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THE DYNAMIC STABILITY OF AN UNBALANCED MASS EXCITER

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ABSTRACT

The dynamic stability of single- and multi-degree-of-freedom unbalanced mass exciter systems is discussed. Previous work concerning this subject by A. Sommerfield, Y. Rocard, R. Mazét, V. O. Kononenko, Y. G. Panovko and I. I. Gubanova is summarized. A single-degree-of-freedom system consisting of a linear mechanical oscillator with a rotating unbalanced mass connected rigidly to it is defined as the basic single-degree-of-freedom system. This system is mathematically equivalent to the one used by Rocard in his analysis. The differential equations of motion for the system are obtained by using Lagrange's Equations. Global stability, stability in the sense of Laplace, is proved using Liapunov's second method. Four separate local stability analyses of this system are developed, two of which assume a constant angular velocity, Ω, of the unbalanced mass and two which allow for periodic variations in Ω. These analyses are termed zero and first order respectively.

The first zero order analysis is based directly on the differential equations of motion and the zero order steady state solution. The steady state torque output of the vibration exciter motor and the steady state torque requirements of the oscillator are obtained as functions of the operating frequency. Stability is determined by examining the behavior of the system in the vicinity of the intersection points of these two functions. The second zero order analysis examines the behavior of small perturbations added to the steady state solution. The system is considered stable if these perturbations disappear with time. The first first order analysis is a perturbation type, but is based on a steady state solution which allows for periodic variations in Ω.

The second first order analysis is also based on the first order perturbed equations of motion but is a Floquet type analysis. Validity criteria for the zero and first order analyses are obtained, and the zero order region of validity is plotted graphically. A representative set of systems is analyzed numerically, and the results are presented in a figure showing the stability boundary as a function of the system parameters in non-dimensional form.

Two distinct types of multi-degree-of-freedom systems are discussed. The first consists of a single oscillator mass that is free to perform planar motion. It is shown that when an unbalanced mass exciter with a uniaxial force output is mounted on the oscillator in such a way that only one mode is excited, the problem reduces to the single-degree-of-freedom problem. The second system consists of a series of linear single-degree-of-freedom oscillators with an unbalanced mass exciter mounted on one of them. The special case of a three oscillator system with equal masses is used to demonstrate that, for systems with widely separated resonances, the "equivalent" single-degree-of-freedom analysis presented by Kononenko is valid. From these results it is concluded that, in any multi-degree-of-freedom system, an "equivalent" single-degree-of-freedom analysis may be used to examine the stability of the system near any resonance as long as that particular mode is the only one which is being significantly excited.

Appendices covering the details of Rocard's analysis, and of the unbalanced mass exciters designed and built at the California Institute of Technology are included.

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I INTRODUCTION

In recent years there has been considerable interest in the dynamic properties of large structures such as dams and buildings, and in their behavior under the forces generated by earthquakes, explosions, and high velocity winds. This interest created a need for vibration exciters capable of supplying relatively high level force outputs at controlled frequencies. When a sinusoidal exciting force is desirable or acceptable for testing, this kind of output may be obtained by means of a rotating unbalanced mass. Such a device is called an unbalanced mass exciter.

A practical difficulty with unbalanced mass exciters, observed as early as 1904 by A. Sommerfeld[(1)], is that local instabilities may occur in the operating speed of such devices. Other investigators have shown that these instabilities occur near the exciter-structure system resonance frequencies. In particular, it has been observed that as the operating speed of an unbalanced mass exciter is increased through a resonance region, a sudden jump to a higher operating speed may occur; similarly, a sudden decrease in operating speed may occur as the operating speed is decreased through the resonance region. Such frequency jumps may be merely annoying as long as the vibration generator is small, but may become dangerous for large vibration exciters. The danger arises when a jump causes the operating speed to exceed the safe limits of the exciter. Possible consequences include damage to or disintegration of the exciter, with attendant injury to personnel and damage to the structure being tested.

The problem of instabilities in the operating speed may be avoided by keeping the unbalanced mass small, avoiding systems where large motions occur, and using motors with the steepest possible speed-torque curves. A feedback control system may be used to provide a very steep effective speed-torque curve. A good example of this type of exciter system is the one designed and built by the California Institute of Technology. Appendix II describes the design and development of this machine.

Several investigators, including Y. Rocard[2], V. O. Konenenko[3, 4], R. Mazẽt[5], Y. G. Panovko and I. I. Gubanova[6] have studied the problem of the stability of the unbalanced mass exciter. They all agree that an unstable condition may occur due to a nonlinear interaction between the motor driving the unbalanced mass and the motion of the structure to which the unbalanced mass is attached. Each of these men used the unbalanced mass system which he considered the most convenient. The three basic systems used, which differ only in detail, consist of rotating unbalanced masses which are coupled to simple single-degree-of-freedom mechanical oscillators. The differences are in the form of coupling used, which affects the equations of motion, and in the location of the motor, which does not affect the equations of motion. Rocard studied a system with a rigid coupling between the rotating unbalanced mass and the oscillator mass, and with the motor mounted on a fixed base driving the unbalanced mass through a flexible shaft. Mazẽt, Panovko, and Gubanova also used this type of system. Kononenko used a system with the motor attached to a fixed base, but with the rotating unbalanced mass coupled to the oscillator with an elastic element. The system considered in this thesis, which will be called a Modified Rocard System, uses a rigid coupling, with the motor mounted directly on the oscillator mass. Figure 1.1 shows sketches of the three types of systems, using the notation developed by their originators. Figure 1.2 lists and defines the parameters and notation used in the following discussion.

The equations of motion for the Modified Rocard System may be obtained by using Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial D}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \tag{1.1}$$

where T, the kinetic energy, is

$$T = \frac{1}{2} M' \dot{x}^2 + \frac{1}{2} m\left[(\dot{x} + \dot{\theta} r \cos\theta)^2 + (\dot{\theta} r \sin\theta)^2\right] + \frac{1}{2} J' \dot{\theta}^2, \tag{1.2}$$

V, the potential energy, is

$$V = \frac{1}{2} k x^2 + m g r (1 + \cos\theta), \tag{1.3}$$

D, the Rayleigh Dissipation Function[7], is

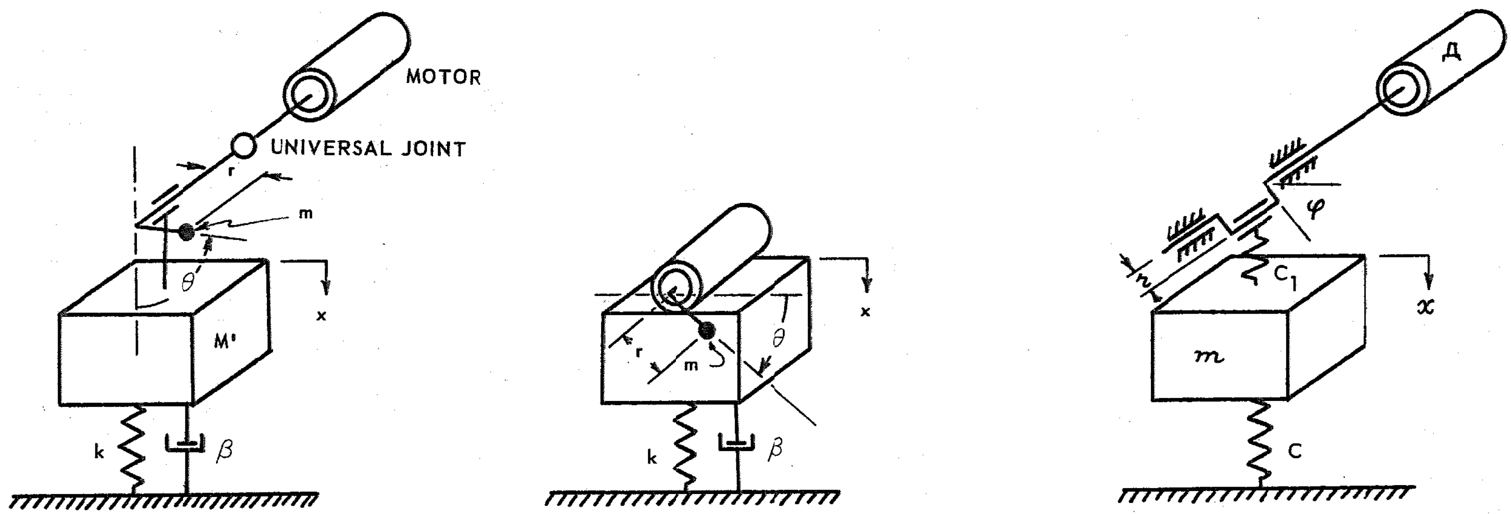
$$D = \frac{1}{2} \beta \dot{x}^2 + \frac{1}{2} \tau (\omega_s - \dot{\theta}), \tag{1.4}$$

and q_i is the generalized coordinate. Applying Lagrange's Equation for the two coordinates $q_i = x$ and $q_i = \theta$ gives the differential equations of motion

$$(M' + m)\ddot{x} + \beta \dot{x} + k x = m r (\dot{\theta}^2 \sin\theta - \ddot{\theta} \cos\theta) \tag{1.5}$$

and

$$(J' + m r^2)\ddot{\theta} + m r (\ddot{x} \cos\theta - g \sin\theta) = \frac{1}{2J}\left[\frac{\partial \tau}{\partial \theta}(\dot{\theta} - \omega_s) + \tau\right]. \tag{1.6}$$



THE ROCARD SYSTEM

THE MODIFIED ROCARD SYSTEM

THE KONONENKO SYSTEM

THE SINGLE-DEGREE-OF-FREEDOM SYSTEMS

FIGURE 1.1

Figure 1.2

Parameters and Notation

D	the Rayleigh Dissipation Function, used in Lagrange's Equation
E	the voltage across the motor
g	the component of the acceleration of gravity acting in the plane of rotation of m
I	the electric current in the motor
J	the sum of J' and the moment of inertia of the unbalanced mass m about its center of rotation
J'	the polar moment of inertia of the motor
k	the oscillator spring constant
K_1	motor constant relating input voltage to set speed
K_2	motor constant relating back EMF to operating speed
m	the unbalanced mass
M	the sum of m and M'
M'	the oscillator mass
q_i	the generalized displacement, used in Lagrange's Equation
r	the distance between the center of mass of m and its center of rotation
R	the electrical resistance of the motor windings
T	the kinetic energy, used in Lagrange's Equation
V	the potential energy when used in Lagrange's Equation; the Liapunov Function when used in the Liapunov Stability analysis

Figure 1.2 (continued)

x	the linear displacement of M', measured from the position of static equilibrium
β	the oscillator damping constant
ζ	the damping ratio of the oscillator
θ	the angular displacement of m, measured clockwise from the vertical
$\varkappa$	$\frac{K_1 K_2}{R}$
τ	the motor output torque which is a function of $\dot{\theta}$ and ω_s
ω	the angular speed of the motor
ω_s	the set, or zero-load angular speed of the motor
ω_o	the natural frequency of the system
Ω	the ratio of ω to ω_o
Ω_s	the ratio of ω_s to ω_o

A dot over a variable indicates a derivative with respect to time. For example,

$$\dot{x} = \frac{dx}{dt} \text{ and } \ddot{x} = \frac{d^2 x}{dt^2} .$$

Defining

$$M = M' + m, \qquad J = J' + mr^2, \qquad \omega_o^{\,2} = \frac{k}{m}, \qquad \text{and } 2\zeta\omega_o = \frac{\beta}{M} \tag{1.7}$$

allows these equations to be put into the standard form

$$\ddot{x} + 2\zeta\omega_o\dot{x} + \omega_o^{\,2}x = \frac{mr}{M}(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta) \tag{1.8}$$

and

$$\ddot{\theta} + \frac{mr}{J}(\ddot{x}\cos\theta - g\sin\theta) = \frac{1}{2}\left[\frac{\partial\tau}{\partial\theta}(\dot{\theta} - \omega_s) + \tau\right]. \tag{1.9}$$

This pair of differential equations is autonomous because it does not explicitly involve the time variable t.

When the stability of the system described by equations (1.8) and (1.9) is examined, the type of stability being considered must be specified. In general, the three types of stability are Laplace Stability, Poincaré Stability, and Liapunov Stability(8). The least stringent type is Laplace Stability, which merely requires that all motions of the system be finite. A system is stable in the sense of Poincaré if its solution trajectory Γ has the property that the positive half paths of all trajectories that are once near it remain near it. Liapunov Stability is the most stringent of the three, and requires that motions which are once near to each other remain close together for all future time, as functions of time.

Laplace Stability of the unbalanced mass exciter system may be established using a technique developed by Liapunov(9). Consider a function

$$V = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2 + m r \dot{x}\dot{\theta} \cos\theta + \frac{1}{2M} (M\dot{x} + \beta x$$

$$+ m r \dot{\theta} \cos\theta)^2 + k x^2 \tag{1.10}$$

which, along the trajectory of the equations of motion, has the time derivative

$$\dot{V} = -\beta\left(\dot{x}^2 + \frac{k}{M} x^2\right) + \frac{1}{2}\left[\frac{\partial \tau}{\partial \theta} (\dot{\theta} - \omega_s) + \tau\right] - \frac{k}{M} m r x \dot{\theta} \cos\theta , \tag{1.11}$$

from which follows

$$\dot{V} \leq -\beta\left(\dot{x}^2 + \frac{k}{M} x^2\right) + \frac{1}{2}\left[\frac{\partial \tau}{\partial \theta} (\dot{\theta} - \omega_s) + \tau\right] + \frac{k}{M} m r |x||\dot{\theta}| . \tag{1.12}$$

Therefore, $\dot{V} < 0$ for x, $\dot{x}$, and $\dot{\theta}$ sufficiently large, but still bounded, and under the restriction that $\partial\tau/\partial\dot{\theta} < 0$. Thus there exists a set, X, such that for x, $\dot{x}$, and $\dot{\theta}$ in its complement, X_c, $\dot{V} < 0$. Hence all solutions are stable in the sense of Laplace because they are ultimately bounded and lie in X, which is a compact set.

The previously published work concerning the stability of an unbalanced mass exciter system has been concerned with local stability. Local stability has been interpreted to mean orbital, or Poincaré, stability, since an autonomous system of this type does not possess asymptotic, or Liapunov, stability.

In 1943 Rocard published an analysis of the stability of an unbalanced mass exciter[(2)]. In that analysis he obtained the equations of motion by using Lagrange's Equation and assuming a linear speed-torque curve for the driving motor. The oscillator equation was found to be

$$\ddot{x} + 2\zeta\omega_o \dot{x} + \omega_o^2 x = \frac{m\,r}{M}\frac{d^2}{dt^2}\cos\theta \,. \; * \tag{1.13}$$

A change of variables,

$$x = \ddot{u} \,, \tag{1.14}$$

was used to transform the equation into the form

$$\ddot{u} + 2\zeta\omega_o \dot{u} + \omega_o^2 = \frac{m\,r}{M}\cos\theta \,. \tag{1.15}$$

A solution, u, was found by slowly varying the amplitude and phase. The value of u was substituted into the motor equation, which was then averaged over one cycle to eliminate the rapidly varying effects. This procedure resulted in Rocard's "steady state" motor equation:

$$\left\{ J - \frac{2(m\,r\Omega)^2}{M}\left[\frac{1-\Omega^2-\zeta^2}{(1-\Omega^2)^2+(2\zeta\Omega)^2}\right]\right\}\ddot{\theta} + \left\{\frac{(m\,r)^2}{M}\left[\frac{\zeta\omega_o\Omega^4}{(1-\Omega^2)^2+(2\zeta\Omega)^2}\right] + K\right\}\dot{\theta} = K\omega_s \,. \tag{1.16}$$

Instability results when the inertia term, the coefficient of $\ddot{\theta}$, becomes negative. Solutions using the same method have been published by Mazét(5) in 1955 and by Panovko and Gubanova(6) in 1964.

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* Mazét and Panovko and Gubanova have shown that the right hand side should have a minus sign. Appendix I contains a complete discussion of this analysis.

In 1961, Kononenko published an analysis of the same problem(3). Instead of integrating the equations of motion, he assumed a steady state solution for the oscillator equation of the form

$$x = A \cos(\phi + \Xi), \qquad \frac{dx}{dt} = -A\,\omega(\phi + \Xi), \qquad \frac{d\phi}{dt} = \theta \tag{1.17}$$

where A, Ξ, and θ are slowly varying functions of ϕ. Equations for dA/dt, $d\Xi/dt$, and $d\theta/dt$ were developed, and solutions were obtained using a perturbation technique. The resulting equations are linear with constant coefficients, and the Routh-Hurwitz criteria were applied to obtain the stability criteria for the system.

In addition to the simple oscillator, Kononenko discussed systems with both hard and soft nonlinear oscillator springs. Multiple oscillator systems were also discussed and it was shown that, if a system has well separated resonances, the stability of the regions near each resonance can be investigated by using an "equivalent" single-degree-of-freedom analysis. This latter idea is discussed more fully in the section on multi-degree-of-freedom systems.

II THE SINGLE-DEGREE-OF-FREEDOM SYSTEM

In this study of the stability of unbalanced mass exciter systems, the usual procedure of starting with the simplest case and proceeding to the more complex cases is followed. The simplest unbalanced mass exciter system consists of a rotating unbalanced mass connected to a single-degree-of-freedom linear mechanical oscillator. Three systems falling within the limits of this definition are illustrated in Figure 1.1.

The following discussion and analysis applies to the Modified Rocard System. It was shown in the introduction that the motion of this system is described by the differential equations

$$\ddot{x} + 2\zeta\omega_o \dot{x} + \omega_o^2 x = \frac{mr}{M}(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) \qquad (2.1)$$

and

$$\ddot{\theta} + \frac{mr}{J}(\ddot{x}\cos\theta - g\sin\theta) = \frac{1}{2J}\left[\frac{\partial\tau}{\partial\dot{\theta}}(\dot{\theta} - \omega_s) + \tau\right]. \qquad (2.2)$$

It was also stated that τ is a function of the operating speed $\dot{\theta}$ and the set speed ω_s, and that $\partial\tau/\partial\dot{\theta}$ must be negative to insure the Laplace stability of the system. In any given situation the exact nature of $\tau(\dot{\theta}, \omega_s)$ is therefore a function of the exciter motor characteristics. Since speed-torque curves are continuous functions, they may be approximated by piecewise linear, continuous curves. The stability analysis can therefore be simplified by considering the torque to be a linear function of $\dot{\theta}$ and ω_s near the operating speed $\dot{\theta}$. If some typical speed-torque curves, such as those shown in Figure 2.1, are examined, it is clear that, for some speed range from zero to a small enough $\dot{\theta}_{max}$, a straight line speed-torque curve is an approximation to the physical situation. As a further example, consider the torque balance equation for a shunt wound d.c. motor:

$$J'\ddot{\theta} + \tau = K_1 I = K_1 \frac{E - K_2\dot{\theta}}{R} \tag{2.3}$$

where K_1 and K_2 are constants of the motor. For steady state operation, in which $\ddot{\theta} = 0$, $K_2\dot{\theta}$ is the back EMF, and $E = K_2\omega_s$, the motor equation is

$$\tau = \frac{K_1 K_2}{R}(\omega_s - \dot{\theta}). \tag{2.4}$$

The motor speed-torque characteristic is defined as $\varkappa$, where $\varkappa = K_1 K_2/R$. Substituting into (2.4) gives

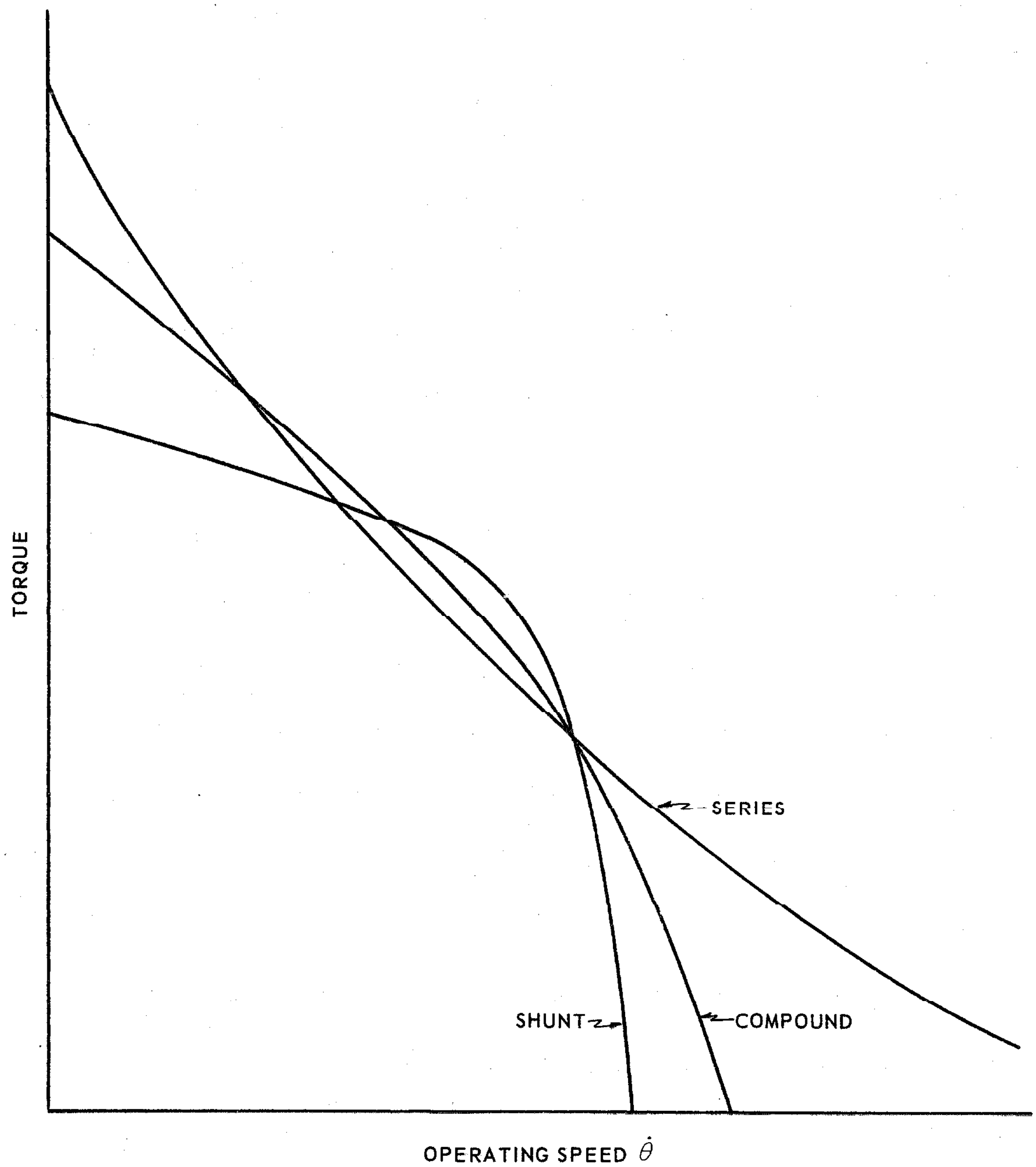
$$\tau = \varkappa(\omega_s - \dot{\theta}). \tag{2.5}$$

Equation (2.5) may be used to describe any linear speed-torque curve which has a slope of magnitude $\varkappa$. Substituting equation (2.5) into the equations of motion (2.1) and (2.2), the simplified equations (2.6) and (2.7) result:

$$\ddot{x} + 2\zeta\omega_o\dot{x} + \omega_o^2 x = \frac{mr}{M}(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta); \tag{2.6}$$

$$\ddot{\theta} + \frac{mr}{J}(\ddot{x}\cos\theta - g\sin\theta) = \frac{\varkappa}{J}(\omega_s - \dot{\theta}). \tag{2.7}$$

It should be noted at this point that, in assuming a linear speed-torque curve, no allowance is made for phenomena that limit the torque or power output of a motor. Two such phenomena that do occur are temperature rise and saturation. Temperature



GENERAL SPEED-TORQUE CHARACTERISTICS FOR SERIES, SHUNT, AND COMPOUND WOUND DC ELECTRIC MOTORS[10]

FIGURE 2.1

rise is caused by the heating of the conductor wire by the current passing through it. As the torque or power output of a motor increases, the internal temperature of the motor also increases. When this temperature exceeds the maximum allowable value, the insulation in the motor begins to degrade, resulting in lower efficiency and still higher temperatures. Continued operation of the motor at excessive power or torque outputs will result in either shorting or an open circuit. The second phenomenon which limits a motor torque or power output is the magnetic saturation which occurs when the magnetic flux density exceeds the capacity of the iron core. When this happens, the torque and power outputs level off. In actual testing, the average operating torque is kept well below the saturation level of the motor. If this is not done, the speed regulation becomes poor and it is difficult, if not impossible, to obtain useful data. One other consequence of these phenomena is that they provide an upper bound to the operating speed $\dot{\theta}$. This is in effect saying that from physical arguments alone, the motions of the unbalanced mass exciter must be ultimately bounded.

Four separate analyses of the dynamic local stability of the system described by equations (2.6) and (2.7) are conducted here. The first two are based on the assumption that the angular speed ω of the unbalanced mass is a constant. These are called zero order analyses. The last two, which allow for periodic variations in ω, are called first order analyses. In the first zero order analysis, the Torque-Slope Analysis, the assumed steady state solution is used directly in the equations of motion to obtain the average speed-torque curves for the system, and the stability of the system is derived from these curves. The second zero order analysis is a standard perturbation type in which the behavior in time of the perturbations is used to determine the stability of the system. The third analysis is a first order version of the perturbation analysis. The fourth analysis is also based on the perturbed equations of motion, but employs a Floquet type analysis to determine the stability of the system. The advantage of this Floquet type analysis is that it will detect instabilities caused by any perturbation, rather than just those specific ones allowed for in the other perturbation analyses. Results obtained using the various analyses are presented in a section following these analyses.

Torque-Slope Analysis:

This analysis is based on the assumption that the speed of rotation of the unbalanced mass is constant. Mathematically this means that

$$\theta = \omega t, \quad \dot{\theta} = \omega, \quad \ddot{\theta} = 0. \tag{2.8}$$

Using equations (2.8) in the differential equation of motion, (2.6), results in a specific differential equation of motion

$$\ddot{x} + 2\zeta\omega_o\dot{x} + \omega_o^{\,2}x = \frac{mr}{M}\,\omega^2 \sin\omega t \tag{2.9}$$

which has the approximate solution

$$x = A\sin(\omega t + \phi) \tag{2.10}$$

where

$$A = \frac{mr}{M}\left[\frac{\Omega^2}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}\right], \quad \phi = \tan^{-1}\left[\frac{-2\zeta\Omega}{1-\Omega^2}\right],$$

$$\text{and } \Omega = \frac{\omega}{\omega_o}. \tag{2.11}$$

Substituting equation (2.10) into (2.7), using the relations (2.11), and averaging the resulting equation over one cycle gives

$$\mathrm{K}(\Omega_s - \Omega) = \frac{\zeta\omega_o m^2 r^2}{M}\left[\frac{\Omega^5}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}\right]. \tag{2.12}$$

The left hand side of this equation is the average torque produced by the motor, and the right hand side is the average torque required by the oscillator. Figure 2.2 shows a plot of the average oscillator torque and of a family of average motor torque curves for various set speeds Ω_s. Examination of this figure shows that, if the maximum possible torque output of the motor is less than the torque required by the oscillator at resonance ($\Omega = 1$), the operating speed Ω will always remain less than unity even though the set speed Ω_s is made very large. However, if the maximum motor torque available is greater than the torque required by the oscillator at resonance, the operating speed can increase as Ω_s is increased up to some limiting value which is appreciably larger than the resonance frequency. We recall that this limit has not been included in the mathematical analysis.

Two questions must now be asked regarding the approximate solution:

1) For what range of the parameters is the solution valid?

2) For what range of the parameters is the solution locally stable?

Consider the question of the region of validity. Assuming for reasons of convenience that the mass is rotating in the horizontal plane, equation (2.7) becomes

$$J\ddot{\theta} + \mathrm{K}(\dot{\theta} - \omega_s) = -mr\ddot{x}\cos\theta. \tag{2.13}$$

Substituting the approximate solution for x (equation (2.10)) and the assumed value of θ (equations (2.8)) into the right hand side of this equation, the value $\theta = \omega t + \psi$ into the left hand side, and choosing ω equal to its average value so that $K(\omega - \omega_s) = (1/2) m r \omega^2 A \sin\phi$ results in the equation

$$J\ddot{\psi} + K\dot{\psi} = \frac{1}{2} m r \omega^2 A \sin(2\omega t + \phi) \qquad (2.14)$$

where ψ is the "distortion" in the θ variable. This equation has a solution of the form

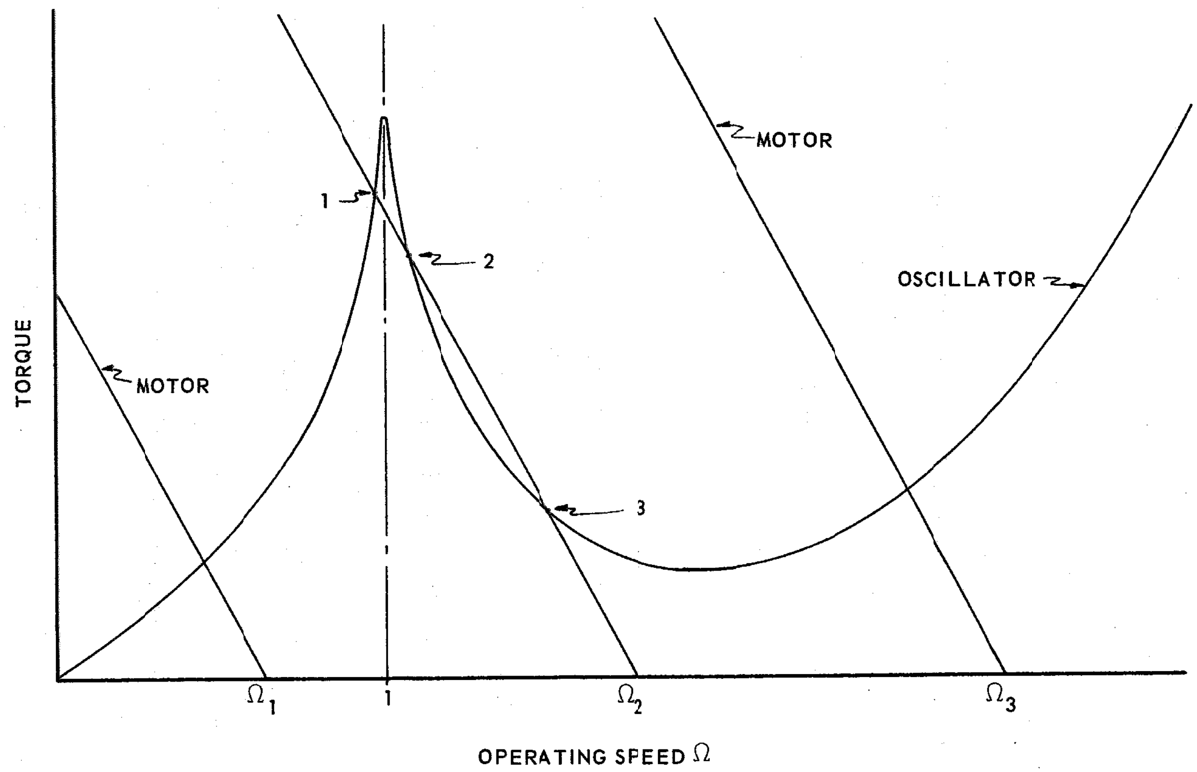
$$\psi = B \sin(2\omega t + \delta)$$

where

$$B = \frac{\left(\frac{1}{2} m r\right)^2 \left(\frac{\Omega^3}{M}\right)}{\sqrt{\left(\frac{K}{\omega_o}\right)^2 + (2J\Omega)^2}\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}} . \qquad (2.15)$$

The magnitude of B is the distortion amplitude on the θ variable. The original assumption that $\theta = \omega t$ implies that $B = 0$, so the approximate solution may be considered valid as long as $B \ll 1$. This is the primary test for the validity of the solution. The magnitude of B may be estimated by considering the worst of all possible situations, that one where Ω is identically one. At this point

$$\left|B(\Omega)\right|_{max} = B(1) = \frac{m}{M}\frac{m r^2}{J}\frac{1}{16\zeta\sqrt{1+\left(\frac{K}{2J\omega_o}\right)^2}} . \qquad (2.16)$$



OSCILLATOR AND MOTOR AVERAGE TORQUE CURVES

FIGURE 2.2

Local stability is examined by means of the speed-torque diagram of Figure 2.2. Consider first the case in which Ω_s is such that the motor speed-torque curve intersects the oscillator speed-torque curve at only one point, e.g. where $\Omega_s = \Omega_1$ and $\Omega_s = \Omega_3$. The operating speed is the Ω defined by the intersection of the motor and oscillator curves. If the operating speed Ω is perturbed by a small increase, $\Delta\Omega$, the oscillator requires more torque than the motor can supply and the operating speed tends to decrease back to Ω. Similarly, if the operating speed is decreased to $\Omega - \Delta\Omega$, the oscillator requires less torque than the motor is producing and the system tends to return to the operating speed Ω. Thus, the system tends to cancel any perturbations in Ω and is therefore stable.

Consider now the case shown in Figure 2.2 for $\Omega_s = \Omega_2$ in which the motor and oscillator curves intersect at three points. It is readily seen that small perturbations in the operating speeds at points 1 and 3 are cancelled in the same manner as for the single point of intersection when $\Omega_s = \Omega_3$. However, a small perturbation in Ω at point 2 has the opposite effect. A small perturbation tends to grow in magnitude rather than to decrease, and therefore the system at point 2 is unstable.

This analysis shows that an unstable region can exist for operating frequencies just above the system resonance frequency. Instability will not occur if the motor characteristic K, which is the magnitude of the slope of the motor speed-torque curve, is greater than the magnitude of the maximum negative slope of the oscillator speed-torque curve. Thus, the minimum value of K required for the complete stability of any given system may be obtained by determining the maximum negative slope of its oscillator speed-torque curve.

Perturbation Analyses:

Steady state solutions, to which small perturbations are added, are assumed in these perturbation analyses. The stability of the system is investigated by determining the behavior of these perturbations with respect to time. Defining x_o and θ_o to be the assumed steady state solutions of the differential equations of motion, and ξ and η to be small perturbations, the relations

$$x = x_o + \xi \quad \text{and} \quad \theta = \theta_o + \eta \tag{2.17}$$

are obtained. Substituting these relations into the differential equations of motion of the system, (2.6) and (2.7), gives the perturbed equations

$$\ddot{\xi} + 2\zeta\omega_o\dot{\xi} + \omega_o^2\xi = \frac{mr}{M}\left[(2\dot{\theta}_o\dot{\eta} + \ddot{\theta}_o\eta)\sin\theta_o + (\dot{\theta}_o^2\eta - \ddot{\eta})\cos\theta\right] \tag{2.18}$$

and

$$\ddot{\eta} + \frac{K}{J}\dot{\eta} - \frac{mr}{J}\left[\ddot{x}_o\sin\theta_o + g\cos\theta_o\right]\eta = -\frac{mr}{J}\ddot{\xi}\cos\theta_o . \tag{2.19}$$

To solve for the perturbed variables ξ and η a vector $\bar{z}$ is defined as

$$\bar{z} = \begin{Bmatrix} \xi \\ \dot{\xi} \\ \eta \\ \dot{\eta} \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} \tag{2.20}$$

Using this definition, the perturbed equations, (2.18) and (2.19), may be written in the form

$$\frac{d\bar{z}}{dt} = A(t)\bar{z} ; \quad A\left(t + \frac{2\pi}{\omega}\right) = A(t). \tag{2.21}$$

Equation (2.21) implies the existence of a solution of the form

$$\begin{aligned} \xi &= e^{\mu t}\phi(t) \\ \eta &= e^{\mu t}\psi(t) \end{aligned} \quad ; \quad \begin{array}{l} \phi(t),\ \psi(t) \text{ periodic} \\ \text{in } t \text{ with period} \\ 2\pi/\omega \text{ or } \pi/\omega, \end{array} \tag{2.22}$$

where μ, the characteristic exponent, is much less than unity. The values of $\phi(t)$ and $\psi(t)$ may be expressed in series form as

$$\begin{aligned} \phi(t) &= A\sin\omega t + B\cos\omega t + D\sin 2\omega t + E\cos 2\omega t + \ldots \\ \psi(t) &= C + F\sin\omega t + G\cos\omega t + \ldots \end{aligned} \tag{2.23}$$

The zero order solution is obtained by assuming a uniform rotational velocity for the unbalanced mass. In the first order solution a periodic component is superimposed on this uniform rotational velocity. These analyses are set forth below.

The zero order perturbation analysis assumes that

$$\theta_o = \omega t \quad \text{and} \quad x_o = \bar{A}\sin[\omega t + \phi(t)]. \tag{2.24}$$

This is the same steady state solution which is used in the torque-slope analysis, and the validity discussion used there applies here also.

Carrying the stability analysis to the same order as the assumed steady state solution gives

$$\phi(t) = A\sin\omega t + B\cos\omega t \quad \text{and} \quad \psi(t) = C \tag{2.25}$$

where A, B, and C are constants. Substituting the values of the perturbed variables into equations (2.18) and (2.19) respectively, and setting the coefficients of like trigonometric terms to zero, gives three algebraic equations. Two of these come from the sine and cosine coefficients of the first equation, and one comes from the constant coefficient of the second equation. The three equations are:

$$
\begin{aligned}
&\left(1 - \Omega^2 + 2\zeta \frac{\mu}{\omega_o}\right)A - 2\Omega\left(\zeta + \frac{\mu}{\omega_o}\right)B - 2\frac{mr\Omega}{M}\frac{\mu}{\omega_o}C = 0 \\
&2\Omega\left(\zeta + \frac{\mu}{\omega_o}\right)A + \left(1 - \Omega^2 + 2\zeta \frac{\mu}{\omega_o}\right)B - \frac{mr\Omega^2}{M}C = 0 \\
&\frac{\mu}{\omega_o}\Omega A - \frac{\Omega^2}{2}B + \left[\frac{J}{mr}\left(\frac{\mu}{\omega_o}\right)^2 + \frac{\kappa}{mr\omega_o}\left(\frac{\mu}{\omega_o}\right)\right. \\
&\qquad\left. + \frac{mr\Omega^4}{2M}\left(\frac{1 - \Omega^2}{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}\right)\right]C = 0 .
\end{aligned}
\tag{2.26}
$$

A non-trivial solution to this set of equations is guaranteed by setting the determinant of the coefficients of A, B, and C equal to zero. This determinant, when expanded, yields a fourth order equation in μ. However, the constant term is identically zero. Thus we have a zero root and a third order equation in μ. The zero root is expected, since the system is nonlinear, autonomous, and has a periodic solution[11]. Setting μ equal to zero in this third order equation gives the boundaries between the stable and the unstable regions. The equation for the stability boundaries is

$$\Delta = 0$$

where

$$
\begin{aligned}
\Delta = &\left[\frac{1 - \Omega^2}{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}\right]\left[(1 + \Omega^2)\frac{2\zeta mr\Omega^4}{M}\right] + \left[(1 - \Omega^2)^2\right. \\
&\left. + (2\zeta\Omega)^2\right]\left[\frac{\kappa}{mr\omega_o}\right] + \frac{3\zeta mr\Omega^4}{M} .
\end{aligned}
\tag{2.27}
$$

In pactice the value of Δ is computed for any given system as a function of Ω, and the zero crossings determine the stability boundaries. The value of Δ may be plotted as a function of Ω to give a qualitative idea of the degree of stability the system has at any given value of Ω.

A variation on this method of determining the stability boundaries is to assume that

$$\bar{z} = \begin{Bmatrix} \xi \\ \dot{\xi} \\ \eta \\ \dot{\eta} \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = e^{\mu t} \begin{Bmatrix} \beta_1 \sin \omega t + \alpha_1 \cos \omega t \\ \beta_2 \sin \omega t + \alpha_2 \cos \omega t \\ a_3 \\ a_4 \end{Bmatrix} \tag{2.28}$$

where again μ is very small compared to unity. The equation $d\bar{z}/dt = A(t)\bar{z}$ is expanded into a set of four equations using (2.28). In this process a point is reached at which $\dot{z}_2$ and $\dot{z}_4$ appear together in two of the equations. It is then possible to substitute directly the values of the z_i and $\dot{z}_i$, or to combine the two equations in such a manner as to produce two new equations, neither of which contains both $\dot{z}_2$ and $\dot{z}_4$. The values of the z_i and $\dot{z}_i$ are then substituted into the resulting equations. The remainder of the procedure follows closely that used before. The coefficients of the various trigonometric terms are set equal to zero independently. The parameters α_2, β_2, and a_4 are eliminated, leaving three algebraic equations involving α_1, β_1, and a_3. Setting the determinant of the coefficients of these parameters to zero guarantees a non-trivial solution. The components of this determinant depend upon which of the two techniques mentioned above is followed. The first technique produces a determinant which is identical to the one obtained by the previous method. The second technique yields a slightly different third row in the determinant. This difference results in part from neglecting $(m r)^2/MJ$ compared to unity, and in part because the equations used to form the determinant differ due to the separation of $\dot{z}_2$ and $\dot{z}_4$.

In the first order perturbation analysis, periodic variations in the θ variable are taken into account by assuming that

$$\theta = \omega t + \epsilon_1 \sin(\omega t + \delta_1) + \epsilon_2 \sin(2\omega t + \delta_2), \tag{2.29}$$

where the constants ϵ_1 and ϵ_2 are small. θ may be written in the equivalent form

$$\theta = \omega t + e_1 \sin\omega t + d_1 \cos\omega t + e_2 \sin 2\omega t + d_2 \cos 2\omega t \tag{2.30}$$

where $d_1 = \epsilon_1 \sin\delta_1$, $e_1 = \epsilon_1 \cos\delta_1$,

$d_2 = \epsilon_2 \sin\delta_2$, $e_2 = \epsilon_2 \cos\delta_2$.

Computational difficulties caused by the $\sin\theta$ and $\cos\theta$ terms are resolved by using the Jacobi-Anger relation[12]. This relation expresses $e^{i z \sin z}$ as a series of Bessel Functions. Using the Bessel Function expansion for small arguments, obtain

$$\sin\theta = \frac{d_1}{2} + \left(1 + \frac{e_2}{2}\right) \sin\omega t + \frac{d_2}{2} \cos\omega t + \frac{e_1}{2} \sin 2\omega t$$
$$+ \frac{d_1}{2} \cos 2\omega t + \frac{e_2}{2} \sin 3\omega t + \frac{d_2}{2} \cos 3\omega t + 0(\epsilon^2) \tag{2.31}$$

and

$$\cos\theta = -\frac{e_1}{2} + \frac{d_2}{2}\sin\omega t + \left(1 - \frac{e_2}{2}\right)\cos\omega t - \frac{d_1}{2}\sin 2\omega t$$

$$+ \frac{e_1}{2}\cos 2\omega t - \frac{d_2}{2}\sin 3\omega t + \frac{e_2}{2}\cos 3\omega t + 0(\epsilon^2). \qquad (2.32)$$

Substituting (2.30) into equation (2.6) and using (2.31) and (2.32) gives an approximate equation of motion which has the solution

$$x = A_1 \sin\omega t + B_1 \cos\omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t$$

$$+ A_3 \sin 3\omega t + B_3 \cos 3\omega t, \qquad (2.33)$$

where

$$A_1 = \frac{mr\Omega^2}{M}\left[\frac{(1-\Omega^2)\left(1+\frac{1}{2}e_2\right) + (2\zeta\Omega)\frac{1}{2}d_2}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}\right],$$

$$B_1 = \frac{mr\Omega^2}{M}\left[\frac{(1-\Omega^2)\frac{1}{2}d_2 - (2\zeta\Omega)\left(1+\frac{1}{2}e_2\right)}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}\right],$$

$$A_2 = \frac{2mr\Omega^2}{M}\left[\frac{(1-4\Omega^2)e_1 + (4\zeta\Omega)d_1}{(1-4\Omega^2)^2 + (4\zeta\Omega)^2}\right],$$

$$B_2 = \frac{2mr\Omega^2}{M}\left[\frac{(1-4\Omega^2)d_1 - (4\zeta\Omega)e_1}{(1-4\Omega^2)^2 + (4\zeta\Omega)^2}\right],$$

$$A_3 = \frac{9mr\Omega^2}{2M}\left[\frac{(1-9\Omega^2)e_2 + (6\zeta\Omega)d_2}{(1-9\Omega^2)^2 + (6\zeta\Omega)^2}\right],$$

$$B_3 = \frac{9mr\Omega^2}{2M}\left[\frac{(1-9\Omega^2)d_2 - (6\zeta\Omega)e_2}{(1-9\Omega^2)^2 + (6\zeta\Omega)^2}\right].$$

The range of validity of this approximate solution may be examined by rewriting equation (2.7) in the form

$$J\ddot{\theta} + K(\dot{\theta} - \omega_s) = mr(g\sin\theta - \ddot{x}\cos\theta) \tag{2.34}$$

and substituting the approximate values of x and θ from (2.30) and (2.33), denoted here as x_o and θ_o, into the right hand side. If we let $\theta = \theta_o + \psi$ in the left hand side, the amplitude of ψ will be the primary test of the validity of the approximate solution. As long as the magnitude of ψ is much smaller than ϵ_1 or ϵ_2, the solution is valid. Taking ψ to contain only components with frequencies higher than those included in the approximate solution for θ, this procedure gives values for d_1, e_1, d_2, and e_2 as functions of the other parameters of the system.

The examination of the local stability of the approximate solution follows the second method described in the zero order analysis. Starting from the same set of perturbed equations, (2.18) and (2.19), and using (2.20), it is assumed that

$$\bar{z} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = e^{\mu t}\begin{Bmatrix} \beta_1 \sin\omega t + \alpha_1 \cos\omega t + \psi_1 \sin 2\omega t + \phi_1 \cos 2\omega t \\ \beta_2 \sin\omega t + \alpha_2 \cos\omega t + \psi_2 \sin 2\omega t + \phi_2 \cos 2\omega t \\ a_3 + \beta_3 \sin\omega t + \alpha_3 \cos\omega t \\ a_4 + \beta_4 \sin\omega t + \alpha_4 \cos\omega t \end{Bmatrix} \tag{2.35}$$

This value of $\bar{z}$ is substituted into the perturbed equation (2.21). The resulting vector equation is divided into its four component scalar equations, and the coefficients of like trigonometric terms are set equal to zero in each equation. In doing this the relations $\dot{z}_1 = z_2$ and $\dot{z}_3 = z_4$ are used, all terms of $0(\epsilon^2)$ are neglected, and it is assumed that $|\mu/\omega_o| \ll \Omega$. This procedure results in a set of seven equations involving the "variables" α_1, β_1, ϕ_1, ψ_1, a_3, α_3, β_3, and coefficients made up of the system parameters.

Again, setting the determinant of the coefficients equal to zero guarantees a non-trivial solution. As before, the "stability determinant" is expected to have a zero root. Consequently, the constant term in the determinant expansion should be zero. It turns out that the root is not identically zero because of the approximate nature of the solution, but it is of second order. Thus the stability of the system is again governed by the coefficient of the μ term in the determinant expansion. This coefficient is obtained by expanding the 7 X 7 determinant, and retaining only those terms which involve the first power of μ. All terms involving powers of ϵ higher than the first are also neglected.

This procedure produces an expression which, when set equal to zero, determines the values of the system parameters for which a solution exists. Because of its complexity, this expression was programmed for solution on a digital computer. The actual computations were performed by specifying a particular physical system and then sweeping Ω through the regions of interest.

Floquet Analysis:

This analysis is based on the same steady state solution that was used in the first order perturbation analysis. The stability of the system described by equations (2.6) and (2.7) is examined by considering the perturbed equations in the form used above, namely

$$\frac{d\bar{z}}{dt} = A(t)\bar{z} \ ; \qquad A\left(t + \frac{2\pi}{\omega}\right) = A(t) \, . \tag{2.21}$$

Equation (2.21) is a set of linear first order differential equations with periodic coefficients of period $T = 2\pi/\omega$. G. Floquet has shown[11] that for a set of equations of this type there is at least one non-zero $\bar{z}$ such that

$$\bar{z}(t + T) = \lambda \bar{z}(t) \ ; \qquad \lambda \neq 0 \, . \tag{2.36}$$

The number λ is called a characteristic factor of the system (2.21), and may be real or complex.

Because the unperturbed system is autonomous and is known to have a periodic solution of period T, the perturbed equation (2.21) will also have a solution of period T[11]. This implies that $\lambda = 1$ is one of the characteristic factors of the perturbed system. The following theorem, attributed to A. Liapunov[11], defines the stability of the unperturbed system in terms of the characteristic factors of the perturbed system. In the notation used above, the theorem states: If all but one of the characteristic factors of the perturbed equation (2.21) have magnitude less than unity, then the periodic solution of the unperturbed system has asymptotic orbital stability and asymptotic phase (Poincaré stability).

To apply this theorem to the problem under discussion, consider the fundamental set of solutions of equation (2.21), $Z(t)$, where $Z(0) = I$, the identity matrix. These solutions satisfy the equation

$$\frac{dZ}{dt} = A(t)Z , \tag{2.37}$$

and also

$$Z(t + T) = CZ(t) , \tag{2.38}$$

where C is a constant matrix. Setting $t = 0$ in (2.38) gives

$$Z(T) = C\mathrm{I} \tag{2.39}$$

and therefore

$$C = Z(T) .$$

The eigenvalues of the $Z(T)$ matrix are the characteristic factors of the system (2.21). Thus, by the theorem stated above, the unperturbed system will be stable if all but one of the eigenvalues have magnitude less than unity. The remaining eigenvalue will be equal to unity for the reasons stated above.

The matrix $Z(T)$ is obtained by integrating equation (2.37) numerically from $t = 0$ to $t = T$ with $Z(0) = \mathrm{I}$. The eigenvalues of $Z(T)$ are then computed using standard techniques to determine the stability of the system at the point in question. Because of the length and complexity of this operation, this analysis is feasible only if a digital computer is available to perform the computations.

The primary advantage of this Floquet analysis is that as long as the assumed steady state solution is valid and accurate, any variations from this solution will appear as an instability. The primary disadvantage is that the analysis must be carried out numerically for each individual case.

Stability Analysis Results:

Any discussion of the results obtained from the analyses presented above must consider three separate points: 1) The region of validity of each analysis; 2) the relative advantages and disadvantages of each type of analysis; and 3) the numerical results obtained from each analysis. The discussion of the region of validity is simplified by using the classification of the analyses developed above, which is on the basis of the steady state solution that is assumed in each. Those analyses which assume a constant angular velocity for the unbalanced mass ($\theta = \omega t$), namely the torque-slope analysis and the zero order perturbation analysis, are zero order analyses. The first order perturbation analysis and the Floquet analysis, which are based on a non-uniform angular velocity of the unbalanced mass of the form $\theta = \omega t + \epsilon_1 \sin(\omega t + \delta_1) + \epsilon_2 \sin(2\omega t + \delta_2)$ are first order analyses.

The criterion for the validity of the zero order analyses is developed in the discussion of the torque-slope analysis. It states that the magnitude of the distortion amplitude $B(\Omega)$ must be much less than unity, i.e.

$$\left| B(\Omega) \right| \ll 1.0 \tag{2.40}$$

where

$$\left| B(\Omega) \right|_{max} = \frac{m}{M} \frac{m r^2}{J} \frac{1}{16\zeta} \frac{1}{\sqrt{1 + \left(\frac{\kappa}{2 J \omega_o}\right)^2}} \, . \tag{2.16}$$

For computational purposes, inequality (2.4) has been taken to mean

$$\left| B(\Omega) \right| < 0.1 \, . \tag{2.41}$$

A conservative estimate of the boundary of the region of validity may be obtained by substituting $\left|B(\Omega)\right|_{max}$ for $\left|B(\Omega)\right|$ in the inequality (2.41), and defining the boundary as

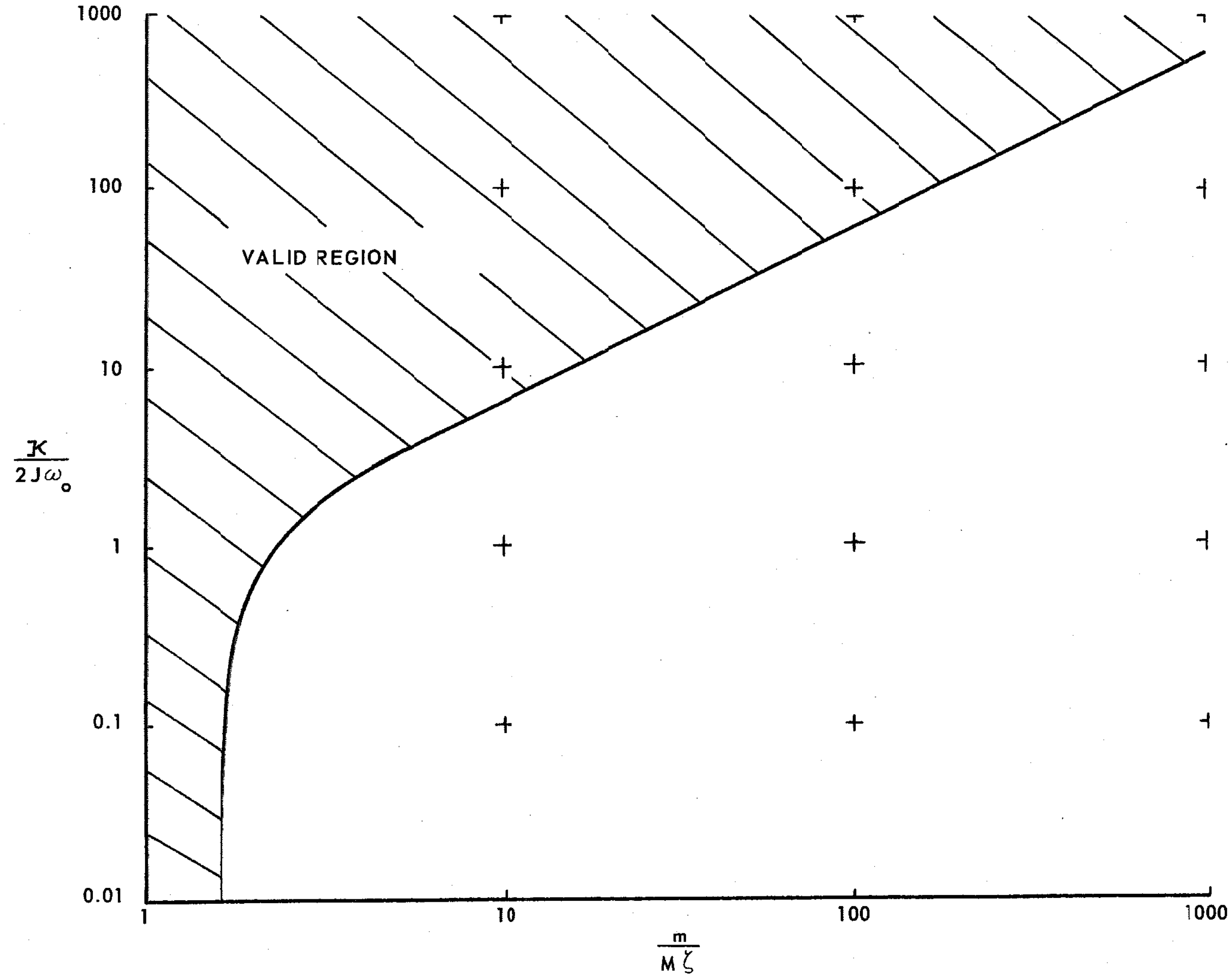
$$\left|B(\Omega)\right|_{max} = 0.1 \; . \tag{2.42}$$

It is convenient to set the value of the non-dimensional parameter $m\,r^2/J$ equal to its maximum possible value, unity. This is justified on the grounds that it gives a conservative estimate of the boundary of the region of validity and that, for the cases investigated in this thesis, this parameter had no effect on the stability of the system. Substituting this value of $m\,r^2/J$ into equation (2.16), substituting the result into equation (2.42) and rearranging the terms, results in the equation

$$\frac{m}{M\zeta} = 1.6\sqrt{1 + \left(\frac{\kappa}{m\,r^2\,\omega_o}\right)^2} \tag{2.43}$$

which defines the boundary of the region of validity for the zero order analyses. A plot of this boundary, with $m/M\zeta$ and $\kappa/m\,r^2\,\omega_o$ as coordinates is presented in Figure 2.3.

The validity of the first order analyses is also determined by computing the magnitude of a "distortion amplitude," the computation of which is a long and tedious process. If the first order analyses prove to be invalid in any given case, a new analysis based on a more complete steady state solution must be developed. Most cases of practical interest today will lie within the region of validity of either the zero or first order analyses presented above.



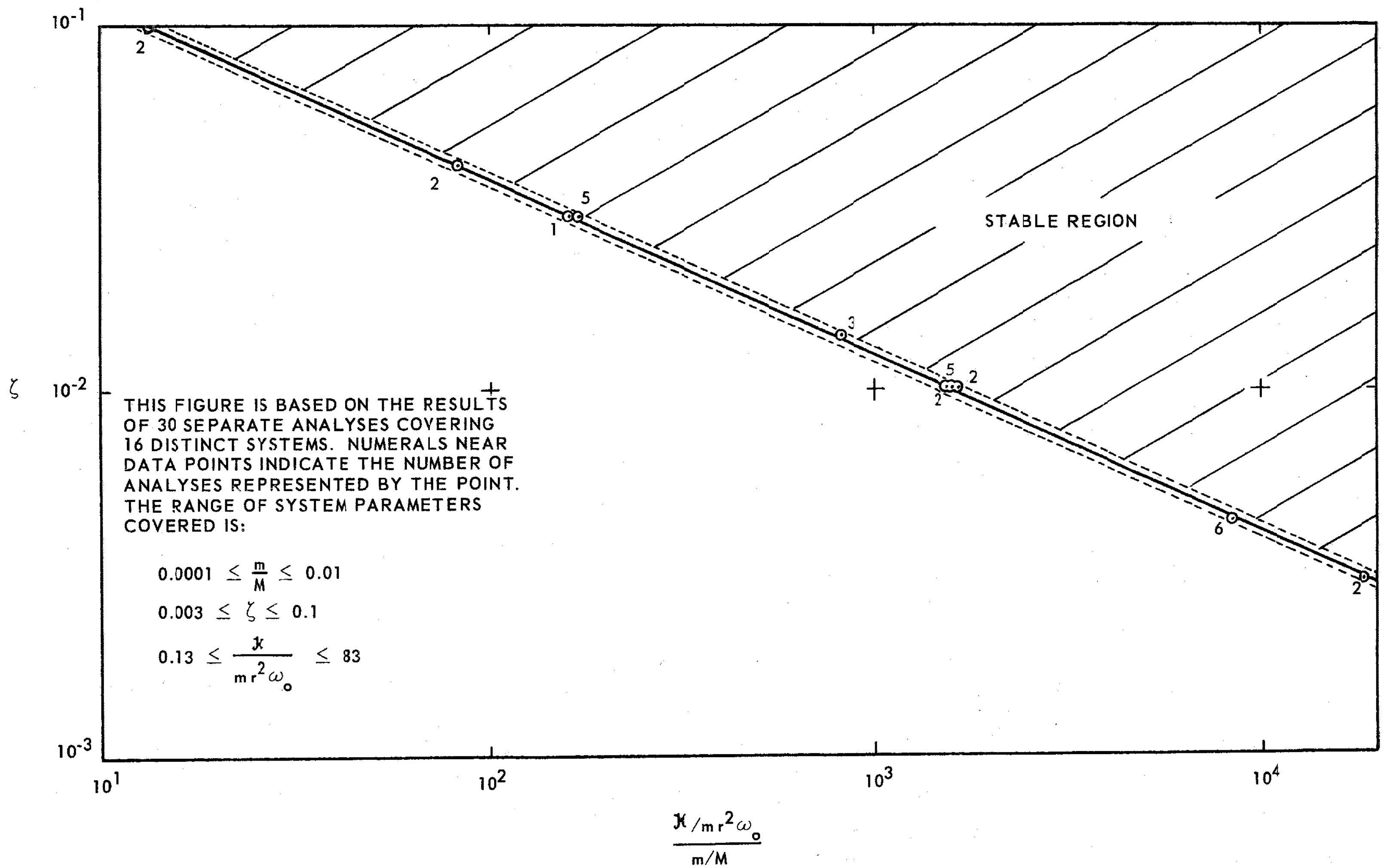
RANGE OF VALIDITY OF THE ZERO ORDER SINGLE-DEGREE-OF-FREEDOM ANALYSES

FIGURE 2.3

It is possible that the Floquet analysis may not be valid for certain cases even though the "distortion amplitude" test indicates validity. This lack of validity is characterized by the failure of the first eigenvalue of the solution matrix to hold constant at unity. This complication appears to be due to the facts that the generating solution is not exact, the numerical method of solution introduces some approximations, and the changes being observed in the eigenvalue are of magnitude less than 0.1%. It was found that this problem becomes significant for values of m/M greater than 0.01, which is not common in test situations.

The type of analysis required for the investigation of the stability of any given system is determined by the parametric values of that system. Where applicable, Figure 2.4 may be used directly. Otherwise, a zero or first order analysis must be selected. Whenever it is valid, a zero order analysis is preferable over a first order analysis because of its relative simplicity and ease of application. The choice of which zero order analysis to use is a matter of personal preference. If a first order analysis is indicated and deemed desirable, a digital computer based analysis must be employed. The first order perturbation solution requires significantly less machine time than the Floquet analysis and is therefore more desirable.

The numerical results obtained using the various analyses are shown in Figure 2.4 with ζ and $(\kappa/m r^2 \omega_o)/(m/M)$ as coordinates. The parameter $m r^2/J$ was varied over the range 0.1 to 1.0 and had no significant effect on the stability of the cases studied. The other parameters were varied over the following ranges: m/M from 0.0001 to 0.01; ζ from 0.003 to 0.1; and $\kappa/m r^2 \omega_o$ from 0.13 to 83. The results of any two analyses are considered to agree if the values of all the parameters under consideration



STABILITY REGION FOR SINGLE-DEGREE-OF-FREEDOM SYSTEMS

FIGURE 2.4

agree to two significant figures. Using this criterion the speed-torque analysis results agree with the zero order perturbation analysis results. The zero and first order perturbation analysis results likewise agree in the region in which they are both valid. Differences occur between the first order perturbation analysis results and the Floquet analysis results when the eigenvalues of the solution matrices in the Floquet analysis are close to the limits of validity described above. These differences increase as the mass ratio m/M becomes larger than 0.01. Within the stated limits of validity, the various analyses agree to within one unit in the second significant figure.

The maximum distribution of 15% found in the data points of Figure 2.4 can be attributed to the fact that the parameters were computed to only two significant figures. The dashed lines in Figure 2.4 on each side of the solid line bound the region of the mean value of $(\varkappa/m r^2 \omega_o)/(m/M) \pm 10\%$. This is the maximum variation that arises from computing the parameters to only two significant figures. All of the valid data points obtained from the various analyses lie within this region.

It is evident from Figure 2.4 that the stability of a system is increased when either ζ or $(\varkappa/m r^2 \omega_o)/(m/M)$ is increased. Since the damping is usually a fixed parameter of the system, the easiest and most practical method of increasing a system's stability is to decrease the size of the unbalanced mass m, which also decreases the vibration generator output force. A second alternative is to increase the magnitude of the motor characteristic $\varkappa$, which may be done by changing motors, changing drive gear ratios, or by the use of a servo control system. Further measures would require a redesign of the vibration generator with a choice of $\varkappa$ such that all desired combinations of m/M, ω_o, and ζ lie in the stable operating region. The use of some type of servo speed control system, such as is used in the machines developed at the California Institute of Technology*, will be needed for systems with a broad range of requirements.

* Described in Appendix II.

III MULTI-DEGREE-OF-FREEDOM SYSTEMS

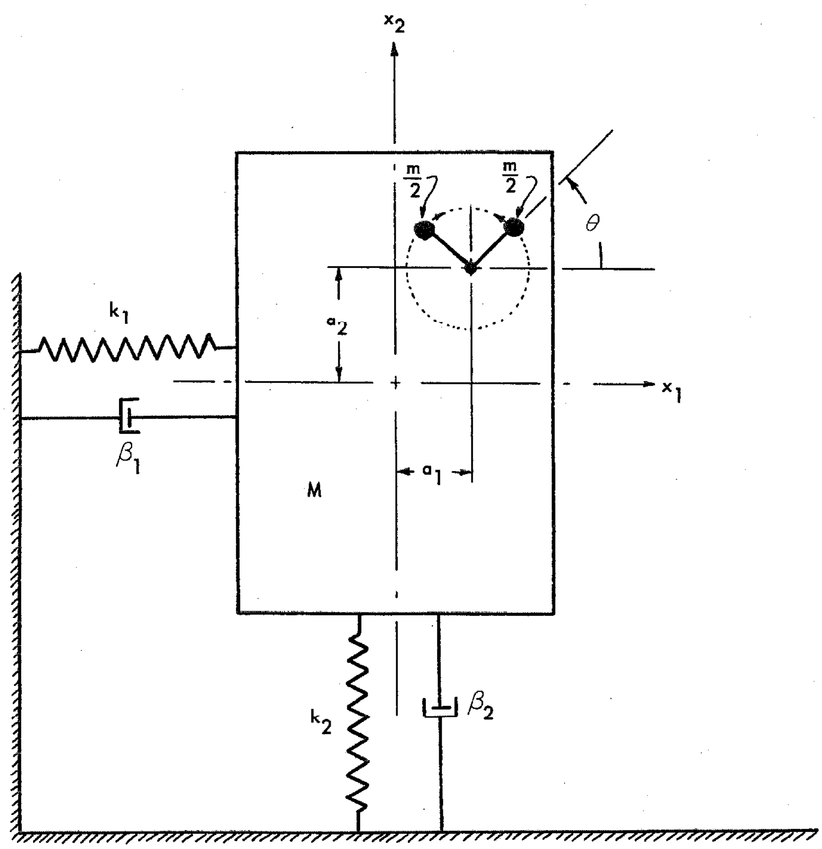
The single-degree-of-freedom systems discussed in the preceding chapter can be expanded into a multi-degree-of-freedom system in several ways. One way is to allow the oscillator mass to move in two or more directions. Another is to combine a group of simple oscillators to form a multiple mass system. Examples of both these types of systems are discussed below.

Single Oscillator Systems:

One logical extension of the single-degree-of-freedom system is to allow the oscillator mass to move in a plane. Such motion is two dimensional and has three degrees of freedom, two translational and one rotational. The differential equations of motion can be found by using a method similar to the one used for the single-degree-of-freedom system. However, the equations are so unwieldy that it is necessary to make a number of simplifying assumptions in order to solve them. These assumptions limit the validity of the solutions to a small number of cases of limited value.

A similar, often used system, which is more easily analyzed in a general manner, is shown in Figure 3. 1. The unbalanced mass exciter in this system uses two equal contra-rotating unbalanced masses to produce a uniaxial force.

As was done in the single-degree-of-freedom case, the differential equations of motion were obtained by using Lagrange's Equation (1. 1) where the kinetic energy is given by



SINGLE OSCILLATOR WITH CONTRA-ROTATING MASSES

FIGURE 3.1

$$T = \frac{1}{2} M' (\dot{x}_1^{\,2} + \dot{x}_2^{\,2}) + \frac{1}{2} m \Big[\dot{\theta}^2 r^2 + 2\dot{x}_2 \dot{\theta} r \cos\theta - 2\dot{x}_1 \dot{x}_3 (a_1 \sin x_3$$

$$+ a_2 \cos x_3) + 2\dot{x}_2 \dot{x}_3 (a_1 \cos x_3 - a_2 \sin x_3) + 2\dot{\theta} r \cos\theta \, \dot{x}_3 (a_1 \cos x_3$$

$$- a_2 \sin x_3) + \dot{x}_3^{\,2} (a_1^{\,2} + a_2^{\,2}) \Big] + \frac{1}{2} J' \dot{\theta}^2 + \frac{1}{2} J'' \dot{x}_3^{\,2} \, , \tag{3.1}$$

the potential energy is given by

$$V = \frac{1}{2} (k_1 x_1^{\,2} + k_2 x_2^{\,2} + k_3 x_3^{\,2}) \, , \tag{3.2}$$

and the Rayleigh Dissipation Function for the system is

$$D = \frac{1}{2} (\beta_1 \dot{x}_1^{\,2} + \beta_2 \dot{x}_2^{\,2} + \beta_3 \dot{x}_3^{\,2} + \kappa \dot{\theta}^2 - 2\kappa \dot{\theta} \omega_s) \, . \tag{3.3}$$

Applying Lagrange's Equation with these values for T, V, and D gives the following differential equations of motion for the system:

$$M\ddot{x}_1 + \beta_1 \dot{x}_1 + k_1 x_1 = m \Big[\ddot{x}_3 (a_1 \sin x_3 + a_2 \cos x_3)$$

$$+ \dot{x}_3^{\,2} (a_1 \cos x_3 - a_2 \sin x_3) \Big] \, , \tag{3.4}$$

$$M\ddot{x}_2 + \beta_2 \dot{x}_2 + k_2 x_2 = m \Big[r\omega^2 \sin\omega t - \ddot{x}_3 (a_1 \cos x_3 - a_2 \sin x_3)$$

$$+ \dot{x}_3^{\,2} (a_1 \sin x_3 + a_2 \cos x_3) \Big] \, , \tag{3.5}$$

$$\Upsilon \ddot{x}_3 + \beta_3 \dot{x}_3 + k_3 x_3 = m \Big\{ \Big[\ddot{x}_1 + \dot{x}_3 (x_2 + \omega r \cos\omega t) \Big] \Big[a_1 \sin x_3$$

$$+ a_2 \cos x_3 \Big] - (\ddot{x}_2 - r\omega^2 \sin\omega t)(a_1 \cos x_3$$

$$- a_2 \sin x_3) \Big\} \, , \tag{3.6}$$

$$J\ddot{\theta} + K(\dot{\theta} - \omega_s) + m\,r\cos\theta\Big[\ddot{x}_2 + \ddot{x}_3(a_1\cos x_3 - a_2\sin x_3)$$

$$- \dot{x}_3^{\,2}(a_1\sin x_3 + a_2\cos x_3)\Big] + m\,r\sin\theta\Big[-\ddot{x}_1 + \ddot{x}_3(a_1\sin x_3$$

$$+ a_2\cos x_3) + \dot{x}_3^{\,2}(a_1\cos x_3 - a_2\sin x_3)\Big] = 0\,, \tag{3.7}$$

where

$$M = M' + m\,,$$

$$J = J' + m\,r^2\,,$$

$$\Upsilon = J'' + m(a_1^{\,2} + a_2^{\,2})\,.$$

Assuming a zero order solution to equation (3.7) of the form

$$\theta = \omega t\,, \tag{3.8}$$

where ω is a constant, gives the approximate solutions to equations (3.5), (3.6), and (3.7):

$$\begin{aligned} x_1 &= A_1\sin(\omega t + \phi_1)\,, \\ x_2 &= A_2\sin(\omega t + \phi_2)\,, \\ x_3 &= A_3\sin(\omega t + \phi_3)\,. \end{aligned} \tag{3.9}$$

For small rotational amplitudes ($x_3 \ll 1$), substituting equations (3.9) into (3.4), (3.5), and (3.6), and setting the coefficients of $\sin\omega t$ and $\cos\omega t$ equal to zero separately gives the set of equations:

$$\left[(k_1 - M\omega^2)\cos\phi_1 - \beta_1\omega\sin\phi_1\right]A_1 + \left[m a_2 \omega^2 \cos\phi_3\right]A_3 = 0,$$

$$\left[(k_1 - M\omega^2)\sin\phi_1 + \beta_1\omega\cos\phi_1\right]A_1 + \left[m a_2 \omega^2 \sin\phi_3\, A_3\right] = 0,$$

$$\left[(k_2 - M\omega^2)\cos\phi_2 - \beta_2\omega\sin\phi_2\right]A_2 + \left[-m a_1 \omega^2 \cos\phi_3\right]A_3 = -m r \omega^2,$$

$$\left[(k_2 - M\omega^2)\sin\phi_2 + \beta_2\omega\cos\phi_2\right]A_2 + \left[-m a_1 \omega^2 \sin\phi_3\right]A_3 = 0, \qquad (3.10)$$

$$\left[m a_2 \omega^2 \cos\phi_1\right]A_1 - \left[m a_1 \omega^2 \cos\phi_2\right]A_2 + \Big[(k_3$$
$$\Big. + \Upsilon\omega^2)\cos\phi_3 - \beta_3\omega\sin\phi_3\Big]A_3 = m a_1 r \omega^2,$$

$$\left[m a_2 \omega^2 \sin\phi_1\right]A_1 - \left[m a_1 \omega^2 \sin\phi_2\right]A_2 + \Big[(k_3$$
$$\Big. + \Upsilon\omega^2)\sin\phi_3 + \beta_3\omega\cos\phi_3\Big]A_3 = 0 .$$

In theory these equations may be solved for A_1, A_2, A_3, ϕ_1, ϕ_2, and ϕ_3. However, in practice this procedure can be extremely involved.

The validity of the solution (3.9) is checked by examining the motor equation, (3.7). Using the same type of analysis as was used in the single-degree-of-freedom discussion, the expression for the distortion amplitude B is found to be

$$B = -\frac{1}{4} m r \omega \left\{ \left[A_2 (2 J \omega \sin\phi_2 + \kappa \cos\phi_2) + a_1 A_3 (2 J \omega \sin\phi_3 + \kappa \cos\phi_3) \right]^2 + \left[2 J \omega - \kappa \right]^2 \left[A_2 \sin\phi_2 + a_1 A_3 \sin\phi_3 \right]^2 \right\} \Bigg/ \left\{ \kappa^2 + \frac{(2 J \omega)^2 (A_2 \sin\phi_2 + a_1 A_3 \sin\phi_3)}{A_2 \cos\phi_2 + a_1 A_3 \cos\phi_3} \right\} . \tag{3.11}$$

The solution is considered valid if $B \ll 1$.

The special case in which the center of mass of the oscillator lies on the thrust axis of the unbalanced mass exciter ($a_1 = 0$) is of interest because of the form taken by the resulting simplified equations of motion. These equations are:

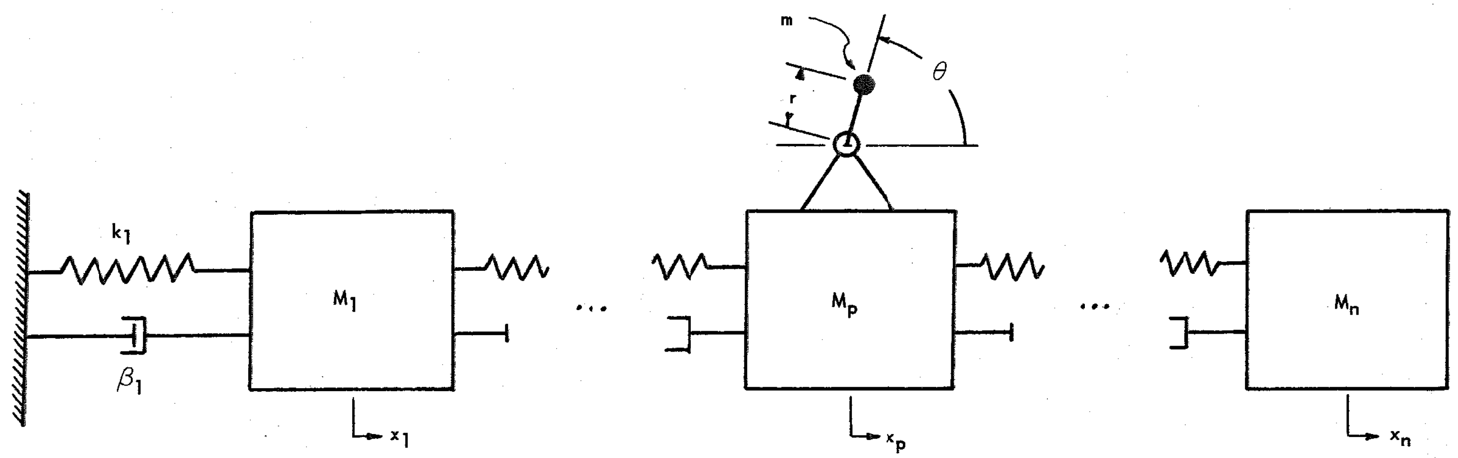
$$\begin{aligned} M\ddot{x}_1 + \beta_1 \dot{x}_1 + k_1 x_1 &= m a_2 (\ddot{x}_3 \cos x_3 - \dot{x}_3^2 \sin x_3) , \\ M\ddot{x}_2 + \beta_2 \dot{x}_2 + k_2 x_2 &= m r (\dot{\theta}^2 \sin\theta - \ddot{\theta} \cos\theta) + m a_2 (\ddot{x}_3 \sin x_3 - \dot{x}_3^2 \cos x_3) , \\ \Upsilon \ddot{x}_3 + \beta_3 \dot{x}_3 + k_3 x_3 &= \frac{d}{dt} \left[m a_2 \dot{x}_1 \cos x_3 + m (\dot{x}_3 + r \dot{\theta} \cos\theta) a_2 \sin x_3 + m a_2 \dot{x}_1 \dot{x}_3 \sin x_3 \right] , \\ J\ddot{\theta} + \kappa (\dot{\theta} - \omega_s) + m r \cos\theta \left[\ddot{x}_2 - a_2 \ddot{x}_3 \sin x_3 - \dot{x}_3^2 a_2 \cos x_3 \right] &= 0 . \end{aligned} \tag{3.12}$$

Examination of these equations shows that x_1 has a non-zero value only if x_3 has a non-zero value. Likewise x_3 has a non-zero value only if the initial values of x_1 and/or x_3 are non-zero. Thus, if the unbalanced mass exciter-oscillator system is started from rest, only the x_2 and θ variables have non-zero values, and the set of equations (3. 12) reduces directly to the equations describing the single-degree-of-freedom system.

Multiple Oscillator Systems:

Systems made up of a series of linear single-degree-of-freedom oscillators are of particular interest because they may be used as approximate representations of large complex structures. Figure 3. 2 shows a general multiple linear oscillator system.

Kononenko[(3)] has suggested that in certain situations the stability of multiple oscillator systems may be examined using an "equivalent" single-degree-of-freedom analysis. The reasoning behind this suggestion becomes apparent when the equations of motion are written in modal form. Each modal equation corresponds to a single-degree-of-freedom oscillator with a resonance frequency equal to the modal resonance frequency, and to a forcing function on the right hand side that is a function of the operating frequency. Unless the operating frequency is near the modal resonance frequency, the modal response, to the first approximation, is negligible. Thus, for a system with well separated resonances operating near one resonance frequency, only the motor equation and the appropriate modal equation need to be considered, in the first approximation. These two equations form the "equivalent" single-degree-of-freedom system, the stability of which may be studied using an appropriate single-degree-of-freedom analysis. This is, in brief, Kononenko's approach to the problem of multi-degree-of-freedom systems with well separated resonances.



MULTIPLE LINEAR OSCILLATOR SYSTEM

FIGURE 3.2

In the following analysis the general equations of motion for a multi-degree-of-freedom linear oscillator are developed in the modal form. A stability analysis of a particular three-oscillator system with equal masses, and with the vibration generator mounted on the second mass, is developed in detail. The results of the analysis are compared with the "equivalent" single-degree-of-freedom analysis as a check on Kononenko's method. The parameters and notation are the same as before, with the additions that double subscript refers to a matrix element, the single subscript p refers to the oscillator on which the vibration generator is mounted, the brackets [ ] indicate a matrix, and the braces { } indicate a vector.

As before, the equations of motion for the system are obtained from Lagrange's Equation (1.1). The kinetic energy is now

$$T = \frac{1}{2} M_1 \dot{x}_1^{\,2} + \frac{1}{2} M_2 \dot{x}_2^{\,2} + \dots + \frac{1}{2} M'_m \dot{x}_m^{\,2} + \dots + \frac{1}{2} M_n \dot{x}_n^{\,2}$$
$$+ \frac{1}{2} m \left((\dot{x}_m + \dot{\theta} r \cos\theta)^2 + (\dot{\theta} r \sin\theta)^2 \right), \tag{3.13}$$

the potential energy is

$$V = \frac{1}{2} k_1 x_1^{\,2} + \frac{1}{2} k_2 (x_2 - x_1)^2 + \dots + \frac{1}{2} k_n (x_n - x_{n-1})^2 , \tag{3.14}$$

and the Rayleigh Dissipation Function is

$$D = \frac{1}{2} \beta_1 \dot{x}_1^{\,2} + \frac{1}{2} \beta_2 (\dot{x}_2 - \dot{x}_1)^2 + \dots + \frac{1}{2} \beta_n (\dot{x}_n - \dot{x}_{n-1})^2$$
$$+ \frac{1}{2} \kappa \dot{\theta}^2 - \kappa \omega_s \dot{\theta} . \tag{3.15}$$

The equations of motion are

$$M_i \ddot{x}_i + (\beta_i + \beta_{i+1})\dot{x}_i$$

$$+ (k_i + k_{i+1})x_i = \beta_i \dot{x}_{i-1} + \beta_{i+1}\dot{x}_{i+1} + k_i x_{i-1} + k_{i+1}x_{i+1}$$

$$+ \delta(i, p)\Big(m r(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta)\Big) \qquad (3.16)$$

and

$$J\ddot{\theta} + \kappa(\dot{\theta} - \omega_s) + m r \ddot{x}_p \cos\theta = 0 ; \qquad (3.17)$$

where

$\delta(i, p) = 0 \quad$ for $i \neq p$

$\delta(i, p) = 1 \quad$ for $i = p$

and

$i = 1, 2, \ldots, n.$

The equation of motion (3.16) may be written in the form

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} . \qquad (3.18)$$

Premultiplying (3.18) by $[M]^{-1}$ gives

$$\{\ddot{x}\} + [M]^{-1}[C]\{\dot{x}\} + [M]^{-1}[K]\{x\} = [M]^{-1}\{f(t)\} \qquad (3.19)$$

where $[M]^{-1}[C]$ and $[M]^{-1}[K]$ are symmetric, positive definite, and differ only by a multiplicative constant. Defining

$$\{x\} = [\tau]\{\alpha\} , \qquad (3.20)$$

where τ is composed of the normalized eigenvectors of $[M]^{-1}[K]$, substituting

(3.20) into (3.19) and premultiplying the resulting equation by the transpose of τ gives

$$\{\ddot{\alpha}\} + [D]\{\dot{\alpha}\} + [E]\{\alpha\} = \{F(t)\}. \tag{3.21}$$

This equation has a solution

$$\{\alpha\} = \begin{Bmatrix} A_1 \sin t + B_1 \cos t \\ \vdots \\ A_i \sin t + B_i \cos t \\ \vdots \\ A_n \sin t + B_n \cos t \end{Bmatrix} \tag{3.22}$$

where

$$\begin{aligned} A_i &= \frac{\tau_{2i} m r \omega^2}{M} \frac{E_{ii} - \omega^2}{(E_{ii} - \omega^2)^2 + (D_{ii}\omega)^2}, \\ B_i &= -\frac{\tau_{2i} m r \omega^2}{M} \frac{D_{ii}\omega}{(E_{ii} - \omega^2)^2 + (D_{ii}\omega)^2} \end{aligned} \tag{3.23}$$

Differentiating equation (3.20) twice with respect to time yields the relation

$$\ddot{x}_2 = \sum_{i=1}^{n} \tau_{2i}\ddot{\alpha}_i \tag{3.24}$$

which, when substituted into equation (3.17) gives

$$\ddot{\theta} + \frac{K}{J}(\dot{\theta} - \omega_s) + \frac{mr}{J}\left(\sum_{i=1}^{n} \tau_{2i}\ddot{\alpha}_i\right)\cos\theta = 0. \tag{3.25}$$

The perturbed equations of motion for the system are obtained by defining

$$\begin{aligned} \alpha_i &= \beta_i + \xi_i \\ \theta &= \theta_o + \eta \end{aligned} \tag{3.26}$$

where β_i and θ_o are the assumed steady state solutions, and ξ and η are small perturbations. Following a procedure similar to the one used in the single-degree-of-freedom case gives the perturbed equations

$$\ddot{\xi}_i + D_{ii}\dot{\xi}_i + E_{ii}\xi_i = \tau_{2i}\left(\frac{mr}{M}\right)(2\dot{\theta}_o\dot{\eta}\sin\theta_o + \dot{\theta}_o^{\,2}\eta\cos\theta_o);$$

$$i = 1, 2, \ldots, n\,. \tag{3.27}$$

and

$$\ddot{\eta} + \frac{K}{J}\dot{\eta} - \frac{mr}{J}\left(\sum_{i=1}^{n}\tau_{2i}\ddot{\beta}_i\right)\sin\theta = -\frac{mr}{J}\left(\sum_{i=1}^{n}\tau_{2i}\ddot{\xi}_i\right)\cos\theta\,. \tag{3.28}$$

Defining the vector

$$\bar{z} = \begin{Bmatrix} z_1 \\ \vdots \\ z_i \\ \vdots \\ z_n \end{Bmatrix} \tag{3.29}$$

allows the equations (3.27) and (3.28) to be written in the form

$$\frac{d\bar{z}}{dt} = A(t)\bar{z}\,. \tag{3.30}$$

For the particular case of a three oscillator system with equal masses, and with the vibration generator mounted on the second mass, the vector $\bar{z}$ becomes

$$\bar{z} = \begin{Bmatrix} \xi_1 \\ \dot{\xi}_1 \\ \xi_2 \\ \dot{\xi}_2 \\ \xi_3 \\ \dot{\xi}_3 \\ \eta \\ \dot{\eta} \end{Bmatrix} . \tag{3.31}$$

A reasonable simplifying assumption to make at this point is that $z_7 \gg z_8$. This is justified on the grounds that equation (3.28), when considered as a variation of the Mathieu Equation, indicates that z_7 will be constant or slowly varying. Using this assumption, the components of the A(t) matrix in equation (3.30) have the following values:

All the A_{ij} are zero except for

$$A_{11} = 1.0$$

$$A_{21} = -E_{11}$$

$$A_{22} = -D_{11}$$

$$A_{27} = \tau_{21} \frac{m r \omega^2}{M} \cos \omega t$$

$$A_{28} = \tau_{21} \frac{2 m r \omega}{M} \sin \omega t$$

$$A_{34} = 1.0$$

$$A_{43} = -E_{22}$$

$$A_{44} = -D_{22}$$

$$A_{47} = \tau_{22} \frac{m r \omega^2}{M} \cos \omega t$$

$$A_{48} = \tau_{22} \frac{2 m r \omega}{M} \sin \omega t$$

$$A_{56} = 1.0$$

$$A_{65} = -E_{33}$$

$$A_{66} = -D_{33}$$

$$A_{67} = \tau_{23} \frac{m r \omega^2}{M} \cos \omega t$$

$$A_{68} = \tau_{23} \frac{2 m r \omega}{M} \sin \omega t$$

$$A_{78} = 1.0$$

$$A_{81} = \tau_{21} \frac{m r}{J} E_{11} \cos \omega t$$

$$A_{82} = \tau_{21} \frac{m r}{J} D_{11} \cos \omega t$$

$$A_{83} = \tau_{22} \frac{m r}{J} E_{22} \cos \omega t$$

$$A_{84} = \tau_{22} \frac{m r}{J} D_{22} \cos \omega t$$

$$A_{85} = \tau_{23} \frac{m r}{J} E_{33} \cos \omega t$$

$$A_{86} = \tau_{23} \frac{m r}{J} D_{33} \cos \omega t$$

$$A_{87} = -(\tau_{21}^{\,2} + \tau_{22}^{\,2} + \tau_{23}^{\,2}) \frac{(m r \omega \cos \omega t)^2}{MJ}$$

$$- \frac{m r \omega^2}{J} \Big(\tau_{21} (A_1 \sin \omega t + B_1 \cos \omega t)$$

$$+ \tau_{22} (A_2 \sin \omega t + B_2 \cos \omega t) + \tau_{23} (A_3 \sin \omega t$$

$$+ B_3 \cos \omega t) \Big) \sin \omega t$$

$$A_{88} = -(\tau_{21}^{\,2} + \tau_{22}^{\,2} + \tau_{23}^{\,2}) \frac{(m r)^2}{MJ} \, 2 \omega \sin \omega t \cos \omega t - \frac{K}{J}$$

Of the types of analyses developed for the single-degree-of-freedom system, only the torque-slope and Floquet analyses may be applied to the multi-degree-of-freedom problem. The perturbation analyses are not valid because the multi-degree-of-freedom problem, unlike the single-degree-of-freedom case, does not have its characteristic exponents approximately equal to each other near the stability boundary. For this reason a Floquet type analysis was programmed for solution on a digital computer for the particular three-oscillator system described above. The stability of this system for a given set of parametric values was determined as a function of the value of the unbalanced mass in the region of each resonance. A Kononenko "equivalent" single-degree-of-freedom analysis was also applied to this particular system, using equation (3.27) with the appropriate value of i as the oscillator equation and equation (3.28) as the motor equation. The results from the two analyses shown no significant differences, thus indicating that in this case Kononenko's approximate analysis is valid.

IV SUMMARY AND CONCLUSIONS

A review of the literature related to the problem of the dynamic stability of unbalanced mass exciter systems discloses that two zero order analyses of the local stability of the single-degree-of-freedom case have been developed. The first analysis, which is based on the direct integration of the differential equations of motion, was originally presented by Y. Rocard[2]. Variations of this analysis have also been developed by R. Mazét[5], and by Y. G. Panovko and I. I. Gubanova[6]. The second analysis, developed by V. O. Kononenko[3, 4], is of the perturbation type in which the stability criteria for the system are developed by applying the Routh-Hurwitz stability criteria to the perturbed equations. The effects of non-linear oscillator springs, and the problems of multiple oscillator systems are also considered by Kononenko. These analyses have shown that an unstable region may exist when the system is operating at a frequency just above the system resonance frequency, and that this instability is due to the non-linear interaction between the oscillator motion and the motor.

This thesis summarizes the previous work done and develops a first order analysis of the single-degree-of-freedom system. A stability diagram showing the stability boundary as a non-dimensional function of the system parameters is presented. In addition, two zero order analyses of specific multi-degree-of-freedom systems are developed and used to verify the "equivalent" single-degree-of-freedom analysis suggested by Kononenko.

Single-Degree-of-Freedom System:

The single-degree-of-freedom system is defined as a linear mechanical oscillator with an unbalanced mass vibration generator mounted on the oscillator mass. The differential equations of motion are obtained by applying Lagrange's Equation to the model. Global stability of the system, that is, stability in the sense of Laplace, is proved using Liapunov's second method.

Four analyses of the stability of the single-degree-of-freedom system are developed. The first two, which are based on the assumption that the angular speed of the unbalanced mass is uniform, are called the zero order analyses. The other two, which allow for variations in the angular speed of the unbalanced mass, are called the first order analyses. The regions of validity of the zero and first order analyses are limited by the validity of their respective assumed solutions for θ, the angular displacement of the unbalanced mass. For the zero order analyses a simple inequality in terms of the system parameters determines the boundary of the region of validity. This boundary is defined by equation (2.16) and is presented graphically in Figure 2.3. The relation is not as simple for the first order analyses, and a validity computation is performed for each case investigated.

The first zero order analysis developed is the Torque-Slope Analysis, which is derived directly from the differential equations of motion. The application of the zero order assumption, that the angular speed, θ, of the unbalanced mass is a constant ω, to the differential equation of motion for the oscillator displacement, x, gives a simple sinusoidal form for x with a known amplitude and phase. The value of x is then used in the differential equation of motion for the angular displacement θ, and the result is averaged to obtain the steady state characteristics. This procedure gives the average motor output torque on one side of the equation, and the average torque required by the oscillator on the other side. When these two quantities are plotted as functions of ω, their intersection points correspond to the operating points of the system. Examination of the torque characteristics at these points, particularly the relative slopes of the two curves, shows that the motor speed-torque curve must have a larger negative slope than the oscillator speed-torque curve for the system to be stable.

The second zero order analysis is based on the determination of the behavior of small perturbations which are added to x and θ. If these perturbations die away in time, the system is considered stable. The differential equations for the perturbed variables are obtained as functions of the system parameters and the assumed steady state solutions for x and θ. The form of the perturbed equations implies the existence of a periodic solution. When this implied solution is used as the solution of the perturbed equations, a third order algebraic equation is obtained, the roots of which define the stability of the system. The point at which the system is marginally stable, i.e. the stability boundary, is determined from this equation.

The third analysis, a first order perturbation analysis, differs from the zero order perturbation analysis in that it is based on an assumed value of θ which allows for periodic variations in the angular operating speed. This value of θ also requires the use of a more complete solution for x. In this case the analysis gives a seventh order algebraic equation which is used to obtain the parametric values for marginal stability, as was done in the zero order perturbation analysis.

The fourth analysis is a Floquet analysis based on the first order perturbed equations. Numerical integration of the differential perturbation equations over one cycle, for the appropriate set of initial conditions, gives the fundamental set of solutions Z(T). The eigenvalues of the matrix Z(T), which are the characteristic factors for the system described by the perturbed equations, are obtained. Using a theorem attributed to A. Liapunov, the stability of the unperturbed system is related to the values of the characteristic multipliers. Because of the length and complexity of the computations involved, a digital computer was used to obtain all numerical results.

The stability of a representative set of systems is examined using the applicable analyses. The results obtained are presented in non-dimensional form in Figure 2.4. All variations in the results are within the limits of accuracy of the analyses. Since as many as nine separate analyses of different systems may have contributed data points for a given non-dimensional stability boundary point, it is concluded that each analysis is valid when used within the restrictions developed during the analysis. Another conclusion which may be drawn from this close agreement of non-dimensional data points is that the stability of any system may be computed using the simplest analysis that is valid for that specific system. When the system parameters lie within the ranges covered by Figure 2.4, the results may be obtained directly from the figure.

Examination of Figure 2.4 shows that the stability of a system may be increased by appropriate changes in the values of certain parameters. Perhaps the easiest of such changes is to decrease either the size of the unbalanced mass or its distance from its center of rotation. Either of these changes decreases the force output of the vibration generator. Other possible changes involve the "fixed" parameters of the system. For example, the resonance frequency can be decreased; or the damping, the total mass, or the magnitude of the slope of the motor speed-torque curve can be increased.

Multi-Degree-of-Freedom Systems:

Two general types of multi-degree-of-freedom systems are considered. The first is a single oscillator which is free to perform planar motion. In this system, a vibration generator using two equal contra-rotating unbalanced masses giving an axial oscillating force is mounted on the oscillator mass. Lagrange's Equation is used to obtain the differential equations of motion for the system, and a zero order steady state solution is developed. An expression for checking the validity of the solution is also developed. It is noted that while the stability of this system can be solved for in a general way, it is much more constructive to consider the special case where the center of mass of the oscillator lies on the thrust axis of the vibration generator. This simplification results in equations of motion which reduce to those for the single-degree-of-freedom system.

The second type considered is composed of a series of simple single-degree-of-freedom oscillators. In many cases this type of system may be used as a model of more complex systems. A brief development of Kononenko's "equivalent" single-degree-of-freedom analysis[3] for multi-degree-of-freedom systems with well separated resonances is presented. A specific system consisting of three identical oscillators with a vibration generator mounted on the second one is analyzed using both a complete zero order Floquet-type analysis and an "equivalent" single-degree-of-freedom analysis. The results show that, for the conditions specified, Kononenko's "equivalent" single-degree-of-freedom analysis is valid.

From these results it is concluded that, when a multi-degree-of-freedom system has well separated resonances, the stability of the system near any particular resonance point is, to the first approximation, a function only of the motion in that mode and of the vibration generator parameters. Thus, a form of Kononenko's "equivalent" single-degree-of-freedom analysis may be used to examine the stability of any system with well separated resonances or one in which only one mode is being excited. Investigation of the stability of all other multi-degree-of-freedom systems requires the development of specific complete analyses.

V APPENDIX I — THE ROCARD ANALYSIS

The following is an analysis of the stability of a simple unbalanced mass exciter system based on the direct integration of the differential equation of motion. This analysis is parallel to the work originally published by Y. Rocard[2], although the Modified Rocard System is used here as a model instead of the Rocard system. The primary difference in these two systems, as far as the equations of motion are concerned, is the difference in the definition of the θ variable. This difference does not affect the final results, although the terms in the intermediate steps do differ. An assumption first made by Rocard, and inherent in this analysis, is that the unbalanced mass rotates in the horizontal plane.

The differential equations of motion for the system are obtained by applying Lagrange's Equation, as was done before. These equations are:

$$\ddot{x} + 2\zeta\omega_o\dot{x} + \omega_o^2 x = \frac{mr}{M}(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) \tag{5.1}$$

and

$$\ddot{\theta} + \frac{mr}{J}\ddot{x}\cos\theta = \frac{K}{J}(\omega_s - \dot{\theta}), \tag{5.2}$$

where the motor torque τ is described by

$$\tau = K(\omega_s - \dot{\theta}). \tag{5.3}$$

The first equation may also be written in the form

$$\ddot{x} + 2\zeta\omega_o\dot{x} + \omega_o^2 x = -\frac{mr}{M}\frac{d^2}{dt^2}(\sin\theta) . \tag{5.4}$$

Defining a new variable u as

$$\ddot{u} = x\,, \tag{5.5}$$

substituting into equation (5.4), and integrating twice with respect to time gives

$$\ddot{u} + 2\zeta\omega_o\dot{u} + \omega_o^2 u = -\frac{mr}{M}\sin\theta + Ft + G\,, \tag{5.6}$$

where F and G are constants of integration. The value of G is not important since it merely changes the zero of u. The value of F must be zero, because the system is stable in the sense of Laplace (shown in Part I above). Thus, equation (5.6) becomes

$$\ddot{u} + 2\zeta\omega_o\dot{u} + \omega_o^2 u = -\frac{mr}{M}\sin\theta\,. \tag{5.7}$$

Assuming that $\theta = \omega t$, where ω is slowly varying and $d\omega/dt$ is assumed known, leads to a solution

$$u = A\sin\theta + B\cos\theta$$

where A and B may vary slowly with time. Defining $\Omega = \omega/\omega_o$, and solving for A and B gives

$$A = -\frac{mr}{M\omega_o^2}\left[\frac{1-\Omega^2}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}\right], \tag{5.8}$$

$$B = \frac{mr}{M\omega_o^2}\left[\frac{2\zeta\Omega}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}\right], \tag{5.9}$$

where

$$\dot{A} = \frac{1}{2}\left(\frac{\zeta B - \Omega A}{\Omega^2 + \zeta^2}\right)\frac{d\Omega}{dt}\,, \tag{5.10}$$

$$\dot{B} = -\frac{1}{2}\left(\frac{\zeta A + \Omega B}{\Omega^2 + \zeta^2}\right)\frac{d\Omega}{dt} \; . \tag{5.11}$$

Formally,

$$x = \frac{d^2}{dt^2}\left(A \sin\theta + B \cos\theta\right) , \tag{5.12}$$

which is valid for A, B, and θ slowly varying parameters in time, t. Equation (5.2) may be written in the form

$$\ddot{\theta} + \frac{\kappa}{J}\dot{\theta} + \frac{m\,r}{M}\cos\ddot{U} = \frac{\kappa}{J}\omega_s \; . \tag{5.13}$$

Looking at the "quasi-static" problem (i.e., considering only the slowly varying part of the equation), gives

$$\ddot{\theta} + \frac{\kappa}{J}\dot{\theta} + \frac{m\,r}{J}\left(\frac{1}{2}B\dot{\theta}^4 - 2\dot{A}\dot{\theta}^3 - 3A\dot{\theta}^2\ddot{\theta}\right) = \frac{\kappa\omega_s}{J} \; . \tag{5.14}$$

Substituting the values of A, $\dot{A}$, B, $\dot{\theta}$, and $\ddot{\theta}$ in equation (5.14) gives

$$\left[1 + \frac{2(m\,r\,\Omega)^2}{J\,M}\;\frac{1 - \Omega^2 - \zeta^2}{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}\right]\ddot{\theta} + \left[\frac{(m\,r)^2}{J\,M}\left(\frac{\zeta\omega_o\Omega^4}{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}\right) + \frac{\kappa}{J}\right]\dot{\theta} = \frac{\kappa\omega_s}{J} \; . \tag{5.15}$$

To the first approximation the effects of the coupling between the motion of the unbalanced mass and that of the oscillator are reflected in the modification of the $\ddot{\theta}$ coefficient.

At certain frequencies close to resonance, provided that ζ is small enough, this modification may result in a negative inertia. It follows from this that, at these frequencies, (5. 15) will lead to an unstable state in which the angular velocity $\dot{\theta}$ will depart from its equilibrium value ω without any possibility of return.

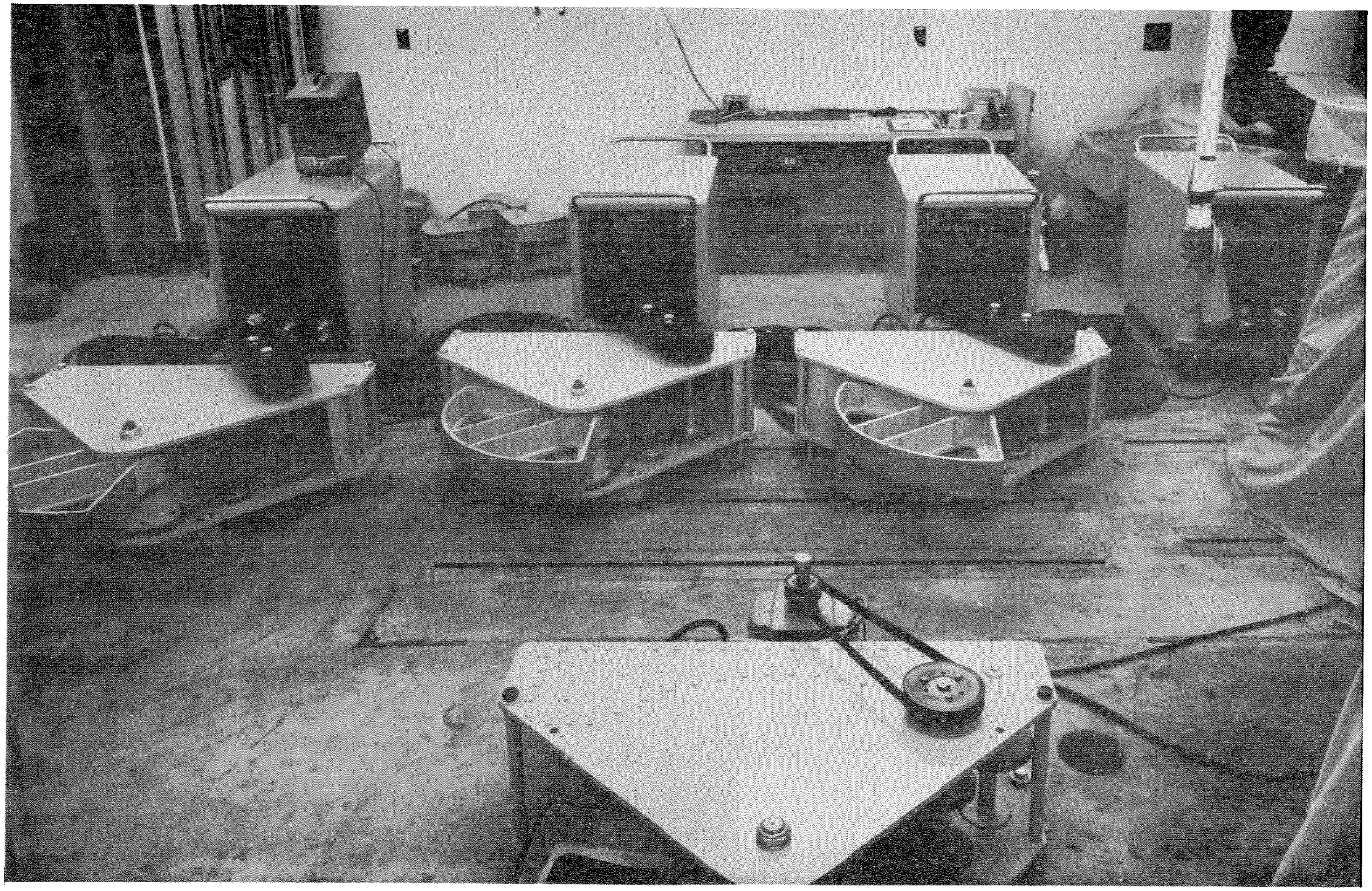
This result does not agree exactly with Rocard's work for two reasons. First of all it appears that, in formulating his version of equation (5.4), Rocard omitted the minus sign in the right hand side of the equation. As a result, the second term in the coefficient of $\ddot{\theta}$ in equation (5. 15) comes out with the opposite sign. This results in the region of instability appearing to be on the left hand side of the resonance curve. The analysis of this problem, but with zero damping, by Panovko and Gubanova[6] along parallel lines also brings out this point.

The second reason for the differences between equation (5. 15) and Rocard's results is a series of algebraic errors, which may be uncorrected typographical errors, that change the coefficient of $\dot{\theta}$ in (5. 15). These "errors" are not important to the stability analysis in the first approximation. With these two errors corrected, his analysis does agree with those presented previously.

VI APPENDIX II — THE C.I.T.-DEVELOPED VIBRATION GENERATORS

Information concerning the response properties of structures subjected to earthquake-induced motions is of great importance to architects and engineers. This information is best obtained from dynamic tests of existing structures. Because of this, the California State Department of Architecture sponsored through the Earthquake Engineering Research Institute the development of a large unbalanced mass exciter system suitable for tests of full size structures. Development of this system was begun in 1959 at the California Institute of Technology. The completed system consists of four separate rotating unbalanced mass vibration generator units which can be operated independently or simultaneously and can be synchronized to excite various modes of vibration in the structure being tested.

Each mechanical vibration generator consists of a pair of contra-rotating unbalanced masses (eccentric weights) arranged so that a rectilinear sinusoidally varying horizontal inertia force is generated[13]. The two unbalanced masses in each generator are chain driven and rotate in opposite directions about a common shaft. Figure 6.1 shows the four C.I.T.-developed vibration generators and their speed control consoles. Lead weights fit into machined compartments in the rotating circular segments. The exciters are driven by 1-1/2 hp d.c. motors which are mounted on the backs of the units and drive the rotating unbalanced masses through a timing belt and chain system. The drive motor and tachometer assembly can be dismounted from the exciter for ease of handling, transportation, and installation.



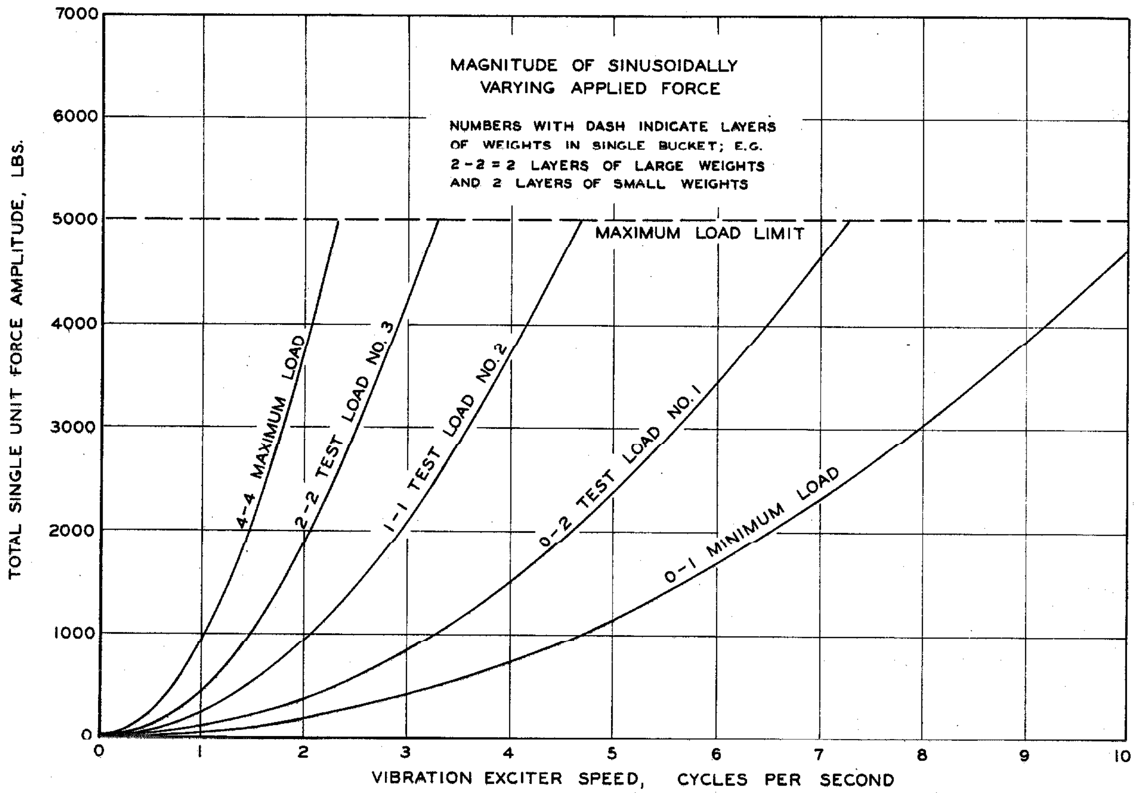
THE C.I.T.-DEVELOPED VIBRATION GENERATORS

FIGURE 6.1

These exciter units are fabricated mainly of aluminum castings and aluminum rolled plate in order to keep the weight to a minimum.

The horizontal force curves for the vibration exciter unit are presented in Figure 6.2. These curves give the uniaxial sinusoidal force amplitude as a function of frequency for the maximum, minimum, and three intermediate loads of lead weights. When the machine is fully loaded with weights, the maximum force produced at a frequency of one cycle per second is 921 lb. The maximum force that the machine can produce is limited by strength considerations to 5000 lb. If the operating frequency is greater than 2.33 cycles per second, the total amount of eccentric weight must be reduced from the maximum to keep the total force below the 5000 lb. limit. With the four units of the system operating together in synchronization, a total horizontal dynamic force of 3684 lb. can be generated at one cycle per second, and a total of 20,000 lb. can be generated at frequencies above 2.33 cycles per second.

The electric drive system requirements for the vibration exciter system are particularly severe because of the variable torques imposed on the system and of the necessity for insuring stability of the whole vibrating system when operating near a resonance point of a lightly damped structure[14]. As has been shown earlier, the ability to maintain accurate speed control at and near resonance requires that the slope of the speed-torque curve of the drive system be unusually steep. In effect, an essentially constant speed needs to be maintained for relatively large torque variations. In addition, the speed control must be operable over a relatively large speed range.



VIBRATION EXCITER FORCE OUTPUT

FIGURE 6.2

To meet these speed control requirements, a d.c. motor is used along with a servo-controlled electronic amplidyne system. A tachometer driven directly from the drive motor supplies a speed signal which is compared with a standard voltage. The difference between these voltages provides an input signal to the amplidyne amplifier, which then acts on the drive motor to adjust its speed to correspond to the set speed. Smooth stable speed adjustments over a 40 to 1 speed range are obtained with this system. An accurate (0.1%) measure of the vibration exciter frequency is read off a digital electronic counter which is energized by a 100 pulse per revolution permanent magnet tachometer generator mounted on the drive motor shaft.

After completion and laboratory testing of the first of the four vibration exciter units in late 1960, a full scale vibration test of the old Encino Dam intake tower was conducted[13]. The opportunity to conduct this particular test arose when the City of Los Angeles began its program of increasing the storage capacity of the reservoir by raising the height of the earth dam. This necessitated replacement of the existing intake tower by a taller tower in a different location. The Department of Water and Power offered to make the old tower available for testing prior to demolition, thus giving the opportunity to proof test the first vibration exciter unit before the rest of the units were completed. The existence of the unstable region and corresponding frequency jump was dramatically illustrated during this test. The vibration exciter was powered by a portable a.c. generator with an inoperative voltage regulator. As resonance was approached from below, the generator output dropped as increased current was required by the exciter drive. The operator, who was new to the job, compensated for this by increasing the set speed of the exciter drive. This enabled him to obtain the low frequency part of the resonance curve, but almost resulted in disaster when he tried to obtain the upper part of the resonance curve. Only quick action prevented damage to the exciter. Replacement of the a.c. generator with a regulated power source, which effectively increased the value of the motor constant $\mathcal{K}$, eliminated the instability in the following tests of the structure, and justified the design of the vibration exciter unit.

Other large structures which have been tested using this system include: Dry Canyon Dam[15], an earth dam 485 ft. long by 60 ft. high by 450 ft. thick at the base; Bouquet Canyon Dam[16], an earth dam 1200 ft. long at the crest by 200 ft. high by 1300 ft. thick at the base; Central Engineering Building of the Jet Propulsion Laboratory[17], a nine story steel frame building; and Olin Hall of Engineering on the campus of the University of Southern California[17], a five story reinforced concrete building.

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