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AN ESTIMATE FOR THE NUMBER OF BOUND STATES OF THE SCHRÖDINGER OPERATOR IN TWO DIMENSIONS

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Abstract. For the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^2$ let $N(V)$ be the number of bound states. One obtains the following estimate:

$$N(V) \leq \; 1 \; + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)| \; |V(y)| \; |C_1 \ln|x-y| + C_2|^2 \; dx\,dy$$

where $C_1 = -\frac{1}{2\pi}$ and $C_2 = \frac{\ln 2 - \gamma}{2\pi}$ (γ is the Euler constant). This estimate holds for all potentials for which the previous integral is finite.

1. Introduction

On $\mathbb{R}^3$, there is a well-known bound for the number of bound states $N(V)$ discovered by Birman [3] and Schwinger [9]:

$$N(V) \leq \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)| \; |V(y)|}{|x-y|^2} \; dx\,dy.$$

The method of proof is the "Birman-Schwinger principle", which states that for a potential $V \leq 0$ and for a number $E < 0$:

$$N_{(-\infty,E]}(-\Delta + V) = N_{[1,\infty)}(|V|^{1/2}(-\Delta - E)^{-1}|V|^{1/2})$$

where $N_I(T)$ denotes the number of eigenvalues (counting multiplicities) of the operator T in the interval I.

The operator $(-\Delta - E)^{-1}$ has integral kernel

$$G_3(x, y, E) = \frac{1}{4\pi}|x-y|^{-1}e^{-\sqrt{-E}|x-y|},$$

which converges when $E \uparrow 0$ (for $x \neq y$). This implies that an estimate for $N(V)$ can be obtained by estimating $N_{(-\infty,E]}(-\Delta + V)$ first and then taking $E \uparrow 0$. A detailed proof of this result can be found in [12].

This proof does not work in two dimensions since the integral kernel of $(-\Delta + V)^{-1}$ contains $\ln(\sqrt{-E}|x-y|)$ which diverges as $E \uparrow 0$.

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Khuri, Martin and Wu conjectured in [5] the following bound for $N(V)$ in two dimensions:

$$
\begin{aligned}
N(V) \leq\ & 1 + C_1 \int_{\mathbb{R}^2} |V|^*(x) \left(\ln\left|\frac{x_0}{x}\right|\right)^+ dx \\
& + C_2 \int_{\mathbb{R}^2} |V_-(x)| \left(\ln\left|\frac{x}{x_0}\right|\right)^+ dx \ + \ C_3 \int_{\mathbb{R}^2} |V_-(x)|\ dx
\end{aligned} \tag{1.1}
$$

where $x_0 \neq 0$ and $|V|^*$ denotes the symmetric decreasing rearrangement of $|V|$.

Our goal is to find bounds similar to (1.1) using the "Birman-Schwinger principle" and a method discovered by Simon in [11]. The idea is to write the integral kernel of $|V|^{1/2}(-\Delta - E)^{-1}|V|^{1/2}$ as the sum of a Hilbert-Schmidt operator and a rank-one perturbation and then apply the Birman-Schwinger method.

The main results in this paper are Theorem 3.3 and Proposition 3.4, which give two bounds for $N(V)$ in two dimensions:

$$B(V) = 1 \ + \ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)|\ \left| -\frac{1}{2\pi}\ln|x-y| + \frac{\ln 2 - \gamma}{2\pi}\right|^2 dx\, dy$$

and

$$
\begin{aligned}
\widetilde{B}(V) = 1 \ + \ C_3\|V\|_1^2 \ + \ C_4\|V\|_1 \int_{\mathbb{R}^2} |V(x)|\,[\ln(1+|x|)]^2\ dx \\
+ \ C_5\|V\|_1 \int_{\mathbb{R}^2} |V|^*(x)\ \ [\ln|x|]^2\ \ \chi_{\{|x|\leq 1\}}\ dx.
\end{aligned}
$$

These bounds are similar to the one conjectured in [5] with the difference that for large coupling constants, $B(\lambda V), \widetilde{B}(\lambda V) \sim \lambda^2$ whereas the bound conjectured by Khuri, Martin and Wu is $\sim \lambda$.

2. Estimate for nice potentials

One can prove the following:

Theorem 2.1. *Let $V \in L^\infty(\mathbb{R}^2)$ be a real-valued compactly supported potential. Then*

$$N(V) \leq \ 1 \ + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)|\ \ |C_1 \ln|x-y| \ + \ C_2|^2\ dx\, dy$$

where $C_1 = -\frac{1}{2\pi}$ and $C_2 = \frac{\ln 2 - \gamma}{2\pi}$ ($\gamma \approx 0.577$ is the Euler constant).

Proof. Since $V \in L^\infty(\mathbb{R}^2)$ and $\operatorname{supp}(V)$ is compact, it follows that $V \in L^2(\mathbb{R}^2)$. So $(-\Delta + V)$ is a well-defined selfadjoint operator with $\sigma_{\text{ess}}(-\Delta+V) = [0,\infty)$.

Without loss of generality, one can assume $V \leq 0$. Indeed, if $V_- = \max(-V, 0)$ is the negative part of V, then

$$-\Delta + (-V_-)\ \leq -\Delta + V.$$

So by the min-max principle, $N(V) \leq N(-V_-)$.

Now, from the Birman-Schwinger principle one gets, for $E < 0$:

$$N_{(-\infty,E]}(-\Delta + V) = N_{[1,\infty)}(|V|^{1/2}(-\Delta - E)^{-1}|V|^{1/2}).$$

The integral kernel of $(-\Delta + E)^{-1}$ is (cf. [1] and [2]):

$$G_2(x,y,E) = \frac{1}{2\pi} K_0(\sqrt{-E}\ |x-y|) \tag{2.1}$$

where K_0 is a modified Bessel function. In particular,

$$K_0(x) = -(\ln x)I_0(x) + h(x) \tag{2.2}$$

where the Bessel function I_0 and the function h (defined on $\mathbb{R}$) are real-valued analytic functions with $I_0(0) = 1$ and $h(0) = \ln 2 - \gamma$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be such that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$. Then, for $f(x) = -\frac{1}{2\pi}I_0(x)\varphi(x)$ and $g(x) = \frac{1}{2\pi}K_0(x) - f(x)\ln(x)$, one gets

$$\frac{1}{2\pi}K_0(x) = (\ln x)f(x) + g(x). \tag{2.3}$$

The functions f and g are in $C^\infty(\mathbb{R})$, with $f(0) = -\frac{1}{2\pi}$ and $g(0) = \frac{\ln 2-\gamma}{2\pi}$. Furthermore, f has compact support and g has exponential decay at infinity (since the modified Bessel function K_0 has exponential decay at infinity). Using (2.1) and (2.3) one gets

$$\begin{aligned}G_2(x,y,E) &= \left[\ln(\sqrt{-E}\ |x-y|)\right] f(\sqrt{-E}\ |x-y|) + g(\sqrt{-E}\ |x-y|)\\ &= \ln\sqrt{-E}\ \left[f(\sqrt{-E}\ |x-y|) + \frac{1}{2\pi}\right] + \ln|x-y|\ f(\sqrt{-E}\ |x-y|)\\ &\quad +\ g(\sqrt{-E}\ |x-y|) + \left[-\frac{1}{2\pi}\ln(\sqrt{-E})\right].\end{aligned} \tag{2.4}$$

One can write $(|V|^{1/2}(-\Delta - E)^{-1}|V|^{1/2}) = A_E + B_E$ where A_E and B_E are operators on $L^2(\mathbb{R}^2)$ defined by

$$\begin{aligned}A_E\varphi(x) = \int_{\mathbb{R}^2} |V(x)|^{1/2}\ \Bigg[\ \ln\sqrt{-E}\ \left(f(\sqrt{-E}\ |x-y|) + \frac{1}{2\pi}\right)\\ +\ln|x-y|f(\sqrt{-E}\ |x-y|) + g(\sqrt{-E}\ |x-y|)\Bigg]\ |V(y)|^{1/2}\varphi(y)\,dy,\end{aligned} \tag{2.5}$$

$$B_E\varphi(x) = |V(x)|^{1/2}\int_{\mathbb{R}^2}\left[-\frac{1}{2\pi}\ln(\sqrt{-E})|V(y)|^{1/2}\right]\varphi(y)\,dy. \tag{2.6}$$

B_E is therefore a selfadjoint rank-one operator with range $\mathbb{C}|V|^{1/2}$.

Now, in order to estimate the number of eigenvalues greater than or equal to 1 of the operator A_E, the following lemma will be useful.

Lemma 2.2. *Let V, E, f, g, A_E be as before, and let F_E be the integral kernel of A_E. Then $F_E \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$.*

Proof. The potential V is compactly supported. So let $R > 0$ be such that $\mathrm{supp}(V) \subset \{x \in \mathbb{R}^2, |x| \leq R\}$.

Since f and g are bounded, one immediately gets that, for any $E < 0$, the functions $\ln(\sqrt{-E})\left(f(\sqrt{-E}\ |x-y|) + \frac{1}{2\pi}\right), f(\sqrt{-E}|x-y|)$ and $g(\sqrt{-E}|x-y|)$ are bounded on $\mathbb{R}^2$.

A simple computation shows that

$$\int_{|x|\leq R}\int_{|y|\leq R}(\ln|x-y|)^2\ dx\,dy\ \leq \pi R^2\int_{|z|\leq 2R}(\ln|z|)^2\,dz < \infty$$

and therefore, since $V \in L^\infty(\mathbb{R}^2)$, one can conclude that $F_E \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$. $\square$

From the previous lemma, it follows immediately that A_E is a selfadjoint Hilbert-Schmidt operator. Therefore, one can estimate $N_{[1,\infty)}(A_E)$ as in the proof of the

Birman-Schwinger theorem and, denoting by S the set of eigenvalues of A_E, one gets

$$\begin{aligned} N_{[1,\infty)}(A_E) &\le \sum_{\lambda\in S} |\lambda|^2 = \|A_E\|_2^2 = \|F_E\|_2^2 \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |F_E(x,y))|^2 \, dx\, dy. \end{aligned}$$

($\|A_E\|_2$ denotes the Hilbert-Schmidt norm of the operator A_E.) Since B_E is a rank-one operator, the eigenvalues of A_E and (A_E+B_E) interlace. So

$$\begin{aligned} N_{(-\infty,E]}(-\Delta+V) &= N_{[1,\infty)}(A_E+B_E) \\ &\le 1 + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |F_E(x,y))|^2 \, dx\, dy. \end{aligned} \tag{2.7}$$

Now, in order to obtain an estimate for $N(V)$ one has to take $E \uparrow 0$. The following lemma shows that the previous integral converges as E approaches 0.

Lemma 2.3. *Let V, E and F_E be as before. Then*

$$\begin{aligned} &\lim_{E\uparrow 0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |F_E(x,y)|^2 \, dx\, dy \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)| \ \left| -\frac{1}{2\pi}\ln|x-y| + \frac{\ln 2-\gamma}{2\pi}\right|^2 \ dx\, dy. \end{aligned} \tag{2.8}$$

Proof. Taking into account the definition of F_E, it suffices to prove the following three statements:

$$\lim_{E\uparrow 0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)| \ \left|\ln(\sqrt{-E})\left(f(\sqrt{-E}\ |x-y|)+\frac{1}{2\pi}\right)\right|^2 \ dx\, dy = 0; \tag{2.9}$$

$$\lim_{E\uparrow 0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)| \ \left|\ln|x-y|\left(f(\sqrt{-E}\ |x-y|)+\frac{1}{2\pi}\right)\right|^2 \ dx\, dy = 0; \tag{2.10}$$

$$\lim_{E\uparrow 0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)| \ \left|g(\sqrt{-E}|x-y|) - \frac{\ln 2-\gamma}{2\pi}\right|^2 \ dx\, dy = 0. \tag{2.11}$$

Since V is compactly supported and f and g are continuous with $f(0) = -\frac{1}{2\pi}$ and $g(0) = \frac{\ln 2-\gamma}{2\pi}$, one gets (2.10) and (2.11). As for (2.9) one can write

$$\begin{aligned} &\ln(\sqrt{-E})\left(f(\sqrt{-E}\ |x-y|)+\frac{1}{2\pi}\right) \\ &\qquad = \ln(\sqrt{-E})\ \sqrt{-E}\ |x-y| \frac{f(\sqrt{-E}\ |x-y|)+\frac{1}{2\pi}}{\sqrt{-E}\ |x-y|}. \end{aligned}$$

Let

$$k(z) = \frac{f(z)+\frac{1}{2\pi}}{z}.$$

Since $f \in C^\infty(\mathbb{R}^2)$, it follows that k is continuous (and, in particular, bounded on compact sets). Therefore (since $|x|\le R$ and $|y|\le R$ imply $|x-y|\le 2R$), there exists an $M>0$ such that

$$k(\sqrt{-E}\ |x-y|)\ \chi_{\mathrm{supp}(V)}(x)\ \chi_{\mathrm{supp}(V)}(y)\ \le M$$

for any $E\in(-1,0)$ and any $x,y\in\mathbb{R}^2$.

Since

$$\lim_{E\uparrow 0} \ln(\sqrt{-E})\ \sqrt{-E} = 0,$$

one gets (2.9). □

Using (2.7) and the previous lemma, one immediately gets

$$N(V) \leq\ 1\ +\ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)|\ \left| -\frac{1}{2\pi}\ln|x-y| + \frac{\ln 2-\gamma}{2\pi}\right|^2\ dx\,dy.$$

□

The proof of Lemma 2.2 also shows that for $V \in L^\infty(\mathbb{R}^2)$ with $\operatorname{supp}(V)$ compact, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)|\ \left| -\frac{1}{2\pi}\ln|x-y| + \frac{\ln 2-\gamma}{2\pi}\right|^2\ dx\,dy\ <\ \infty. \tag{2.12}$$

3. Estimate for a larger class of potentials

The result of Theorem 2.1 can be extended to any potential V for which (2.12) holds. In order to prove this, the following lemma will be useful.

Lemma 3.1. *Let V be a real-valued measurable function such that*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)|\ |C_1\ln|x-y| + C_2|^2\ dx\,dy\ <\ \infty.$$

Then $V \in L^1(\mathbb{R}^2)$.

Proof. Let V be a nonzero measurable function. Then there exists $a > 0$ such that for $M = \{x \in \mathbb{R}^2, |V(x)| \geq a\}$ one has $\lambda_2(M) > 0$ (λ_2 is the Lebesgue measure in $\mathbb{R}^2$). Furthermore, there exists a bounded open set $N \subset \mathbb{R}^2$ such that $\lambda_2(M \cap N) = b > 0$.

Let $\epsilon > 0$ be small enough such that

$$\lambda_2\left(\{z \in \mathbb{R}^2, |z| \in [e^{-C_2/C_1} - \epsilon, e^{-C_2/C_1} + \epsilon]\}\right) \leq \frac{b}{2}. \tag{3.1}$$

For any $x \in \mathbb{R}^2$, let $N_x = \{y \in \mathbb{R}^2, |y - x| \in [e^{-C_2/C_1} - \epsilon, e^{-C_2/C_1} + \epsilon]\}$. From (3.1) it follows that $\lambda_2(N_x) \leq \frac{b}{2}$ for any $x \in \mathbb{R}^2$.

Since $C_1 \ln|z| + C_2 = 0$ if and only if $|z| = e^{-C_2/C_1}$, it follows that there exists a $c > 0$ such that for a fixed $x \in \mathbb{R}^2$, one has

$$|C_1 \ln|x-y| + C_2| > c \ \text{ for any }\ y \notin N_x. \tag{3.2}$$

Now, for any $x \in \mathbb{R}^2$, since $\lambda_2(M \cap N) = b > 0$ and $\lambda_2(N_x) \leq \frac{b}{2}$ one gets $\lambda_2(P_x) > \frac{b}{2}$, where $P_x = (M \cap N)\backslash N_x$. So

$$\begin{aligned}\int_{\mathbb{R}^2} |V(y)|\,|C_1\ln|x-y| + C_2|^2\ dx\,dy &\geq \int_{P_x} |V(y)|\,|C_1\ln|x-y| + C_2|^2\ dx\,dy \\ &\geq\ a\cdot c^2\cdot\lambda_2(P_x)\ \geq\ \frac{ac^2b}{2}\ >\ 0.\end{aligned} \tag{3.3}$$

Let $d = ac^2b/2$. Since

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\ |V(y)|\ |C_1\ln|x-y| + C_2|^2\ dx\,dy$$

$$= \int_{\mathbb{R}^2} |V(x)| \left(\int_{\mathbb{R}^2} |V(y)| \ \left|C_1 \ln|x-y| + C_2\right|^2 \ dy \right) dx \geq \ d\|V\|_1,$$

one gets $\|V\|_1 < \infty$; so $V \in L^1(\mathbb{R}^2)$. □

Proposition 3.2. *Let V be a real-valued measurable function such that*

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V(x)| \ |V(y)| \ \left|C_1 \ln|x-y| + C_2\right|^2 \, dx\, dy \ < \ \infty.$$

Then V is a relatively form compact perturbation of $-\Delta$ that defines $-\Delta + V$ with $\sigma_{\mathrm{ess}}(-\Delta + V) = [0, \infty)$.

Proof. $C_1 \neq 0$ implies that $|V(x)| \ |V(y)| \ \left|\ln|x-y| + D\right|^2 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ where $D = C_2/C_1$. Since $V \in L^1(\mathbb{R}^2)$, one gets

$$\begin{aligned} &\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ |V(x)|\,|V(y)| \ \left|\ln|x-y|\right|^2 \, dx\, dy \\ &\qquad \leq 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V(x)|\,|V(y)|\,\left|\ln|x-y| + D\right|^2 \, dx\, dy \ + \ 2D^2\|V\|_1^2 \ < \ \infty. \end{aligned} \tag{3.4}$$

From previous considerations, the integral kernel of $(-\Delta + 1)^{-1}$ is

$$\frac{1}{2\pi} K_0(|x-y|) = (\ln|x-y|) f(|x-y|) + g(|x-y|),$$

and therefore the integral kernel of $|V|^{1/2}(-\Delta + 1)^{-1}|V|^{1/2}$ is

$$K(x,y) = |V(x)|^{1/2} |V(y)|^{1/2} \left[(\ln|x-y|) f(|x-y|) + g(|x-y|) \right].$$

Since f and g are bounded, (3.4) implies that $K \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$. Therefore, $|V|^{1/2}(-\Delta + 1)^{-1}|V|^{1/2}$ is Hilbert-Schmidt. One can now write $(-\Delta + 1)^{-1} = (-\Delta + 1)^{-1/2}(-\Delta + 1)^{-1/2}$ and, using trace class ideals methods (see [10]), it follows that $(-\Delta + 1)^{-1/2}|V|(-\Delta + 1)^{-1/2}$ is Hilbert-Schmidt. This implies (cf. [8]) that V is a relatively form compact perturbation of $-\Delta$ and therefore V has relative form bound zero and $\sigma_{\mathrm{ess}}(-\Delta + V) = \sigma_{\mathrm{ess}}(-\Delta) = [0, \infty)$. □

Theorem 3.3. *Let V be a real-valued measurable function such that*

$$B(V) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V(x)| \ |V(y)| \ \left| -\frac{1}{2\pi} \ln|x-y| + \frac{\ln 2 - \gamma}{2\pi} \right|^2 \, dx\, dy \ < \ \infty.$$

Then

$$N(V) \leq \ 1 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V(x)| \ |V(y)| \ \left| -\frac{1}{2\pi} \ln|x-y| + \frac{\ln 2 - \gamma}{2\pi} \right|^2 \, dx\, dy.$$

Proof. As was shown before, one can assume, without loss of generality, that $V \leq 0$. For any positive integer N, let

$$V_N(x) = \max(V(x), -N) \ \chi_{(-N,N)}(x).$$

Obviously, $V_N \in L^\infty$ and $\mathrm{supp}(V_N) \subset [-N, N]$ for any N. So from Theorem 2.1, one gets

$$N(V_N) \leq 1 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V(x)| \ |V(y)| \ \left| -\frac{1}{2\pi} \ln|x-y| + \frac{\ln 2 - \gamma}{2\pi} \right|^2 \, dx\, dy.$$

Since $(-\Delta + V_N)\varphi \to (-\Delta + V)\varphi$ for any φ in the domain of $(-\Delta)^{1/2}$ it follows, using a result from [6], that $(-\Delta + V_N) \to (-\Delta + V)$ in strong resolvent sense, which implies that $N(V) \leq \limsup_{N\to\infty} N(V_N)$. Since $|V_N| \uparrow |V|$ one gets, using the monotone convergence theorem, that

$$N(V) \;\leq\; 1 + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\; |V(y)| \;\left| -\frac{1}{2\pi}\ln|x-y| + \frac{\ln 2 - \gamma}{2\pi}\right|^2 dx\,dy.$$

□

Proposition 3.4. *Let $V \in L^1(\mathbb{R}^2)$ be such that $V\left[\ln(1+|x|)\right]^2 \in L^1(\mathbb{R}^2)$ and $|V|^*\left[\ln|x|\right]^2 \chi_{\{|x|\leq 1\}} \in L^1(\mathbb{R}^2)$ where $|V|^*$ is the symmetric decreasing rearrangement of $|V|$. Then the integral in Theorem 3.3 is finite (i.e., $B(V) < \infty$) and*

$$\begin{aligned} N(V) \;\leq\; 1 \;+\; C_3\|V\|_1^2 \;+\; C_4\|V\|_1 \int_{\mathbb{R}^2} |V(x)|\left[\ln(1+|x|)\right]^2 dx \\ +\; C_5\|V\|_1 \int_{\mathbb{R}^2} |V|^*(x)\;\left[\ln|x|\right]^2\;\chi_{\{|x|\leq 1\}}\,dx \end{aligned} \tag{3.5}$$

where C_3, C_4 and C_5 are positive constants.

Proof. As in (3.4),

$$\begin{aligned} B(V) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\;|V(y)|\;|C_1\ln|x-y| + C_2|^2\,dx\,dy \\ &\leq 2C_1^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)|\;|V(y)|\;|\ln|x-y||^2\,dx\,dy + 2C_2^2\|V\|_1^2. \end{aligned} \tag{3.6}$$

Let $S_1 = \{(x,y) \in \mathbb{R}^2\times\mathbb{R}^2, |x-y| \geq 1\}$ and $S_2 = (\mathbb{R}^2\times\mathbb{R}^2)\setminus S_1$. Then, since $|x-y| \leq (1+|x|)(1+|y|)$, one gets that

$$|\ln|x-y||^2 \;\leq\; 2\left[\ln(1+|x|)\right]^2 + 2\left[\ln(1+|y|)\right]^2$$

for any $(x,y)\in S_1$ and therefore

$$\int_{S_1} |V(x)|\;|V(y)|\;|\ln|x-y||^2\,dx\,dy \leq 4\|V\|_1\int_{\mathbb{R}^2} |V(x)|\left[\ln(1+|x|)\right]^2 dx. \tag{3.7}$$

Now let $h(z) = |\ln|z||^2\chi_{\{|z|\leq 1\}}$. Using the Brascamp-Lieb-Luttinger inequality [4] one gets:

$$\begin{aligned} \int_{S_2} |V(x)||V(y)|\,|\ln|x-y||^2\,dx\,dy &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)||V(y)|h(x-y)\,dx\,dy \\ &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V|^*(x)|V|^*(y)h^*(x-y)\,dx\,dy \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V|^*(x)|V|^*(y)h(x-y)\,dx\,dy. \end{aligned} \tag{3.8}$$

Since for any $y\in\mathbb{R}^2$,

$$\int_{\{|x-y|\leq 1\}} |V|^*(x)\,|\ln|x-y||^2\,dx \leq \int_{\{|x|\leq 1\}} |V|^*(x)(\ln|x|)^2\,dx,$$

one gets, using (3.8),

$$\int_{S_2} |V(x)||V(y)|\,|\ln|x-y||^2\,dx\,dy \leq \|V\|_1\int_{\mathbb{R}^2} |V|^*(x)\left[\ln|x|\right]^2\chi_{\{|x|\leq 1\}}\,dx. \tag{3.9}$$

Combining (3.6), (3.7) and (3.9) one gets (3.5) with $C_3 = 2C_2^2 = (\ln 2-\gamma)^2/2\pi^2$, $C_4 = 8\;C_1^2 = 2/\pi^2$ and $C_5 = 2C_1^2 = 1/2\pi^2$. □

Remarks. 1. Since $N(V) \leq N(-V_-)$, one can improve the estimates in Theorem 3.3 and Proposition 3.4 by replacing (on the right-hand side) V with V_-.

2. As mentioned before, for large coupling constants, $B(\lambda V), \widetilde{B}(\lambda V) \sim \lambda^2$. Using the trace class ideals methods developed in [10], it should be possible to get bounds $\sim \lambda^{1+\epsilon}$, for any $\epsilon > 0$. However, it is not clear how to get a bound $\sim \lambda$ as conjectured by Khuri, Martin and Wu in [5].

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