

ON THERMODYNAMICS OF RATIONAL MAPS. II: NON-RECURRENT MAPS

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ABSTRACT

The pressure function $p(t)$ of a non-recurrent map is real analytic on some interval $(0, t_*)$ with t_* strictly greater than the dimension of the Julia set. The proof is an adaptation of the well known tower techniques to the complex dynamics situation. In general, $p(t)$ need not be analytic on the whole positive axis.

1. Introduction and results

1.1. Introduction

In this paper we study analyticity properties of the pressure function of non-recurrent maps. Our approach is based on the well known tower techniques adapted to the complex dynamics situation.

The pressure function $p(t)$, which is defined in terms of the Poincaré series (see (1.4)), carries essential information about ergodic and dimensional properties of the maximal measure. In particular, it characterizes the dimension spectrum of harmonic measure on the Julia set in the case of a polynomial dynamics. According to the classical theory of Sinai, Ruelle and Bowen, $p(t)$ is real analytic if the dynamics is *hyperbolic*, that is, expanding on the Julia set. This fact is closely related to the so called ‘spectral gap’ phenomenon, which also implies other important features of hyperbolic dynamics such as the existence of equilibrium states, exponential decay of correlations, etc. The problem of extending (some parts of) the classical theory to the non-hyperbolic case has become one of the central themes in the ergodic theory of conformal dynamics.

In the first part of this work [8], we provided a detailed analysis of the negative part $t < 0$ of the pressure function for general rational maps. The case $t > 0$ is substantially more complicated (and more important). The main difficulty arises from the presence of singularities (critical points) on the Julia set. To circumvent this difficulty, we propose to use a tower construction which forces the dynamics to be expanding on some auxiliary space. The tower method has been widely used in the general theory of dynamical systems with some degree of hyperbolicity (see especially [12]), and in particular in 1-dimensional real dynamics, where the construction is known as *Hofbauer’s tower*. To apply this method in the complex case, it is natural to use some basic elements of the Yoccoz jigsaw puzzle structure (see [9]).

We will discuss only the simplest type of non-hyperbolic behavior: the case of non-recurrent dynamics (every critical point in the Julia set is non-recurrent, that is, is away from its iterates) without parabolic cycles (see [2] for various characterizations

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in the polynomial case). We will see that the tower method provides an almost automatic way to establish real analyticity of the pressure function on some interval $(0, t_*)$ with t_* strictly greater than the dimension of the Julia set. We will also show that it is not always possible to get analyticity on the whole positive axis. It might be interesting to apply the tower method to some other classes of conformal dynamical systems, for instance to Collet–Eckmann maps (see [3]). The main problem in the general case is that of controlling the distortion (as in Lemma 1).

1.2. Branched Cantor dynamics

We will consider the following model example where the puzzle piece structure is already explicit. This kind of dynamics appears, for example, if all but one critical points of a polynomial escape to infinity.

Let U_0 be an open Jordan domain, and suppose that we have a finite number of open topological discs ('generation 1 puzzle pieces') P_1 such that their closures \bar{P}_1 are disjoint and are contained in U_0 . Let

$$F : \bigcup P_1 \longrightarrow U_0$$

be an analytic function such that the restriction of F to one of the pieces, which we denote by U_1 , is a 2-fold branched cover $U_1 \longrightarrow U_0$, and the restriction of F to each of the other P_1 is a conformal isomorphism $P_1 \longrightarrow U_0$. For each $n \geq 0$, consider the open set

$$F^{-n}U_0 = \left\{ z : F^j z \in \bigcup P_1 \text{ for } j = 0, 1, \dots, n-1 \right\};$$

its components P_n are called *puzzle pieces* of generation n . The Julia set $J = J(F)$ is defined by the usual Cantor procedure

$$J = \bigcap_n \bigcup P_n.$$

If $z \in J$, then $P_n(z)$ is the notation for the puzzle piece of generation n containing z . Clearly,

$$FP_n(z) = P_{n-1}(Fz). \quad (1.1)$$

Let c denote the (unique) critical point of F . We will assume that $c \in J$, and call the components

$$U_n = P_n(c)$$

critical puzzle pieces. If the dynamics F is non-recurrent on J , we can assume without loss of generality that

$$c_n := F^n c \notin U_1, \quad \forall n \geq 1. \quad (1.2)$$

In this case F is expanding on the postcritical set, and so we have

$$\chi_c := \liminf_{n \rightarrow \infty} \frac{1}{n} \log |F'_n(c_1)| > 0, \quad (1.3)$$

where, as a general rule, we write F'_n instead of $(F^n)'$. (As mentioned in [2], (1.3) is a corollary of Mañé's lemma, and it holds for arbitrary non-recurrent maps without parabolic cycles.)

1.3. Statement of results

We define the *pressure function* of F by the formula

$$p(t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z \in F^{-n}c} |F'_n(z)|^{-t}. \quad (1.4)$$

Note that instead of c we can take any other point in U_1 .

THEOREM A. *Exactly one of the following occurs:*

(1) $p(t) > -(\chi_c/2)t$ for all $t > 0$, in which case the pressure function is real analytic on $(0, \infty)$.

(2) There exists a point $t_c > \dim J$ such that $p(t)$ is a real analytic function on $(0, t_c)$, and $p(t) \equiv -(\chi_c/2)t$ on $[t_c, \infty)$.

With some minor changes in the statement and proof, the dichotomy of Theorem A holds for arbitrary non-recurrent maps without parabolic cycles. To avoid unnecessary technical details, we have decided to give the argument only for the model example of generalized conformal Cantor sets. We also believe that the dichotomy should extend to arbitrary Collet–Eckmann maps, that is, the maps satisfying $p'(\infty) < 0$. We are somewhat less certain about whether the pressure function is always real analytic on $[0, t_0)$ for general rational maps, where t_0 is the first zero of $p(t)$.

The real dynamical version of Theorem A is known for smooth unimodal interval maps satisfying the Collet–Eckmann condition. Young [11] and Keller and Nowicki [5] established the exponential decay of correlations (the latter paper is based on Hofbauer’s tower), and paper [1] contains an explicit proof of analyticity of the pressure function for t close to 1.

THEOREM B. *For generalized conformal Cantor sets, both of the alternatives in Theorem A are possible.*

2. Proof of Theorem A

This proof will follow the standard argument applied to the tower dynamics. Without loss of generality, we will assume that the domains U_0 and P_1 have real analytic boundaries.

2.1. Critical annuli

For $k \geq 1$, the critical annulus D_k is the domain

$$D_k = U_k \setminus \bar{U}_{k+1}.$$

By assumption (1.2), $F^k : U_k \rightarrow U_0$ is a 2-cover branched over c , and so

$$F^k : D_k \rightarrow U_0 \setminus \bar{P}_1(c_k)$$

is an unbranched covering map of degree 2. If N denotes the total number of non-critical generation 1 pieces, then there are precisely $2N$ puzzle pieces of generation $k+1$ inside D_k , and F^k maps each of them univalently onto the corresponding piece $P_1 \neq P_1(c_k)$. Let us denote

$$A_n = |F'_n(c_1)|^{1/2}, \quad (2.1)$$

and

$$Q = \liminf_{n \rightarrow \infty} A_n^{1/n} \stackrel{(1.3)}{=} e^{\lambda c/2}. \quad (2.2)$$

Note that $Q > 1$.

LEMMA 1. *With constants independent of k , we have*

$$z \in D_k \Rightarrow |F'_k(z)| \asymp A_k, \quad |F''_k(z)| \lesssim A_k^2; \quad (2.3)$$

in particular,

$$|F'_k(z)|^{-2} |F''_k(z)| \lesssim 1. \quad (2.4)$$

Proof. Since we have assumed that the boundaries of U_0 and the P_1 are real analytic, the Koebe distortion theorem implies that

$$\text{diam } P_{k-1}(c_1) \asymp A_k^{-2},$$

and

$$|F'_{k-1}(w)| \asymp A_k^2, \quad \forall w \in P_{k-1}(c_1).$$

We also have

$$z \in D_k \Rightarrow |z - c| \asymp [\text{diam } P_{k-1}(c_1)]^{1/2} \asymp A_k^{-1}.$$

Applying the chain rule, we get the first relation in (2.3). The second relation then follows from the Cauchy formula. \square

2.2. Tower dynamics

We will use a version of Hofbauer's construction in 1-dimensional real dynamics. The tower space \mathcal{T} is defined as a subset of the direct product $U_0 \times \mathbb{N}$:

$$\mathcal{T} = \bigsqcup_{k \geq 1} \mathcal{T}_k,$$

where

$$\mathcal{T}_1 = (\text{union of all } P_1, 1), \quad \mathcal{T}_k = (U_k, k), \quad k \geq 2.$$

The tower map $T : \mathcal{T} \rightarrow \mathcal{T} \cup (U_0, 1)$ acts according to the rule

$$T : (z, k) \mapsto \begin{cases} (z, k+1) & z \in U_{k+1} \\ (F^k z, 1) & z \notin U_{k+1}. \end{cases}$$

Let us introduce a (Riemann) metric on \mathcal{T} as follows. Fix a number $Q_* > 1$ which we call the *tower constant*; this constant will vary in the course of the proof, but we will always assume that

$$Q_* < Q. \quad (2.5)$$

We define the metric on the n th floor \mathcal{T}_n so that the natural projection $\mathcal{T}_n \rightarrow U_0$ decreases distances by the factor of Q_*^{n-1} . By definition, the distance between any two floors is equal to infinity. On each floor of the tower, the map T is continuous and differentiable almost everywhere with respect to the Lebesgue measure, namely

$$T'(z, k) = \begin{cases} Q_* & z \in U_{k+1} \\ Q_*^{1-k} F'_k(z) & z \notin U_{k+1}. \end{cases}$$

By (2.2), (2.3) and (2.5), the tower dynamics is expanding:

$$|T'_n(y)| \geq \text{const } q^n, \quad \exists q > 1, \forall n, \quad (2.6)$$

provided that T^n is defined and differentiable at y .

LEMMA 2. *There is a constant C independent of n such that if T^n is defined and differentiable at y , then*

$$\frac{|T''_n(y)|}{|T'_n(y)|^2} \leq C.$$

Proof. The function $|T'|^{-2}|T''|$ is zero if we ‘go up’. If T sends $y = (z, k)$ to the first floor, then $z \in D_k$, and by (2.4) we have

$$\frac{|T''(z, k)|}{|T'(z, k)|^2} = \frac{|F''_k z|}{|F'_k z|^2} \lesssim 1. \quad (2.7)$$

It follows that $|T'|^{-2}|T''|$ is a bounded function. Let X_n denote the supremum of $|T'_n|^{-2}|T''_n|$ on the set where this expression is defined. From the identity

$$\frac{T''_n}{(T'_n)^2} = \frac{T''_{n-1} \circ T}{(T'_{n-1} \circ T)^2} + \frac{T''}{(T')^2} \frac{1}{T'_{n-1} \circ T}$$

(differentiate the equation $T^n = T^{n-1} \circ T$ twice), and from (2.6) and (2.7), we have

$$X_n \leq X_{n-1} + Cq^{-n},$$

and so the lemma follows. \square

2.3. Ruelle–Perron–Frobenius transfer operator

For $t > 0$, the operator L_t is given by the formula

$$L_t f(x) = \sum_{Ty=x} f(y) |T'(y)|^{-t}, \quad x \in \mathcal{T}.$$

Note that the values $T'(y)$ are well defined, and the sum is finite for bounded functions. We will study the action of L_t on the following Banach spaces:

- (1) $C(\mathcal{T})$, the space of bounded continuous functions with the L^∞ -norm $\|f\|_\infty$;
- (2) $W^{1,\infty}(\mathcal{T})$, the Sobolev space with the norm $\|f\|_* = \|f\|_\infty + \|\nabla f\|_\infty$.

Note that the norm in the latter space depends on the choice of Q_* .

LEMMA 3. *L_t is a bounded operator in $C(\mathcal{T})$ and in $W^{1,\infty}(\mathcal{T})$.*

Proof. The first statement is clear by construction. The boundedness in $W^{1,\infty}(\mathcal{T})$ follows from Lemma 2. (For more details, see the proof of Lemma 5.) \square

Define $\lambda(t) = \rho(L_t, C(\mathcal{T}))$, where the latter is the notation for the spectral radius of the operator $L_t : C(\mathcal{T}) \rightarrow C(\mathcal{T})$. We want to compare $\lambda(t)$ with the spectral radius $\rho(L_t, W^{1,\infty}(\mathcal{T}))$, and with the essential spectral radius

$$\rho_{\text{ess}}(L_t, W^{1,\infty}(\mathcal{T})) = \inf\{\rho(L_t - K, W^{1,\infty}(\mathcal{T})) : \text{rank } K < \infty\}.$$

PROPOSITION 1. *If $t > 0$ is such that*

$$p(t) > -(\chi_c/2)t, \quad \text{that is, } e^{p(t)} > Q^{-t}, \quad (2.8)$$

then there exists a tower constant $Q_ \in (1, Q)$ such that, for the corresponding Ruelle–Perron–Frobenius operator L_t , we have the following:*

- (1) $\lambda(t) = e^{p(t)}$.
- (2) $\rho_{\text{ess}}(L_t, W^{1,\infty}(\mathcal{T})) < \rho(L_t, W^{1,\infty}(\mathcal{T})) = \lambda(t)$.
- (3) $\lambda(t)$ is a simple eigenvalue of L_t in $W^{1,\infty}(\mathcal{T})$: $\dim \ker(L_t - \lambda(t)) = 1$.

Once we have verified this proposition, we can apply the usual argument of analytic perturbation theory to establish real analyticity of $p(t)$ in the interval $(0, t_c)$, where

$$t_c = \inf\{t : p(t) = (\chi_c/2)t\}.$$

Theorem A then follows from the general properties of the pressure function. The rest of this section is devoted to the proof of the proposition.

2.4. Pressure and spectral radius

Here we prove that $\lambda(t) = e^{p(t)}$ for every $Q_* \in (1, Q)$. To show that $\lambda(t) \geq e^{p(t)}$, we first observe that

$$z \in U_1 \Rightarrow T^{-n}(z, 1) \cap \mathcal{T}_1 = (F^{-n}z, 1). \quad (2.9)$$

Indeed, if $F^n w = z$, then T^n sends $(w, 1)$ to the first floor, for if $T^n(w, 1) = (v, k)$ with $k \geq 2$, then $v \in U_k$, and

$$T^{n-k+1}(w, 1) = (v, 1),$$

so $v = F^{n-k+1}w$ and $z = F^{k-1}v \in F^{k-1}U_k = P_1(c_{k-1}) \cap U_1 = \emptyset$.

Let $\mathbf{1}$ denote the constant function $x \mapsto 1$. From (2.9), we have

$$[L_t^n \mathbf{1}](c, 1) \geq \sum_{T^n(w, 1) = (c, 1)} |T'_n(w, 1)|^{-t} = \sum_{F^n w = c} |F'_n w|^{-t},$$

and therefore

$$\begin{aligned} \log \lambda(t) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|L_t^n \mathbf{1}\|_{\infty} \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log [L_t^n \mathbf{1}](c, 1) \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{F^n w = c} |F'_n w|^{-t} \geq p(t). \end{aligned}$$

In the proof of the opposite inequality, we will use assumption (2.8). Let us estimate the value of $L_t^n \mathbf{1}(x)$ at $x = (z, 1)$ with $z \in U_1$. If $w \in U_k$, then $T^n(w, k) = (z, 1)$ if and only if $F^{n+k-1}w = z$. (Apply (2.9) to the equation $T^{n+k-1}(w, 1) = (z, 1)$). In this case we also have

$$|F'_{n+k-1}w| = |T'_{n+k-1}(w, 1)| = Q_*^{k-1} |T'_n(w, k)|,$$

and so

$$|T'_n(w, k)| = Q_*^{1-k} |F'_{n+k-1}w|.$$

It follows that

$$\sum_{k=1}^{\infty} \sum_{T^n(w,k)=(z,1)} |T'_n(w,k)|^{-t} = \sum_{k=1}^{\infty} Q_*^{t(k-1)} \sum_{w \in U_k \cap F^{-(n+k-1)}z} |F'_{n+k-1}(w)|^{-t}. \quad (2.10)$$

By (1.2), every point $w \in U_k \cap F^{-(n+k-1)}z$ belongs to one of the critical annuli D_{k+l} with $l \leq n$. Since

$$F^{k+l-1} : D_{k+l} \longrightarrow U_1 \setminus P_2(c_{k+l-1})$$

is a 2-cover, from $F^{n+k-1} = F^{n-l} \circ F^{k+l-1}$, we derive

$$\begin{aligned} \sum_{w \in D_{k+l} \cap F^{-(n+k-1)}z} |F'_{n+k-1}(w)|^{-t} &\stackrel{(2.3)}{\lesssim} A_{k+l-1}^{-t} \sum_{y \in F^{-(n-l)}z} |F'_{n-l}(y)|^{-t} \\ &\lesssim e^{o(n)} A_{k+l-1}^{-t} e^{(n-l)p(t)} \\ &\lesssim e^{o(n)} e^{o(k)} Q^{-t(k+l)} e^{(n-l)p(t)}, \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{w \in U_k \cap F^{-(n+k-1)}z} |F'_{n+k-1}(w)|^{-t} &\lesssim e^{o(n)} e^{o(k)} Q^{-tk} e^{np(t)} \sum_{l=1}^n Q^{-tl} e^{-lp(t)} \\ &\stackrel{(2.8)}{\lesssim} e^{o(n)} e^{o(k)} Q^{-tk} e^{np(t)}. \end{aligned}$$

Returning to (2.10), we have

$$[L_t^n \mathbf{1}](z, 1) \lesssim e^{np(t)+o(n)} \sum_{k=1}^{\infty} e^{o(k)} Q_*^{tk} Q^{-tk} = e^{np(t)+o(n)},$$

and it readily follows that

$$\|L_t^n \mathbf{1}\|_{\infty} \lesssim e^{np(t)+o(n)},$$

so $\lambda(t) \leq e^{p(t)}$.

2.5. Two-norm inequality

Next we establish the Ionescu-Tulcea and Marinescu type inequality for the tower map (see [4]). Fix $t > 0$ satisfying (2.8). We will write L for L_t , λ for $\lambda(t)$, etc.

LEMMA 4. *There exist an $o(1)$ -sequence and a constant C such that, for all integer numbers n and for all $f \in W^{1,\infty}(T)$, we have*

$$\|\nabla(L^n f)\|_{\infty} \leq o(1)\lambda^n \|f\|_* + C \|L^n |f|\|_{\infty}.$$

Proof. We estimate

$$|\nabla(L^n f)(x)| \leq I + II,$$

where

$$\begin{aligned} I &\leq \sum_{y \in T^{-n}x} |\nabla f(y)| |T'_n(y)|^{-t-1} \\ &\leq \sup_y |T'_n(y)|^{-1} \|L^n \mathbf{1}\|_{\infty} \|f\|_* \\ &\stackrel{(2.6)}{=} o(1)\lambda^n \|f\|_*, \end{aligned}$$

and

$$\begin{aligned} II &= \sum_{y \in T^{-n}x} |f(y)| |T'_n(y)|^{-t} \frac{|T''_n(y)|}{|T'_n(y)|^2} \\ &\leq C \|L^n f\|_\infty \quad (\text{by Lemma 2}). \end{aligned} \quad \square$$

For an integer $\tau \geq 1$ and for $f \in C(\mathcal{T})$, we denote

$$f_\tau = \begin{cases} f & \text{on } T_1 \cup \dots \cup \mathcal{T}_\tau \\ 0 & \text{on } T_{\tau+1} \cup \dots \end{cases}$$

LEMMA 5. *There are sequences $\tau(n)$ and $C(n)$ such that, for all $f \in C(\mathcal{T})$, we have*

$$\|L^n f\|_\infty \leq o(1)\lambda^n \|f\|_\infty + C(n) \|f_{\tau(n)}\|_\infty.$$

Proof. Fix $n \gg 1$ and choose a number $\tau = \tau(n) \gg n$ which we will be able to specify at the end of the argument. Denote $h = \mathbf{1} - \mathbf{1}_\tau$. We have

$$\|L^n f\|_\infty \leq \|L^n f h\|_\infty + \|L^n f_\tau\|_\infty \leq \|L^n h\|_\infty \|f\|_\infty + \|L^n\| \|f_\tau\|_\infty,$$

and so we only need to show $\|L^n h\|_\infty = o(\lambda^n)$. To estimate

$$L^n h(x) = \sum_{T^n y = x, \tau(y) > \tau} |T'_n(y)|^{-t},$$

let $x = (z, k)$. We consider two cases.

Case 1 ($k > n$): The only point $y \in T^{-n}x$ is $y = (z, k - n)$, and $T'_n(y) = Q_*$, so

$$L^n h(x) = Q_*^{-nt} = o(\lambda^n),$$

provided that Q_* has been chosen sufficiently close to Q . (We use here the assumption that $\lambda > Q^{-t}$.)

Case 2 ($k \leq n$): Let $y = (w, m) \in T^{-n}x$ with $m > \tau$. Under n iterations of T , there is at least one drop to the ground floor from level $> \tau$. By (2.6), it follows that $|T'_n(y)| \gtrsim q^\tau$ with $q > 1$, and therefore if $\tau(n) \gg n$, then

$$\sum_{T^n y = x, \tau(y) > \tau} |T'_n(y)|^{-t} \lesssim q^\tau = o(\lambda^n). \quad \square$$

COROLLARY 1. $\|L^n f\|_* \leq o(1)\lambda^n \|f\|_* + C(n) \|f_{\tau(n)}\|_\infty$.

In particular, it follows that $\lambda(t)$ is the spectral radius of L_t in $W^{1,\infty}(\mathcal{T})$: iterate the inequality $\|L^N f\|_* \leq (1/2)\lambda^N \|f\|_* + C\|f\|_\infty$, which is true for sufficiently large N .

2.6. Quasiconpactness

LEMMA 6. *Fix an integer number $\tau \geq 1$. Given $\delta > 0$, there is a finite rank operator*

$$K : W^{1,\infty}(\mathcal{T}) \longrightarrow W^{1,\infty}(\mathcal{T})$$

such that

$$\|K\|_* \leq A, \quad A \geq 1 \text{ independent of } \delta, \quad (2.11)$$

$$\|(Kf - f)_\tau\|_\infty \leq \delta \|f_\tau\|_*, \quad \forall f. \quad (2.12)$$

Proof. Given τ , δ and f , we define Kf as follows. Set $Kf(\cdot, n) = 0$ for all $n > \tau$. For each $n \leq \tau$, we extend $f(\cdot, n)$ to the whole plane with Sobolev's norm (relative to the n th floor metric) $\lesssim \|f\|_*$, and consider a grid of equilateral triangles Δ of size δ . We define $Kf(\cdot, n)$ to be a continuous function such that $Kf(z, n) = f(z, n)$ at all vertices z of the grid and such that $Kf(\cdot, n)$ is a linear function in each triangle Δ . Properties (2.11) and (2.12) follow from the construction. \square

We can now show that $\rho_{\text{ess}}(L_t, W^{1,\infty}(T)) < \lambda$. By Corollary 1, we can find n and τ such that the following inequality holds:

$$\|L^n f\|_* \leq \frac{1}{5A} \lambda^n \|f\|_* + C \|f_\tau\|_\infty. \quad (2.13)$$

For this τ , and for $\delta \ll 1$ depending on n and C , we choose a finite rank operator K according to the last lemma. Then

$$\begin{aligned} \|L^n(I - K)f\|_* &\stackrel{(2.13)}{\leq} \frac{1}{5A} \lambda^n \|f - Kf\|_* + C \|(f - Kf)_\tau\|_\infty \\ &\leq \frac{2}{5} \lambda^n \|f\|_* + C \delta \|f_\tau\|_* \quad (\text{by (2.11) and (2.12)}) \\ &\leq \frac{3}{5} \lambda^n \|f\|_*, \end{aligned}$$

and so the essential spectral radius is strictly less than λ .

To complete the proof of Theorem A, we still need to show that $\lambda = \lambda(t)$ is a simple eigenvalue of L_t in $W^{1,\infty}(\mathcal{T})$. The proof of this fact follows from a standard argument (see, for example, [8, Section 3]), as soon as we can construct a t -conformal measure on the tower, that is, a probability measure ν satisfying

$$L_t^* \nu = \lambda \nu,$$

where L_t^* is the conjugate of $L_t : C(\mathcal{T}) \rightarrow C(\mathcal{T})$. The difficulty lies, of course, in the non-compactness of the tower space. The usual construction nevertheless works: we can take for ν any weak*-limit point of the sequence of probability measures ν_n ,

$$\nu_n = \frac{\mu_n}{\|\mu_n\|}, \quad \mu_n := \sum_{k=0}^n \lambda^{-k} (L_t^{*k} \delta),$$

where δ is the delta-measure at $(c, 1)$. The existence of a weak*-limit point follows from the estimate

$$\nu_n \left(\bigcup_{j \geq \tau} \mathcal{T}_j \right) = o(1) \quad \text{as } \tau \rightarrow \infty \text{ uniformly in } n,$$

which is implicitly contained in the computation in Subsection 2.4.

3. Proof of Theorem B

3.1. Sufficient conditions

We start with two conditions that are sufficient for the analyticity of the pressure function (alternative (1) in Theorem A). For a periodic point $a = F^m a$, let $\chi(a)$ denote its Lyapunov exponent:

$$\chi(a) = \frac{1}{m} \log |F'_m(a)|.$$

Clearly

$$\inf\{\chi(a) : a \in \text{Per}(F)\} \leq \frac{\chi_c}{2}.$$

PROPOSITION 2. *The pressure function is analytic on $(0, \infty)$ if*

$$\chi(a) \leq \frac{\chi_c}{2}, \quad \exists a \in \text{Per}(F).$$

Proof. If we consider a hyperbolic subset containing a , we have

$$p(t) > -t\chi(a) \geq -\frac{\chi_c}{2}t, \quad \forall t > 0,$$

so one can apply Theorem A. □

PROPOSITION 3. *Let $\{A_n\}$ and Q be as in (2.1) and (2.2). The pressure function is analytic on $(0, \infty)$ if*

$$\sum \left(\frac{Q^n}{A_n} \right)^t = \infty, \quad \forall t > 0.$$

We give a proof in Subsection 3.3. This proposition implies that the pressure function is real analytic in the critically finite case, a fact first established in [7]. Using either of the above sufficient conditions, it is easy to construct a critically infinite dynamics without phase transition.

3.2. Itineraries

We will need some further notation. For each $w \in J$, we define

$$l(w) = \inf\{l \geq 0 : F^l w \in U_1\},$$

$$k(w) = k, \quad \text{if } w \in D_k.$$

Thus $l(w) = 0$ if and only if $w \in U_1$, and $k(w) = 0$ if and only if $w \notin U_1$; $k(w) = \infty$ if $w = c$. Note that

$$w \in F^{-m}c \Rightarrow l(w) \leq m, \quad k(w) \leq m.$$

To see the latter, assume that $w \in P_{m+1}(c)$. Then $c = F^m w \in P_1(c_m) \cap U_{m+1} = \emptyset$, a contradiction.

Let us now fix $n \gg 1$. To each $z \in F^{-n}c$, we assign a finite sequence of integer numbers

$$I(z) = (k_1, l_1; \dots; k_v, l_v)$$

as follows. Starting with $z_1 = z$, we define inductively

$$k_j = k(z_j), \quad z'_j = F^{k_j} z_j, \quad l_j := l(z'_j),$$

$$z_{j+1} = F^{l_j} z'_j = F^{k_1 + l_1 + \dots + k_j + l_j} z.$$

We stop as soon as we get $z_{v+1} = c$. Clearly, $k_j \geq 1$ for all $j \geq 2$, and

$$k_1 + l_1 + \dots + k_v + l_v = n. \tag{3.1}$$

3.3. Proof of Proposition 3

We will only consider the points with itineraries

$$(k_1, 0; k_2, 0; \dots; k_v, 0), \quad \sum_{j=1}^v k_j = n, k_j \geq 1.$$

To each such an itinerary there correspond precisely 2^v points $z \in F^{-n}c$, and by (2.3) we have

$$|F'_n(z)|^{-t} \geq C^{-tv} A_{k_1}^{-t} \dots A_{k_v}^{-t} = Q^{-nt} \alpha_{k_1} \dots \alpha_{k_v},$$

where

$$\alpha_{k_j} = \left(\frac{Q^{k_j}}{C A_{k_j}} \right)^t.$$

It follows that

$$\sum_{z \in F^{-n}c} |F'_n(z)|^{-t} \geq Q^{-nt} \sum_v 2^v c_{nv},$$

where

$$c_{nv} = \sum_{k_1 + \dots + k_v = n} \alpha_{k_1} \dots \alpha_{k_v}, \quad (3.2)$$

the sum being taken over ordered sequences (k_1, \dots, k_v) of positive integers. The result now follows from Theorem A and the following computation. Let $\{\alpha_k\}$ be a sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \alpha_k > 1, \quad \limsup_{k \rightarrow \infty} \alpha_k^{1/k} = 1.$$

If the numbers c_{nv} are defined by (3.2), and if $c_n = \sum_v c_{nv}$, then $\limsup c_n^{1/n} > 1$.

To see this, we note that the generating function

$$\psi(z) = \sum_{k=1}^{\infty} \alpha_k z^k$$

is analytic in the unit disc. We have

$$\psi^v(z) = \sum_n c_{nv} z^n,$$

and

$$\frac{1}{1 - \psi(z)} = \sum_v \psi^v(z) = \sum_n c_n z^n.$$

Since ψ takes the value 1 on the real interval $(0, 1)$, the radius of convergence of the latter series is strictly less than 1. \square

The above argument is almost reversible except that we need to keep track of the constants. This can be done in the following simple example.

3.4. An example

We now turn to the second part of Theorem B: we will construct a non-recurrent Cantor dynamics such that the pressure function has a phase transition point.

We start with the unit disc U_0 and two disjoint round discs of radii $\alpha < \beta$ inside $\{|z| < 1/2\}$. On each of two small discs define F to be a linear conformal map onto U_0 (so F maps the centers to 0). Let \tilde{J} denote the corresponding Cantor set. We choose a point $c_1 \in \tilde{J}$ so that the number of times its orbit visits the disc α in the time interval $[1, n]$ is $\asymp O(\sqrt{n})$. If we denote, as usual, $A_n = |F'_n(c_1)|^{1/2}$, then $Q := \lim A_n^{1/n} = \beta^{-1/2}$, and

$$\sum \frac{Q^n}{A_n} \lesssim \sum \left(\frac{\alpha}{\beta}\right)^{C\sqrt{n}} < \infty. \quad (3.3)$$

Next we add a disjoint critical piece $U_1 = B(c_0, \delta)$ with δ satisfying

$$\delta < 10^{-1}\alpha, \quad (3.4)$$

and define $F : U_1 \rightarrow U_0$ by the equation

$$F(z) = B\left(\frac{(z - c_0)^2}{\delta^2}\right), \quad B(w) := \frac{w + c_1}{1 + \bar{c}_1 w}.$$

Note that $c := c_0$ is the critical point and c_1 the critical value.

PROPOSITION 4.

$$\sum_{z \in F^{-n}c} |F'_n(z)|^{-t} \sim Q^{-nt}, \quad t \gg 1.$$

It follows that the pressure function $p(t)$ is a linear function for $t \gg 1$, and so alternative (2) of Theorem A occurs.

We first establish some lemmas. Denote

$$m_k = \inf_{D_k} |F'|,$$

where the D_k are critical annuli (see Subsection 2.1). With this notation, we have

$$|F'_k(z)| \geq m_k A_{k-1}^2, \quad \forall z \in D_k. \quad (3.5)$$

Indeed, F takes D_k to $P_{k-1}(c_1) \setminus P_k(c_1)$, and the map $F^{k-1} : P_{k-1}(c_1) \rightarrow U_0$ is just a dilation by A_{k-1}^2 (plus a translation).

LEMMA 7. $m_k \geq 2QA_{k-1}^{-1}$.

Proof. If $z \in D_k$, then

$$\begin{aligned} |Fz - c_1| &\geq \text{dist}(c_1, \partial P_k(c_1)) \\ &= A_k^{-2} \text{dist}(c_{k+1}, \partial U_0) \geq \frac{1}{2} A_k^{-2}. \end{aligned}$$

On the other hand,

$$|Fz - c_1| = |B(w) - B(0)| \leq 2|w|, \quad w := \frac{\delta^2}{(z - c_0)^2},$$

and therefore

$$|z - c_0| \geq \frac{\delta}{2A_k}.$$

Since $|B'| \geq 1/3$, we have

$$\begin{aligned} |F'(z)| &\geq \frac{2|z - c_0|}{3\delta^2} \geq \frac{1}{3\delta A_k} \\ &\geq \frac{\sqrt{\alpha}}{3\delta} \frac{1}{A_{k-1}} \geq \frac{2Q}{A_{k-1}}. \end{aligned}$$

The latter follows by our choice of parameters (see (3.4)):

$$\sqrt{\alpha}/(3\delta) \geq 2/\sqrt{\alpha} > 2Q. \quad \square$$

We will apply this lemma to the estimation of the derivative $F'_n(z)$ in terms of the itinerary $I(z) = (k_1, l_1; \dots; k_v, l_v)$ of a point $z \in F^{-n}c$. Using the notation of Subsection 3.2, we apply the chain rule to the sequence of maps

$$z = z_1 \xrightarrow{F^{k_1}} z'_1 \xrightarrow{F^{l_1}} z_2 \xrightarrow{\quad} \dots \xrightarrow{\quad} z_v \xrightarrow{F^{k_v}} z'_v \xrightarrow{F^{l_v}} c.$$

Since the l -maps are all linear, and estimate (3.5) applies to the k -maps, we get

$$|F'_n(z)|^{-t} \leq m_{k_1}^{-t} A_{k_1-1}^{-2t} \dots m_{k_v}^{-t} A_{k_v-1}^{-2t} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t}.$$

If we denote $L = l_1 + \dots + l_v$ and

$$a_k(t) = 2m_k^{-t} A_{k-1}^{-2t} Q^{kt}, \quad (3.6)$$

then by (3.1) we have

$$|F'_n(z)|^{-t} \leq Q^{Lt-nt} a_{k_1}(t) \dots a_{k_v}(t) 2^{-v} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t}. \quad (3.7)$$

LEMMA 8. *If the functions $a_k(t)$ are defined by (3.6), then*

$$\sum a_k(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. By dominating convergence, it is enough to show that the series

$$\sum \frac{Q^k}{m_k A_{k-1}^2}$$

converges and that all terms are at most $1/2$. The convergence follows from (3.3) and Lemma 7. By Lemma 7, we also have

$$\frac{Q^k}{m_k A_{k-1}^2} \leq \frac{Q^{k-1}}{2A_{k-1}} \leq \frac{1}{2}. \quad \square$$

LEMMA 9. *Denote $\tilde{p}(t) = \log(\alpha^t + \beta^t)$. Then*

$$\sum_{I(z)=I} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t} \leq 2^v e^{L\tilde{p}(t)},$$

where the sum is taken over all points with a given itinerary I .

Proof. Observe first that $\tilde{p}(t)$ is the pressure function of the dynamics \tilde{F} that generates the Cantor set \tilde{J} (see Subsection 3.1) (that is, \tilde{F} is the restriction of F to the union of discs α and β). There are 2^n \tilde{F} -pieces of generation n , namely

$$\binom{n}{k}$$

discs of radii $\alpha^k \beta^{n-k}$, and so if $z_0 \in U_1$, then

$$\sum_{z \in \tilde{F}^{-n} z_0} |F'_n(z)|^{-t} = \sum_{k=0}^n \binom{n}{k} \alpha^{tk} \beta^{t(n-k)} = (\alpha^t + \beta^t)^n = e^{\tilde{p}(t)n}.$$

We can code the points z such that $I(z) = I$ with sequences $[z_1, z'_1, \dots, z_v, z'_v]$ defined inductively as follows. Start with $z_{v+1} := c$ and proceed in the inverse order:

$$z'_j \in \tilde{F}^{-l_j} z_{j+1} \setminus P_1(c_{k_j}), \quad z_j \in D_{k_j} \cap F^{-k_j} z'_j.$$

Note that, given z'_j and k_j , there are only two choices for z_j . Thus we have

$$\begin{aligned} \sum_{I(z)=I} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t} &= \sum_{[z_1, z'_1, \dots, z_v, z'_v]} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t} \\ &= 2 \sum_{[z'_1, z_2, \dots, z_v, z'_v]} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t} \\ &= 2 \sum_{[z_2, z'_2, \dots, z_v, z'_v]} |F'_{l_2}(z'_2)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t} \\ &\quad \times \sum_{z'_1 \in \tilde{F}^{-l_1} z_2 \setminus P_1(c_{k_1})} |F'_{l_1}(z'_1)|^{-t} \\ &\leq 2e^{l_1 \tilde{p}(t)} \sum_{[z_2, z'_2, \dots, z_v, z'_v]} |F'_{l_2}(z'_2)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t} \\ &\quad \dots \\ &\leq 2^v e^{(l_1 + \dots + l_v) \tilde{p}(t)}. \end{aligned}$$

□

3.5. Proof of Proposition 4

Define $\varepsilon = \varepsilon(t) = e^{\tilde{p}(t)} Q^t$ so $\varepsilon < 1$ for large t . We have

$$\begin{aligned} &\sum_{F^n z=c} |F'_n(z)|^{-t} \quad (\text{by (3.7)}) \\ &\leq Q^{-nt} \sum_v 2^{-v} \sum_{v(I)=v} Q^{Lt} a_{k_1} \\ &\quad \dots a_{k_v} \sum_{I(z)=I} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_v}(z'_v)|^{-t} \quad (\text{by Lemma 3}) \\ &\leq Q^{-nt} \sum_v \sum_{v(I)=v} Q^{Lt} e^{L\tilde{p}(t)} a_{k_1} \dots a_{k_v} \\ &= Q^{-nt} \sum_v \sum_{v(I)=v} \varepsilon^{l_1} \dots \varepsilon^{l_v} a_{k_1} \dots a_{k_v}. \end{aligned}$$

Denote

$$c_{nv} = \sum_{n(I)=n, v(I)=v} \varepsilon^{l_1} \dots \varepsilon^{l_v} \alpha_{k_1} \dots \alpha_{k_v}.$$

We need to show that

$$c_n := \sum_v c_{nv} \lesssim 1.$$

Consider the series

$$\psi(z) = a_1 z + a_2 z^2 + \dots,$$

and

$$\chi(z) = 1 + \varepsilon z + \varepsilon^2 z^2 + \dots = \frac{1}{1 - \varepsilon z}.$$

The series converge in the unit disc. Observe that

$$(1 + \psi)\psi^{v-1}\chi^v = \dots + c_{nv}z^n + \dots,$$

because the coefficient of z^n comes from all possible choices for the term

$$a_{k_1}\varepsilon^{l_1} \dots a_{k_v}\varepsilon^{l_v} \dots, \quad k_1 + l_1 + \dots + k_v + l_v = n;$$

k_1 can be zero. Summing over v , we obtain the power series $\sum c_n z^n$, and it remains to show that its radius of convergence is at least 1. It is enough to prove this for

$$\sum \varphi^v(z) = \frac{1}{1 - \varphi(z)}, \quad \varphi := \psi\chi.$$

The latter is true because, by Lemma 2,

$$|\varphi(z)| = \frac{1}{1 - \varepsilon} \sum a_j < 1, \quad |z| < 1,$$

provided that t is large enough.

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