Energy Landscape of 3D Spin Hamiltonians with Topological Order

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We explore the feasibility of a quantum self-correcting memory based on 3D spin Hamiltonians with topological quantum order in which thermal diffusion of topological defects is suppressed by macroscopic energy barriers. To this end we characterize the energy landscape of stabilizer code Hamiltonians with local bounded-strength interactions which have a topologically ordered ground state but do not have stringlike logical operators. We prove that any sequence of local errors mapping a ground state of such a Hamiltonian to an orthogonal ground state must cross an energy barrier growing at least as a logarithm of the lattice size. Our bound on the energy barrier is tight up to a constant factor for one particular 3D spin Hamiltonian.

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Topologically ordered phases of matter display a variety of fascinating properties having no counterpart in the classical physics. Most notable ones are topological invariants such as the Hall conductance, ground state degeneracy, and topological entanglement entropy [1] which are insensitive to generic local perturbations [2–4]. The intrinsic stability against perturbations motivated interest in topological phases as a storage medium for a reliable quantum memory [3,5,6] and as a platform for building a topological quantum computer [3,7].

A big open question in the theory of topological quantum order (TQO) concerns the feasibility of a nonvolatile, or, self-correcting, quantum memory [3,8]. Such a memory would permit reliable long-term storage of quantum information in the presence of sufficiently weak thermal noise without need for active stabilization and error correction during the storage period. The main challenge in designing Hamiltonians with self-correcting properties is to combine TQO with an energy landscape that could prevent errors caused by thermal fluctuations from accumulating. This could guarantee that the error density remains sufficiently small during the entire storage period and the encoded information can be safely extracted from the memory by performing an active error correction at the readout phase.

In spite of being intrinsically stable against perturbations at the zero temperature, TQO models display extreme fragility against thermal fluctuations [9], suggesting impossibility of quantum self-correction. A thermal stability analysis involving finite-temperature extensions of the topological entanglement entropy has been undertaken for the 2D and 3D toric code models by Castelnovo and Chamon [10], and by Iblisdir *et al.* [10]. These models were shown to undergo a transition from a topologically ordered phase at $T = 0$ to a different phase with either partial or no topological order at any positive temperature [10].

The first rigorous analysis of self-correcting properties for the toric code models was carried out Alicki *et al.* [11,12]. It showed that the 4D toric code Hamiltonian has self-correcting properties for sufficiently low temperature, while 2D and 3D toric codes are not self-correcting at any finite temperature. The ideas of [11,12] were developed further by Kay [13], Chesi *et al.* [14,15] and Pastawski *et al.* [16].

The main feature of the 4D toric code model responsible for self-correction is the macroscopic energy barrier that must be crossed by any sequence of local errors whose combined action on encoded states cannot be corrected at the final readout phase [12]. The height of this barrier grows linearly with the lattice size due to a finite string-tension characterizing boundaries of membranes associated with errors. It is analogous to the energy barrier separating ground states with positive and negative magnetization in the ferromagnetic 2D Ising model. Unfortunately, this behavior cannot be reproduced in any known 2D or 3D model due to a presence of pointlike excitations carrying a nontrivial topological charge, or, pointlike defects. These defects are analogous to domain-walls in the 1D Ising model—a single isolated defect has only a constant energy cost, but its creation requires a highly nonlocal operation affecting a macroscopic number of qubits (spins). Whether or not the presence of pointlike defects rules out self-correcting properties may depend on how fast these defects can diffuse across the system. For example, Hamma *et al.* [17] used a coupling with a bosonic field to create an effective long-range attractive interaction between defects whereby suppressing the diffusion. A different possibility is realized in the 3D Chamon's model [18,19]. This model offers a topological protection against diffusion of some types of defects (but not all of them). These defects, called monopoles in [19], can be created at corners of rectangular shaped membranes. A hopping of a single isolated monopole between adjacent lattice sites is a highly nonlocal operation affecting a macroscopic number of qubits; see [19] for details.

In the present Letter we propose yet another possibility to suppress the diffusion of defects that can be realized in a certain class of 3D spin Hamiltonians with strictly local bounded-strength interactions. The Hamiltonians in this class, associated with stabilizer error correcting codes [20], have a peculiar property that isolated defects cannot move further than a constant distance away without creating other defects. For brevity, we shall refer to this property (stated more formally below) as a no-strings rule because it is closely connected to the absence of logical stringlike operators capable of moving the defects. Let us point out that the first example of a 3D spin Hamiltonian with TQO obeying the no-strings rule has been found only quite recently by one of us [21]. This example is very special and may not be realized in nature with its precise form. However, it could be the case that at least one model satisfying the no-strings rule can be found in a laboratory.

We prove that any sequence of local errors creating an isolated defect from the vacuum with no other defects within distance R must cross an energy barrier at least $c\log R$ for some constant c. The same bound applies to creation of any isolated cluster of defects with a nontrivial total topological charge. It shows that although defects do not interact directly, their diffusive motion is suppressed by the logarithmic energy barriers preventing the defects from spreading (the concept of a diffusive motion must be used with care in our case because individual defects can only move a constant distance away).

We also prove a similar logarithmic lower bound on the energy barrier for implementing any logical operator. More precisely, we prove that any sequence of local errors mapping a ground state to an orthogonal ground state must cross the energy barrier at least $c\log L$, where L is the lattice size and c is some constant. For the Hamiltonian discovered in [21] this bound is tight up to a constant factor. Although the scaling of the energy barrier is not as favorable as the one in the 4D toric code, we point out that the energy barrier does not grow with the lattice size at all for all previously studied TQO Hamiltonians in the 2D and 3D geometry. A naive estimate of the storage time τ for a memory with an energy barrier B operating at a temperature T can be made using the Arrhenius law, namely, $\tau \sim e^{B/T}$. Since in our case $B = c\log L$ for some constant c, we arrive at $\tau \sim L^{c/T}$. Although this "derivation" gives only polynomial scaling of τ, the degree of the polynomial can be made arbitrarily large by choosing sufficiently low temperature.

It is worth mentioning that a 2D Hamiltonian with TQO always have stringlike logical operators [22,23] and thus 3 is the smallest spatial dimension for constructing Hamiltonians obeying the no-strings rule. Indeed, it was shown by Terhal and one of us [22] that for any 2D local stabilizer-type Hamiltonians the energy barrier for implementing at least one logical operator is constant. It should also be noted that a 3D translation-invariant stabilizer Hamiltonian with TQO can obey the no-strings rule only if the ground state degeneracy is not invariant under changing lattice dimensions [24,25]. Let us now state our main results more formally.

Stabilizer code Hamiltonians.—We consider a regular D-dimensional cubic lattice Λ of linear size L with periodic boundary conditions, that is, $\Lambda = \mathbb{Z}_L^D$. Each site $u \in \Lambda$ is populated by a finite number of qubits. A stabilizer Hamiltonian is defined as

$$H = -\sum_{a=1}^{M} G_a, \tag{1}$$

where each term G_a is a multiqubit Pauli operator (a tensor product of I, X, Y, Z with an overall ± 1 sign) and different terms commute with each other. The Abelian group $\mathcal{G}$ generated by $G_1, \ldots, G_M$ is called a stabilizer group of the code. Elements of $\mathcal{G}$ are called stabilizers. We assume that each generator G_a acts nontrivially (by X, Y, or Z) only on a set of qubits located at vertices of an elementary cube. It is allowed to have more than one generator per cube. Any short-range stabilizer Hamiltonian can be written in this form after a coarse graining of the lattice. The Hamiltonian may or may not be translation invariant.

We assume that H is frustration-free, that is, ground states ψ_0 of H obey $G_a\psi_0 = \psi_0$ for all a. Consider any multiqubit Pauli operator E. A state $\psi = E\psi_0$ is an excited eigenstate of H. Obviously, $G_a\psi = \pm\psi$ where the sign depends on whether G_a commutes (plus) or anticommutes (minus) with E. Any flipped generator ($G_a\psi = -\psi$) will be referred to as a defect. An eigenstate with m defects has energy $2m$ above the ground state. For brevity, we use the term vacuum for any ground state of H whenever its choice is not important. A Pauli operator E whose action on the vacuum creates no defects is either a stabilizer ($E \in \mathcal{G}$), or a logical operator ($E \notin \mathcal{G}$, but E commutes with $\mathcal{G}$). In the former case any ground state of H is invariant under E. In the latter case E maps some ground state of H to an orthogonal ground state.

Topological order.—Our definition of TQO depends on a length scale L_{tqo} that must be bounded as $L_{\mathrm{tqo}} \geq L^{\beta}$ for some constant $\beta > 0$. Our first TQO condition concerns ground states.

Condition 1: If a Pauli operator E creates no defects when applied to the vacuum and its support can be enclosed by a cube of linear size L_{tqo}, then E is a stabilizer, $E \in \mathcal{G}$.

Our second TQO condition concerns excited states. A cluster of defects S will be called neutral if it can be created from the vacuum by a Pauli operator E whose support is enclosed by a cube of linear size L_{tqo} without creating any other defects. Otherwise we say that S is a charged cluster. Given a region $A \subseteq \Lambda$ we shall use a notation $\mathcal{B}_r(A)$ for the r neighborhood of A, that is, a set of all points that have distance at most r from A. Here and below we use l_∞ distance on $\mathbb{Z}_L^D$; i.e., the distance between a pair of sites

is the maximum of their coordinatewise distances. We shall need the following condition saying that neutral clusters of defects can be created from the vacuum locally.

Condition 2: Let S be a neutral cluster of defects and $C_{\min}(S)$ be the smallest cube that encloses S. Then S can be created from the vacuum by a Pauli operator supported on $\mathcal{B}_1[C_{\min}(S)]$.

In the last condition $\mathcal{B}_1[C_{\min}(S)]$ can be replaced by $\mathcal{B}_\rho[C_{\min}(S)]$ for any constant ρ if one performs a coarse-graining of the lattice.

No-strings rule.—Informally, the rule says that applying an operator with a 'stringlike' support to the vacuum cannot create charged defects at the end points of the string. Let us now define this property rigorously. Let E be any Pauli operator whose support is enclosed by a cube of linear size L_{tqo} and S be a cluster of defects obtained by applying E to the vacuum. Let A_1, A_2 be any pair of disjoint cubes of the same linear size ρ. We shall say that E is a logical string segment with anchor regions A_1, A_2 iff S is contained in the union $A_1 \cup A_2$. We will say that a logical string segment E has aspect ratio α iff the distance between A_1 and A_2 is at least $\alpha\rho$. A logical string segment E is called trivial iff the cluster of defects contained inside any anchor region is neutral.

Definition: A stabilizer Hamiltonian obeys the no-strings rule iff there exists a constant α such that all logical string segments with aspect ratio greater than α are trivial.

We note that a 3D stabilizer code (Code 1) discovered in [21] obeys our topological order conditions with $L_{\text{tqo}} \sim L$ and obeys the no-strings rule with $\alpha = 15$.

Energy barrier.—Let us consider a process of building a logical operator $\bar{P}$ from local errors. It can be described by an error path—a finite sequence of local Pauli errors $E_1, \ldots, E_T$ such that $\bar{P} = E_T \cdots E_2 E_1$. For simplicity we shall assume that each local error E_t is a single-qubit Pauli operator X, Y, or Z. Applying this sequence of errors to a ground state ψ_0 generates a sequence of states $\{\psi(t)\}_{t=0,\ldots,T}$, where $\psi(0) = \psi_0$ and $\psi(T) = \bar{P}\psi_0$ are ground states of H, while the intermediate states $\psi(t) = E_t \cdots E_1 \psi_0$ are typically excited. We say that a logical operator $\bar{P}$ has energy barrier ω iff for any error path implementing $\bar{P}$ at least one of the intermediate states $\psi(t)$ has more than ω defects. Note that we do not impose any restrictions on the length of the path T (as long as it is finite). In particular, an error may be repeated in the error path several times at different time steps. We shall also consider an energy barrier for creating a cluster of defects S from the vacuum. We will say that S has energy barrier ω iff for any Pauli operator E that creates S from the vacuum and for any error path implementing E at least one of the intermediate states has more than ω defects.

Our main results are the following theorems. Both theorems apply to any stabilizer Hamiltonian Eq. (1) on a D-dimensional lattice that obeys the topological order conditions and the no-strings rule.

Theorem 1.—The energy barrier for any logical operator is at least $c \log L$, where L is the lattice size, and c is a constant coefficient.

Theorem 2.—Let S be a neutral cluster of defects containing a charged cluster $S' \subseteq S$ of diameter r such that there are no other defects within distance R from S'. If $r + R < L_{\text{tqo}}$, then the energy barrier for creating S from the vacuum is at least $c \log R$, where $c = O(1)$.

The constant c depends only on the spatial dimension D, the constant α in the no-strings rule, and the constant β in the bound $L_{\text{tqo}} \geq L^\beta$. The bounds on the energy barrier are optimal up to a constant factor [26]. Below we focus on proving Theorem 1. Proof of Theorem 2 requires only minor modifications, see [26] for details.

Proof of Theorem 1.—A configuration of defects created by applying a Pauli operator E to the vacuum will be called a syndrome caused by E. The process of building up a logical operator $\bar{P}$ by a sequence of local errors $E_1, \ldots, E_T$ can be described by a syndrome history $\{S(t)\}_{t=0,\ldots,T}$. Here $S(t)$ is the syndrome caused by the product $E_t \cdots E_1$, that is, the partial implementation of $\bar{P}$ up to a step t. The syndrome history starts and ends with the vacuum, i.e., $S(0) = S(T) = \varnothing$. Without loss of generality all intermediate syndromes $S(t)$ are nonempty. For any integer $p \geq 0$ define a level-p unit of length as

$$\xi(p) = (10\alpha)^p, \qquad p = 0, 1, \ldots.$$

Let $S(t)$ be any nonempty syndrome. Recall that each defect in $S(t)$ can be associated with some elementary cube of the lattice.

Definition 1.—A syndrome $S(t)$ is called sparse at level p iff the set of elementary cubes occupied by $S(t)$ can be partitioned into a disjoint union of clusters such that each cluster has diameter at most $\xi(p)$ and any pair of distinct clusters combined together has diameter larger than $\xi(p+1)$. Otherwise $S(t)$ is called dense at level p.

For example, suppose all defects in $S(t)$ occupy the same elementary cube. Since an elementary cube has diameter 1, such a syndrome $S(t)$ is sparse at any level $p \geq 0$. If $S(t)$ occupies a pair of adjacent cubes, $S(t)$ is sparse at any level $p \geq 1$, and is dense at level $p = 0$.

Lemma 1.—Suppose a nonempty syndrome $S(t)$ is dense at all levels $q = 0, \ldots, p$. Then $S(t)$ contains at least $p + 2$ defects.

Proof.—Let $C_1^{(0)}, \ldots, C_g^{(0)}$ be elementary cubes occupied by $S(t)$. Obviously, $S(t)$ contains at least g defects. Since $S(t)$ is nonempty and dense at level 0, we have $g \geq 2$ and there exists a pair of cubes $C_a^{(0)}$, $C_b^{(0)}$ such that the union $C_a^{(0)} \cup C_b^{(0)}$ has diameter at most $\xi(1)$. Combining the pair $C_a^{(0)}$, $C_b^{(0)}$ into a single cluster we obtain a partition $S(t) = C_1^{(1)} \cup \ldots \cup C_{g-1}^{(1)}$ where each cluster $C_a^{(1)}$ has diameter at most $\xi(1)$. Suppose $S(t)$ is dense at level 1. Then $g \geq 3$ and there exists a pair of clusters $C_a^{(1)}$, $C_b^{(1)}$ such that the union $C_a^{(1)} \cup C_b^{(1)}$ has diameter at most $\xi(2)$. Combining the pair

$C_a^{(1)}$, $C_b^{(1)}$ into a single cluster and proceeding in the same way we arrive at $g \geq p + 2$.

Define a level-p syndrome history as a subsequence of the original syndrome history $\{S(t)\}_{t=0,\ldots,T}$ that includes only those syndromes $S(t)$ that are dense at all levels $q = 0, \ldots, p - 1$. The level-0 syndrome history includes all syndromes $S(t)$. The syndrome history starts and ends with the vacuum (empty syndrome) at any level p. Let $S(t')$ and $S(t'')$ be a consecutive pair of level-p syndromes. We define a level-p error E connecting $S(t')$ and $S(t'')$ as the product of all single-qubit errors E_j that occurred between $S(t')$ and $S(t'')$. The following lemma shows that E can be regarded as an approximately local error on a coarse-grained lattice characterized by the unit of length $\xi(p)$. Let m be the maximum number of defects in the syndrome history, such that any $S(t)$ contains at most m defects.

Lemma 2.—Let $S' \equiv S(t')$ and $S'' \equiv S(t'')$ be a consecutive pair of syndromes in the level-p syndrome history. Let E be the product of all errors E_j that occurred between S' and S''. If $4m[2 + \xi(p)] < L_{\mathrm{tqo}}$, then there exists an error $\tilde{E}$ supported on $\mathcal{B}_{\xi(p)}(S' \cup S'')$ such that $E\tilde{E}$ is a stabilizer.

The proof of the lemma, presented in [26], uses induction in the level p and relies crucially on the scale-invariance of the no-strings rule. The latter asserts that an isolated charged cluster belonging to some sparse level-p syndrome cannot be moved by local errors further than distance $\alpha\xi(p)$ away without making the syndrome dense. Any such movement can be accounted for by local errors on the coarse-grained lattice with a unit of length $\xi(p + 1)$. As for isolated neutral clusters, they can be created (annihilated) locally in the beginning (end) of each sparse period of the syndrome history; see [26] for details.

Let $p_{\max}$ be the highest RG level, that is, the smallest integer p such that a single level-p error E maps the vacuum to itself. We claim that $p_{\max} = \Omega(\log L)$. Indeed, suppose that $4m(\xi(p_{\max}) + 2) < L_{\mathrm{tqo}}$. Then we can apply Lemma 2 to the level-$p_{\max}$ syndrome history with $S' = S'' = \varnothing$ (vacuum). Lemma 2 would imply $\tilde{E} = I$, that is, E must be a stabilizer. On the other hand, E is equivalent to a logical operator modulo stabilizers. Hence we obtain a contradiction unless $4m[\xi(p_{\max}) + 2] \geq L_{\mathrm{tqo}}$. We can assume that the maximum number of defects is $m \ll \log L$ (if not, there is nothing to prove). Since L_{tqo} grows as a power of L, we conclude that $p_{\max} = \Omega(\log L)$. The syndrome history must contain at least one syndrome $S(t)$ which is dense at all levels $q = 0, \ldots, p_{\max} - 2$ since otherwise $p_{\max}$ could not be the highest RG level. Lemma 1 then implies that such syndrome $S(t)$ contains $\Omega(\log L)$ defects proving Theorem 1.

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