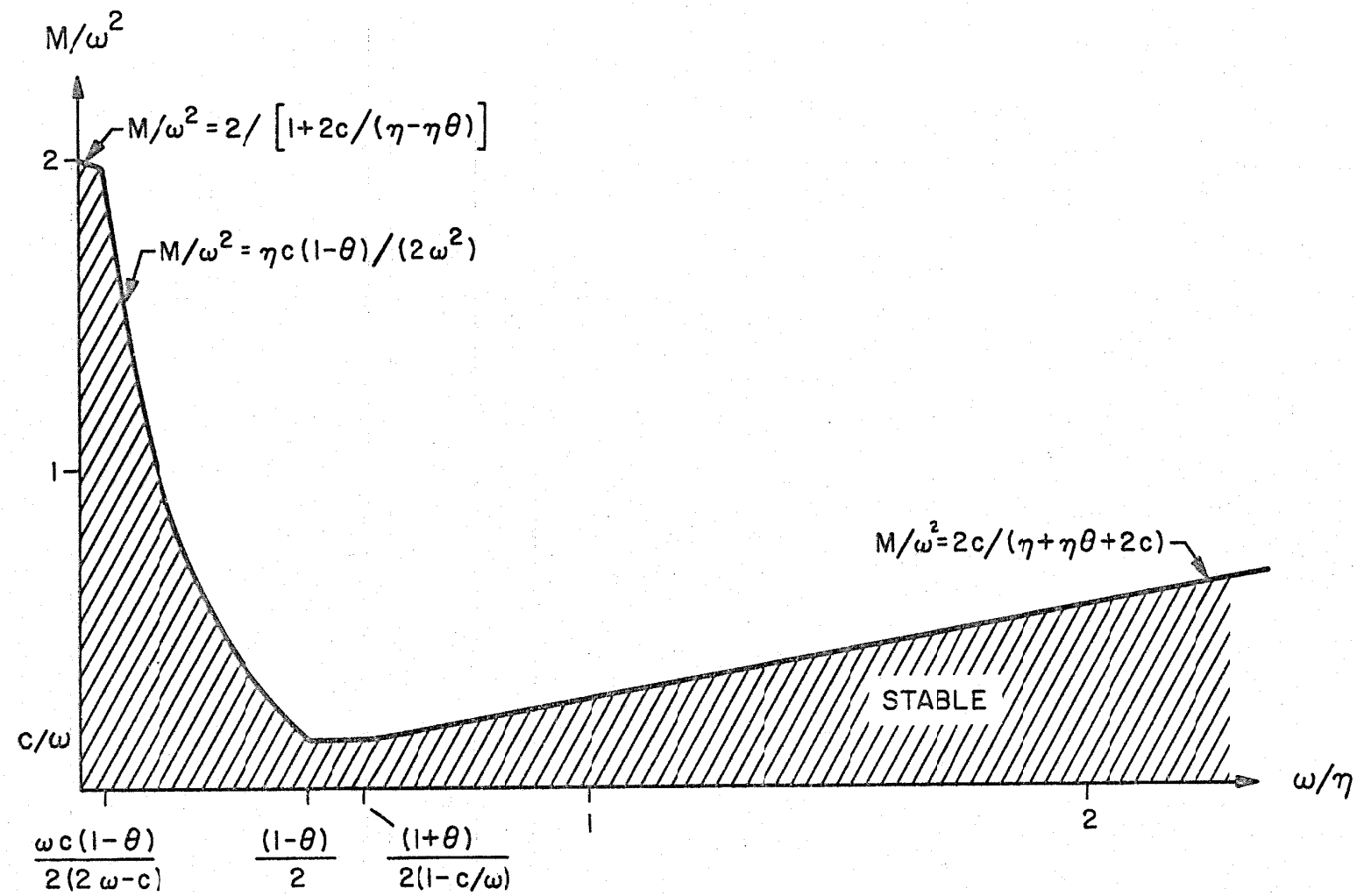

Fig.1 SUFFICIENT STABILITY BOUNDARY FOR EXAMPLE 1.1

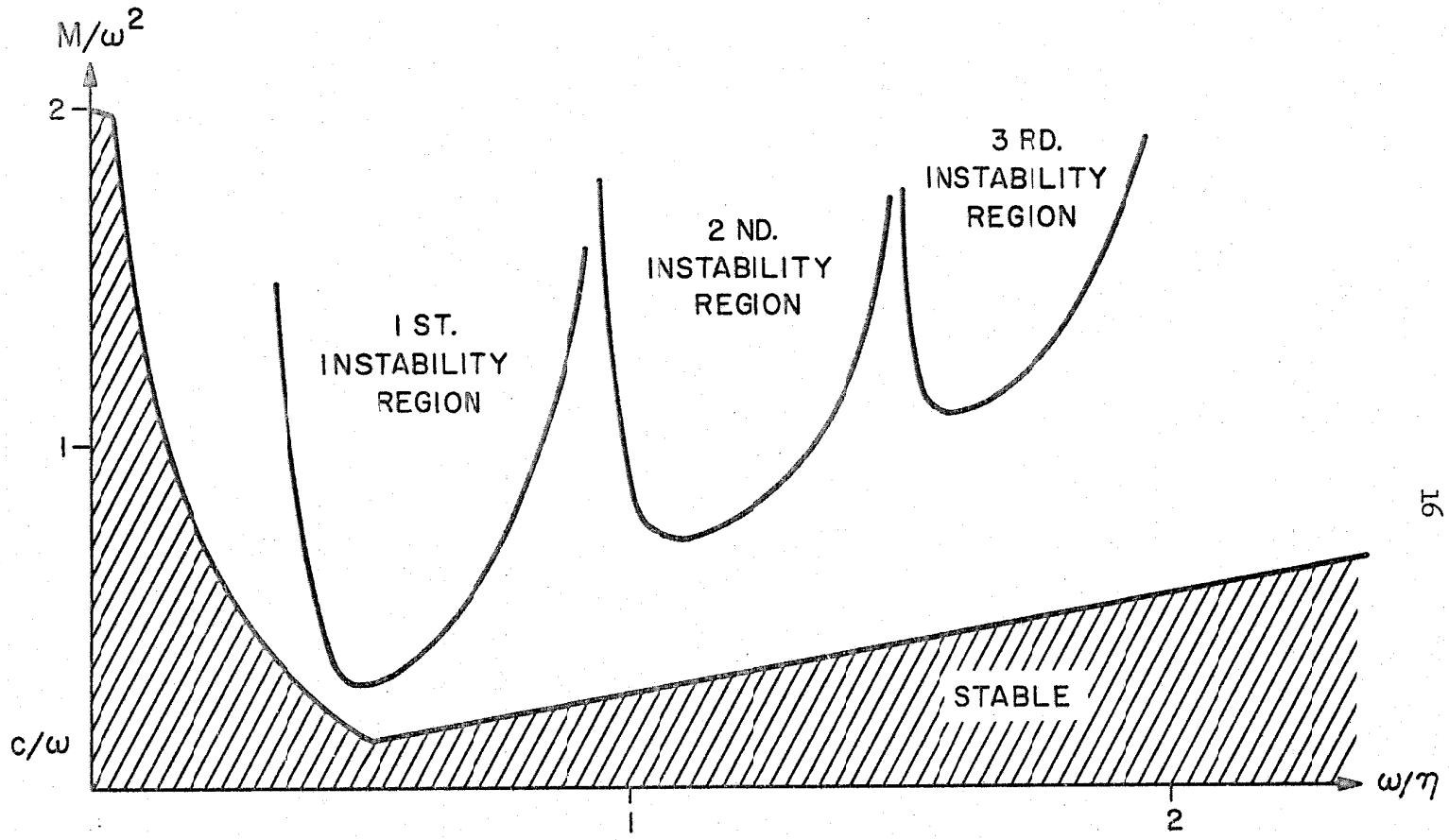

Fig.2 SUFFICIENT STABILITY BOUNDARY OF THE MATHIEU EQUATION SHOWN WITH THE ACTUAL STABILITY BOUNDARY

Example 1.2

Consider the system of linear equations:

$$\left.\begin{aligned} &I\ddot{\bar{y}} + C\dot{\bar{y}} + K\bar{y} + \ddot{s}(t)\ P\bar{y} = 0 \\ &\bar{y}(0) = \bar{y}_o\ , \qquad \dot{\bar{y}}(0) = \dot{\bar{y}}_o \end{aligned}\right\} \quad (1.49)$$

where C, K and P are commutable, symmetric $n \times n$ matrices, C and K are positive definite and $\ddot{s}(t)$ is a scalar function of time given by 1.45.

Since C, K and P commute, there exists an orthogonal matrix such that:

$$\left.\begin{aligned} &\text{i)} \quad \Phi^T\Phi = I \\ &\text{ii)} \quad \Phi^T C\Phi = [\ulcorner C_i \lrcorner] \\ &\text{iii)} \quad \Phi^T K\Phi = [\ulcorner \omega_i^2 \lrcorner],\ 0 < \omega_1 \le \omega_2 \le \dots \le \omega_n < \infty \\ &\text{iv)} \quad \Phi^T P\Phi = [\ulcorner \lambda_i \lrcorner] \end{aligned}\right\} \quad (1.50)$$

Using the transformation $\bar{y} = \Phi\bar{x}$, equation 1.49 is reduced to the set of $\underline{n}$ uncoupled equations:

$$\left.\begin{aligned} &\ddot{x}_i + C_i\ \dot{x}_i + [\omega_i^2 + \lambda_i\ \ddot{s}(t)]x_i = 0 \\ &i = 1,\ 2,\ \dots\ n \\ &\bar{x}(0) = \Phi^T\bar{y}_o\ , \qquad \dot{\bar{x}}(0) = \Phi^T\dot{\bar{y}}_o \end{aligned}\right\} \quad (1.51)$$

Two special cases will be examined.

Case 1

i) $\lambda_i = \omega_i$ (parametric term proportional to the frequency)

ii) $C_i = 2\omega_i\zeta$ (same percent of critical damping in each mode)

The criteria for asymptotic stability for this example reduce to:

"Low Frequency Stability Boundaries"

From 1.13:

$$\left.\begin{aligned} &\text{i)}\quad M < \omega_1 \le \omega_i \ \ldots \le \omega_n \\ &\text{ii)}\quad M < \frac{4\zeta\,\omega_1}{[\frac{1+\theta}{\omega_1/\eta}] + 4\zeta} \le \frac{4\zeta\,\omega_2}{[\frac{1+\theta}{\omega_2/\eta}] + 4\zeta} \quad \ldots\ldots \end{aligned}\right\} \quad (1.52)$$

"High Frequency Stability Boundaries"

From 1.24:

$$\left.\begin{aligned} &\text{ii)}\quad M < \zeta\eta(1-\theta) \qquad \forall i,\ \zeta < 1 \\ &\text{iii)}\quad M < \frac{2}{\omega_1^{-1}+2\zeta\eta^{-1}(1-\theta)^{-1}} \le \frac{2}{\omega_2^{-1}+2\zeta\eta^{-1}(1-\theta)^{-1}} \ \ldots\ldots \end{aligned}\right\} \quad (1.53)$$

"Universal Stability Boundaries"

From 1.38:

$$\text{i)}\quad M < 2\omega_1\zeta/(1+2\zeta) \le 2\omega_2\zeta/(1+2\zeta) \ \ldots\ldots \qquad (1.54)$$

In this case it is seen that the lowest mode, $i=1$, determines the stability of the whole system. Since Φ is a matrix with bounded elements, and since each x_i, $\dot{x}_i$ is bounded and tends to zero as $\underline{t}$ tends to infinity, it follows that $\|\bar{y}\|$ is bounded and tends

to zero as $\underline{t}$ tends to infinity; thus, under conditions 1.52, 1.53 and 1.54, system 1.49 is asymptotically stable at the origin.

Case 2

i) $\lambda_i = \omega_i^2$ (parametric term proportional to the frequency squared)

ii) $c_i = 2\omega_i\zeta$ (same percent of critical damping for each mode)

The criteria for asymptotic stability for this example reduce to:

"Low Frequency Stability Boundaries"

From 1.13:

$$\left.\begin{aligned} &\text{i)}\quad M < 1 \qquad \forall i \\ &\text{ii)}\quad M < \frac{2\zeta}{\eta(1+\theta)\omega_1^{-1}+2\zeta} \quad \frac{2\zeta}{\eta(1+\theta)\omega_2^{-1}+2\zeta} \cdots\cdots \end{aligned}\right\} \quad (1.55)$$

"High Frequency Stability Boundaries"

From 1.24:

$$\left.\begin{aligned} &\text{ii)}\quad M < \zeta(1-\theta)\omega_n^{-1} \le \zeta(1-\theta)\omega_{n-1}^{-1} \cdots\cdots \zeta < 1 \\ &\text{iii)}\quad M < \frac{2}{1+2\omega_n\zeta\eta^{-1}(1-\theta)^{-1}} \quad \frac{2}{1+2\omega_{n-1}\zeta\eta^{-1}(1-\theta)^{-1}} \cdots\cdots \end{aligned}\right\} \quad (1.56)$$

"Universal Stability Boundaries"

From 1.38:

$$\text{i)}\quad M < 2\zeta/(1+2\zeta) \qquad \forall i \qquad (1.57)$$

In this case the "low frequency" and the universal stability boundaries are determined by the lowest mode. However, the "high frequency"

stability boundary is determined by the highest mode $i=n$.

As in Case 1, it is easily shown that conditions 1.55, 1.56 and 1.57 are sufficient to guarantee that system 1.49 is asymptotically stable at the origin. Figures 3 and 4 show the stability boundaries for Case 1 and Case 2 respectively.

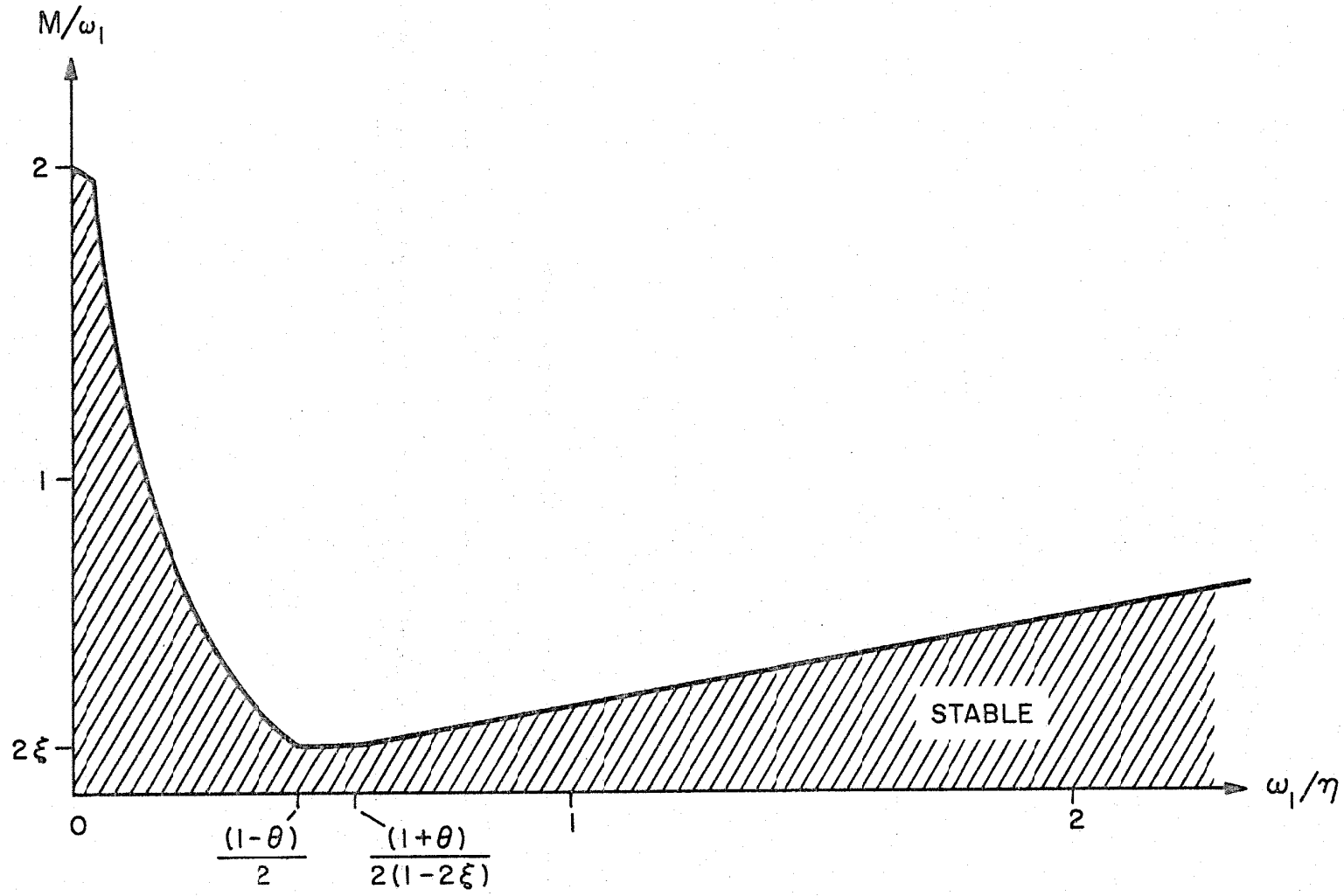

Fig.3 SUFFICIENT STABILITY BOUNDARY FOR CASE I, EXAMPLE I.2

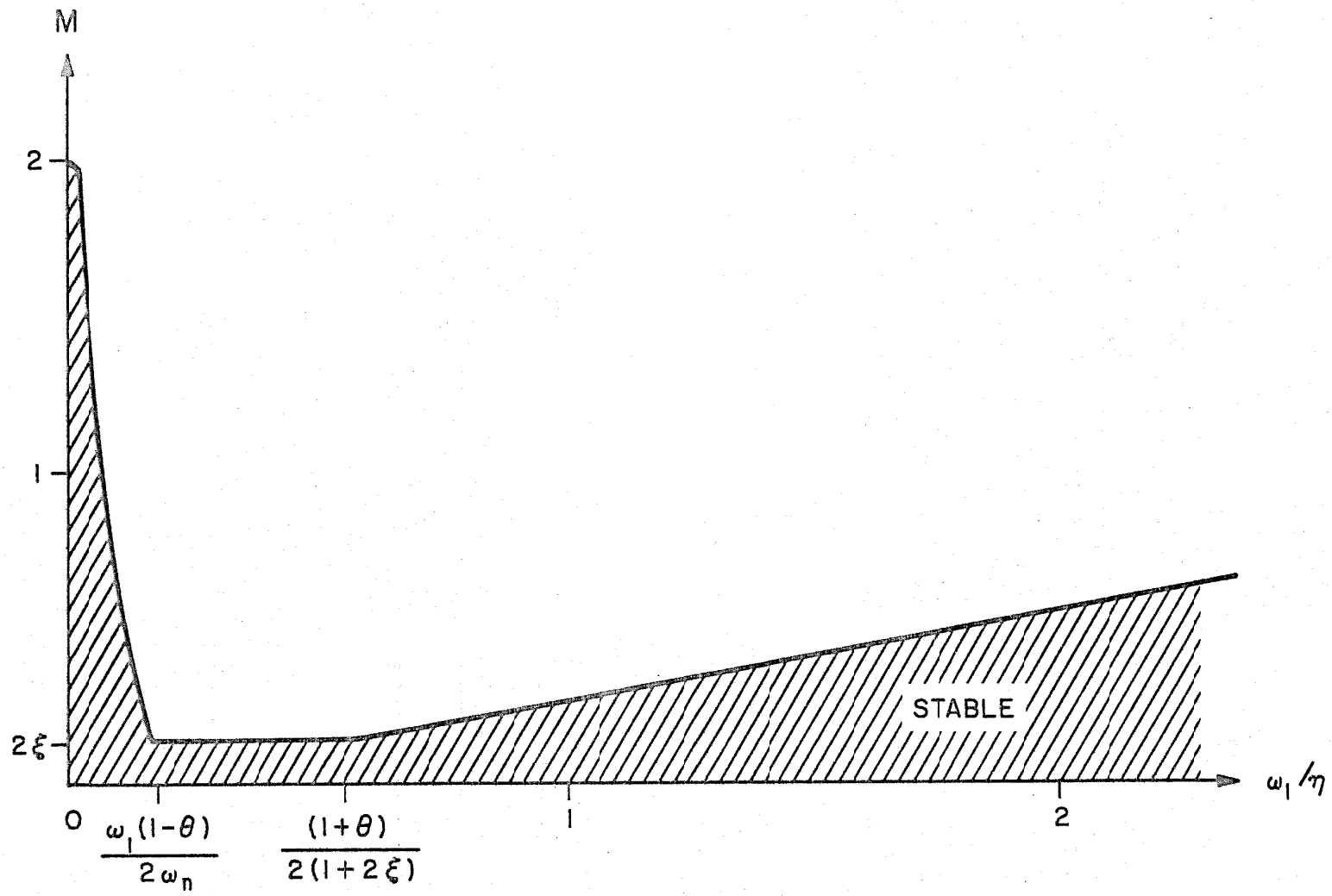

Fig. 4 SUFFICIENT STABILITY BOUNDARY FOR CASE 2, EXAMPLE 1.2

CHAPTER II

N'TH ORDER LINEAR SYSTEMS

INTRODUCTION

The stability analysis of n-dimensional systems of equations of the form: $\frac{d\bar{x}}{dt} = A(t)\bar{x}$ is developed in this chapter along the same lines as for the equation considered in Chapter I. The criteria for stability are developed for a very general class of problems and the basic concern is to demonstrate validity of the methods and the existence of stability boundaries. Therefore the bounds developed are expected to be crude but the methods would accommodate, in a particular example, better approximation on the bounds of quadratic forms, etc. Also, for classical systems the approach of Example 2 of Chapter I would give sharper results.

Sufficient stability boundary for small derivative bounds on the parameters is developed in Theorem E by a Liapunov method.

Sufficient stability boundaries for small integral bounds on the parameters are developed using four independent methods. Theorem F first transforms the equation to introduce the integral of the original parameters into the parameters of the transformed equation and then utilizes Gronwall's Lemma (Kozin's (11) approach). Theorem G introduces the same transformation as above but utilizes the Liapunov approach (Caughey's (6) approach). Theorem H attacks the integral representation of the original differential equation directly and makes use of Gronwall's Lemma. Theorem I demonstrates the stability

by a Liapunov method applied directly to the differential equation.

Sufficient stability boundaries for small bounds on the parameters are demonstrated in Theorem J, developed by Kozin (11) utilizing Gronwall's Lemma, and Theorem K, developed by Caughey and Grey (6) utilizing a Liapunov approach. Proof of these theorems will not be included since they are available in the literature.

Theorem E "Small Derivative Bounds"

This theorem considers the stability of the parametric equation wherein the parameters are "slowly varying." The basic idea is that if the system is asymptotically stable for any fixed value of the parameters then one might expect the actual system to be stable if the time derivatives of the parameters are sufficiently small. Theorem E, under suitable restrictions demonstrates this concept.

Given the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= A(t)\bar{x} \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \qquad (2.1)$$

Let $\alpha_{ij} \le A_{ij}(t) \le \beta_{ij}$ where α_{ij} and β_{ij} are bounded and independent of time. Let S_{Ao} be the compact n^2 dimensional space with elements denoted as A_o where $\alpha_{ij} \le A_{oij} \le \beta_{ij}$. Clearly $A(t) \epsilon S_{Ao}$ $\forall t$.

Then if i) $A_{ij}(t)$ possesses a bounded derivative for all t

ii) A_o possesses eigenvalues with negative real parts everywhere in the parametric space S_{Ao}

iii) $\|\frac{dA}{dt}\|$ is sufficiently small (bounds developed in the proof of the theorem)

Then the system 2.1 is asymptotically stable in the sense of Liapunov and further there exists an A and $\alpha > 0$ such that $\|\bar{x}\| \leq Ae^{-\alpha t}$.

Proof:

Liapunov's Theorem shows that ii implies there exists a unique matrix P_o symmetric and positive definite such that $A_o^T P_o + P_o A_o = -I$. Clearly then there exists a $P(t)$ such that $A^T(t)\ P(t) + P(t)\ A(t) = -I$ for all t and further $P(t) \epsilon S_{Po} \equiv \{P_o | A_o^T P_o + P_o A_o = -I\ ,\ A_o \epsilon S_{Ao}\}$. The uniqueness of P_o and the relation $A_o^T dP_o + dP_o A_o = -dA_o^T P_o - P_o dA_o$ implies that P_o is continuous on A_o . Since the eigenvalues of a matrix are always continuously related to the parameters of the matrix it is seen that the eigenvalues of P_o are continuous on the space S_{Ao} . This, along with the fact that the eigenvalues of P_o are positive everywhere on S_{Ao} , which is a closed and bounded set, implies, since a continuous function on a compact set assumes its minimum, that there exists an $\epsilon > 0$ such that $\lambda_{Po} \geq \epsilon$ where λ_{Po} is any eigenvalue of $P_o \epsilon S_{Po}$.

Let the Liapunov function $V(\bar{x},t)$ be:

$$V(\bar{x},t) = \bar{x}^T P(t) \bar{x} \geq c\bar{x}^T \bar{x} \qquad \forall t \ . \tag{2.2}$$

Taking the time derivative of $V(\bar{x},t)$ along the trajectory of the system:

$$\frac{dV}{dt} = \dot{\bar{x}}^T P x + \bar{x}^T \dot{P} \bar{x} + \bar{x}^T P \dot{\bar{x}} = -\bar{x}^T \bar{x} + \bar{x}^T \dot{P} \bar{x} \leq -\bar{x}^T \bar{x} + \bar{x}^T \bar{x} \, \|\dot{P}\| = -\bar{x}^T \bar{x}(1-\|\dot{P}\|) \tag{2.3}$$

The n-dimensional matrix equation $A_o^T P_o + P_o A_o = -I$ can be formulated in n^2 space as $[\mathcal{A}_o]\{\mathcal{P}_o\} = -\{\mathcal{Q}\}$ and the uniqueness of P_o implies $\{\mathcal{P}_o\} = -[\mathcal{A}_o]^{-1}\{\mathcal{Q}\}$. Due to the continuity between a matrix, its inverse and its modulus, one can conclude, as above, that $\|[\mathcal{A}_o]^{-1}\| \leq M < \infty$ everywhere in S_{Ao} .

Therefore if $\|\dot{P}\| < 1$

and since $V(\bar{x},t) \leq \bar{x}^T \bar{x} \, \|P\| \leq \bar{x}^T \bar{x} \, M$ (2.4)

implies $\frac{dV}{dt} \leq -\frac{V}{M}(1 - \|\dot{P}\|)$

or $V \leq V_o \exp -[\frac{(1-\|\dot{P}\|)}{M}]$ (2.5)

which implies from 2.2 that $\|\bar{x}(t)\|$ is asymptotically stable, and using Cauchy's inequality

$$\|\bar{x}\| \leq \sqrt{\frac{V_o \cdot n}{\epsilon}} \exp[-\frac{1}{2}(\frac{1-\|\dot{P}\|}{M})] .$$

To demonstrate that iii) implies $\|\dot{P}\| < 1$, the following argument suffices.

$$A^T \dot{P} + \dot{P} A = -\dot{A}^T P - P\dot{A} \tag{2.6}$$

or $$\{\dot{\mathcal{P}}\} = -[\mathcal{A}]^{-1} [\dot{\mathcal{A}}] \{\mathcal{P}\} \tag{2.7}$$

and $$\|\{\dot{\mathcal{P}}\}\| < \|[\mathcal{A}]^{-1}\| \, \|[\dot{\mathcal{A}}] \{\mathcal{P}\}\| \leq M \, \|\dot{A}^T P + P\dot{A}\| \tag{2.8}$$

equivalently, since $\|P\| < M$, $\|\dot{P}\| \leq 2M^2 \|\dot{A}\|$ and for $\|\dot{A}\| < 1/2M^2$

implies $\|\dot{P}\| < 1$.

This completes the proof of the theorem. However it is felt, but cannot be proved, that the condition ii should more reasonably be replaced by the less restrictive condition of only the eigenvalues of $A(t)$ possessing negative real parts bounded away from zero for all times.

Theorem F "Small Integral Bounds No. 1"

This theorem as well as the next three theorems G, H, I considersthe case where the parametric terms exhibit rapid variation about a zero mean and further the system, neglecting the parametric terms, is stable. Physically this stability is realizable when one considers a finite elastic system subjected to parametric excitation of a frequency much greater than its highest eigenfrequency. This parametric excitation has no effect on the stability if damping is present. Theorem F makes use of a matrix exponential transformation to cast the equation into a form that is applicable to the theorem demonstrated in Kozin's paper (11). Lemmas B and C, which are invoked in the proof of this theorem, are matrix theorems that are obvious for the scalar cases.

Given the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= A\bar{x} + B(t)\bar{x} \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \quad (2.9)$$

Then if: i) A is a stability matrix, that is $\exists\ a\ \&\ b > 0$

$$\|Y\| \le b\ \exp(-at) \quad \text{where} \quad \frac{dY}{dt} = AY,\ Y(0) = I\ .$$

ii) $$\|S\|\ \exp\|S\|\ \{2\|A\| + \|A\|\ \|S\|\ \exp\|S\| + 2\|B\|\ \exp\|S\|\} < \frac{a}{b}$$

where $S = \int_{t_o}^{t} B(\tau)d\tau$

Then the system 2.9 is asymptotically stable in the sense of Liapunov.

<u>Proof</u>:

$$\frac{d\bar{x}}{dt} = A\bar{x} + B(t)\bar{x}$$

let $$\bar{x} = \exp[\int_{t_o}^{t} B(\tau)d\tau]\bar{w} \tag{2.10}$$

taking the derivative of $\bar{x}$:

$$\frac{d\bar{x}}{dt} = \frac{d}{dt}\ \{\exp[\int_{t_o}^{t} B(\tau)d\tau]\}\ \bar{w} + \exp[\int_{t_o}^{t} B(\tau)d\tau]\ \frac{d\bar{w}}{dt} \tag{2.11}$$

replacing this into the equation of motion 2.9 and pre-multiplying by

$\exp[-\int_{t_o}^{t} B(\tau)d\tau]$ the equation of motion in terms of $\bar{w}(t)$ is:

$$\frac{d\bar{w}}{dt} = A\bar{w} + \{\exp(-\int_{t_o}^{t} B(\tau)d\tau)\ A\ \exp(\int_{t_o}^{t} B(\tau)d\tau) - A\}\bar{w}$$

$$\left.\begin{aligned} &+ \{\exp(-\int_{t_o}^{t} B(\tau)d\tau)\ B\ \exp(\int_{t_o}^{t} B(\tau)d\tau) \\ &- \exp(-\int_{t_o}^{t} B(\tau)d\tau)\ \frac{d}{dt}\ [\exp(\int_{t_o}^{t} B(\tau)d\tau)]\}\ \bar{w} \end{aligned}\right\} \qquad (2.12)$$

By writing $\frac{d\bar{w}}{dt} = A\bar{w} + Q(t)\bar{w}$ and from Lemmas B and C (Appendix F), the modulus of $Q(t)$ can be bounded above as follows:

$$\left.\begin{aligned} \|Q\| \le \{&\|A\|\ \|\int_{t_o}^{t} B(\tau)d\tau\|\ \exp\|\int_{t_o}^{t} B(\tau)d\tau\| \\ &\cdot[2 + \|\int_{t_o}^{t} B(\tau)d\tau\|\ \exp\|\int_{t_o}^{t} B(\tau)d\tau\|] \\ &+ 2\|B\|\ \|\int_{t_o}^{t} B(\tau)d\tau\|\ \exp\ 2\|\int_{t_o}^{t} B(\tau)d\tau\|\} \end{aligned}\right\} \qquad (2.13)$$

From this it can be seen that for bounded $\|A\|$ and $\|B\|$ the modulus of $Q(t)$ can be made small with the $\|\int_{t_o}^{t} B(\tau)d\tau\|$.

The proof of the theorem is then just a direct application of Kozin's Theorem denoted as Theorem J, as the stability of $\bar{w}(t)$ implies the stability of $\bar{x}(t)$ since the transformation $\exp[\int_{t_o}^{t} B(\tau)d\tau]$ is

bounded under condition ii .

It should be noted that the stability boundary is dependent on the ratio a/b, which as pointed out in Caughey's paper (6), tends to zero for the second order linear equation as the damping ratio tends to one. The following theorem overcomes this weakness.

Theorem G "Small Integral Bounds No. 2"

This theorem utilized the same matrix exponential transformation as Theorem F but then applies Caughey's Theorem on the modulus of $Q(t)$.

Given the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= A\bar{x} + B(t)\bar{x} \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \quad (2.14)$$

Then if: i) A is a stability matrix, that is, $\exists$ a matrix P symmetric and positive definite $\ni$

$A^T P + PA = -I$; λ^p_{max} = maximum eigenvalue of P

ii) $\|S\| \exp\|S\| \{2\|A\| + \|A\| \|S\| \exp\|S\| + 2\|B\| \exp\|S\|\}$

$$< \frac{1}{2\lambda^p_{max} \|P^{1/2}\| \|P^{-1/2}\|} ;$$

where $S = \int_{t_o}^{t} B(\tau)d\tau$

Then the system 2.14 is asymptotically stable in the sense of Liapunov.

Proof:

Utilizing the same transformation as in Theorem F $\bar{x} = [\exp S(t)]\bar{w}$ the equation 2.12 in $\bar{w}$ is $\frac{d\bar{w}}{dt} = A\bar{w} + Q(t)\bar{w}$ where $\|Q\| \leq \|S\| \exp\|S\| \{2\|A\| + \|A\| \|S\| \exp\|S\| + 2\|B\| \exp\|S\|\}$ and the modulus of $\|P^{-1/2}Q^T P^{1/2} + P^{1/2}QP^{-1/2}\|$, that appears in Caughey's Theorem, can be bounded above by $2\|P^{-1/2}\| \|P^{1/2}\| \|Q\|$. By Theorem K the criteria ii) will imply asymptotic stability of $\bar{w}$ and thereby the stability of $\bar{x}$. Appendix E shows that the $\|P^{1/2}\|$ and $\|P^{-1/2}\|$ can be bounded above by $n^{7/2}\sqrt{\text{Trace }(P)}$ and respectively $n^{7/2}\sqrt{\text{Trace }(P^{-1})}$, where n is the dimension of the system of equations, i.e., the size of P.

Theorem H "Small Integral Bounds No. 3"

This theorem utilized Gronwall's Lemma by first putting the equation into the integral formulation and then integration by parts introduces the integral of the parametric terms into the integral equation. Gronwall's Lemma is then used to establish bounds on the integral of the parameters for asymptotic stability of the equation of motion.

Consider the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= A\bar{x} + B(t)\bar{x} \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \qquad (2.15)$$

Then if: i) A is a stability matrix, that is, $\exists$ a & b > 0 $\ni$

$\|Y\| \leq b \exp(-at)$ where $\frac{dY}{dt} = AY$; $Y(0) = I$.

ii) S(t) is sufficiently small such that $[I-S(t)]^{-1}$ exists $\forall t$ and is bounded by M, i.e., $\|[I-S(t)]^{-1}\| < M \ \forall t$.

where $S(t) = \int_0^t B(\tau)d\tau$

iii) $M \|A\| \|S\| \{2 + \frac{\|B\|}{\|A\|}\} < a/b$

Then the system 2.15 is asymptotically stable in the sense of Liapunov.

Proof:

It can be shown that the equation of motion can be put into the integral form:

$$\bar{x}(t) = Y(t)\bar{x}_o + \int_0^t Y(t-\tau)\ B(\tau)\ \bar{x}(\tau)d\tau \tag{2.16}$$

by integration by parts $\bar{x}(t)$ can be put in the form:

$$\left.\begin{aligned} \bar{x}(t) &= Y(t)\bar{x}_o + Y(t-\tau) \int_0^\tau B(\eta)d\eta\ \bar{x}(\tau) \Big|_0^t \\ &- \int_0^t \frac{dY(t-\tau)}{d\tau} \int_0^\tau B(\eta)d\eta\ \bar{x}(\tau)d\tau - \int_0^t Y(t-\tau) \int_0^\tau B(\eta)d\eta \frac{d\bar{x}(\tau)}{d\tau}\ d\tau \end{aligned}\right\} \tag{2.17}$$

by defining the integral of B(t) as:

$$S(\tau) \equiv \int_0^\tau B(\eta)d\eta \tag{2.18}$$

$$\left.\begin{aligned} \text{then:} \quad \bar{x}(t) &= Y(t)\bar{x}_o + S(t)\bar{x}(t) + \int_o^t AY(t-\tau)\, S(\tau)\bar{x}(\tau)d\tau \\ &- \int_o^t Y(t-\tau)\, S(\tau)\, [A + B(\tau)]\bar{x}(\tau)d\tau \end{aligned}\right\} \quad (2.19)$$

or:

$$[I-S(t)]\bar{x}(t) = Y(t)\bar{x}_o + \int_o^t \{AY(t-\tau)\, S(\tau) - Y(t-\tau)\, S(\tau)\, [A+B(\tau)]\}\bar{x}(\tau)d\tau \quad (2.20)$$

For $\|S(t)\|$ sufficiently < 1 implies $[I-S(t)]^{-1} = U$ exists and there exists an M such that $\|[I-S(t)]^{-1}\| < M$, $\forall t$; note $\{\|[I-S(t)]^{-1}\| < \frac{1}{1-\|S(t)\|}\}$.

Replacing $[I-S(t)]^{-1}$ by $U(t)$ yields:

$$\bar{x}(t) = UY\bar{x}_o + U \int_o^t \{AY(t-\tau)\, S(\tau) - Y(t-\tau)\, S(\tau)\, [A+B(\tau)]\}\, \bar{x}(\tau)d\tau \quad (2.21)$$

To establish bounds:

$$\|\bar{x}(t)\| < M\, [\|Y\|\, \|\bar{x}_o\| + \|A\| \int_o^t \|Y(t-\tau)\|\, \|S(\tau)\|\, \{2 + \frac{\|B(\tau)\|}{\|A\|}\}\, \|\bar{x}(\tau)\|d\tau] \quad (2.22)$$

and since $\|Y\| \le be^{-at}$ one gets:

$$\|\bar{x}(t)\|e^{at} \le M\, [b\|\bar{x}_o\| + \|A\|b \int_o^t \|S(\tau)\|\, \{2 + \frac{\|B(\tau)\|}{\|A\|}\}\, \|\bar{x}(\tau)\|\, e^{a\tau}d\tau] \quad (2.23$$

and by Gronwall's Lemma:

$$\|\bar{x}(t)\|e^{at} \leq M\ b\ \|\bar{x}_o\| \exp[M\ b\ \|A\| \int_o^t \|S(\tau)\| \{2 + \frac{\|B(\tau)\|}{\|A\|}\}\ d\tau] \tag{2.24}$$

or

$$\|\bar{x}(t)\| \leq M\ b\ \|\bar{x}_o\| \exp[-a + M\ b\ \|A\|\ \|S\|\ \{2 + \frac{\|B\|}{\|A\|}\}]t \tag{2.25}$$

hence if $M\ b\ \|A\|\ \|S\|\ \{2 + \frac{\|B\|}{\|A\|}\} < a$ (2.26)

Then equation 2.15 is asymptotically stable.

Here again it is seen that for bounded $\|A\|$ and $\|B\|$ the equation can always be made stable by taking $\|S\|$ sufficiently small. As in the comments following Theorem F the ratio a/b may tend to zero undesirably.

<u>Theorem I</u> <u>"Small Integral Bounds No. 4"</u>

This theorem uses a Liapunov approach to show stability but does not develop the stability of the equation on the basis of the time derivative of the Liapunov function being negative. The idea is to bound $V(\bar{x})$ above by an exponentially decreasing function in time. This then would imply asymptotic stability.

Consider the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= A\bar{x} + B(t)\bar{x} \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \tag{2.27}$$

Then if: i) A is a stability matrix, that is, $\exists$ a matrix P symmetric and positive definite $\ni$ $A^TP + PA = -I$

ii) $$\|S'\| \left[2\|A\| + 2\|B\| + \frac{1}{\lambda^P_{min}} + \frac{2\|B\|\ \|P\|}{\lambda^P_{min}}\right] < \frac{\lambda^P_{min}}{\lambda^P_{max}}$$

where λ^P are the eigenvalues of P, and

where $S'(t) = \int_0^t (B^TP + PB)d\tau$

Then the system 2.27 is asymptotically stable.

<u>Proof</u>:

Let: $$V = \bar{x}^TP\bar{x} \tag{2.28}$$

$$\frac{dV}{dt} = -\bar{x}^T\bar{x} + \bar{x}^T(B^TP + PB)\bar{x} \tag{2.29}$$

and since $\bar{x}^TP\bar{x} < \lambda^P_{max}\ \bar{x}^T\bar{x}$ it follows that

$$\frac{dV}{dt} \le -\frac{V}{\lambda^P_{max}} + \frac{\bar{x}^T(B^TP + PB)\bar{x}}{\bar{x}^TP\bar{x}}\cdot V \tag{2.30}$$

or: $$\int_{V_o}^{V} \frac{dV}{V} \le \int_0^t \frac{-1}{\lambda^P_{max}} + \frac{\bar{x}^T(B^TP + PB)\bar{x}}{\bar{x}^TP\bar{x}}\,d\tau \tag{2.31}$$

let $$S'(t) = \int_0^t (B^TP + PB)d\tau \tag{2.32}$$

and integrating 2.31 by parts yields:

$$V \le V_o \exp\{- \frac{t}{\lambda^P_{max}} + \frac{x^T S' x}{x^T P x} - \int_o^t \frac{2x^T S' \dot{x}}{x^T P x} - \frac{x^T S' x \dot{V}}{(x^T P x)^2} d\tau\} \tag{2.33}$$

and taking the modulus of the matrices and utilizing the equation of motion 2.27:

$$\left.\begin{aligned} V(\bar{x}) \le V_o \exp\{- \frac{t}{\lambda^P_{max}} + \frac{\|S'\|}{\lambda^P_{min}} \\ + [\frac{2\|S'\|}{\lambda^P_{min}} (\|A\| + \|B\|) + \frac{\|S'\|}{(\lambda^P_{min})^2} (1 + 2\|B\| \, \|P\|)]t\} \end{aligned}\right\} \tag{2.34}$$

hence if:

$$\|S'\| \, [2\|A\| + 2\|B\| + \frac{1}{\lambda^P_{min}} + \frac{2\|B\| \, \|P\|}{\lambda^P_{min}}] < \frac{\lambda^P_{min}}{\lambda^P_{max}} \tag{2.35}$$

Then the system 2.27 is asymptotically stable.

It should be noted that the modulus of S' can be bounded above by a constant times the S matrix of the previous three theorems. That is:

$$\|S'\| \le 2\|P\| \, \|S\| \tag{2.36}$$

hence the results 2.35 can be put in terms of $\|S\|$ as follows:

$$\text{ii)} \quad 2\|S\| \, \|P\| \, [2\|A\| + 2\|B\| + \frac{1}{\lambda^P_{min}} + \frac{2\|B\| \, \|P\|}{\lambda^P_{min}}] < \frac{\lambda^P_{min}}{\lambda^P_{max}} \tag{2.37}$$

Also this theorem illustrates the inherent difficulty that will be encountered in Chapter IV concerning continuous systems.

Clearly for $\lambda^P_{min}/\lambda^P_{max}$ to be bounded away from zero requires the A matrix to have bounded eigenvalues. For most continuous operators, however, this is not the case and therefore the methods for demonstrating stability under small integral bounds do not seem applicable.

Theorem J "Universal Bounds No. 1"

This theorem is taken from an article by Kozin (11) entitled "On the Almost Sure Stability of Linear Systems with Random Coefficients." It utilizes a direct Gronwall Lemma approach. It is included since Theorem F utilizes its results as well as the fact that under certain conditions the universal bounds will be better than those bounds developed in Theorems F, G, H and K.

Given the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= A\bar{x} + B(t)\bar{x} \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \quad (2.38)$$

Then if: i) A is a stability matrix, that is $\exists$ a & b > 0 such that $\|Y\| \leq b \exp(-at)$ where

$$\frac{dY}{dt} = AY \quad \text{with} \quad Y(0) = I$$

ii) $\|B\| < a/b$

Then the system 2.38 is asymptotically stable.

Theorem K "Universal Bounds No. 2"

This theorem is taken from an article by Caughey and Grey (6) entitled "On the Almost Sure Stability of Linear Dynamic Systems with Stochastic Coefficients." A direct Liapunov approach is used. The results, at least in the case of a second order dynamic system, are stronger than those in Theorem J. (See reference (6)).

Given the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= A\bar{x} + B(t)\bar{x} \\ \bar{x}(0) &= \bar{x}_0 \end{aligned}\right\} \quad (2.39)$$

Then if: i) A is a stability matrix, that is $\exists$ a matrix P symmetric and positive definite $\ni$ $A^T P + PA = -I$

ii) $\|P^{-1/2} B^T P^{1/2} + P^{1/2} B P^{-1/2}\| < \frac{1}{\lambda^P_{max}}$

where λ^P_{max} is the maximum eigenvalue of P .

Then the system 2.39 is asymptotically stable.

Application of the Theorems of Chapter II

The following example, the Mathieu equation, is used to demonstrate Theorems E, F, G, H, I, J, K. Figure 5 shows these boundaries along with the previously derived results (Figure 2, Chapter I). It should be noted that the only region where the bounds are improved is in the vicinity of $\omega/\eta = 0$. One expects, however,

that as the generality of a theorem increases the quality of the bounds would decrease.

Consider the equation:

$$\ddot{x} + 2\xi\omega\dot{x} + (\omega^2 + M \sin \eta t)x = 0$$

$$x(0) = x_o \; ; \; \dot{x}(0) = \dot{x}_o$$

with $\quad 0 < \xi < 1 \; , \; 0 < \omega^2$

This can be written in the following form:

$$\frac{d}{dt}\begin{pmatrix} x \\ \frac{dx}{dt} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 - 2\xi\omega & \end{bmatrix}\begin{pmatrix} x \\ \frac{dx}{dt} \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ -M \sin \eta t & 0 \end{bmatrix}\begin{pmatrix} x \\ \frac{dx}{dt} \end{pmatrix}$$

For stability of the above equation:

Theorem E requires:

i) $M < \omega^2$

ii) For this example the $P(t)$ matrix can be easily developed and is (See Appendix D):

$$P(t) = \begin{bmatrix} \frac{\omega^2 + M \sin \eta t}{4\xi\omega} + \frac{1}{4\xi\omega} + \frac{\xi\omega}{(\omega^2 + M \sin \eta t)} & \frac{1}{2(\omega^2 + M \sin \eta t)} \\ \frac{1}{2(\omega^2 + M \sin \eta t)} & \frac{1}{4\xi\omega} + \frac{1}{4\xi\omega(\omega^2 + M \sin \eta t)} \end{bmatrix}$$

and the requirement that $\|\frac{dA}{dt}\|$ be sufficiently small is satisfied if $\|\frac{dP}{dt}\| < 1$ (See Proof of Theorem E). This condition is met if:

$$\eta M \left[\frac{1}{4\xi\omega} + \frac{\xi\omega}{(\omega^2-M)^2} + \frac{1}{(\omega^2-M)^2} + \frac{1}{4\xi\omega(\omega^2-M)^2} \right] < 1$$

It is easily seen that for $M < \omega^2$ and η sufficiently small the system is stable.

Theorem F requires:

i) The A matrix is always stable under the requirement of ω^2 and ξ being greater than zero. The $Y(t)$ matrix for the problem, as well as values for a and b, are given in Appendix C.

$$b \leq \frac{[2|\gamma| \cos \gamma\theta + (2|\alpha| + 1 + \omega^2) \sin \gamma\theta]}{\gamma}$$

where: $\gamma = \omega\sqrt{1-\xi^2}$

$\alpha = \omega\xi$

$\tan \theta = \frac{2|\alpha| + 1 + \omega^2}{2|\alpha|} \qquad 0 \leq \theta \leq \pi/2$

$a = \omega\xi$

ii) The modulus of $S(t) = \frac{M}{\eta}$ and therefore:

$$\frac{M}{\eta} \exp(\frac{M}{\eta}) [2(1+\omega^2+2\xi\omega)+(1+\omega^2+2\xi\omega) \frac{M}{\eta} \exp(\frac{M}{\eta})+2M \exp(\frac{M}{\eta})] < \frac{a}{b}$$

implies stability. Note that for fixed M, η can always be chosen sufficiently large to insure stability.

Theorem G requires:

i) A is always a stability matrix under the conditions on ω^2 and ξ and the P matrix and its eigenvalues take the form given in Appendix D. Values for the modulus of $\|P^{1/2}\|$ and $\|P^{-1/2}\|$ are developed in Appendix E.

ii) $\|S\|$ is less than $\frac{M}{\eta}$ and therefore the criteria is of the form:

$$\frac{M}{\eta}\exp(\frac{M}{\eta})[2(1+\omega^2+2\xi\omega)+(1+\omega^2+2\xi\omega)\frac{M}{\eta}\exp(\frac{M}{\eta})+2M\exp(\frac{M}{\eta})]$$

$$< \frac{1}{2\lambda^P_{max}\|P^{1/2}\|\,\|P^{-1/2}\|}$$ which implies stability.

Theorem H requires:

i) A is always stable and the values for a and b are given in Appendix C.

ii) The bound on $\|[I-S(t)]^{-1}\|$ is $\leq \frac{1}{1-M/\eta}$

iii) $$(\frac{1}{1-\frac{M}{\eta}})\ (1+\omega^2+2\xi\omega)\ \frac{M}{\eta}\ [2+\frac{M}{(1+\omega^2+2\xi\omega)}] < a/b$$

and therefore for any M there is an η sufficiently large so that ii) and iii) are satisfied and the system is stable.

Theorem I requires:

i) A is always stable and the P matrix along with its eigenvalues is given in Appendix D.

ii) $\|s\| < M/\eta$ and therefore

$$2\,\frac{M}{\eta}\,\|P\|\,\left[2(1+\omega^2+2\xi\omega) + 2M + \frac{1}{\lambda^P_{min}} + \frac{2M\|P\|}{\lambda^P_{min}}\right] \le \frac{\lambda^P_{min}}{\lambda^P_{max}}$$

will imply stability.

Theorem J requires:

i) The values for a and b are developed in Appendix C and therefore: $M < a/b$ will imply stability.

Theorem K requires:

i) The P matrix along with its eigenvalues are developed in Appendix D and bounds on the modulus of $P^{1/2}$ and $P^{-1/2}$ are given in Appendix E. Therefore $M < \dfrac{1}{2\lambda^P_{max}\,\|P^{1/2}\|\,\|P^{-1/2}\|}$ implies stability.

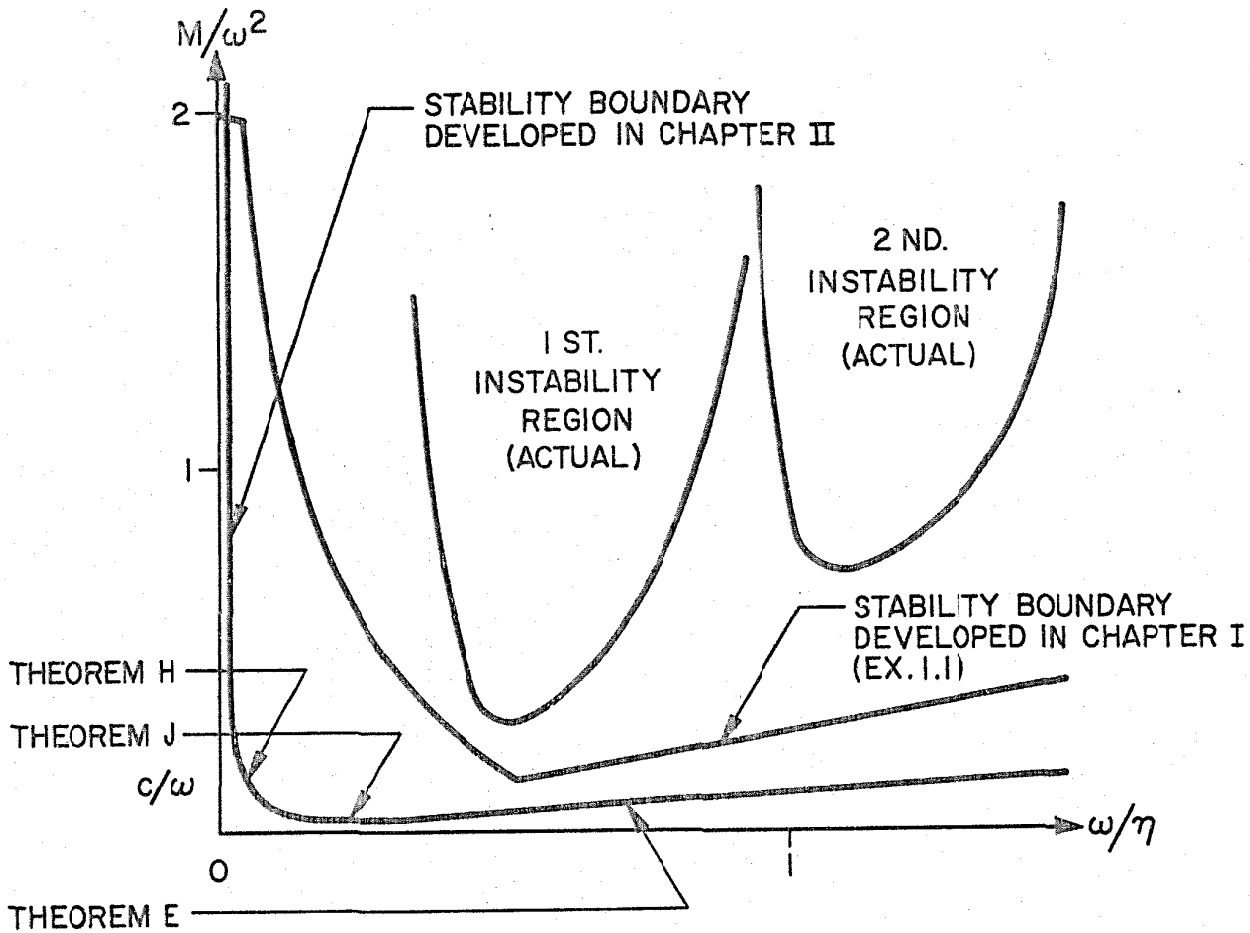
M/ω²
2
1
STABILITY BOUNDARY
DEVELOPED IN CHAPTER II
1 ST.
INSTABILITY
REGION
(ACTUAL)
2 ND.
INSTABILITY
REGION
(ACTUAL)
STABILITY BOUNDARY
DEVELOPED IN CHAPTER I
(EX. 1.1)
THEOREM H
THEOREM J
c/ω
0
1
ω/η
THEOREM E

Fig. 5

CHAPTER III

N'TH ORDER NONLINEAR SYSTEMS

INTRODUCTION

The following three theorems develop a method of attack for the parametric stability analysis of nonlinear equations. They are generalizations of some of the previous theorems for linear systems. One important weakness of the theorems is that for application one must find an "exponential Liapunov function" relating to the equation. In the linear case this can be done as illustrated in Chapter II and therefore certain equations of the type $\frac{d\bar{x}}{dt} = A(t)\bar{x} + \bar{g}(\bar{x})$ can be analyzed, for example, using the following theorems. Theorem L concerns small derivative bounds on the parameters, Theorem M concerns small integral bounds on the parameters, and Theorem N concerns small bounds of the amplitude of the parameters. This chapter vividly illustrates the basic ideas of all the theorems of this thesis. Although the other chapters concern only linear systems, conceptually the methods are in no way restricted to linear systems. However, for application, the existence of the "exponential Liapunov function" is assured only for the discrete linear systems by the Liapunov Theorem.

Theorem L "Small Derivative Bounds"

The idea, again, is that if the system is exponentially stable for each value of the parametric terms then the parametric system is asymptotically stable for sufficiently slowly varying parameters.

Consider the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{f}(\bar{x},t) \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \qquad (3.1)$$

Then if: i) $\exists$ a Liapunov function $V(\bar{x},t) > 0 \quad \bar{x} \neq 0$ and $V(\bar{0},t) = 0$ such that $\nabla V \cdot \bar{f}(\bar{x},t) \leq -\alpha V(\bar{x},t)$ $\forall t,\ \alpha > 0$

ii) $\left|\frac{\partial \log V}{\partial t}\right| < \alpha$

$\Big\}\ \|\bar{x}\| < R$

Then the system 3.1 will be asymptotically stable at the origin.

<u>Proof</u>:

Let the Liapunov function be $V(\bar{x},t)$.

$V(\bar{x},t) > 0$

The time derivative along the trajectory of the system is:

$$\left.\begin{aligned} \frac{dV}{dt} &= \nabla V \cdot \bar{f}(\bar{x},t) + \frac{\partial V}{\partial t}(\bar{x},t) \\ &\leq -\alpha V(\bar{x},t) + \frac{\partial V}{\partial t}(\bar{x},t) \end{aligned}\right\} \qquad (3.2)$$

Therefore:

$$V \leq V_o \exp \int_o^t \left\{-\alpha + \frac{\partial \log V}{\partial t}\right\} d\tau \qquad (3.3)$$

Since $V(\bar{x},t)$ is continuous at the origin one can choose for any $R' < R$ an $\bar{x}_o \ni V(\bar{x}) > V_o$ for $\|\bar{x}\| = R'$.

Then if $\sup_t \left|\frac{\partial \log V}{\partial t}\right| < \alpha$, $\forall \|\bar{x}\| < R'$, then V is decreasing and therefore $\|\bar{x}\| < R'$ and further $V(\bar{x})$ going to zero implies $\|\bar{x}\|$ goes to zero, hence the system 3.1 is asymptotically stable at the origin.

<u>Theorem M</u> <u>"Small Integral Bounds"</u>

This theorem concerns the stability of the parametric system under the condition of sufficiently small integral bounds.

Consider the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{f}(\bar{x}) + B(t)\bar{g}(\bar{x}) \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \qquad (3.4)$$

Then if: i) There exists an exponential Liapunov function $V(\bar{x})$, $\|\bar{x}\| < R$, for the system with $B(t) \equiv 0$ of order $-\alpha$, that is $\nabla V(\bar{x})\cdot\bar{f}(\bar{x}) \le -\alpha V(\bar{x})$.

ii) $\left|\frac{\nabla V\cdot S(t)g(\bar{x})}{V}\right|$

iii) $\left\|\nabla\left(\frac{\nabla V\cdot\bar{f}(\bar{x})+\nabla V\cdot B\bar{g}(\bar{x})}{V}\right)\right\|\cdot\|\bar{g}(\bar{x})\|$

iv) $\left\|\frac{\nabla V}{V}\right\|\cdot\left\|\frac{d\bar{g}}{d\bar{x}}(\bar{f} + B\bar{g})\right\|$

are continuous bounded functions at the origin

v) $\|S\| \equiv \left\|\int_o^t B(\tau)d\tau\right\|$ is sufficiently small (bounds given in proof)

Then the system 3.4 is asymptotically stable at $\bar{x} = 0$.

Proof:

Let the Liapunov function be $V(\bar{x})$

$$V(\bar{x}) > 0 \ , \quad \bar{x} \neq 0 \qquad \|\bar{x}\| < R \tag{3.5}$$

then
$$\frac{dV}{dt}(\bar{x}) \leq -\alpha V(\bar{x}) + \nabla V \cdot B(t)\bar{g}(\bar{x}) \tag{3.6}$$

dividing by $V(\bar{x})$ and integrating 3.6 yields:

$$V \leq V_o \exp\{-\alpha t + \int_o^t \frac{\nabla V}{V} \cdot B(\tau)\bar{g}(\bar{x})d\tau\} \tag{3.7}$$

and integration 3.7 by parts:

$$\left.\begin{aligned} V \leq V_o \exp\{-\alpha t + \frac{\nabla V}{V} \cdot \int_o^\tau B(\eta)d\eta \ \bar{g}(\bar{x}) \Big|_o^t \\ - \int_o^t [\frac{d}{dt}(\frac{\nabla V}{V}) \cdot \int_o^\tau B(\eta)d\eta \ \bar{g}(\bar{x}) + \frac{\nabla V}{V} \cdot \int_o^\tau B(\eta)d\eta \ \frac{d\bar{g}}{dt}]d\tau\} \end{aligned}\right\} \tag{3.8}$$

let
$$S(\tau) = \int_o^\tau B(\eta)d\eta \tag{3.9}$$

then:

$$\left.\begin{aligned} V \leq V_o \exp\{-\alpha t + \frac{\nabla V}{V} \cdot S(t)\bar{g}(\bar{x}) \\ - \int_o^t [\nabla(\frac{\nabla V \cdot \frac{d\bar{x}}{dt}}{V}) \cdot S\bar{g} + \frac{\nabla V}{V} \cdot S \frac{d\bar{g}}{d\bar{x}} \frac{d\bar{x}}{dt}]d\tau \} \end{aligned}\right\} \tag{3.10}$$

or

$$\left.\begin{aligned} V(\bar{x}) \le V_o \exp\{|\tfrac{\nabla V}{V}\cdot S(t)\bar{g}(\bar{x})|\} \exp\{\int_o^t |\nabla(\tfrac{\nabla V\cdot\bar{f}(\bar{x})+\nabla V\cdot B\bar{g}(\bar{x})}{V})S\bar{g}| \\ + |\tfrac{\nabla V}{V}\cdot S\,\tfrac{d\bar{g}}{d\bar{x}}\,(f(x) + Bg(x))|dt\} \exp\{-\alpha t\} \end{aligned}\right\} \quad (3.11)$$

From ii)-iv) there exists an $R' > 0$ and $M_1, M_2, M_3 < \infty$ such that

$$\left.\begin{aligned} &|\tfrac{\nabla V\cdot S(t)\bar{g}(\bar{x})}{V}| < M_1 \\ &\|\nabla(\tfrac{\nabla V\cdot\bar{f}(\bar{x})+\nabla V\cdot B\bar{g}(\bar{x})}{V})\|\cdot\|\bar{g}(\bar{x})\| < M_2 \\ &\|\tfrac{\nabla V}{V}\|\cdot\|\tfrac{d\bar{g}}{d\bar{x}}\,(\bar{f} + B\bar{g}(\bar{x}))\| < M_3 \end{aligned}\right\} \|\bar{x}\| < R' , \quad \forall t \qquad (3.12)$$

Then one can pick for any $R'' \ni 0 < R'' < \min(R,R')$ an $\bar{x}(0) \ni$

$V(\bar{x}) > V_o \exp M_1$ for $\|\bar{x}\| = R''$

hence if:

$$\|S\| < \frac{\alpha}{M_2 + M_3} \qquad \forall t \qquad (3.13)$$

then V is decreasing and therefore $\|\bar{x}\|$ remains less than R'' and further $V(\bar{x})$ going to zero implies $\|\bar{x}\|$ goes to zero, hence the system 3.4 is asymptotically stable at the origin.

Theorem N "Universal Bounds"

The concept of the bounds on the parametric terms being sufficiently small is developed in this theorem.

Consider the system of equations:

$$\left.\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{f}(\bar{x}) + B(t)\bar{g}(\bar{x}) \\ \bar{x}(0) &= \bar{x}_o \end{aligned}\right\} \quad (3.14)$$

Then if: i) $\exists$ an exponential Liapunov function $V(\bar{x})$ for the system with $B(t) \equiv 0$ of order $-\alpha$, that is

$\nabla V(\bar{x})\cdot\bar{f}(\bar{x}) \leq -\alpha V(\bar{x})$ with $\|\bar{x}\| < R$.

ii) $\frac{\|\nabla V\|}{V}\|\bar{g}(\bar{x})\|$ is continuous and bounded at $\bar{x} = 0$.

iii) $\|B\|$ is sufficiently small. (Bounds given in proof)

Then the system 3.14 is asymptotically stable at $\bar{x} = 0$.

Proof:

Let the Liapunov function be $V(\bar{x})$

$V(\bar{x}) < 0$

$$\left.\begin{aligned} \frac{dV(\bar{x})}{dt} &= \nabla V\cdot\bar{f}(\bar{x}) + \nabla V\cdot B(t)\bar{g}(\bar{x}) \quad \|\bar{x}\| < R \\ &\leq -\alpha V(\bar{x}) + \nabla V\, B(t)\bar{g}(\bar{x}) \end{aligned}\right\} \quad (3.1$$

$$V \le V_o \exp\{-\alpha t + \int_o^t \frac{\nabla V}{V} B(t)\bar{g}(\bar{x})dt\} \tag{3.16}$$

From ii) there exists an $R' < R$ such that

$$\frac{\|\nabla V\|}{V} \|\bar{g}(\bar{x})\| < M < \infty \quad \text{for} \quad \|\bar{x}\| < R' \tag{3.17}$$

then one can always choose for any $R'' < R'$ an $\bar{x}_o$ such that

$V(\bar{x}) > V_o$ for $\|\bar{x}\| = R''$.

Therefore if:

$$\|B\| < \frac{\alpha}{M} \qquad \forall t \tag{3.18}$$

then from 3.16 V is decreasing, hence $\|\bar{x}\|$ remains less than R'' and further $V(\bar{x})$ going to zero implies $\bar{x}$ goes to zero, hence the system 3.14 is asymptotically stable at the origin.

DISCUSSION

No examples are included for this chapter for two reasons. First, the purpose is to illustrate the generality of the Liapunov approach to demonstrate asymptotic stability of ordinary differential equations, linear or nonlinear, and second, the author was unable to find an exponential Liapunov function relating to any worthwhile example.

The criteria for stability developed within Theorems L, M and N basically require that the vector $\bar{f}(\bar{x})$ is not dominated by $\bar{g}(\bar{x})$ in the neighborhood of the origin. That is to say, that the type of stability is characterized by the nature of $\bar{f}(\bar{x})$.

CHAPTER IV

CONTINUOUS DYNAMIC SYSTEMS

INTRODUCTION

This chapter demonstrates sufficient asymptotic stability conditions for a class of continuous parametric systems by Liapunov's Direct Method. The particular Liapunov function used is a somewhat logical extension of that used in Theorem D, Chapter I. However, it is conjectured that perhaps there exists a better quadratic functional for the system in contrast to the discrete system where the quadratic form $\bar{x}^T P \bar{x}$ appears to be unique in regard to the quality of the results. The one used nevertheless proves quite useful as demonstrated in the example problems.

Theorem O demonstrates a universal stability bound on the parametric terms and Theorem P concerns bounds on the derivative of the parametric terms. No results could be obtained for the case of small bounds on the integral of the parametric term (see discussion after Theorem I). The examples include both classical and non-classical systems and demonstrate the generality of the functional approach in contrast to a modal approach.

Preliminaries

Let S be a bounded subset of an M-dimensional Euclidean space $\{\bar{x}\}$ with boundary denoted by Γ. The equation under consideration will be:

$$u_{tt} + L_1 u_t + L_2(t)u + g(u) = 0 \tag{4.1}$$

with initial conditions:

$$\left.\begin{aligned} u(0,\bar{x}) &= u_o(\bar{x}) \\ u_t(0,\bar{x}) &= u_{t_o}(\bar{x}) \end{aligned}\right\} \tag{4.2}$$

with boundary conditions:

$$Bu(t,\bar{x}) = 0 \ , \ \text{for} \ \ \bar{x}\epsilon\Gamma \tag{4.3}$$

The solution of 4.1, 4.2 and 4.3 will be assumed to exist and to possess sufficient smoothness so that $L_1 u$ and $L_2 u$ are continuous. An inner product is defined on this space as $\int_S u(\bar{x})v(\bar{x})d\bar{x} \equiv \langle u,v\rangle$. L_1 and $L_2(t)$ are linear spatial operators where $L_2(t)$ can be expressed in the following form:

$$L_2(t) = a_1(t)L_{21} + a_2(t)L_{22} + \dots a_m(t)L_{2m} \tag{4.4}$$

where L_{2j} are independent of time.

Main Problem

Given the continuous differential equation:

$$\left.\begin{aligned} &u_{tt} + L_1 u_t + L_2(t)u + g(u) = 0 \\ &u(0,\bar{x}) = u_o(\bar{x}) \qquad u_t(0,\bar{x}) = u_{t_o}(\bar{x}) \end{aligned}\right\} \tag{4.5}$$

where: i) $\langle u, L_1 u\rangle$, $\langle u, L_2 u\rangle$, $\langle L_1 u, L_2 U\rangle$ and

$\langle 1, \int_0^u g(u)du\rangle$ are positive functionals for all t

and all acceptable functions $u(t,\bar{x})$.

ii) $\frac{da_i}{dt}$ from equation 4.4 is defined and

$$\dot{L}_2 \equiv \sum_{i=1}^{m} \frac{da_i(t)}{dt} L_{2i} .$$

<u>Theorem 0</u> "<u>Universal Stability Bound</u>"

The particular case where 4.5 can be put in the form:

$$\left.\begin{aligned} & u_{tt} + L_1 u_t + L_2 u + L_3(t)u + g(u) = 0 \\ & u(0,\bar{x}) = u_o(x) \qquad u_t(0,\bar{x}) = u_{t_o}(\bar{x}) \end{aligned}\right\} \quad (4.6)$$

Then if: i) $\langle u, L_2 u\rangle > 0$

ii) $\langle u, L_1 u\rangle \geq \lambda \langle u,u\rangle \qquad \lambda > 0$

iii) $\dfrac{\frac{1}{\lambda}|\langle L_3 u, L_3 u\rangle| + |\langle L_3 u, L_1 u\rangle|}{|\langle L_1 u, L_2 u\rangle|} \leq M_1(t)$

iv) $\sup_t M_1(t) \leq \frac{1}{1+2\epsilon}$ where $0 \leq t$ and $0 < \epsilon < \frac{1}{2}$

and there exists an M_2

v) $M_2 \langle L_1 u, g(u)\rangle \geq 2\langle 1, \int_0^u g(u)du\rangle$ (4.7)

$$\left.\begin{array}{l} \text{vi)} \quad M_2 \langle L_1 u, L_2 u\rangle \geq \frac{3}{4} \langle L_1 u, L_1 u\rangle + \langle u, L_2 u\rangle \\ \text{vii)} \quad M_2 \lambda \geq 2 \ . \end{array}\right\}$$

Then the equation is asymptotically stable in the following sense:

$$\left.\begin{array}{l} \text{i)} \quad |\langle u, L_2 u\rangle| \leq A_1 \exp(-\alpha_1 t) \\ \text{ii)} \quad |\langle L_1 u, L_1 u\rangle| \leq A_1 \exp(-\alpha_1 t) \\ \text{iii)} \quad |\langle (\frac{L_1 u}{2} + u_t)^2\rangle| \leq A_1 \exp(-\alpha_1 t) \\ \text{where} \quad A_1 = [\frac{\langle L_1 u_o, L_1 u_o\rangle}{4} + \langle u_o, L_2 u_o\rangle + \langle (\frac{L_1 u_o}{2} + u_{o_t})^2\rangle \\ \qquad + \langle 1, 2 \int_o^{u_o} g(u) du\rangle] \\ \text{and} \quad \alpha_1 = \frac{\epsilon}{M_2} \{1-(1+2\epsilon) \sup_t M_1(t)\} \end{array}\right\} \quad (4.8)$$

<u>Proof</u>:

Let $$V(t) = \frac{\langle L_1 u, L_1 u\rangle}{4} + \langle u, L_2 u\rangle + \langle (\frac{L_1 u}{2} + u_t)^2\rangle + \langle 1, 2 \int_o^u g(u) du\rangle \quad (4.9)$$

and along the trajectory:

$$\frac{dV}{dt} = -[\langle u_t, L_1 u_t\rangle + \langle L_1 u, L_2 u\rangle + \langle L_1 u, L_3 u\rangle + \langle L_1 u, g(u)\rangle + \langle 2u_t, L_3 u\rangle] \ .$$

Using equation 4.7:

$$\frac{dV}{dt} \le -[\langle L_1u,L_2u\rangle + \langle L_1u,L_3u\rangle + \langle L_1u,g(u)\rangle + \langle \epsilon\lambda u_t,u_t\rangle$$

$$+ \lambda(1-\epsilon)\ \langle [u_t + \frac{L_3u}{\lambda(1-\epsilon)}]^2\rangle - \langle \frac{L_3u,L_3u}{\lambda(1-\epsilon)}\rangle]$$

and from 4.7

$$\frac{dV}{dt} \le -\frac{1}{M_2}\ [2\langle 1, \int_0^u g(u)du\rangle + 2\epsilon\langle u_t,u_t\rangle$$

$$+ (\tfrac{3}{4}\ \langle L_1u,L_1u\rangle + \langle u,L_2u\rangle)\ (1-M_1(t) - 2\epsilon M_1(t))]\ . \qquad (4.10)$$

However, from 4.9

$$V \le \langle 1,2\int_0^u g(u)du\rangle + \langle \frac{L_1u,L_1u}{4}\rangle + \langle u,L_2u\rangle + \langle 2(\frac{L_1u}{2})^2\rangle + \langle 2u_t^2\rangle$$

$$= \langle 1,2\int_0^u g(u)du\rangle + \langle \tfrac{3}{4}\ L_1u,L_1u\rangle + \langle u,L_2u\rangle + \langle 2u_t,u_t\rangle\ . \qquad (4.11)$$

Hence from 4.10 and 4.11 and 4.7:

$$\frac{dV}{dt} \le = \frac{\epsilon}{M_2}\ [1 - (1 + 2\epsilon)M_1(t)]V$$

or

$$V \le V_o(\bar{x}_o)\ \{\exp - \frac{\epsilon t}{M_2}\ [1 - (1 + 2\epsilon)\ \sup_t M_1(t)]\}\ . \qquad (4.12)$$

From equation 4.9 and 4.12 the results 4.8 follow directly.

Theorem P "Small Derivative Bounds"

The particular case where 4.5 can be put in the form:

$$\left.\begin{aligned} & u_{tt} + L_1 u_t + L_2(t)u + g(u) = 0 \\ & u(0,\bar{x}) = u_o(\bar{x}) \qquad u_t(0,\bar{x}) = u_{t_o}(\bar{x}) \end{aligned}\right\} \qquad (4.13)$$

$$\left.\begin{aligned} \text{Then if:}\quad & \text{i)}\ \ \langle u, L_2(t)u\rangle > 0 \qquad \forall t \\ & \text{ii)}\ \ \langle u, L_1 u\rangle \geq \lambda\langle u,u\rangle \qquad \lambda > 0 \\ & \text{iii)}\ \ \frac{|\langle u, \dot{L}_2 u\rangle|}{|\langle L_1 u, L_2 u\rangle|} \leq M_1(t) \\ & \text{iv)}\ \ \sup_t M_1(t) < 1 \quad \text{for} \quad 0 \leq t \\ & \text{and there exists an } M_2 \\ & \text{v)}\ \ M_2\ \langle L_2 u, g(u)\rangle \geq 2\langle 1, \int_o^u g(u)du\rangle \\ & \text{vi)}\ \ M_2\ \langle L_1 u, L_2 u\rangle \geq \frac{3}{4}\langle L_1 u, L_1 u\rangle + \langle u, L_2 u\rangle \\ & \text{vii)}\ \ M_2\lambda \geq 2 \end{aligned}\right\} \qquad (4.14)$$

Then the equation 4.13 is asymptotically stable in the following sense:

$$\left.\begin{aligned} & \text{i)}\ \ |\langle u, L_2 u\rangle| \leq A_2 \exp(-\alpha_2 t) \\ & \text{ii)}\ \ |\langle L_1 u, L_1 u\rangle| \leq A_2 \exp(-\alpha_2 t) \\ & \text{iii)}\ \ |\langle (\frac{L_1 u}{2} + v_t)^2\rangle| \leq A_2 \exp(-\alpha_2 t) \end{aligned}\right\} \qquad (4.15)$$

where:

$$A_2 = [\frac{\langle L_1 u_o, L_1 u_o\rangle}{4} + \langle u_o, L_2(0) u_o\rangle + \langle (\frac{L_1 u_o}{2} + u_o)^2\rangle$$

$$+ \langle 1, 2\int_o^{u_o} g(u)du\rangle]$$

and

$$\alpha_2 = \frac{1}{M_2}(1 - \sup_t M_1(t))$$

Proof:

Let $V(\bar{x},t) = \frac{\langle L_1 u, L_1 u\rangle}{4} + \langle u, L_2 u\rangle + \langle (\frac{L_1 u}{2} + u_t)^2\rangle + \langle 1, 2\int_o^u g(u)du\rangle$ (4.16)

and along the trajectory:

$$\frac{dV}{dt} = -[\langle u_t, L_1 u_t\rangle + \langle L_1 u, L_2 u\rangle + \langle L_1 u, g(u)\rangle - \langle u, \dot{L}_2 u\rangle]$$

$$\leq -[\langle \lambda u_t, u_t\rangle + \langle L_1 u, L_2 u\rangle (1-M_1(t)) + \langle L_1 u, g(u)\rangle]$$

Using equations 4.14

$$\frac{dV}{dt} \leq -\frac{1}{M_2}[2\langle 1, \int_o^u g(u)du\rangle + 2\langle u_t, u_t\rangle$$

$$+ (\frac{3}{4}\langle L_1 u, L_1 u\rangle + \langle u, L_2 u\rangle)(1-M_1(t))] \qquad (4.17)$$

However, from 4.16

$$V \le \langle 1, 2\int_0^u g(u)du\rangle + \langle \tfrac{3}{4} L_1 u, L_1 u\rangle + \langle u, L_2 u\rangle + \langle 2u_t, u_t\rangle \tag{4.18}$$

and with 4.17 and 4.18

$$\frac{dV}{dt} \le -\frac{(1-M_1(t))}{M_2} V \tag{4.19}$$

$$V \le V_o(\bar{x}_o)\ \{\exp - \frac{t}{M_2}(1-\sup_t M_1(t))\} \tag{4.20}$$

From 4.16 and 4.20 the results 4.15 follow directly.

Example 1

Consider the equation of a string with parametric excitation proportional to the slope:

$$\left.\begin{array}{l} u_{tt} + 2z\, u_t - u_{xx} + V(t)\, u_x = 0 \\ \\ u(0,x) = u_o(x) \qquad u_t(0,x) = u_{t_o}(x) \end{array}\right\} \tag{4.21}$$

with boundary conditions:

$$\left.\begin{array}{lllll} & \text{i)} & u(t,0) = 0 & u(t,1) = 0 & \text{fixed-fixed} \\ \\ \text{or} & \text{ii)} & u(t,0) = 0 & u_x(t,1) = 0 & \text{fixed-free} \end{array}\right\} \tag{4.22}$$

Part A

Conditions 4.7 from Theorem 0 are satisfied if:

$$\sup_t \left[\frac{\frac{|V(t)|^2}{2z} + 2z\,|V(t)|}{2z} \right] < 1 \,. \tag{4.23}$$

Then since $|u|^2_{max} \le \langle u_x, u_x \rangle$, see 4.25, the theorem implies $\lim_{t\to\infty} |u|_{max} = 0$, providing $\sup_t |V(t)|$ is sufficiently small.

Part B

Conditions 4.14 from Theorem P using 4.25 are satisfied if:

$$\sup_t \left[\frac{|V_t(t)|}{2z[1-|V(t)|]} \right] < 1 \,. \tag{4.24}$$

Then as before $\lim_{t\to\infty} |u|_{max} = 0$, providing $\sup_t |V(t)|$ is less than 1 and $\sup_t V_t(t)$ is sufficiently small.

Note:

$$\left.\begin{aligned} &\langle u, u_{xx} \rangle = u u_x 1_o' - \langle u_x, u_x \rangle = - \langle u_x, u_x \rangle \\ &|u|_{max} \le \langle 1, |u_x| \rangle \le \langle u_x, u_x \rangle^{1/2} \end{aligned}\right\} \tag{4.25}$$

Example 2 Buckling of a Beam with Force $p(t)$

$$\left.\begin{aligned} &u_{tt} + 2zu_t + u_{xxxx} + p(t)\, u_{xx} = 0 \\ &u(o,x) = u_o(x) \qquad \dot{u}(0,x) = \dot{u}_o(x) \end{aligned}\right\} \tag{4.26}$$

Boundary Conditions allowed:

i)	$u(0) = 0$	$u_x(0) = 0$	$u(1) = 0$	$u_x(1) = 0$	fixed-fixed
ii)	$u(0) = 0$	$u_x(0) = 0$	$u(1) = 0$	$u_{xx}(1) = 0$	fixed-pinned
iii)	$u(0) = 0$	$u_x(0) = 0$	$u_{xx}(1) = 0$	$u_{xxx}(1) = 0$	fixed-free
iv)	$u(0) = 0$	$u_x(0) = 0$	$u_x(1) = 0$	$u_{xxx}(1) = 0$	fixed-semifixed
v)	$u(0) = 0$	$u_{xx}(0) = 0$	$u_x(1) = 0$	$u_{xxx}(1) = 0$	pinned-semifixed

Part A

Conditions 4.7 from Theorem 0 are satisfied if:

$$\sup_t \left[\frac{\frac{|p(t)|^2}{2z} + 2z|p(t)|}{2z} \right] < 1 \tag{4.27}$$

or if one excludes Boundary Condition iii) then 4.27 can be put in the form:

$$\sup_t \left[\frac{\frac{|p(t)|^2}{2z} + \frac{2z|p(t)|}{\lambda_{min}}}{2z} \right] < 1 \tag{4.28}$$

where λ_{min} is the minimum eigenvalue of $\frac{d^2}{dx^2}$ with the respective boundary conditions. Then since $|u|^2_{max} \leq \langle u_{xx} u_{xx} \rangle$, see 4.31, the theorem implies $\lim_{t\to\infty} |u|_{max} = 0$, providing $\sup_t |p(t)|$ is sufficiently small.

Part B

Conditions 4.14 from Theorem P are satisfied if:

$$\sup_t \left[\frac{|p_t(t)|}{2z(1-|p(t)|)} \right] < 1 \tag{4.29}$$

or if one excludes Boundary Condition iii) then 4.29 can be put in the form:

$$\sup_t \left[\frac{|p_t(t)|}{2z(\lambda_{min}-|p(t)|)} \right] < 1 \ . \tag{4.30}$$

Then as before $\lim_{t\to\infty} |u|_{max} = 0$, providing $\sup_t |p(t)|$ is less than 1 or λ_{min} and $\sup_t |p_t(t)|$ is sufficiently small.

Note:

$$\left.\begin{aligned}
&\langle u,u_{xxxx}\rangle = u,u_{xx}\Big|_0^1 - u_x u_{xx}\Big|_0^1 + \langle u_{xx},u_{xx}\rangle \\
&|u|_{max} \le \langle 1,|u_x|\rangle \le \langle u_x u_x\rangle^{1/2} < |u_x|_{max} \le \langle u_{xx}u_{xx}\rangle^{1/2} \\
&\frac{\langle u,u_{xx}\rangle}{\langle u,u_{xxxx}\rangle} < \frac{1}{\lambda_{min}} \text{ with the exception of Boundary} \\
&\text{Condition iii.}
\end{aligned}\right\} \tag{4.31}$$

Example 3 Buckling of a Plate

$$\left.\begin{aligned} & u_{tt} + 2z\, u_t + \nabla\cdot\nabla(\nabla\cdot\nabla u) + \eta_x(t)\,\frac{\partial^2 u}{\partial x^2} + \eta_y(t)\,\frac{\partial^2 u}{\partial y^2} = 0 \\ & u(0,\bar{x}) = u_o(x,y) \qquad u_t(0,\bar{x}) = u_{t_o}(x,y) \end{aligned}\right\} \quad (4.32)$$

with boundary condition:

$$u(\Gamma) = 0 \quad \text{and} \quad \int_\Gamma \frac{\nabla}{2}(\nabla u\cdot\nabla u)\cdot\bar{n}\,d\Gamma = 0 \qquad (4.33)$$

Part A

Conditions 4.7 from Theorem Q with $N(t) = \max\{|\eta_x(t)|\ ,\ |\eta_y(t)|\}$ requires:

$$\sup_t \left[\frac{\dfrac{N^2(t)}{2z} + \dfrac{2zN(t)}{\lambda_{min}}}{2z} \right] < 1\ , \qquad (4.34)$$

where λ_{min} is defined in 4.36.

Then since $\lim_{t\to\infty}\left\langle [u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2]\right\rangle = 0$, Appendix B and Sobolev's Lemma (10)

$$|u|_{max}^2 \le \text{const} \int_S [u^2 + u_x^2 + u_y^2 + u_{xx}^2 + u_{xy}^2 + u_{yy}^2]\,dS$$

with the only restriction that S satisfy the "cone condition" (10), the $\lim_{t\to\infty} |u|_{max} = 0$, providing $\sup_t |N(t)|$ is sufficiently small.

Part B

Conditions 4.14 from Theorem P with $N_t(t) = \max\{|\dot{\eta}_x(t)|\ ,\ |\dot{\eta}_y(t)|\}$ requires:

$$\sup_t \left[\frac{N_t(t)}{2z(\lambda_{min}-N(t))} \right] < 1 \ . \tag{4.35}$$

Then by the previous argument $\lim_{t\to\infty} |u|_{max} = 0$, providing $\sup_t |N(t)|$ is less than λ_{min} and $\sup_t |N_t(t)|$ is sufficiently small.

Note: (see Appendix A and B)

$$\left.\begin{aligned}
&\frac{\int_S \nabla u\cdot\nabla u\ dS}{\int_S u\nabla\cdot\nabla(\nabla\cdot\nabla u)dS} \le \frac{1}{\lambda_{min}} \\
&\text{and} \\
&\frac{\int_S u^2\ dS}{\int_S u\nabla\cdot\nabla u\ dS} \le \frac{1}{\lambda_{min}} \\
&\text{where } \lambda_{min} \text{ is the minimum eigenvalue of the} \\
&\text{operator } \nabla\cdot\nabla u \text{ where } u(\Gamma) = 0 \ . \\
&\text{and} \quad \int_S u\nabla\cdot\nabla(\nabla\cdot\nabla u)dS = -\int_\Gamma \frac{\nabla}{2}\ (\nabla u\cdot\nabla u)\cdot\bar{n}\ d\Gamma \\
&\qquad\qquad + \int_S (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)dS
\end{aligned}\right\} \tag{4.36}$$

Comments

The Theorems O and P considered only global stability but the argument for stability of the equilibrium solution, $u(t,\bar{x}) = 0$, when the conditions 4.7 and 4.14 are not global can in most examples, by the following argument, be demonstrated. Whenever the displacement u can be bounded above by the Liapunov function, as in all the examples presented, then since $V(u)$ is always less than $V(u_o)$ one can choose the initial conditions sufficiently small so that u always remains in the region of validity of conditions 4.7 and 4.14 about the equilibrium position.

The author has been unable to obtain stability under the requirement of sufficiently small integral bounds on the value $\left|\int_o^t a_i(t)dt\right|$ in the operator $L_2(t)$, i.e., "high frequency bounds." This is supported from investigations on discrete systems (see Theorem K) and indicates that for systems where L_2 possesses an unbounded spectrum of eigenvalues, as is the case for most continuous dynamic systems, the approach fails.

DISCUSSION

The theorems developed within demonstrate the existence of sufficient exponential asymptotic stability boundaries in the sense of Liapunov for homogeneous parametric ordinary and partial linear differential equations.

For many applications the bounds will not be good enough, that is, the parameters of the system will not be included in the sufficient stability region, and design requirements will not permit conservatisms in the parameters to assure stability under the theorems developed. However, the theorems do demonstrate the existence of stability boundaries and perhaps enlarge the presently known stability region by considering more properties of the system. (Caughey and Grey considered stability under conditions on the maximum of $\|B(t)\|$ whereas the developed theorems consider the effect of the maximum of $\|\int_0^t B(\tau)d\tau\|$ and $\|\frac{dB(t)}{dt}\|$) .

Besides the physical problems to which the considered equations apply directly (i.e., the example problems), the application to the study of local stability of a non-trivial solution of nonlinear equations appears promising. Under certain restrictions (see Struble (12)) the stability of a trajectory $\bar{x}(t)$ of the equation $\frac{d\bar{x}}{dt} = f(\bar{x},t)$ is determined by the asymptotic stability of the trivial solution of $\frac{d\bar{y}}{dt} = \bar{f}_{\bar{x}}(\bar{x}(t),t)\bar{y}$ (i.e., the equation of first variation) which is of the form treated in the theorems developed for linear

systems with parametric coefficients. If $\bar{x}(t)$ is not exactly known but one is able to set bounds on its modulus or the modulus of its integral or derivative, the theorems will still be applicable.

For nonhomogeneous equations (the existence of forcing terms $\bar{f}(t)$) the Liapunov functions used to prove asymptotic stability can, in most cases, be used to show boundedness of the solution for bounded forcing terms. This can be done by recognizing that in many cases, $\exists R \ni -\alpha V(\bar{x})$ will dominate $|\nabla V \cdot \bar{f}(t)|$ for $\|\bar{x}\| > R$ and hence the Liapunov function will be exponentially decreasing for $\|\bar{x}\| > R$ and therefore the solution will be bounded within R for all times providing the initial conditions are sufficiently small. It may even be possible to improve R by consideration of the frequency content of $\bar{f}(t)$.

The theorems were developed with deterministic parameters in mind but the concept of almost sure asymptotic stability can in most cases be deduced by comparable statements in the stability requirements.

APPENDIX A

A proof is presented of the following inequality:

$$\frac{\int_S \nabla u \cdot \nabla u \, dS}{\int_S u \nabla \cdot \nabla (\nabla u \cdot \nabla u) dS} \leq \frac{1}{\lambda_{min}}$$

where λ_{min} is the minimum eigenvalue of the continuous operator $-\nabla \cdot \nabla$, i.e., the Laplacian, and further that $u(\Gamma) \equiv 0$ where Γ is the boundary of S.

By suitable restrictions on $u(\bar{x})$, it may be expressed as $u = \sum_{i=1}^{\infty} a_i e_i(\bar{x})$ where $\nabla \cdot \nabla e_i + \lambda_i e_i = 0$, that is e_i is an eigenvector of $-\nabla \cdot \nabla$ and λ_i is the associated eigenvalue. Clearly then:

$$0 = \sum_{i=1}^{\infty} a_i \int_S \nabla u \cdot [\nabla (\nabla \cdot \nabla e_i + \lambda_i e_i)] dS$$

$$= \int_S \nabla u \cdot \nabla (\nabla \cdot \nabla u) dS + \sum_{i=1}^{\infty} a_i \lambda_i \int_S \nabla u \cdot \nabla e_i \, dS$$

and by Green's Theorem:

$$0 = \int_\Gamma u \nabla (\nabla \cdot \nabla u) \cdot \bar{n} d\Gamma - \int_S u \nabla \cdot \nabla (\nabla \cdot \nabla u) dS$$

$$+ \sum_{i=1}^{\infty} a_i \lambda_i \int_S \nabla u \cdot \nabla e_i \, dS$$

since $u(\Gamma) \equiv 0$, then:

$$\frac{\sum_{i=1}^{\infty} a_i \lambda_i \int_S \nabla u \cdot \nabla e_i \, dS}{\int_S u \nabla \cdot \nabla (\nabla \cdot \nabla u) dS} = 1$$

and since the eigenvectors of $-\nabla \cdot \nabla$ are orthogonal and the eigenvalues are positive it can be shown that:

$$\frac{\int_S \nabla u \cdot \nabla u \, dS}{\int_S u \nabla \cdot \nabla (\nabla \cdot \nabla u) dS} \leq \frac{1}{\lambda_{min}} .$$

APPENDIX B

This appendix demonstrates the application of Green's Theorem to the functional $\int_S u(\nabla\cdot(\nabla(\nabla\cdot\nabla u)))dS$.

The following identities can easily be shown:

i) $\nabla\cdot(u\nabla(\nabla\cdot\nabla u)) = u(\nabla\cdot(\nabla(\nabla\cdot\nabla u))) + (\nabla u)\cdot(\nabla(\nabla\cdot\nabla u))$

ii) $\nabla\cdot((\nabla\cdot\nabla u)(\nabla u)) = (\nabla\cdot\nabla u)(\nabla\cdot\nabla u) + (\nabla(\nabla\cdot\nabla u))\cdot(\nabla u)$

iii) in two-dimensional rectangular coordinates:

$$\frac{1}{2}\nabla\cdot\nabla(\nabla u\cdot\nabla u) = (\nabla u)\cdot(\nabla(\nabla\cdot\nabla u)) + (\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 + 2(\frac{\partial^2 u}{\partial x\partial y})^2$$

Therefore by Green's Theorem

$$\int_S u(\nabla\cdot(\nabla(\nabla\cdot\nabla u)))dS = \int_\Gamma [u\nabla(\nabla\cdot\nabla u) - (\nabla\cdot\nabla u)\ (\nabla u)]\cdot\bar{n}\ d\Gamma$$

$$+ \int_S (\nabla\cdot\nabla u)\ (\nabla\cdot\nabla u)dS$$

or

$$\int_S u(\nabla\cdot(\nabla(\nabla\cdot\nabla u)))dS = \int_\Gamma [u\nabla(\nabla\cdot\nabla u) - \frac{\nabla}{2}\ (\nabla u\cdot\nabla u)]\cdot\bar{n}\ d\Gamma$$

$$+ \int_S [(\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 + 2(\frac{\partial^2 u}{\partial x\partial y})^2]dS$$

APPENDIX C

The $Y(t)$ matrix for the equation:

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \qquad \omega > 0 \quad \text{and} \quad 0 < \xi < 1$$

is just the solution of the matrix equation:

$$\frac{dY}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix} Y \quad \text{with} \quad Y(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and is easily shown to be of the form

$$Y(t) = \begin{bmatrix} \gamma \cos \gamma t - \eta \sin \gamma t & \sin \gamma t \\ (-\gamma^2 - \eta^2) \sin \gamma t & \gamma \cos \gamma t - \eta \sin \gamma t \end{bmatrix} \frac{e^{\eta t}}{\gamma}$$

where: $\eta = -\omega$ and $\gamma = \omega\sqrt{1-\xi^2}$

clearly:

$$\|Y(t)\| \le [2\gamma|\cos \gamma t| + (2|\eta| + 1 + \omega^2)|\sin \gamma t|] \frac{e^{\eta t}}{\gamma}$$

$$\le [2\gamma \cos \gamma\theta + (2|\eta| + 1 + \omega^2) \sin \gamma\theta] \frac{e^{\eta t}}{\gamma}$$

where $\tan \quad = \dfrac{2|\eta| + 1 + \omega^2}{2\gamma} \qquad 0 \le \theta \le \pi/2$

This expression then allows one to pick values for a and b such that $\|Y\| \le be^{-at}$.

APPENDIX D

For a matrix A of the form:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

the associated P matrix that gives $A^T P + PA = -I$ is as follows:

$$P = \begin{bmatrix} \frac{\omega}{4\xi} + \frac{1}{4\xi\omega} + \frac{\xi}{\omega} & \frac{1}{2\omega^2} \\ \frac{1}{2\omega^2} & \frac{1}{4\xi\omega} + \frac{1}{4\xi\omega^3} \end{bmatrix}$$

The eigenvalues of P are:

$$\lambda = \frac{1}{2}\left(\frac{1}{2\xi\omega} + \frac{\omega}{4\xi} + \frac{\xi}{\omega} + \frac{1}{4\xi\omega^3}\right)$$

$$\pm \frac{1}{2}\sqrt{\left(\frac{1}{2\xi\omega} + \frac{\omega}{4\xi} + \frac{\xi}{\omega} + \frac{1}{4\xi\omega^3}\right)^2 - 4\left[\left(\frac{\omega}{4\xi} + \frac{1}{4\xi\omega} + \frac{\xi}{\omega}\right)\frac{1}{4\xi\omega} + \left(\frac{\omega}{4\xi} + \frac{1}{4\xi\omega}\right)\left(\frac{1}{4\xi\omega^3}\right)\right]}$$

APPENDIX E

This appendix is to develop bounds on $P^{-1/2}$ and $P^{1/2}$ for use in the theorems that involve these expressions. Since P is a symmetric positive definite matrix, there exists an orthogonal transformation Φ such that

$$\Phi^T\Phi = I$$

$$\Phi^T P \Phi = [\ulcorner \lambda_i^P \lrcorner]$$

where λ_i^P are the positive eigenvalues of P. Clearly then:

$$P = \Phi[\ulcorner \lambda_i^P \lrcorner]\Phi^T$$

and

$$P^{1/2} = \Phi[\ulcorner (\lambda_i^P)^{1/2} \lrcorner]\Phi^T$$

$$P^{-1/2} = \Phi[\ulcorner (\lambda_i^P)^{-1/2} \lrcorner]\Phi^T$$

If P is an nxn matrix, then by Cauchy's Inequality:

$$\left(\sum_{i=1}^{n} |\Phi_{ij}|\right)^2 \leq \left(\sum_{i-1}^{n} \Phi_{ij}^2\right)n$$

and since the modes are normal $\left(\sum_{i=1}^{n} \Phi_{ij}^2\right) = 1$

therefore:

$$\sum_{i=1}^{n} |\Phi_{ij}| \leq \sqrt{n}$$

hence: $$\|P^{1/2}\| \le n^3 \sum_{i=1}^{n} (\lambda_i^P)^{1/2}$$

$$\|P^{-1/2}\| \le n^3 \sum_{i=1}^{n} (\lambda_i^P)^{-1/2}$$

Again using Cauchy's Inequality:

$$\sum_{i=1}^{n} (\lambda_i^P)^{1/2} \le \sqrt{(\sum_{i=1}^{n} \lambda_i^P)n} = \sqrt{\mathrm{Trace}(P)n}$$

and $$\sum_{i=1}^{n} (\lambda_i^P)^{-1/2} \le \sqrt{(\sum_{i=1}^{n} (\lambda_i^P)^{-1}n} = \sqrt{\mathrm{Trace}(P^{-1})n}$$

Thus we get:

$$\|P^{1/2}\| \le n^{7/2} \sqrt{\mathrm{Trace}(P)}$$

and $$\|P^{-1/2}\| \le n^{7/2} \sqrt{\mathrm{Trace}(P^{-1})}$$

This is presented as a simpler alternative to calculating all the eigenvalues and eigenvectors of P .

APPENDIX F

Lemma B

This lemma demonstrates that $\|\exp(-S)T\exp(S)-T\|$ goes to zero continuously with $\|S\|$.

Let:

$$\|U\| = \|\exp(-S)T\exp(S)-T\| = \|[I + \sum_{n=1}^{\infty} \frac{(-S)^n}{\lfloor n}]T\,[I + \sum_{n=1}^{\infty} \frac{(S)^n}{\lfloor n}]-T\|$$

$$= \|T \sum_{n=1}^{\infty} \frac{(S)^n}{\lfloor n} + \sum_{n=1}^{\infty} \frac{(-S)^n}{\lfloor n} T + \sum_{n=1}^{\infty} \frac{(-S)^n}{\lfloor n} T \sum_{n=1}^{\infty} \frac{(S)^n}{\lfloor n}\|$$

but $\quad \|\sum_{n=1}^{\infty} \frac{(S)^n}{\lfloor n}\| = \|S \sum_{n=0}^{\infty} \frac{(S)^n}{\lfloor n+1}\| \le \|S\| \sum_{n=0}^{\infty} \frac{(S)^n}{\lfloor n} = \|S\|\, e^{\|S\|}$

hence $\quad \|U\| \le 2\|T\|\,\|S\|\, e^{\|S\|} + \|T\|\,\|S\|^2\, e^{2\|S\|}$

therefore $\|U\| \to 0$ as $\|S\| \to 0$.

Lemma C

This lemma demonstrates that

$\|\exp(-S)\frac{dS}{dt}\exp(S) - \exp(-S)\frac{d\exp(S)}{dt}\|$ goes to zero continuously with $\|S\|$.

$$\|U\| = \|\exp(-S)\ \dot{S}\ \exp(S) - \exp(-S)\ \frac{d\exp(S)}{dt}\|$$

$$\leq e^{\|S\|}\ \|\dot{S} \sum_{n=1}^{\infty} \frac{(S)^n}{\lfloor n} - \frac{d}{dt} \sum_{n=2}^{\infty} \frac{(S)^n}{\lfloor n}\ \|$$

$$\leq e^{\|S\|}\ \|\dot{S} \sum_{n=1}^{\infty} \frac{(S)^n}{\lfloor n} - \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \frac{S^k\ \dot{S}\ S^{n-k-1}}{\lfloor n}\|$$

$$\leq e^{\|S\|}\ \|\dot{S}\|\ [\sum_{n=1}^{\infty} \frac{\|S\|^n}{\lfloor n} + \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \frac{\|S\|^{n-1}}{\lfloor n}]$$

$$\leq e^{\|S\|}\ \|\dot{S}\|\ [\|S\|\ e^{\|S\|} + \sum_{n=1}^{\infty} \frac{\|S\|^n}{\lfloor n}]$$

$$\leq 2\|S\|\ \|\dot{S}\|\ e^{2\|S\|}$$

Hence

$$\|U\| < 2\|S\|\ \|\dot{S}\|\ e^{2\|S\|}$$

and therefore $\|U\| \to 0$ as $\|S\| \to 0$.

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