The Hexagonal Resistive Network and the Circular Approximation

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The Hexagonal Resistive Network and The Circular Approximation

A mathematical analysis
of the finite-difference equation
which the idealized network satisfies.

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Preface

This report explores resistive networks using mathematical analysis. The work arose from a discussion in which Professor Carver Mead showed me the hexagonal network and the circular approximation and observed that it was remarkably tough to solve. He wondered whether I might enjoy struggling with it too. Whatever contemporary relevance this work may claim, it owes to Professor Mead and his soon to be published book, Analog VLSI and Neural Systems. The hexagonal resistive network models the horizontal layer of a silicon retina which the book describes.

This document was written with a sincere desire to communicate; the text is almost tutorial. Anyone familiar with Kirchhoff's current law and comfortable with mathematics at the level of first year calculus should be able to follow sections (1) and (2). Section (1) develops the circular approximation and provides an elementary analysis. Section (2) compares different resistive networks using a novel approach. Section (3) comes to grips with the circular approximation as a boundary value problem. This kind of global analysis tends to be more involved, and so section (3) is more sophisticated than the first two sections; here an acquaintance with complex analysis is helpful.

David Feinstein
May, 1988

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1 The Circular Approximation: Asymptotic Analysis

This section introduces the circular approximation of the hexagonal resistive network. Also, this section presents an elementary analysis of the circular approximation. Figure (1) shows a piece of a hexagonal resistive network with the elements of the circular approximation emphasized. Each of the resistors in the hexagonal network has resistance R. Not shown, but equally important, are the admittances to ground. These admittances, allocated one per node, all have conductance G. The challenge is to find what happens when a voltage source drives a single node of the network. What happens is called the spatial impulse response. Calculating the spatial impulse response is quite difficult because the two dimensions of the hexagonal network are inseparable. The circular approximation addresses this difficulty; it mimics the two dimensions of the hexagonal net with a one dimensional network. To see how 2-D collapses to 1-D, organize the nodes of the hexagonal net in concentric hexagons about the driven node. This organization inspires the circular approximation: All nodes on the perimeter of the nth concentric hexagon have the same voltage V_n. If the hexagonal network had perfect circular symmetry then the circular approximation would be exact. Since the hexagonal net does not have perfect circular symmetry, the circular approximation is not exact.

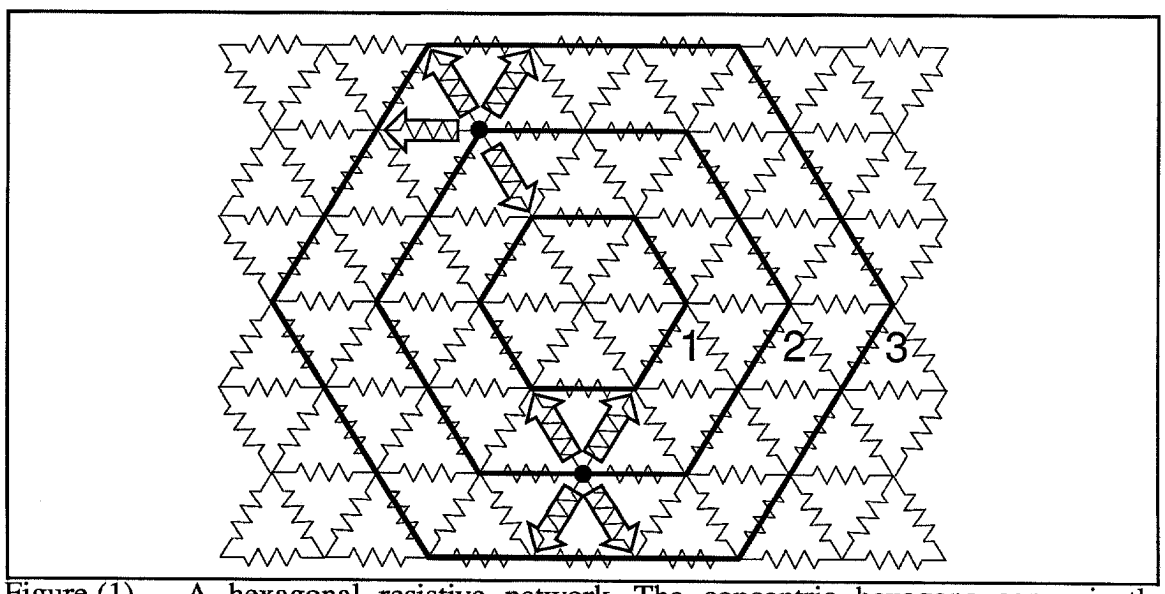

Figure (1) A hexagonal resistive network. The concentric hexagons conveniently group sets of nodes. The circular approximation assumes the voltage is constant around the perimeter of each concentric hexagon. The arrows help to count the number of connections between adjacent hexagons.

Figure (1) makes it easy to find Kirchhoff's current law for the circular approximation. Figure (1) suggests that concentric hexagon n has $6n$ nodes on its perimeter. Of these nodes, 6 are vertices, and the remaining $6(n\text{-}1)$ lie along the edges. The arrows in Figure (1) highlight the paths along which currents flow for a typical

vertex node and for a typical edge node. Consider the paths from hexagon n to its outside neighbor. Each of the 6 vertex nodes makes 3 outside connections, while each of the $6(n\text{-}1)$ edge nodes makes 2 outside connections. The total number of connections with the outside neighbor is $6\times 3+6(n-1)\times 2=12n+6$. By hypothesis, all along hexagon n the potential is constant. Also along hexagon n+1 the potential is constant. Thus hexagon n connects to hexagon n+1 through $(12n+6)$ parallel resistors. The impedance connecting hexagon n to hexagon n+1 is therefore $R/(12n+6)$. Similarly, the impedance connecting hexagon n to hexagon n-1 is $R/(12n\text{-}6)$. Meanwhile, along hexagon n there are $6n$ conductances to ground, making a net admittance to ground of $6nG$. Kirchhoff's current law balances the current flowing into hexagon n from its neighbors with the current flowing out of hexagon n to ground:

$$\frac{V_{n+1}-V_n}{\frac{R}{(12n+6)}}+\frac{V_{n-1}-V_n}{\frac{R}{(12n-6)}}=6nG\cdot V_n\ .$$

Multiply this equation by $R/6$ to simplify it and to convert it to units of voltage. Then simplify the fractions and introduce the dimensionless parameter $a=RG$.

$$(2n+1)V_{n+1}-n(4+a)V_n+(2n-1)V_{n-1}=0\ . \qquad (1)$$

The remainder of this document makes frequent reference to equation (1), often calling it the Kirchhoff current law. In the circular approximation, the node voltages of the network satisfy equation (1) and two boundary conditions: V_0 is given, and V_n is bounded in the limit as $n\to\infty$. The goal is to find how V_n depends on the discrete index n. Because (1) has coefficients which depend on n, it resists a closed form solution. However, (1) is amenable to asymptotic analysis in the limit where n is large. The idea here is that the extent of the network is infinite, and the solution is simple at infinity, although it is complicated closer to the origin. Usually an asymptotic expansion derived at infinity is good for n smaller than infinity too. As a guide to the reader, here is a brief description of what we shall find in equation (1). The asymptotic approximation to V_n has a layered structure. The first layer is that part of the asymptotic approximation which varies most rapidly with n. In the case of V_n, this layer turns out to be an exponentially varying factor. Dividing away the layer of rapid variation exposes another layer. The exposed layer varies more gradually with n. For V_n, this more gradual layer of variation is a factor proportional to $1/\sqrt{n}$. Dividing away this layer as well reveals layers of even slower variation. These slowly varying layers are less important than the first two layers. For V_n, the first two layers together comprise the leading-order asymptotic approximation. The remaining layers contribute only higher-order corrections.

1.1 The First Layer

Start the asymptotic analysis for large n by recasting the Kirchhoff current law, (1), into a more suggestive form. Partition the terms of (1) according to their explicit dependence on n, and scale the entire equation by $1/(2n)$.

$$V_{n+1}-\left(2+\frac{a}{2}\right)V_n+V_{n-1}=\frac{-1}{2n}(V_{n+1}-V_{n-1}) \quad . \tag{2}$$

As n becomes large, the right hand side of (2) becomes small because of the factor $1/n$. If n is large enough, then perhaps the right hand side of (2) will be negligible compared to the left hand side. Equation (2), with its right hand side neglected, accounts for the most rapidly varying factor of the asymptotic expansion of V_n. Dropping the right hand side of (2) leaves an equation that is easy to solve.

$$V_{n+1}-\left(2+\frac{a}{2}\right)V_n+V_{n-1}=0\,. \tag{3}$$

Equation (3) is equation (2) with the right hand side summarily set to zero. (3) is a constant-coefficient, finite-difference equation. To solve (3), assume a solution of the form $V_n \propto r^n$, where r is a constant. Substitute this trial solution into (3) and simplify to obtain

$$r^2-\left(2+\frac{a}{2}\right)r+1=0\,. \tag{4}$$

(4) is a quadratic equation which the constant r must satisfy. This particular quadratic is destined to become familiar, as it pops up rather frequently in the circular approximation to the hexagonal net. Familiar things should have names; name the quadratic Q so that equation (4) becomes $Q(r)=0$. Section (3.2) studies the function Q and characterizes its two roots. One of the roots of Q is greater than unity. This root would cause V_n to be unbounded as $n \to \infty$. The other root is less than unity, (but positive), and is ideal. Denote this smaller root by the symbol r_-. $r=r_-$ is a solution to (4), and so $V_n \propto r_-^n$ is a solution to (3).

Recall that equation (3) is equation (2) stripped of its right hand side. Thus the function r_-^n, which exactly solves (3), is not an exact solution to (2). Carver's book[1], chapter 7, shows that r_-^n is the exact form of solution of the one-dimensional resistive net. The function r_-^n is an important factor in the asymptotic solution of the two-dimensional resistive net. Dividing away the factor r_-^n in equation (2) leaves an equation which comes to grips with the difference between the one-dimensional net and the two-dimensional net. Dividing away the factor r_-^n uncovers the second layer of the asymptotic expansion of V_n. To divide out this factor, assume that the solution to (2) has the form

[1] Carver A. Mead, *Analog VLSI and Neural Systems*, Addison Wesley, 1988.

$$V_n = r_-^n U_n \,, \tag{5}$$

where U_n replaces V_n as the unknown function. Substitute (5) into (2) to obtain the relation which U_n must satisfy.

$$r_-^n\left[r_- U_{n+1} - \left(2+\frac{a}{2}\right)U_n + \frac{1}{r_-}U_{n-1}\right] = -\frac{r_-^n}{2n}\left[r_- U_{n+1} - \frac{1}{r_-}U_{n-1}\right]. \tag{6}$$

Equation (6) for U_n can be simplified. First, cancel the factor r_-^n which is common to both sides. Next, multiply both sides of the quadratic equation (4) by U_n/r ; substitute r_- in place of r in this product and subtract the resulting equation from (6). The result is

$$r_-(U_{n+1} - U_n) - \frac{1}{r_-}(U_n - U_{n-1}) = \frac{-1}{2n}\left(r_- U_{n+1} - \frac{1}{r_-}U_{n-1}\right). \tag{7}$$

Equation (7) might properly be termed the reduced Kirchhoff equation. This name is appropriate because the sequence of steps which converts Kirchhoff's current law, equation (1), into equation (7) is essentially the method of reduction of order. Reduction of order has left both sides of (7) in a special form. The left hand side of (7) involves differences of U_n. These differences are small when U_n is a slowly varying function of n. The right hand side of (7) involves the factor $1/n$. This factor is small when n is large. In the asymptotic regime, as $n \to \infty$, the two sides of (7) have the potential of balancing one another. Taking a continuum limit in (7) is one way to make this balance explicit. Another way is to expand (7) in a formal asymptotic series. Of the two ways, the continuum limit is algebraically simpler, but conceptually somewhat more subtle.

1.2 The Second Layer

The leading order solution of the reduced Kirchhoff equation (7) is the second layer of the asymptotic approximation to V_n. This section solves (7) at leading order by means of a continuum limit. The continuum limit requires an ordering parameter. Assume that n is in the neighborhood of some large number N, and let $1/N$ be the ordering parameter. Introduce the continuous variable x, where

$$x = \frac{n}{N}. \tag{8}$$

x can replace the discrete index n since $n = xN$. The discrete function U_n becomes the continuous function $U(x)$. Incrementing the index n by one corresponds to increasing the variable x by $1/N$, and so

$$U_{n+1} \Rightarrow U\left(x+\frac{1}{N}\right) = U(x) + \frac{1}{N}\frac{dU}{dx} + O\left(\frac{1}{N^2}\right). \tag{9a}$$

Similarly,

$$U_{n-1} \Rightarrow U\left(x-\frac{1}{N}\right) = U(x) - \frac{1}{N}\frac{dU}{dx} + O\left(\frac{1}{N^2}\right). \tag{9b}$$

Substitute into the reduced Kirchhoff equation, (7), using the correspondence between n and x, and U_n and $U(x)$.

$$\frac{1}{N}\left(r_- - \frac{1}{r_-}\right)\frac{dU}{dx} = \frac{-1}{2xN}\left[\left(r_- - \frac{1}{r_-}\right)U\right] + O\left(\frac{1}{N^2}\right). \tag{10}$$

Cancel common factors in (10) to obtain, at leading-order, the continuum limit of equation (7).

$$\frac{dU}{dx} + \frac{U}{2x} = 0. \tag{11}$$

The nice appearance of equation (11) justifies its rather lengthy derivation. Why not a short cut to (11)? For example, why not take the continuum limit directly in equation (1)? Briefly, because the solution to (1) varies too rapidly for the continuum limit to apply. All of the terms on both sides of (7) contribute at leading order in (10) and in (11). This balanced contribution of all terms is due to the special form of (7). Recall that equation (7) is the Kirchhoff current law, equation (1), but with the exponential solution divided out. Equation (1), however, does not have the same balance as equation (7). The same continuum limit which yields such a nice result when applied to (7), does not yield so nice a result when applied to (1). Applied directly to (1), the continuum limit yields only the trivial equation $V(x) = 0$.

The solution to the simple differential equation (11) is $U(x) \propto 1/\sqrt{x}$. Recall that U_n is V_n stripped of the factor r_-^n, (equation 5), and so

$$V_n \propto \frac{r_-^n}{\sqrt{n}}. \tag{12}$$

Equation (12) is the leading-order asymptotic expansion of V_n, at least to within a constant of proportionality. To evaluate the constant of proportionality requires reasonably sophisticated techniques; this is the subject of section (3.4). (See equation (25) at the end of section (3.4) for a good approximate formula.) Figure (2) shows how the constant depends on the parameter RG. Notice that the constant lies

between $V_0/2$ and V_0 for reasonable values of RG. From where does the constant come? Ultimately, the constant of proportionality arises because the hexagonal network poses a second-order, boundary-value problem. Equation (1), the Kirchhoff current law, is born with two auxiliary conditions: V_0 is specified, and V_n is bounded as $n \to \infty$. The boundedness condition at infinity guides the choice of roots in equation (4). In principle, V_0 should determine the constant of proportionality. Unfortunately, equation (12) is not valid for small n, and so there is no direct way to connect the constant in (12) with the boundary condition at the origin.

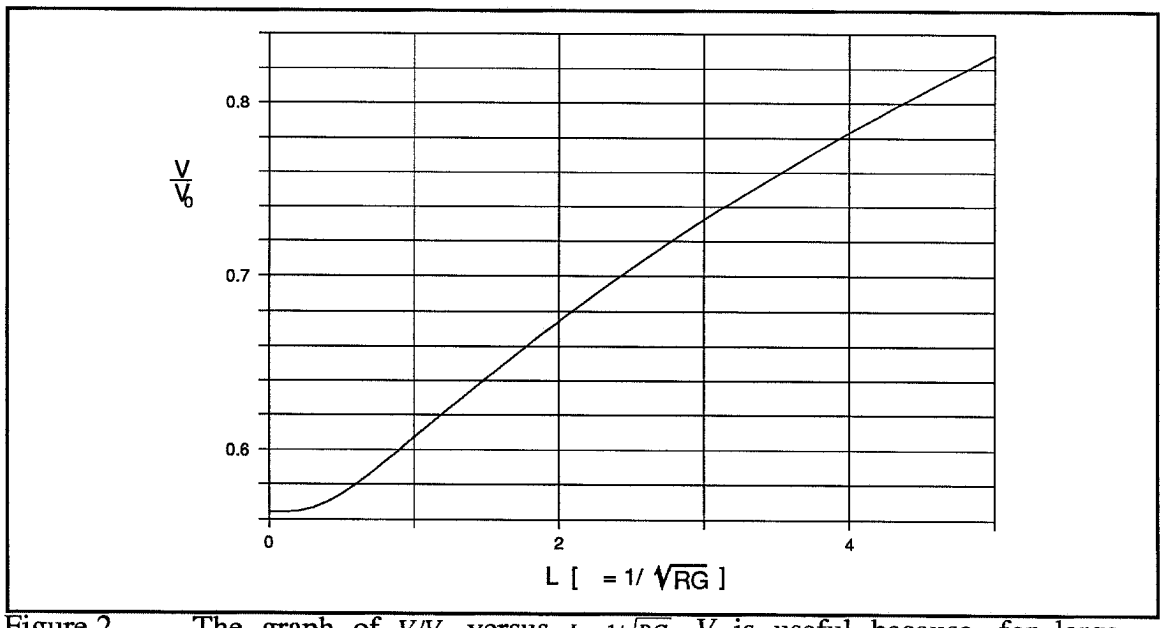

Figure 2 The graph of V/V_0 versus $L = 1/\sqrt{RG}$. V is useful because, for large n, $V_n \sim V r_-^n/\sqrt{n}$.

How does the asymptotic solution for the circular approximation, equation (12), compare with the solution of the uniform, one-dimensional resistive net in Carver's book? The factor $1/\sqrt{n}$ is all that separates the two solutions in form; this factor is a direct consequence of the two-dimensional connectivity of the hexagonal net. How does the asymptotic solution (12) compare with that of the two-dimensional continuous resistive sheet, (again in Carver's book)? The two solutions match one another provided $r=n$ and $\exp(-\alpha) = r_-$, where r is the measure of distance in the continuous net and α is the continuous analog of $a = \sqrt{RG}$.[1]

[1]Possibly there is more substance to the agreement between continuous and discrete nets. The condition for agreement is

$$e^{-\alpha r} = r_-^n \Rightarrow r = n\frac{\ln r_+}{\alpha}$$

This condition relates distance in the continuous net to distance in nodes. It would be interesting to see whether this relation is consistent with what α and a otherwise imply about area versus nodes. The parameter α has units of reciprocal area since $\alpha = \sqrt{\rho\sigma}$, and ρ is the sheet resistance in ohms (per-square) and σ is the conductivity to ground in mhos/unit-area. The parameter $a=RG$, and there is one admittance to ground, G, per node.

1.3 Constructing a Formal Series

Pause here a moment, reader, and decide whether you really want to see the details of constructing an asymptotic series. Unless you're quite interested, consider skipping this section (1.3) and especially the next (1.4). The results of this section appear near its end, in equation (15) and Figure (3). The benefits of equation (15) and Figure (3) are yours, even without their derivation.

The reduced Kirchhoff equation, (7), can be used to construct a formal asymptotic series. Posit a solution to (7) of the form

$$U_n = n^p\left(1+\frac{u_1}{n}+\frac{u_2}{n^2}+\ldots\right). \tag{13}$$

The idea is to substitute this series into (7) and expand every term in powers of reciprocal n. Doing a few preliminary expansions can ease the process considerably. Begin with

$$\begin{aligned} U_{n+1} &= (n+1)^p\left(1+\frac{u_1}{n+1}+\frac{u_2}{(n+1)^2}+\ldots\right) \\ &= n^p\left(1+\frac{1}{n}\right)^p\left(1+\frac{u_1}{n(1+\frac{1}{n})}+\frac{u_2}{n^2(1+\frac{1}{n})^2}+\ldots\right), \end{aligned}$$

and expand in powers of reciprocal n to obtain

$$\begin{aligned} U_{n+1} &= n^p\left(1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots\right)\left(1+\frac{u_1}{n}+\frac{u_2-u_1}{n^2}+\ldots\right) \\ &= n^p\left(1+\frac{u_1+p}{n}+\frac{p(p-1)+2(u_2-u_1+pu_1)}{2n^2}+\ldots\right). \end{aligned}$$

Similarly,

$$U_{n-1} = n^p\left(1+\frac{u_1-p}{n}+\frac{p(p-1)+2(u_2+u_1-pu_1)}{2n^2}+\ldots\right).$$

These equations shift the index of U_n; they are useful for expanding the right hand side of (7). For expanding the left hand side of (7), it is handy to know the first-differences of U_n. Thus

$$U_{n+1}-U_n=n^p\left(\frac{p}{n}+\frac{(p-1)(p+2u_1)}{2n^2}+\ldots\right),$$

and

$$U_n-U_{n-1}=n^p\left(\frac{p}{n}-\frac{(p-1)(p-2u_1)}{2n^2}+\ldots\right).$$

Substitute these intermediate results into (7), and group the terms by powers of n.

$$\frac{(r_-^2-1)(2p+1)}{n}+\frac{(r_-^2+1)p^2+(r_-^2-1)(2p-1)u_1}{n^2}+\ldots=0. \tag{14}$$

Notice that (14) commences at $O(1/n)$. The absence of any $O(1)$ term in (14) is a direct consequence of the special form of (7). The remarks following equation (11) apply to (14) as well. (14) is actually several equations in one since the coefficient of each separate power of n must be zero. Since $r_-<1$, requiring that the coefficient of $1/n$ be zero in (14) implies that $p=-1/2$. Happily, this result agrees with the answer from the previous section, (1.2). Requiring that the coefficient of n^{-2} be zero in (14) implies that

$$u_1=\frac{p^2}{2p-1}\frac{1+r_-^2}{1-r_-^2}.$$

Gather together these results on the asymptotic expansion of U_n. Use equation (5) to relate them to V_n, and thereby obtain the first two terms in the asymptotic expansion of V_n.

$$V_n\sim V\frac{r_-^n}{\sqrt{n}}\left(1-\frac{1}{8n}\frac{1+r_-^2}{1-r_-^2}+\ldots\right). \tag{15}$$

In (15), V is a constant of proportionality. Determining V is the subject of section (3.4). (See also the discussion following equation (12).)

Figure (3) checks the accuracy of the one and two term asymptotic expansions. Each pair of curves corresponds to a different value of RG. Within a pair, the hollow markers show V_n normalized by the one term asymptotic approximation, (equation 12), and the solid markers show V_n normalized by the two term expansion, (equation 15). Figure (3) takes the constant of proportionality, V, to be unity for all RG. Taking V as unity forces the horizontal asymptote of each pair of curves to the true value of V for that pair. Check the horizontal asymptotes by comparing them with the graph of V versus RG, Figure (2).

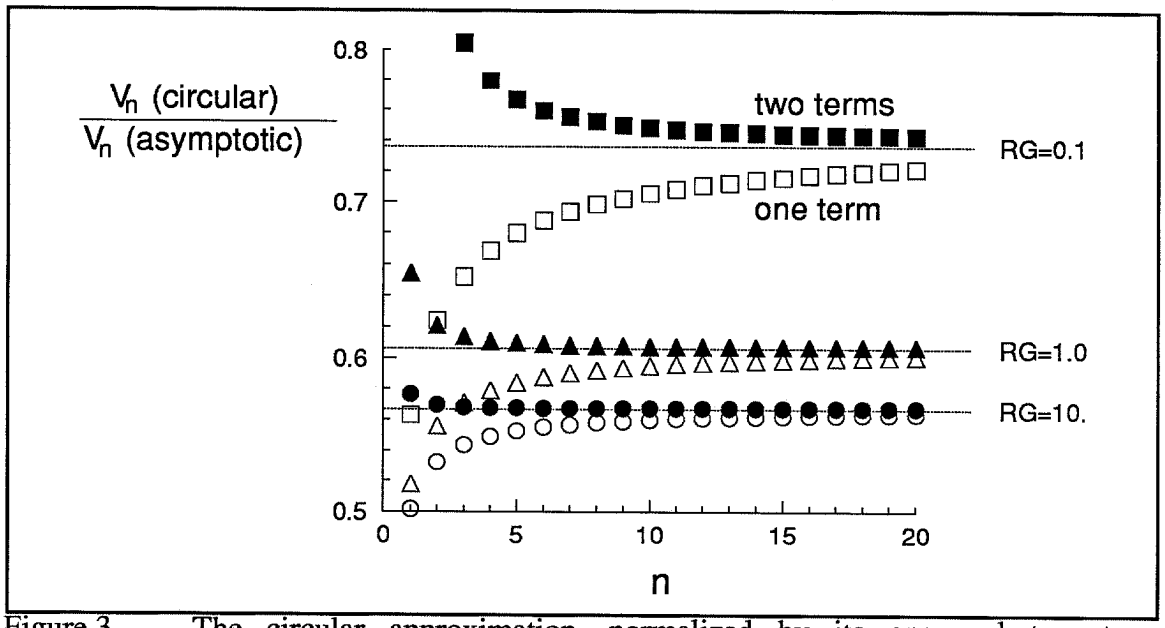

Figure 3 The circular approximation, normalized by its one and two term asymptotic expansions. The flatter the curve, the better the expansion.

The faster a curve flattens with increasing n, the more accurate is the expansion. For large n, the two term asymptotic expansion is more accurate than the one term expansion. Notice that, for all n, the one and two term expansions bracket the true value. On account of this bracketing, the second term of the expansion bounds the error of the one term expansion.

1.4 The Complete Asymptotic Expansion

It is also possible to find the complete asymptotic expansion of V_n. Again, the reduced Kirchhoff equation, (7), is the point of departure. First, put (7) in standard form: multiply by $2n$, rearrange terms, and shift indices so that only positive displacements occur. Also, for convenience, use the symbol r_+ in place of the reciprocal of r_-.

$$r_-(2n+3)U_{n+2}-(r_++r_-)(2n+2)U_{n+1}+r_+(2n+1)U_n=0 \tag{16}$$

The reduced Kirchhoff equation has almost reached standard form in (16). To finish the job, introduce the discrete forward difference operator **D**, where $\mathbf{D}\,U_n=U_{n+1}-U_n$. (16), as written, makes implicit use of the one step advance operator **E**, where $\mathbf{E}U_n=U_{n+1}$. It is easy to see that $\mathbf{E}=\mathbf{D}+1$ and so $\mathbf{E}^2=\mathbf{D}^2+2\mathbf{D}+1$. But, $\mathbf{E}^2U_n=U_{n+2}$ and so $U_{n+2}=(\mathbf{D}^2+2\mathbf{D}+1)U_n$. Using these identities, introduce **D** in (16) and obtain

$$\{r_-(2n+3)D^2+[(r_--r_+)2n+4r_--2r_+]D+r_--r_+\}\,U_n=0 \tag{17}$$

The unpleasant amount of algebra in section (1.3) suggests that the complete asymptotic expansion of V_n is difficult to obtain. In fact, how hard the expansion is to obtain depends on the choice of basis (and on the equation). The use of reciprocal powers of n as the basis of the expansion in section (1.3) is the source of the difficulty. When the coefficients of an equation are polynomials in n, a better choice of expansion is

$$U_n = \sum_{k=0}^{\infty} u_k \frac{\Gamma(n)}{\Gamma(n+k+p)}. \tag{18}$$

Equations (18) and (13) have the same purpose, they establish a basis for the expansion of U_n. Although (18) and (13) appear different, in fact they are similar when n is large because $\Gamma(n)/\Gamma(n+k+p) \sim n^{-(k+p)}$. (Indeed, requiring $u_0 \neq 0$ will ultimately force $p=0.5$ and so, at least to leading order, this section will agree with the previous two.) One virtue of equation (18) as a choice of expansion is that $\Gamma(n)/\Gamma(n+k+p)$ has very nice properties with respect to the forward difference operator **D**.

$$D \frac{\Gamma(n)}{\Gamma(n+k)} = -k \frac{\Gamma(n)}{\Gamma(n+k+1)} \tag{19}$$

The difference law (19) is easy to remember because it corresponds so closely with the law for derivatives of powers: $dx^k/dx = -kx^{-(k+1)}$.

Now comes the real work: determining the coefficients u_k in the expansion (18). The strategy, which is generic in these circumstances, is to substitute the expansion into whatever constitutive relation is available, and then somehow to isolate the separate terms of the summation. Isolating the terms is generally where the labor is. This isolation always depends on the fact that the expansion must satisfy the constitutive relation for every value of the independent variable (which in our case is n). To implement the first part of the strategy, substitute (18) into (17) and use the difference law, (19).

$$\begin{aligned} 0 = \sum_{k=0}^{\infty} u_k \{ & (k+p)(k+p+1)(2n+3) r_- \frac{\Gamma(n)}{\Gamma(n+k+p+2)} \\ & + (k+p)[2(r_+ - r_-)n + 2r_+ - 4r_-] \frac{\Gamma(n)}{\Gamma(n+k+p+1)} \\ & - (r_+ - r_-) \frac{\Gamma(n)}{\Gamma(n+k+p)} \} \end{aligned} \tag{20}$$

Equation (20) must hold for all n; this is the central principle. To fruitfully apply this principle, the dependence on n of the terms of (20) must be streamlined. The Γ functions are not objectionable, the floating factors of n are objectionable. To absorb the floating factors of n, use two Γ function identities.

$$n\frac{\Gamma(n)}{\Gamma(n+k+p+2)}=-(k+p+1)\frac{\Gamma(n)}{\Gamma(n+k+p+2)}+\frac{\Gamma(n)}{\Gamma(n+k+p+1)}$$

$$n\frac{\Gamma(n)}{\Gamma(n+k+p+1)}=-(k+p)\frac{\Gamma(n)}{\Gamma(n+k+p+1)}+\frac{\Gamma(n)}{\Gamma(n+k+p)}$$

These identities are easy to verify; both rely on the basic law that $\Gamma(z+1)=z\Gamma(z)$. These identities confine all of the n-dependence in (20) inside the Γ functions.

$$\begin{aligned}0=\sum_{k=0}^{\infty}\{&(k+p)(k+p+1)[1-2(k+p)]r_-\frac{u_k}{\Gamma(n+k+p+2)}\\&+(k+p)[(4r_--2r_+)(k+p)+2(r_+-r_-)]\frac{u_k}{\Gamma(n+k+p+1)}\\&+(r_+-r_-)[2(k+p)-1]\frac{u_k}{\Gamma(n+k+p)}\}\end{aligned}\tag{21}$$

Shifting summation indices further streamlines the n dependence of (21). Shift the index k so that for each distinct k the sum involves only the single Γ function ratio $\Gamma(n)/\Gamma(n+k+p)$.

$$\begin{aligned}0=\sum_{k=0}^{\infty}\frac{\Gamma(n)}{\Gamma(n+k+p)}\{&(2k+2p-1)(r_+-r_-)u_k\\&+2(k+p-1)[(2k+2p-3)r_-+(2-k-p)r_+]u_{k-1}\\&+(k+p-2)(k+p-1)(5-2p-2k)r_-u_{k-2}\}\end{aligned}\tag{22}$$

Equation (22) is the index shifted version of (21) provided that $u_{-2}=0$ and $u_{-1}=0$. The central principle, that equation (22) hold for each n, now finds direct application. The functions $f_k(n)\equiv\Gamma(n)/\Gamma(n+k+p)$ are independent of one another for each k. Thus, for each different value of k, the bracketed factor in (22) must be zero in order that the whole sum be zero for every n. Equating the bracketed factor to zero for each k produces equations for the u_k. The first of these equations, for $k=0$, is

$$(2p-1)u_0=0\,.$$

For u_0 to be other than zero, $p=1/2$. The general equation, for arbitrary k, is

$$\begin{aligned}&2k(1-r_-^2)u_k+(1-2k)\left[2(1-k)r_-^2+\left(k-\frac{3}{2}\right)\right]u_{k-1}\\&+(1-2k)\left(\frac{3}{2}-k\right)(2-k)r_-^2u_{k-2}=0\,.\end{aligned}\tag{23}$$

Taking u_0 to be a constant, say V, it is easy to solve for the next several u_k. Thus

$$u_0 = V, \qquad u_1 = -\frac{V}{4(1-r_-^2)}, \qquad u_2 = \frac{3}{8}\frac{1-4r_-^2}{1-r_-^2}V, \ \ldots \tag{24}$$

Gather together equation (24) for u_k, and equation (18) for U_n, to obtain

$$U_n \sim \frac{\Gamma(n)}{\Gamma(n+.5)}\left\{1 + \frac{u_1}{n+\frac{1}{2}} + \frac{u_2}{(n+\frac{1}{2})(n+\frac{3}{2})} + \ldots\right\} \tag{25}$$

Does the asymptotic expansion, (25), agree with the result of the previous section, equation (15)? Yes, the two agree through the $1/n$ term which is as far as (15) goes.[1] Presumably, if (15) had more terms, then these terms would also agree.

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[1] To most easily demonstrate their agreement, first show that

$$\Gamma(n+.5) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}, \text{and so } \frac{\Gamma(n)}{\Gamma(n+.5)} = \frac{2^{2n}(n!)^2}{\sqrt{\pi}\,n(2n)!}.$$

Use the Stirling series for $n!$ to obtain $\dfrac{\Gamma(n)}{\Gamma(n+.5)} \sim \dfrac{1}{\sqrt{n}}\left(1 + \dfrac{1}{8n} + \ldots\right)$.

Substitute this relation into (25), expand in powers of reciprocal n, and use (5) to relate to V_n. Compare the result to (15) and observe they agree through order $1/n$.

2 Comparing Hexagonal, Rectangular, & Continuous Nets

How do hexagonal and rectangular resistive networks compare with one another, and how do these compare with a resistive sheet? There are many interesting facets to these comparisons; sadly, most are too hard to answer analytically. For example, what effects do six-fold versus four-fold symmetry have on spatial impulse response? Often, effects of symmetry appear as angle dependence; these effects are hard to analyze. Interestingly though, symmetry also effects distance measure, and this effect is relatively easy to analyze. This section studies the measure of distance in several resistive structures.

Behavior is one means of comparing resistive networks. Unfortunately, learning the behavior of a network generally implies solving the network. As solutions are hard to obtain, so behaviors are hard to compare. Network equations offer another means of comparison. Although different networks obey different equations, in the continuum limit, many networks obey the same equation. Transforming the equation of a network to the continuum limit reveals the measure of distance in the network. This section finds the continuum limit of three different discrete networks.

A brief guide to section (2): Section (2.1) finds the continuum limit of the rectangular and hexagonal discrete networks. Section (2.2) determines the continuum limit of the circular approximation. Interesting comparison of the different networks happens in section (2.3). Finally, section (2.4) compares the circular approximation to the hexagonal network. The reader who is uninterested in detailed mathematics might reasonably skim the first two paragraphs of (2.1) and skip (2.2) altogether.

2.1 Discrete Networks in The Continuum Limit

Kirchhoff's current law is a good starting place for finding the continuum limit of a general discrete network. Equation (1) is Kirchhoff's current law applied to node *(m,n)* of a resistive network. The total current flowing from the neighboring nodes into *(m,n)* just balances the current flowing out of *(m,n)* to ground. The sum in (1) runs over a template N of neighbors. The template N contains the relative offsets of all the nodes to which *(m,n)* connects. Because the template contains only relative offsets, and these are independent of m and n, the network is topologically homogeneous. Node *(m,n)* connects to each neighbor in N through a resistor with resistance R, and to ground through an admittance G. Neither R nor G has subscripts; R and G are constant across the network.

$$\sum_{i,j \in N} \frac{V_{m+i,n+j} - V_{m,n}}{R} = G\, V_{m,n} \tag{1}$$

Figure (1) shows the template of neighbors for a rectangular net and for a hexagonal net. Each node of the rectangular net connects to four neighbors, while each node of the hexagonal net connects to six. Figure (1) also shows the coordinate system used in each net to locate neighbors. N_{rect}, the template for the rectangular net, consists of the set of offsets {(1,0), (0,1), (-1,0), (0,-1)}. Notice that the hexagonal net uses coordinate axes which are not perpendicular to one another. N_{hex}, the template for the hexagonal net, is {(1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)}.

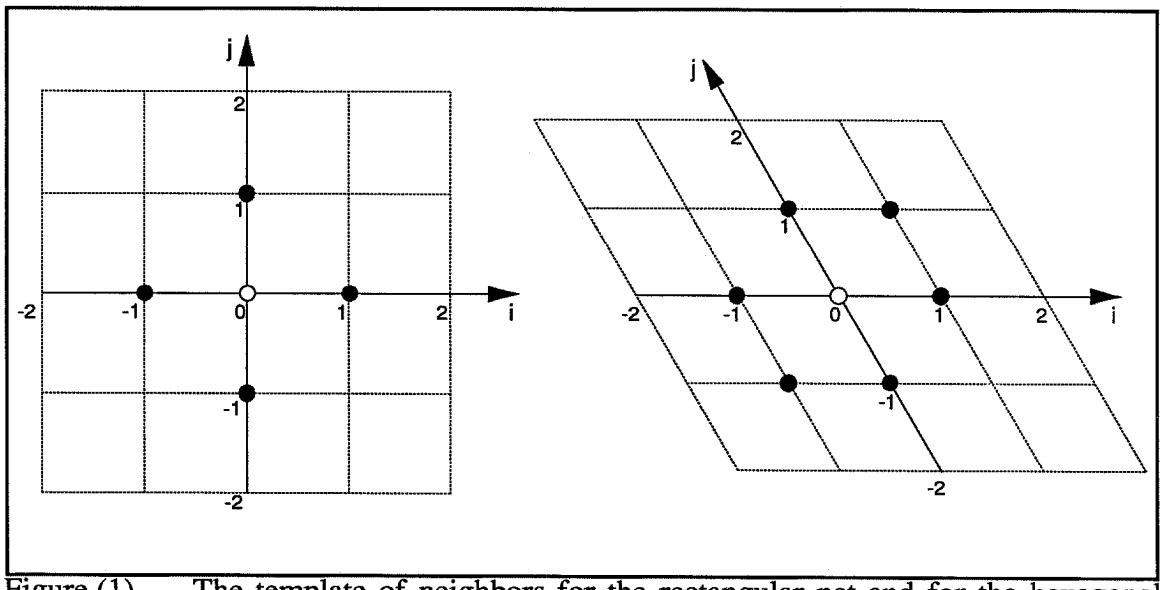

Figure (1) The template of neighbors for the rectangular net and for the hexagonal net. For each template, the coordinates i and j tell the relative displacement of each neighbor. If position (i,j) has a dot, then node (m,n) connects to node $(m+i,n+j)$.

In the continuum limit, the voltages of adjacent nodes approach one another. The lateral resistances drop little voltage, and the admittances to ground conduct little current. To arrange this state of affairs, the lateral resistance, R, and the admittance to ground, G, should be small. When RG is small, a significant change in voltage can occur only over a significant span of nodes. How many nodes form a significant span, if the product RG is a small number ε? Experience suggests that the number will grow as some fractional power of reciprocal ε. In fact, for the rectangular and hexagonal networks, the number of nodes forming a significant span grows as $\varepsilon^{-1/2}$.[1] The ultimate check of this scaling law is that the continuum limit be consistent. To implement the scaling law, introduce the small parameter ε and rescale the independent variables m and n.

$$\varepsilon \equiv RG \qquad x = \sqrt{\varepsilon}m, \quad y = \sqrt{\varepsilon}n \qquad V_{m,n} \to \phi(x,y)$$

[1] We do not expect this to be self-evident. To anticipate this scaling law requires familiarity with singular perturbation theory, and the notion of a distinguished limit. However hard anticipating the scaling law may be, understanding its application is easy. The calculation which follows is self-contained and readily grasped.

The coordinates (x, y) replace the indices m and n as the new independent variables. The scaling law which carries (m, n) to (x, y) maps "significant spans" of m and n to unit intervals of x and y. Lesser spans of m and n correspond to smaller intervals of x and y. For example, the index pair $(m+1, n)$, corresponds to the coordinates $(x+\sqrt{\varepsilon}, y)$. To proceed with the continuum limit, substitute the coordinates x, y and the function ϕ into the Kirchhoff current law, (1).

$$\sum_{i,j \in N} \frac{\phi(x+i\sqrt{\varepsilon},\ y+j\sqrt{\varepsilon})-\phi(x,y)}{R} = G\,\phi(x,y) \tag{2}$$

It is an article of faith that for sufficiently small ε the node voltages $V_{m,n}$ approach a smooth function $\phi(x,y)$. When ε is small, the distance between (x, y) and $(x+i\sqrt{\varepsilon},\ y+j\sqrt{\varepsilon})$ is small. Approximate the voltages at these small offsets using a two-dimensional Taylor series:

$$\phi(x+i\sqrt{\varepsilon},\ n+j\sqrt{\varepsilon}) = \phi(x,y)+\sqrt{\varepsilon}(i\phi_x+j\phi_y)+\frac{\varepsilon}{2!}(i^2\phi_{xx}+2ij\phi_{xy}+j^2\phi_{yy})+\ldots$$

How many terms does the Taylor series need to adequately approximate these nearby voltages? Leaving too few terms affects equation (2) disastrously. For example, retaining only the zeroth order term, $\phi(x, y)$, causes all the voltages to look alike and reduces (2) to $\phi(x, y)=0$. The symmetry of the rectangular and hexagonal networks plays a role here. Because of the symmetry, retaining the linear terms ϕ_x and ϕ_y does not help, these cancel too and (2) still degenerates to $\phi(x, y)=0$. Extending the Taylor series so it includes the quadratic terms ϕ_{xx}, ϕ_{xy} and ϕ_{yy} yields the first meaningful approximation of (2). For the rectangular net, use the Taylor series and the template of neighbors to obtain the continuum limit.

$$\begin{aligned} &N_{\text{rect}} = \{(1,0),(0,1),(-1,0),(0,-1)\} \Rightarrow \\ &\frac{\varepsilon}{2!}(2\phi_{xx}+2\phi_{yy}) = \varepsilon\phi \quad \rightarrow \quad \nabla^2\phi = \phi\,. \end{aligned} \tag{3}$$

Similarly, use the Taylor series and the appropriate template of neighbors to obtain the continuum limit of the hexagonal net.

$$N_{\text{hex}} = \{(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\} \Rightarrow$$

$$\frac{\varepsilon}{2!}(4\phi_{xx}+2\cdot 2\phi_{xy}+4\phi_{yy}) = \varepsilon\phi\,.$$

This equation is not in simplest form. The discrete hexagonal net uses indices which refer to a non-orthogonal basis. The coordinates x, and y refer to a scaled version of this same basis. Transforming x and y to a coordinate system with an orthogonal basis simplifies the equation.

$$\left.\begin{aligned} x &= X\sqrt{2} \\ y &= X\frac{1}{\sqrt{2}} + Y\sqrt{\frac{3}{2}} \end{aligned}\right\} \quad \phi_{XX} + \phi_{YY} = \phi \quad \rightarrow \quad \nabla^2\phi = \phi\,. \tag{4}$$

Notice that equations (3) and (4) match one another exactly. Physically the continuum limit of either the rectangular net or the hexagonal net amounts to the same structure. For either network, the continuum limit corresponds to a resistive sheet with an admittance to ground per unit area.

2.2 The Circular Approximation in The Continuum Limit

This section finds the continuum limit of the circular approximation to the hexagonal net. Not surprisingly, with the right transformation, the resistive sheet equation emerges. This continuum limit is useful because it facilitates comparison of the circular approximation with the true network. Begin the derivation by writing the Kirchhoff current law for the circular approximation (equation 1, section 1) in the following suggestive way.

$$2n(V_{n+1} - 2V_n + V_{n-1}) + (V_{n+1} - V_{n-1}) - nRGV_n = 0 \tag{5}$$

The first set of parentheses contains a second difference, while the second set contains a first difference. Transforming to continuum coordinates converts these differences to derivatives.

$$\varepsilon \equiv RG\,, \qquad r = n\sqrt{\frac{\varepsilon}{2}}\,, \qquad V_n \rightarrow \phi(r)\,.$$

Because the circular approximation is one-dimensional, only a single coordinate is necessary. As above, index shifts map to derivatives according to a Taylor series.

$$V_{n\pm1} \Rightarrow \phi\left(r \pm \frac{\sqrt{\varepsilon}}{\sqrt{2}}\right) = \phi(r) \pm \frac{\sqrt{\varepsilon}}{\sqrt{2}}\frac{d\phi}{dr} + \frac{\varepsilon}{4}\frac{d^2\phi}{dr^2} + \ldots$$

Substitute these transformations into the Kirchhoff current law for the circular approximation, (5), and obtain

$$2\frac{\sqrt{2}}{\sqrt{\varepsilon}}r\left(\frac{\varepsilon}{2}\frac{d^2\phi}{dr^2}\right) + \left(2\frac{\sqrt{\varepsilon}}{\sqrt{2}}\frac{d\phi}{dr}\right) - \frac{\sqrt{2}}{\sqrt{\varepsilon}}r\varepsilon\phi = 0.$$

This equation simplifies and becomes

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}-\phi=0.$$

But this last equation is just the polar coordinate form of the familiar resistive sheet,

$$\nabla^2\phi=\phi\,.$$

2.3 Comparing Network Measures of Distance

The preceding sections show that in the continuum limit, the rectangular net, the hexagonal net, and the circular approximation to the hexagonal net all obey the equation $\nabla^2\phi=\phi$. Not surprisingly, a resistive sheet with a leakage to ground per unit area obeys this same equation.[1] The solution to the equation $\nabla^2\phi=\phi$ is the modified Bessel function of the second kind, K_0.[2] The solutions are most useful when expressed in the natural coordinates of each network. The natural coordinates of a network are the discrete indices which label the nodes of that network. Writing the solution for each network in natural coordinates shows how connectivity alters distance in that network. The following table summarizes the findings.

Network	Euclidean Distance to Origin	Continuum Limit Solution[3]
Rectangular Net	$\|\|(m,n)_{\text{rect}}\|\|=\sqrt{m^2+n^2}$	$V_{m,n}\sim K_0(\sqrt{\varepsilon}\,\|\|(m,n)_{\text{rect}}\|\|)$
Hexagonal Net	$\|\|(m,n)_{\text{hex}}\|\|=\sqrt{m^2-mn+n^2}$	$V_{m,n}\sim K_0\left(\sqrt{\frac{2\varepsilon}{3}}\,\|\|(m,n)_{\text{hex}}\|\|\right)$
Circular Approximation	(see text) $\frac{\sqrt{3}}{2}\tilde{n}\le\|\|\tilde{n}\|\|\le\tilde{n}$	$V_{\tilde{n}}\sim K_0\left(\frac{\sqrt{\varepsilon}}{\sqrt{2}}\tilde{n}\right)$

Remember that the natural coordinates measure distance in nodes. Note that $\|(m,n)_{\text{rect}}\|=\|(m,n)_{\text{hex}}\|$ if $(m,n)_{\text{rect}}$ and $(m,n)_{\text{hex}}$ refer to nodes at equal distances from the origin. Solutions match one another only if their arguments agree. If the potential at

[1] Carver covers the resistive sheet in his book, *Analog VLSI and Neural Systems.*

[2] Also the modified Bessel function of the first kind, I_0, is a solution. But I_0 is infinite at infinity, and so conventional wisdom discards it. On the other hand, K_0 is infinite at the origin, and so the wisdom of convention seems unclear. To marshal stronger support for K_0, consider current, rather than voltage. The radial current, which goes like $r\partial\phi/\partial r$, is finite everywhere with the K_0 solution, (including the origin), but diverges at infinity with I_0.

[3] Strictly, these solutions deserve constants of proportionality in front of them. However, the constants do not appear because they don't affect the distance measure issue.

radius r in the rectangular net matches the potential at radius $r\sqrt{2/3} \doteq 0.82r$ in the hexagonal net, then the two nets share a common solution. Thus on a per node basis, voltage falls more rapidly with distance in the hexagonal net than in the rectangular net.

For the circular approximation there is a question about Euclidean distance. Recall that n in the circular approximation labels the nth concentric hexagon about the origin. The nodes on the perimeter of this hexagon do not all lie at the same distance from the origin. The vertices of nth hexagon lie the furthest from the origin; these do lie at distance n. But the other nodes of the nth hexagon lie closer than n. Where on the nth hexagon does the circular approximation predict the voltage? To answer this question, compare the continuum limit solution for the true hexagonal net with the continuum limit circular approximation solution. As above, the two solutions match only if their arguments agree. Thus the potential at $\tilde{n}$ in the circular approximation matches the potential at $(m,n)_{hex}$ in the hexagonal net if $||(m,n)_{hex}|| = \tilde{n}\sqrt{3}/2$. But $\tilde{n}\sqrt{3}/2$ is exactly the radius of the circle which inscribes the $\tilde{n}^{th}$ concentric hexagon about the origin. In other words, in the continuum limit, the circular approximation predicts the potential at the midpoints of the sides of the concentric hexagons.

2.4 The Accuracy of The Circular Approximation

How accurate is the circular approximation? Numerical computation of the exact and circular approximate spatial impulse response answers this question. Figure (2) displays the results. The diagram in the upper right shows the meaning, for Figure (2), of the indices m and n. n counts concentric hexagons out from the driven node, while m labels different nodes along the perimeter of this hexagon. The remaining graphs of Figure (2) are three scatter plots, corresponding to 3 different values of RG. Each scatter plot shows the response of the true network, $V_{m,n}$, normalized by the circular approximate response, V_n, versus n, for different values of m. The scatter happens at each n because the perimeter of the nth concentric hexagon is not all at one potential. Thus with n fixed, varying m generates the scatter of points.

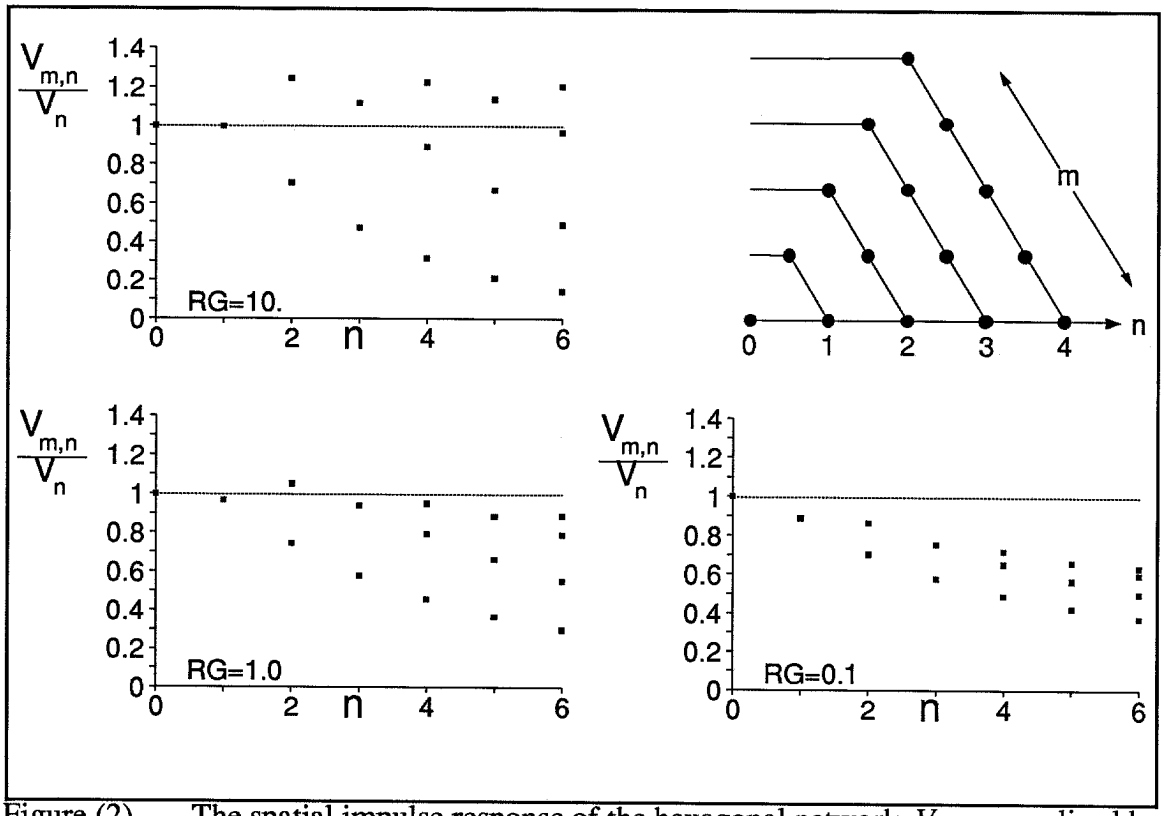

Figure (2) The spatial impulse response of the hexagonal network, $V_{m,n}$, normalized by the circular approximate response, V_n. Diagram at upper right shows that (m, n) is the mth node from vertex along the nth concentric hexagon.

If all the points were at unity height in the graphs of Figure (2), then the circular approximation would agree exactly with the true network. The nodes nearest the middle of an edge always account for the highest points in Figure (2). If these highest points all reached unity height, then the circular approximation would agree with the true network at the centers of edges, as it does in the continuum limit. Figure (2) shows that the circular approximation is only approximate. Near the continuum limit, (small RG), the

circular approximation tends to overestimate the true response, predicting a potential which exceeds even that found at the midpoint of an edge. Well away from the continuum limit, (large RG), the circular approximation predicts an intermediate potential which lies somewhere between the midpoint and the vertex of an edge.

3 The Circular Approximation; A Global Analysis

Consider the Kirchhoff current law for the circular approximation,

$$(2n+1)V_{n+1} - n(4+a)V_n + (2n-1)V_{n-1} = 0 . \tag{1}$$

In the circular approximation, the node voltages of the network satisfy equation (1) and two boundary conditions: V_0 is given and V_n is bounded in the limit as $n \to \infty$. To make further progress in the solution of (1), apply the z-transform, also known as the method of generating-functions. Introduce the function *G(z)*, where

$$G(z) \equiv \sum_{n=1}^{\infty} V_n z^n . \tag{2}$$

G(z) is known as the z-transform of the sequence V_n. Notice that $G(0) = 0$. What else can be said about the function *G(z)?* On account of the boundedness of the V_n, *G(z)* is analytic on $|z| < 1$. The z transform thus converts a local property, the behavior of the point at infinity, (V_n bounded at $n = \infty$), to a global property, the analyticity of *G(z)* on $|z| < 1$. The hard part of solving the hexagonal network is connecting the boundedness condition at infinity with the boundary condition at the origin. The z-transform is effective for this problem because it synthesizes a domain where the two boundary conditions can interact. The analyticity of *G(z)* is the mechanism which allows the two boundary conditions to commingle.

3.1 The z-Transform of The Hexagonal Net

To take the z-transform of the Kirchhoff current law, equation (1), multiply this equation by z^n and sum on *n*.

$$\sum_{n=1}^{\infty} \{z^n(2n+1)V_{n+1} - z^n n(4+a)V_n + z^n(2n-1)V_{n-1}\} = 0 . \tag{3}$$

Equation (3) can be further reduced with the aid of the following three straightforward identities:

$$z^n(2n+1)V_{n+1} = \left[2\frac{d}{dz}(z^{n+1}) - \frac{1}{z}z^{n+1}\right]V_{n+1} ,$$

$$z^n n(4+a)V_n = \left[z\frac{d}{dz}(z^n)\right](4+a)V_n ,$$

$$z^n(2n-1)V_{n-1} = \left[2z^2\frac{d}{dz}(z^{n-1}) + zz^{n-1}\right]V_{n-1} .$$

Notice that the power of z matches the subscript of V on the right hand sides of these identities. Substitute these identities into (3) to obtain

$$\sum_{n=1}^{\infty}\left\{\left[2\frac{d}{dz}-\frac{1}{z}\right]z^{n+1}V_{n+1}-(4+a)\left[z\frac{d}{dz}\right]z^{n}V_{n}+\left[2z^{2}\frac{d}{dz}+z\right]z^{n-1}V_{n-1}\right\}=0\,. \tag{4}$$

Two simple relations help to convert (4) into a useful statement about $G(z)$:

$$\sum_{n=1}^{\infty} z^{n+1}V_{n+1}=G(z)-zV_1\,,$$

$$\sum_{n=1}^{\infty} z^{n-1}V_{n-1}=G(z)+V_0\,.$$

Notice that the agreement between powers and subscripts in (4) is the basis of these relations. Substituting these relations into (4) yields

$$\left[2\frac{d}{dz}-\frac{1}{z}\right](G(z)-zV_1)-(4+a)\left[z\frac{d}{dz}\right]G(z)+\left[2z^{2}\frac{d}{dz}+z\right](G(z)+V_0)=0\,. \tag{5}$$

Collect like terms in (5) and obtain at last the z transform of equation (1).

$$2\left(z^{2}-\left[2+\frac{a}{2}\right]z+1\right)\frac{dG}{dz}+\left(z-\frac{1}{z}\right)G=V_1-zV_0\,. \tag{6}$$

Pause a moment and absorb the structure of the differential equation, (6). (6) is a first-order, inhomogeneous, linear differential equation which the function $G(z)$ must satisfy. The coefficients in (6) are functions of the independent variable z. In particular, the coefficient multiplying the derivative is twice the destined-to-become-familiar quadratic $Q(z)$, last seen in section (1.1). Observe that the right hand side of (6) is a linear function of z. Observe too that the right hand side of (6) involves the unknown voltage V_1.

3.2 The Ubiquitous Quadratic *Q(z)*

The quadratic factor $z^2-[2+a/2]z+1$ multiplying the derivative in equation (6) is ubiquitous. It is advantageous to become familiar with this quadratic and its roots.

$$Q(z)\equiv z^{2}-\left(2+\frac{a}{2}\right)z+1=(z-r_{+})(z-r_{-})\,, \tag{7}$$

where r_+ and r_- are the roots of the quadratic $Q(z)$, so $Q(r_+)=0$ and $Q(r_-)=0$. The quadratic formula, applied to equation (7), gives expressions for these roots.

$$r_{\pm} = \frac{2 + a/2}{2} \pm \frac{1}{2}\sqrt{2a + \left(\frac{a}{2}\right)^2}\ , \tag{8}$$

where the subscripts "+" and "-" correspond to the choice of sign in the quadratic formula.

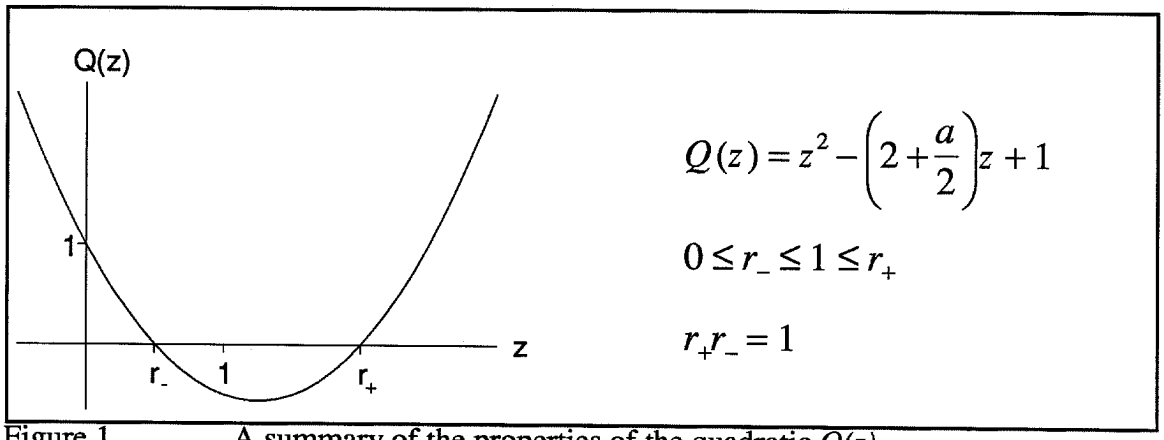

Figure 1 A summary of the properties of the quadratic $Q(z)$.

In general, what can be said about the roots r_+ and r_-? Since the constant term of the quadratic is unity, $r_+ r_- = 1$. Thus one root lies inside the unit circle (in the complex plane) and the other lies outside. Since $a \ge 0$, the discriminant in equation (8) is positive and so both roots are real. It is possible to manipulate (8) into a more revealing form:

$$r_- = 1 - \frac{2}{1 + \sqrt{1 + 8L^2}}, \text{ where } L \equiv \frac{1}{\sqrt{a}}, \text{ and } L \ge 0\ . \tag{9}$$

Equation (9) makes it easy to see that $0 \le r_- \le 1$. Also, since the product of the roots is unity, $r_+ \ge 1$. Figure (1) summarizes the important properties of $Q(z)$.

3.3 Solving for *G(z)*. Determining V_1

To find $G(z)$, the z transform of V_n, solve the differential equation (6) subject to the condition that $G(z)$ be analytic throughout $|z| < 1$. Solve (6) in two steps. First, find a solution to the homogeneous version of (6)[1]:

[1] Equations (10) and higher make frequent use of the absolute-value and signum functions. So long as z is real, the absolute-value and signum functions are a convenient way of keeping signs straight. Signums and absolute-values would not suffice if it became necessary to extend the solution off the real axis and into the complex plane. To extend the solution into the complex plane, remove the signum functions and erase the absolute-value signs and keep careful track of the branches of the various square-root functions.

$$2Q(z)\frac{dG_H}{dz}+\left(z-\frac{1}{z}\right)G_H=0 \qquad \Rightarrow \qquad G_H=\sqrt{\frac{z}{|Q(z)|}}. \tag{10}$$

Notice that $G_H(z)$ is singular at $z=r_-$ because Q is zero there. Now use the homogeneous solution, $G_H(z)$, in combination with the method of reduction of order to find a particular solution of (6). Posit a solution of the form

$$G(z)=G_H(z)g(z)\,, \tag{11}$$

where $g(z)$ has yet to be determined. The inhomogeneous version of (6) determines the unknown function $g(z)$. Thus

$$G(z)=G_H(z)g(z)\rightarrow(6)\ \rightarrow\ 2Q(z)G_H\frac{dg}{dz}=V_1-zV_0 \tag{12}$$

$$\rightarrow\ g=\frac{1}{2}\int_0^z \mathrm{sgn}[Q(\zeta)]\frac{V_1-\zeta V_0}{\sqrt{\zeta|Q(\zeta)|}}d\zeta\ .$$

A slight subtlety in (12) concerns the choice of the lower limit of the integral for $g(z)$. This lower limit controls the amount of the homogeneous solution, $G_H(z)$, which is included in the particular solution. The analyticity of $G(z)$ at the origin is the guideline which determines what amount of $G_H(z)$ is the right amount. It is easy to verify that $G(z)$ is analytic at $z=0$ when the lower limit of the integral in (12) is zero. Note that the condition $G(0)=0$ fails to determine $G(z)$ uniquely because $G_H(0)=0$.

The condition that $G(z)$ be analytic on $|z|<1$ also determines V_1. Since $r_-<1$, $G(z)$ must be analytic at $z=r_-$. But, $G(z)=G_H(z)g(z)$ and $G_H(z)$ is singular at $z=r_-$. In order that $g(z)$ cancel the singularity in $G_H(z)$ at $z=r_-$, $g(z)$ must be no larger than $o(\sqrt{r_--z})$ as $z\rightarrow r_-$. In other words, $g(r_-)=0$. But, considering (12), $g(r_-)=0$ implies that

$$V_1\int_0^{r_-}\frac{1}{\sqrt{\zeta Q(\zeta)}}d\zeta=V_0\int_0^{r_-}\frac{\zeta}{\sqrt{\zeta Q(\zeta)}}d\zeta\ . \tag{13}$$

Neither of the integrals in (13) is amenable to direct evaluation in terms of elementary functions. Nevertheless, some manipulation of (13) is helpful. Notice that the integrands of (13) are singular at both endpoints of the integration range. The trigonometric substitution $\zeta=r_-\sin^2(\theta)$ absorbs the singularities and puts (13) into a nice form.

$$\frac{V_1}{V_0}=r_+\left(1-\frac{\mathrm{E}(r_-^2)}{\mathrm{K}(r_-^2)}\right)\ , \tag{14}$$

where

$$K(r_-^2) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r_-^2 \sin^2(\theta)}} d\theta \ \ \& \ \ E(r_-^2) = \int_0^{\pi/2} \sqrt{1 - r_-^2 \sin^2(\theta)}\, d\theta \, .$$

$K(r_-^2)$ and $E(r_-^2)$ are the complete elliptic integrals of the first and second kind, evaluated at the parameter value r_-^2. Although equation (14) is not especially illuminating, it is an effective way to evaluate V_1 when combined with the algorithm of the arithmetic and geometric mean for computing elliptic integrals. Consult Abromowitz and Stegun[1] for details. Figure (1) shows how V_1, (and also V_2, V_3, and V_4), depend on the parameter RG.

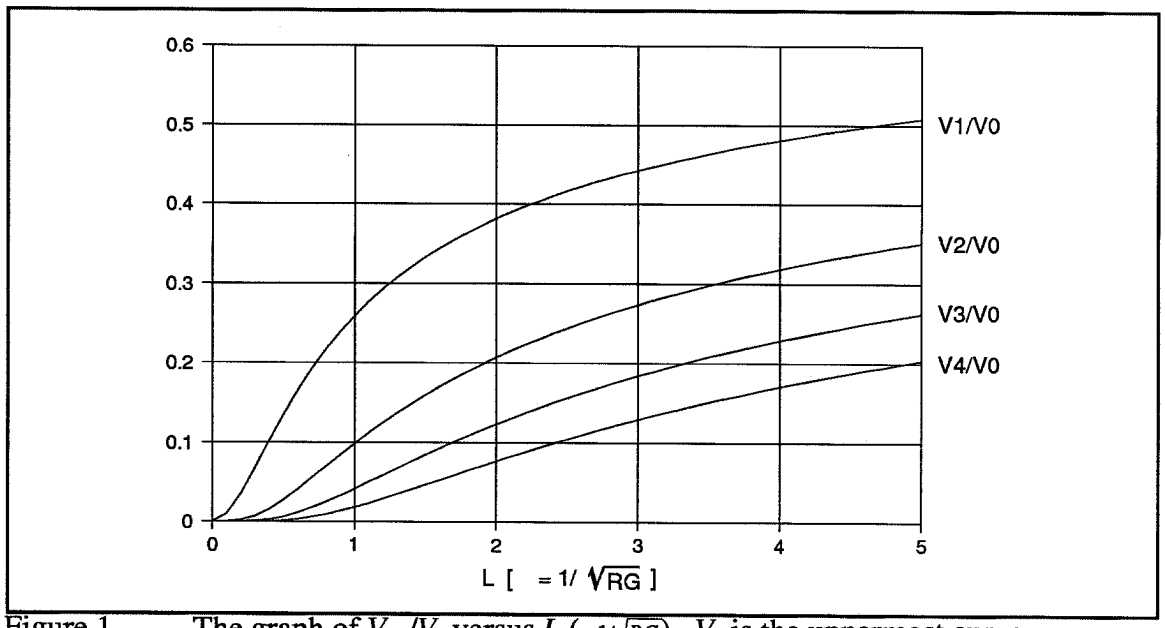

Figure 1 The graph of $V_{1\text{-}4}/V_0$ versus L $(= 1/\sqrt{RG})$. V_1 is the uppermost curve.

In principal, V_1 and equations (10, 11 & 12) together determine $G(z)$. Indeed, $G(z)$ can be expressed in terms of incomplete elliptic integrals. However $G(z)$ is not of much interest, in and of itself. It is the coefficients in the power series expansion of $G(z)$ that are of interest. These coefficients, which are the V_n, do not seem to be particularly accessible from any of the forms of $G(z)$ which we have found.[2] Nevertheless, $G(z)$ is useful. In the next section, (3.4), adroit use of $G(z)$ determines the constant in the asymptotic approximation for V_n.

3.4 *G(z)* And The Asymptotic Expansion of V_n

In section (1.2), asymptotic analysis of the point at infinity yielded the leading-order result

[1] *Handbook of Mathematical Functions*, edited by Milton Abromowitz and Irene A. Stegun, issued June 1964; 10th printing Dec 1972, National Bureau of Standards Applied Mathematics Series 55.

[2] Neither repeated differentiation nor the Cauchy integral formula delivers anything useful.

$$V_n \sim V \frac{r_-^n}{\sqrt{n}} \text{ as } n \to \infty . \tag{15}$$

The analysis was unable to make contact with the boundary condition at $n = 0$, and so the constant V remained undetermined. $G(z)$ provides the necessary contact with the boundary conditions and allows the determination of V. In a nut shell, find V by calculating $G(r_+)$ in two different ways and equating the answers. One way of calculating $G(r_+)$ uses the asymptotic result, equation (15); the other way uses the exact expression for G, equation (12). Actually the procedure is a bit more complicated because $G(r_+)$ is infinite. This infinity is crucial because it is what makes it possible to use the asymptotic result (15) to calculate $G(z)$.

Consider the definition of $G(z)$, equation (2), repeated here for convenience.

$$G(z) \equiv \sum_{n=1}^{\infty} V_n z^n \quad .$$

How does $G(z)$ behave as $z \to r_+$? Considering the asymptotic result, (15), observe that the terms in the series for $G(z)$ decay only as fast as $1/\sqrt{n}$ when $z = r_+$. Since the sum $\sum_{n=1}^{\infty} 1/\sqrt{n}$ is infinite, $G(z)$ diverges as $z \to r_+$. Briefly, $G(z)$ diverges when $z = r_+$ because the terms in the series do not become small enough fast enough. In other words, the tail of the series determines the fate of $G(r_+)$. Suppose now that z is just slightly less than r_+. Then $G(z)$ is very large and, again, the tail of the series forms the dominant contribution to $G(z)$. So long as the tail of the series dominates, it is appropriate to use the asymptotic form of V_n in calculating $G(z)$. Thus

$$G(z) \sim V \sum_{n=1}^{\infty} \frac{(zr_-)^n}{\sqrt{n}} \text{ as } z \to r_+ . \tag{16}$$

As long as z is near r_+, the individual terms in the tail of $G(z)$ change little from one term to the next, and the sum in (16) closely approximates an integral.[1]

$$G(z) \sim V \int_0^{\infty} \frac{(zr_-)^n}{\sqrt{n}} dn = V \int_0^{\infty} \exp\{n \ln(zr_-)\} n^{-1/2} dn, \text{ as } z \to r_+. \tag{17}$$

Change the variable of integration in (17) so that the integral resembles the Γ function. Let $m = -n \ln(zr_-)$. [2]

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[1] In (18) we choose the lower limit of integration to be zero for convenience. Although the integrand is singular at zero, the integral itself is not. Any lower limit that is finite and positive is acceptable in (18) since all such integrals are asymptotic to G(z) as $z \to r_+$.

[2] Recall that
$$\Gamma(1/2) = \int_0^{\infty} e^{-m} m^{-1/2} dm = \sqrt{\pi} .$$

$$G(z) \sim \frac{V\Gamma(\frac{1}{2})}{\sqrt{\ln(r_+/z)}} = \sqrt{\frac{\pi}{\ln(r_+/z)}}V \text{ as } z \to r_+ . \tag{18}$$

As advertised, *G(z)* diverges as $z \to r_+$. Suppose that $z = r_+(1-\delta)$, where $0 < \delta \ll 1$. Substitute this *z* into (18) and conclude that

$$\lim_{\delta \to 0} \sqrt{\delta} G[r_+(1-\delta)] = V\sqrt{\pi} . \tag{19}$$

On the other hand, it is possible to evaluate *G(z)* directly by using equation (12). If $r_- \le z < r_+$ then, according to (12),

$$G(z) = \frac{1}{2}\sqrt{\frac{z}{|Q(z)|}} \int_{r_-}^{z} \frac{\zeta V_0 - V_1}{\sqrt{\zeta |Q(\zeta)|}} d\zeta . \tag{20}$$

Notice in (20) that the lower limit of integration is r_-, even though this lower limit is zero in equation (12). The discrepancy between the two lower limits is illusory because the integral in (20) is zero when taken over the range zero to r_-.[1] When *z* is very near r_+, all the action in (20) happens inside the *Q(z)*. Suppose that $z = r_+(1-\delta)$ where, as before, $0 < \delta \ll 1$. Substitute this *z* into (20) and use the definition $Q(z) = (z - r_+)(z - r_-)$, equation (7). As expected, *G(z)* diverges as $1/\sqrt{\delta}$ when $\delta \to 0$. Conclude that

$$\lim_{\delta \to 0} \sqrt{\delta} G[r_+(1-\delta)] = \frac{1}{2\sqrt{r_+ - r_-}} \int_{r_-}^{r_+} \frac{\zeta V_0 - V_1}{\sqrt{\zeta(r_+ - \zeta)(\zeta - r_-)}} d\zeta . \tag{21}$$

Compare (21) with (19). If these two equations are to be consistent, then

$$V = \frac{1}{2\sqrt{\pi(r_+ - r_-)}} \int_{r_-}^{r_+} \frac{\zeta V_0 - V_1}{\sqrt{\zeta(r_+ - \zeta)(\zeta - r_-)}} d\zeta , \tag{22}$$

where V_1 satisfies equation (13). Equation (22) reduces to elliptic integrals just as the expression for V_1, equation (13), did. The integrand in equation (22) is singular at both end points of the integration range. The trigonometric substitution $\zeta = r_+ - (r_+ - r_-)\cos^2(\theta)$ absorbs the singularities and expresses (22) in terms of the elliptic integrals K and E. Thus

$$V = \frac{1}{\sqrt{\pi(1 - r_-^2)}} \{V_0 \mathrm{E}(1 - r_-^2) - r_- V_1 \mathrm{K}(1 - r_-^2)\} . \tag{23}$$

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[1] Recall that $g(r_-) = 0$; see the discussion immediately preceding equation (13) for more detail.

This expression for V simplifies dramatically. Substitute for V_1 using equation (14) and employ *Legendre's relation,* that $\mathrm{K}(p)\mathrm{E}(1-p)+\mathrm{K}(1-p)\mathrm{E}(p)-\mathrm{K}(p)\mathrm{K}(1-p)=\pi/2$, to reduce (23) to the simple expression

$$V=\frac{V_0}{2\,\mathrm{K}(r_-^2)}\sqrt{\frac{\pi}{1-r_-^2}} \quad . \tag{24}$$

An alternative to (24) which avoids elliptic integrals altogether is desirable. Weierstrass has found expressions for elliptic integrals in terms of power series with astonishingly rapid convergence.[1] Using Weierstrass, substitute for K and obtain

$$V\approx\frac{V_0}{4\sqrt{\pi}}\left[1+(1-r_-^2)^{-1/4}\right]^2 \text{, where } r_-=1-\frac{2}{1+\sqrt{1+8L^2}}. \tag{25}$$

Strictly, equation (25) is valid only in the limit as $L\to 0$. In fact, (25) is quite accurate even for $L\gg 1$. The relative error in (25) is less than 0.2% when L=15. The relative error has grown to 0.5% when L=30, to 1% when L=65, and to 2% when L=130.

[1] See "Theory of Functions of a Complex Variable," E. T. Copson, Oxford University Press, 1957.