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DIFFERENTIAL FLATNESS AND ABSOLUTE EQUIVALENCE OF NONLINEAR CONTROL SYSTEMS

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Abstract

This paper presents a formulation of differential flatness—a concept originally introduced by Fliess, Levine, Martin, and Rouchon—in terms of absolute equivalence between exterior differential systems. Systems which are differentially flat have several useful properties which can be exploited to generate effective control strategies for nonlinear systems. The original definition of flatness was given in the context of differential algebra, and required that all mappings be meromorphic functions. The formulation of flatness presented here does not require any algebraic structure and allows one to use tools from exterior differential systems to help characterize differentially flat systems. In particular, it is shown that, under regularity assumptions, in the case of single input control systems (i.e., codimension 2 Pfaffian systems), a system is differentially flat if and only if it is feedback linearizable via static state feedback. However, in higher codimensions feedback linearizability about an equilibrium point and flatness are *not* equivalent: one must be careful with the role of time as well as the use of prolongations which may not be realizable as dynamic feedbacks in a control setting. Applications of differential flatness to nonlinear control systems and open questions are also discussed.

1. Introduction

The problem of equivalence of nonlinear systems (in particular to linear systems, that is, feedback linearization) is traditionally approached in the context of differential geometry [15, 20]. A complete characterization of static feedback linearizability in the multi-input case is available, and for single input systems it has been shown that static and dynamic feedback linearizability are equivalent [5]. Some special results have been obtained for dynamic feedback linearizability of multi-input systems, but the general problem remains unsolved. Typically, the conditions for

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feedback linearizability are expressed in terms of the involutivity of distributions on a manifold.

More recently it has been shown that the conditions on distributions have a natural interpretation in terms of exterior differential systems [13, 22]. In exterior differential systems, a control system is viewed as a Pfaffian module. Some of the advantages of this approach are the wealth of tools available and the fact that implicit equations and non-affine systems can be treated in a unified framework. For an extensive treatment of exterior differential systems we refer to [1].

Fliess and coworkers [8, 9, 16] studied the feedback linearization problem in the context of differential algebra and introduced the concept of *differential flatness*. In differential algebra, a system is viewed as a differential field generated by a set of variables (states and inputs). The system is said to be differentially flat if one can find a set of variables, called the flat outputs, such that the system is (non-differentially) algebraic over the differential field generated by the set of flat outputs. Roughly speaking, a system is flat if we can find a set of outputs (equal in number to the number of inputs) such that all states and inputs can be determined from these outputs without integration. More precisely, if the system has states $x \in \mathbb{R}^n$, and inputs $u \in \mathbb{R}^m$ then the system is flat if we can find outputs $y \in \mathbb{R}^m$ of the form

$$y = y(x, u, \dot{u}, \ldots, u^{(l)}) \tag{1}$$

such that,

$$\begin{aligned} x &= x(y, \dot{y}, \ldots, y^{(q)}) \\ u &= u(y, \dot{y}, \ldots, y^{(q)}). \end{aligned} \tag{2}$$

Differentially flat systems are useful in situations where explicit trajectory generation is required. Since the behaviour of flat systems is determined by the flat outputs, we can plan trajectories in output space, and then map these to appropriate inputs. A common example is the kinematic car with trailers, where the xy position of the last trailer provides flat outputs [18]. This implies that all feasible trajectories of the system can be determined by specifying only the trajectory of the last trailer. Unlike other approaches in the literature (such as converting the kinematics into a normal form), this technique works globally.

A limitation of the differential algebraic setting is that it does not provide tools for regularity analysis. The results are given in terms of meromorphic functions in the variables and their derivatives, without characterizing the solutions. In particular, solutions to the differential polynomials may not exist. For example, the system:

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1^2 \end{aligned} \tag{3}$$

is flat in the differentially algebraic sense with flat output $y = x_2$. However, it is clear that the derivative of x_2 always has to be positive, and therefore we cannot follow an arbitrary trajectory in y space.

To treat time as a special variable in the relations (2), one can to resort to Lie-Bäcklund transformations on infinite dimensional spaces [10, 11]. The latter paper distinguishes between "orbital (or topological) flatness" where time scalings are allowed, and "differential flatness" where they are not.

In the beginning of this century, the French geometer E. Cartan developed a set of powerful tools for the study of equivalence of systems of differential equations [3, 4, 22]. Equivalence need not be restricted to systems of equal dimensions. In particular a system can be *prolonged* to a bigger system on a bigger manifold, and equivalence between these prolongations can be studied. This is the concept of *absolute equivalence* of systems. Prolonging a system corresponds to dynamic feedback, and it is clear that we can benefit from the tools developed by Cartan to study the feedback linearization problem. The connections between Cartan prolongations and feedback linearizability for single input systems were studied in [23].

In this paper we reinterpret flatness in a differential geometric setting. We make extensive use of the tools offered by exterior differential systems, and the ideas of Cartan. This approach allows us to study some of the regularity issues, and also to give an explicit treatment of time dependence. Moreover, we can easily make connections to the extensive body of theory that exists in differential geometry. We show how to recover the differentially algebraic definition, and give an exterior differential systems proof for a result proven by Martin [16, 17] in differential algebra: a flat system can be put into Brunovsky normal form by dynamic feedback in an open and dense set (this set need not contain an equilibrium point).

We also give a complete characterization of flatness for systems with a single input. In this case, flatness in the neighborhood of an equilibrium point is equivalent to linearizability by static state feedback around that point. This result is stronger than linearizability by endogenous feedback as indicated by Martin et al, [16, 9], since the latter only holds in an open and dense set. We also treat the case of time varying versus time invariant flat outputs, and show that in the case of a single input, time invariant system the flat output can always be chosen time independent. In exterior differential systems, the special role of the time coordinate is expressed as an independence condition, i.e., a one-form that is not allowed to vanish on any of the solution curves. A fundamental problem with exterior differential systems is that most results only hold on open dense sets [14]. It requires extra effort to obtain results in the neighborhood of a point, see for example [19]. In this paper too, we can only get local results by introducing regularity assumptions, typically in the form of rank conditions.

The organization of the paper is as follows. In Section 2 we introduce the definitions pertaining to absolute equivalence and their interpretation in control theory. In Section 3 we introduce our definition of differential flatness and show how to recover the differential algebraic results. In Section 4 we study the connections between flatness and feedback linearizability. In Section 5 we present our main theorems characterizing flatness for single input systems, and in Section 6 we summarize our results and point out some open questions.

2. Prolongations and Control Theory

This section introduces the concept of prolongations, and states some basic theorems. It relates these concepts to control theory. Proofs of most of these results can be found in [22]. We assume that all manifolds and mappings are smooth (C^∞) unless explicitly stated otherwise.

Definition 1 (Pfaffian system)**.** A *Pfaffian system* I on a manifold M is a submodule of the module of differential one-forms $\Omega^1(M)$ over the commutative ring of smooth functions $C^\infty(M)$. A set of one-forms $\omega^1, \dots, \omega^n$, generates a Pfaffian system $I = \{\omega^1, \dots, \omega^n\} = \{\Sigma f_k \omega^k | f_k \in C^\infty(M)\}$.

In this paper, we restrict attention to finitely generated Pfaffian systems on finite dimensional manifolds. It is important to distinguish between a Pfaffian system and its set of generators or the algebraic ideal $\mathcal{I}$ in $\Lambda(M)$ generated by I. Since we are only dealing with Pfaffian systems the term *system* will henceforth mean a Pfaffian system.

For a Pfaffian system I we can define its *derived system* $I^{(1)}$ as $I^{(1)} = \{\omega \in I | d\omega \equiv 0 \bmod \mathcal{I}\}$, where $\mathcal{I}$ is the algebraic ideal generated by I. The derived system is itself a Pfaffian system, so we can define the sequence $I, I^{(1)}, I^{(2)}, \dots$ which is called the *derived flag* of I.

Assumption 1 (Regularity of Pfaffian systems)**.** Unless explicitly otherwise stated, we will assume throughout this paper that the system is *regular*, i.e.

1. the system and all its derived systems have constant rank.
2. for each k, the exterior differential system generated by $I^{(k)}$ has a degree 2 part with constant rank.

If the system is regular the derived flag is decreasing, so there will be an N such that $I^{(N)} = I^{(N+1)}$. This $I^{(N)}$ is called the *bottom derived system*.

When one studies the system of one-forms corresponding to a system of differential equations, the independent variable time becomes just another coordinate on the manifold along with the dependent variables. Hence the notion of an independent variable is lost. If x denotes the dependent variables, a solution to such a system $c : s \to (t(s), x(s))$ is a curve on the manifold. But we are only interested in solution curves which correspond to graphs of functions $x(t)$. Hence we need to reject solutions for which $\frac{dt}{ds}$ vanishes at some point. This is done by introducing dt as an *independence condition*, i.e., a one-form that is not allowed to vanish on any of the solution curves. An independence condition is well defined only up to a nonvanishing multiple and modulo I. We will write a system with independence condition τ as (I, τ). The form τ is usually exact, but it does not have to be. In this paper we shall always take τ exact, in agreement with its physical interpretation as time.

Definition 2 (Control System)**.** A Pfaffian system with independence condition (I, dt) is called a *control system* if $\{I, dt\}$ is integrable.

In local coordinates, control systems can be written in the form:

$$I = \{dx_1 - f_1(x, u, t)dt, \dots, dx_n - f_n(x, u, t)dt\} \tag{4}$$

with states $\{x_1, \dots, x_n\}$ and inputs $\{u_1, \dots, u_p\}$. Note that a control system is always assumed to have independence condition dt. If the functions f are independent of time then we speak of a *time invariant* control system.

Definition 3 (Cartan Prolongation)**.** Let (I, dt) be a Pfaffian system on a manifold M. Let B be a manifold such that $\pi : B \to M$ is a fiber bundle. A Pfaffian system $(J, \pi^* dt)$ on B is a *Cartan prolongation* of the system (I, dt) if the following conditions hold:

1. $\pi^*(I) \subset J$
2. For every integral curve of I, $c : (-\epsilon, \epsilon) \to M$, there is a unique lifted integral curve of J, $\tilde{c} : (-\epsilon, \epsilon) \to B$ with $\pi \circ \tilde{c} = c$.

Assumption 2 (Regularity of Cartan prolongations)**.** In this paper we only look at Cartan prolongations that preserve codimension.

Note that all prolongations are required to preserve the independence condition of the original system. The above definition implies that there is a smooth 1-1 correspondence between the integral curves of a system and of its Cartan prolongation. Cartan prolongations are useful to study equivalence between systems of differential equations that are defined on manifolds of different dimensions. This occurs in dynamic feedback extensions of control systems. We increase the dimension of the state by adding dynamic feedback, but the extended system is still in some sense equivalent to the original system.

This allows us to define the concept of absolute equivalence introduced by Elie Cartan:

Definition 4 (Absolute Equivalence)**.** Two systems I_1, I_2 are called *absolutely equivalent* if they have Cartan prolongations J_1, J_2 respectively that are equivalent in the usual sense, i.e., there exists a diffeomorphism ϕ such that $\phi^*(J_2) = J_1$. This is illustrated in the following diagram:

$$
\begin{tikzcd}
J_1 \arrow[r, leftrightarrow, "\phi"'] \arrow[d, "\pi_1"'] & J_2 \arrow[d, "\pi_2"] \\
I_1 & I_2
\end{tikzcd}
$$

An interesting subclass of Cartan prolongations is formed by *prolongations by differentiation*: If (I, dt) is a system with independence condition on M, and du an exact one-form on M that is independent of $\{I, dt\}$, and if y is a fiber coordinate of B, then $\{I, du - ydt\}$ is called a *prolongation by differentiation* of I. Note that we have omitted writing $\pi^*(du - ydt)$ where $\pi : B \to M$ is the surjective submersion. We will make this abuse in the rest of the paper for notational convenience. Prolongations by differentiation correspond to adding integrators to a system. In the context of control systems, the coordinate u is the input that is differentiated.

If we add integrators to all controls, we obtain a *total prolongation*: Let (I, dt) be a system with independence condition, where $\dim\, I = n$. Let $\dim M = n + p + 1$. Let $u_1, \ldots, u_p$ be coordinates such that $du_1, \ldots, du_p$ are independent of $\{I, dt\}$, and let $y_1, \ldots, y_p$ be fiber coordinates of B, then $\{I, du_1 - y_1 dt, \ldots, du_p - y_p dt\}$ is called a *total prolongation* of I. Total prolongations can be defined independent of coordinates, and are therefore intrinsic geometric objects. It can be shown that in codimension 2 (i.e., a system with n generators on an $n+2$ dimensional manifold), all Cartan prolongations are locally equivalent to total prolongations, [22].

We will call *dynamic feedback* a feedback of the form

$$
\begin{aligned}
\dot{z} &= a(x, z, v, t) \\
u &= b(x, z, v, t).
\end{aligned}
$$

If t does not appear in (a, b) we call (a, b) a *time invariant* dynamic feedback. The dynamic feedback is called *regular* if for each fixed x and t the map $b(x, ., ., t) : (z, v) \mapsto u$ is a submersion. An important question is what type of dynamic feedback corresponds to what type of prolongation. Clearly, prolongations by differentiation correspond to dynamic extension (adding integrators to the inputs).

Cartan prolongations provide an intrinsic geometric way to study dynamic feedbacks. We shall show that Cartan prolongations that extend a control system to another control system can be expressed as dynamic feedbacks in local coordinates. The following example shows that not every dynamic feedback corresponds to a Cartan prolongation:

Example 1 (Dynamic Feedback vs. Cartan prolongation)**.** Consider the control system

$$\dot{x}_1 = u,$$

with feedback

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= -z_1\\ u &= g(z)v.\end{aligned}$$

This dynamic feedback introduces harmonic components which can be used to asymptotically stabilize nonholonomic systems (see [6] for a description of how this might be done). It is not a Cartan prolongation since (z, v) cannot be uniquely determined from (x, u).

It must be said that the feedback in Example 1 is somewhat unusual, in that most theorems concerning dynamic feedback are restricted to adding some type of integrator to the inputs of the system.

Definition 5 (Endogenous Feedback)**.** Let $\dot{x} = f(x, u, t)$ be a control system. A dynamic feedback

$$\begin{aligned}\dot{z} &= a(x, z, v, t)\\ u &= b(x, z, v, t).\end{aligned} \tag{5}$$

is said to be *endogenous* if z and v satisfying (5) can be expressed as functions of x, u, t and a finite number of their derivatives:

$$\begin{aligned}z &= \alpha(x, u, \dots, u^{(l)}, t)\\ v &= \beta(x, u, \dots, u^{(l)}, t).\end{aligned} \tag{6}$$

An endogenous feedback is called *regular* if for each fixed x and t the map $b(x, ., ., t) : (z, v) \mapsto u$ is a submersion.

Note that this differs slightly from the definition given in [16, 17] due to the explicit time dependence used here. The relationship between Cartan prolongations and endogenous dynamic feedback is given by the following two theorems. The first says that a regular endogenous feedback corresponds to a Cartan prolongation.

Theorem 1 (Endogenous feedbacks are Cartan prolongations)**.** *Let I be a control system on an open set $T \times X \times U$ which in coordinates (t, x, u) is given by $\dot{x} = f(x, u, t)$. Let J denote the control system on the open set $T \times X \times Z \times V$ which is obtained from the above system by adding a regular endogenous dynamic feedback. Then J is a Cartan prolongation of I.*

Proof. Define the mapping $F : T \times X \times Z \times V \to T \times X \times U$ by $F(t, x, z, v) = (t, x, b(x, z, v, t))$. Since b is regular, F is a submersion. Furthermore b is surjective since the feedback is endogenous. Therefore F is surjective too. Since F is a surjective submersion, $T \times X \times Z \times V$ is fibered over $T \times X \times U$. Hence we have that solutions $(t, x(t), z(t), v(t))$ of J project down to solutions $(t, x(t), b(x(t), z(t), v(t), t))$ of I. Therefore the first requirement of being a Cartan prolongation is satisfied.

The second requirement of unique lifting is trivially satisfied by the fact that z and v are obtained uniquely by equation (6). □

Conversely, a Cartan prolongation can be realized by endogenous dynamic feedback in an open and dense set, if the resulting prolongation is a control system:

Theorem 2 (Cartan prolongations are locally endogenous feedbacks)**.** *Let I be a control system on a manifold M with p inputs, $\{u_1, \dots, u_p\}$. Every Cartan prolongation $J = \{I, \omega_1, \dots, \omega_r\}$ on B with independence condition dt such that J is again a control system is realizable by endogenous regular feedback on an open and dense set of B.*

Proof. Let r denote the fiber dimension of B over M, and let $\{w_1, \dots, w_r\}$ denote the fiber coordinates. Since I is a control system, $\{I, dt\}$ is integrable, and we can find n first integrals $x_1, \dots, x_n$. Preservation of the codimension and integrability of $\{J, dt\}$ means that we can find r extra functions $a_1, \dots, a_r$ such that $J = \{I, dz_1 - a_1 dt, \dots, dz_r - a_r dt\}$. Here the z_i are first integrals of $\{J, dt\}$ that are not first integrals of $\{I, dt\}$. Pick p coordinates $v(u, w)$ such that $\{t, x, z, v\}$ form a set of coordinates of B. The v coordinates are the new control inputs. Clearly $a_i = a_i(x, z, v, t)$ since we have no other coordinates. Also since $\{t, x, z, v\}$ form coordinates for B, and u is defined on B, there has to be a function b such that $u = b(x, z, v, t)$. Since both (t, x, u, w) and (t, x, z, v) form coordinates on B, there has to be a diffeomorphism ϕ between the 2. From the form of the matrix $\frac{\partial \phi}{\partial (t,x,z,v)}$ it can be seen that $\frac{\partial b}{\partial (z,v)}$ is full rank, and hence b is regular. This recovers the form of equation (5). Since J is a Cartan prolongation, every (x, u, t) lifts to a unique (x, z, v, t). From Lemma 1 , to be presented in the next section, it then follows that we can express (z, v) as functions of x and u and its derivatives in an open and dense set. We thus obtain the form of equation (6). □

3. Differentially Flat Systems

In this section we present a definition of flatness in terms of prolongations. Our goal is to establish a definition of flatness in terms of differential geometry, while capturing the essential features of flatness in differential algebra [8, 9]. We build our definition on the minimal requirements needed to recover these features, namely the one to one correspondence between solution curves of the original system and an unconstrained system, while maintaining regularity of the various mappings. Our definition makes use of the concept of an absolute morphism [22].

Definition 6 (Absolute morphism)**.** An *absolute morphism* from a system (I_1, dt) on M_1 to a system (I_2, dt) on M_2 consists of a Cartan prolongation (J_1, dt) on $\pi : B_1 \to M_1$ together with surjective submersion $\phi : B_1 \to M_2$ such that $\phi^*(I_2) \subset J_1$. This is illustrated below:

$$\begin{tikzcd} J_1 \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ I_1 & I_2 \end{tikzcd}$$

Definition 7 (Invertibly absolutely morphic systems)**.** Two systems (I_1, dt) and (I_2, dt) are said to be *absolutely morphic* if there exist absolute morphisms from (I_1, dt) to (I_2, dt) and from (I_2, dt) to (I_1, dt). This is illustrated below:

$$\begin{tikzcd} J_1 \arrow[d, "\pi_1"'] \arrow[dr, "\phi_2"' near start] & J_2 \arrow[d, "\pi_2"] \arrow[dl, "\phi_1" near start] \\ I_1 & I_2 \end{tikzcd}$$

Two systems (I_1, dt) and (I_2, dt) are said to be *invertibly absolutely morphic* if they are absolutely morphic and the following inversion property holds: let $c_1(t)$ be an integral curve of I_1 with $\tilde{c}_1$ the (unique) integral curve of J_1 such that $c_1 = \pi \circ \tilde{c}_1$, and let $\gamma(t) = \phi_2 \circ \tilde{c}_1(t)$ be the projection of $\tilde{c}_1$. Then we require that $c_1(t) = \phi_1 \circ \tilde{\gamma}(t)$, where $\tilde{\gamma}(t)$ is the lift of γ from I_2 to J_2. The same property must hold for solution curves of I_2.

If two systems are invertibly absolutely morphic, then the integral curves of one system map to the integral curves of the other and this process is invertible in the sense described above. If two systems are absolutely equivalent then they are also absolutely morphic, since they can both be prolonged to systems of the same dimension which are diffeomorphic to each other. However, for two systems to be absolutely morphic we do not require that any of the systems have the same dimension.

A differentially flat system is one in which the "flat outputs" completely specify the integral curves of the system. More precisely:

Definition 8 (Differential Flatness)**.** A system (I, dt) is *differentially flat* if it is invertibly absolutely morphic to the trivial system $I_t = (\{0\}, dt)$.

Notice that we require that the independence condition be preserved by the absolute morphisms, and hence our notion of time is the same for both systems. Since an independence condition is only well defined up to nonvanishing multiples and modulo the system, we do allow time scalings between the systems We also allow time to enter into the absolute morphisms which map one system onto the other.

If the system (I, dt) is defined on a manifold M, then we can restrict the system to a neighborhood around a point in M, which is again itself a manifold. We will call a system flat in that neighborhood if the restricted system is flat.

The following discussion leans heavily on a theorem due to Sluis and Shadwick, [22, 21], which we recall here for completeness:

Theorem 3. *Let I be a system on a manifold M and J a Cartan prolongation of I on $\pi : B \to M$. On an open and dense subset of B, there exists a prolongation by differentiation of J that is also a prolongation by differentiation of I.*

Proof. See [22], Theorem 24. □

In order to establish the relationship between our definition and the differential algebraic notion of flatness, we need the following straightforward corollary to Theorem 3. This lemma expresses the dependence of the fiber coordinates of a Cartan prolongation on the coordinates of the base space:

Lemma 1. *Let (I, dt) be a system on a manifold M with local coordinates $(t, x) \in \mathbb{R}^1 \times \mathbb{R}^n$ and let (J, dt) be a Cartan prolongation on the manifold B with fiber coordinates $y \in \mathbb{R}^r$. Assume the regularity assumptions 1, 2 hold. Then on an open dense set, each y_i can be uniquely determined from t, x and a finite number of derivatives of x.*

Proof. By Theorem 3 there is a prolongation by differentiation, on an open and dense set, say I_2, of J, with fiber coordinates z_i, that is also a prolongation by differentiation of the original system I, say with fiber coordinates w_i. This means that the (x, y, z, t) are diffeomorphic to (x, w, t): $y = y(x, w, t)$. The w are derivatives of x, and therefore the claim is proven. □

This lemma allows us to explicitly characterize differentially flat systems in a local coordinate chart. Let a system in local coordinates (t, x) be differentially flat and let the corresponding trivial system have local coordinates (t, y). Then on an open and dense set there are surjective submersions h and g with the following property: Given any curve $y(t)$, then

$$x(t) = g(t, y(t), \ldots, y^{(q)}(t))$$

is a solution of the original system and furthermore the curve $y(t)$ can be obtained from $x(t)$ by

$$y(t) = h(t, x(t), \ldots, x^{(l)}(t)).$$

This follows from using definitions of absolute morphisms, the invertibility property, and Lemma 1, stating that fiber coordinates are functions of base coordinates and their derivatives and the independent coordinate.

This local characterization of differential flatness corresponds to the differential algebraic definition except that h and g need not be algebraic or meromorphic. Also, we do not require the system equations to be algebraic or meromorphic. The explicit time dependence corresponds to the differential algebraic setting where the differential ground field is a field of functions and not merely a field of constants. The functions g and h now being surjective submersions enables us to link the concept of flatness to geometric nonlinear control theory where we usually impose regularity. We emphasize that we only required a one to one correspondence of solution curves *a priori* for our definition of flatness, and not that this dependence was in the form of derivatives. The particular form of this dependence followed from our analysis.

Finally, the following theorem allows us to characterize the notion of flatness in terms of absolute equivalence.

Theorem 4. *Two systems are invertibly absolutely morphic if and only if they are absolutely equivalent.*

Proof. Sufficiency is trivial. We shall prove necessity. For convenience we shall not mention independence conditions, but they are assumed to be present and do not affect the proof. Let I_1 on M_1 and I_2 on M_2 be invertibly absolutely morphic. Let J_1 on B_1 be the prolongation of I_1 with $\pi_1 : B_1 \to M_1$ and similarly J_2 on B_2 be the prolongation of I_2 with $\pi_2 : B_2 \to M_2$. Let the absolute morphisms be $\phi_1 : B_2 \to M_1$ and $\phi_2 : B_1 \to M_2$.

We now argue that J_2 is a Cartan prolongation of I_1 (and hence I_1 and I_2 are absolutely equivalent). By assumption ϕ_1 is a surjective submersion and every solution $\tilde{c}_2$ of J_2 projects down to a solution c_1 of I_1 on M_1. The only extra requirement for J_2 on $\phi_1 : B_2 \to M_1$ to be a (Cartan) prolongation is that every solution c_1 of I_1 has a unique lift $\tilde{c}_2$ (on B_2) which is a solution of J_2.

To show existence of a lift, observe that for any given c_1 which is a solution of I_1, we can obtain its unique lift $\tilde{c}_1$ on B_1 (which solves J_1), and get its projection c_2 on M_2 (which solves I_2) and then consider its unique lift $\tilde{c}_2$ on B_2. Now it follows

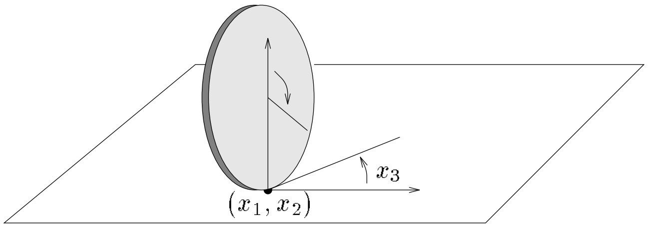

FIGURE 1. Rolling penny

from the invertibility property that $\phi_1 \circ \tilde{c}_2 = c_1$. In other words, $\tilde{c}_2$ projects down to c_1.

To see the uniqueness of this lift, suppose $\tilde{c}_2$ and $\tilde{c}_3$ which are solutions of J_2 on B_2, both project down to c_1 on M_1. Consider their projections c_2 and c_3 (respectively) on M_2. When we lift c_2 or c_3 to B_2 and project down to M_1 we get c_1. Which when lifted to B_1 gives, say $\tilde{c}_1$. By the requirement of the absolute morphisms being invertible $\tilde{c}_1$ should project down to (via ϕ_2) c_2 as well as c_3. Then uniqueness of projection implies that c_2 and c_3 are the same. Which implies $\tilde{c}_2$ and $\tilde{c}_3$ are the same.

Hence J_2 is a Cartan prolongation of I_1 as well. Hence I_1 and I_2 are absolutely equivalent. □

Using this theorem we can completely characterize differential flatness in terms of absolute equivalence:

Corollary 1. *A system (I, dt) is differentially flat if and only if it is absolutely equivalent to the trivial system $I_t = (\{0\}, dt)$.*

Note that we require the feedback equivalence to preserve time, since both systems have the same independence condition. In the classical feedback equivalence we only consider diffeomorphisms of the form $(t, x, u) \mapsto (t, \phi(x), \psi(x, u))$. For flatness we allow diffeomorphisms of the form $(t, x, u) \mapsto (t, \phi(t, x, u), \psi(t, x, u))$. We could allow time scalings of the form $t \mapsto s(t)$ but this does not change the independence condition and does therefore not gain any generality. In Cartan's notion of equivalence all diffeomorphisms are completely general. This is akin to the notion of *orbital* flatness presented in [10], where one allows time scalings dependent on all states and inputs.

Example 2. Consider the motion of a rolling penny, as shown in Figure 1. Let (x_1, x_2) represent the xy position of the penny on the plane, x_3 represent the heading angle of the penny relative to a fixed line on the plane, and x_4 represent the rotational velocity of the angle of Lincoln's head, i.e., the rolling velocity. We restrict $x_3 \in [0, \pi)$ since we cannot distinguish between a positive rolling velocity x_4 at a heading angle x_3 and a negative rolling velocity at a heading angle $x_3 + \pi$.

The dynamics of the penny can be written as a Pfaffian system described by

$$
\begin{aligned}
\omega^1 &= \sin x_3\, dx_1 - \cos x_3\, dx_2 \\
\omega^2 &= \cos x_3\, dx_1 + \sin x_3\, dx_2 - x_4 dt \\
\omega^3 &= dx_3 - x_5 dt \\
\omega^4 &= dx_4 - u_1 dt \\
\omega^5 &= dx_5 - u_2 dt
\end{aligned}
\tag{7}
$$

where $x_5 = \dot{x}_3$ is the velocity of the heading angle. The controls u_1 and u_2 correspond to the torques around the rolling and heading axes. We take dt as the independence condition.

This system is differentially flat away from $x_4 = 0$ using the outputs x_1 and x_2 plus knowledge of time. If not both $dx_1 = 0$ and $dx_2 = 0$ we can solve for x_3 using ω_1. Given these three variables plus time, we can solve for all other variables in the system by differentiation with respect to time. This argument also shows that the system is time independent differentially flat, since we only need to know (x_1, x_2) and their derivatives up to order three in order to solve for all of the states of the system.

Often we will be interested in a more restricted form of flatness that eliminates the explicit appearance of time that appears in the general definition.

Definition 9. An absolute morphism from a time invariant control system (I_1, dt) to a time invariant control system (I_2, dt) is a *time-independent absolute morphism* if locally the maps $\pi : B_1 \to M_1$ and $\phi : B_1 \to M_2$ in definition 6 have the form $(t, x, u) \mapsto (t, \eta(x, u), \psi(x, u))$, i.e. the mappings between states and inputs do not depend on time. A system (I, dt) is *time-independent differentially flat* if it is differentially flat using time-independent absolute morphisms.

Note that the example given above is time-independent differentially flat. One might be tempted to think that if the control system I is time invariant and knowing that the trivial system is time invariant, we can assume that the absolute morphism $x = \phi(t, y, y^{(1)}, \ldots, y^{(q)})$ has to be time independent as well. That this is not true is illustrated by the following example.

Example 3. Consider the system $\dot{y} = ay$, and the coordinate transformation $y = x^2 e^{t+x}$. Then $\dot{x} = \frac{(a-1)x}{2+x}$. Both systems are time invariant, but the coordinate transformation depends on time.

4. Linear Systems and Linearizability

The differential algebra approach to control has given rise to new interpretations of linearity [7, 18]. Rather than overloading the concept of linearity we feel it increases clarity if we stick with the conventional notion of linearity (see for example [20]) and introduce a new term for the broader concept of linearity as exposed in [7, 18]. We will try to clarify the different notions and indicate what the underlying approaches and assumptions are. This will enable us to elucidate the connections with flatness and prolongations. The following definitions are widely accepted and taken from [2].

Definition 10. A *dynamical system* is a 5-tuple (X, U, Y, T, ρ). Here X is the set of states, U is a set of allowable input functions and $U(T)$ denotes the possible values of the inputs at a fixed time. Y is the set of outputs functions and T is the set of times over which the system evolves. The map $\rho : (X, U, T, T) \to Y$, $\rho(x_0, u_{[t_0,t_1]}, t_0, t_1) = y_1$ is the response function that maps an initial state x_0 at an initial time t_0 given an input on an interval $[t_0, t_1]$, to an output y_1 at a final time t_1.

Definition 11 (Linear system)**.** A dynamical system is said to be *linear* if

1. The sets X, U and Y are linear vector spaces over the same field.

2. For each fixed initial and final time (t_0, t_1) respectively, the response function $\rho(.,.,t_0,t_1)$ is a linear map from (X, U) into Y.

The linearity of the response function implies in particular that the origin of the space X is an equilibrium point.

Definition 12 (Time invariant system)**.** Let S_τ denote the delay map from a function space onto itself: $(S_\tau f)(t) = f(t-\tau)$. A dynamical system is said to be *time invariant* if

1. The input, output, and time spaces are closed under operation of S_τ for all $\tau \in \mathbb{R}$.
2. $\rho(x_0, u, t_0, t_1) = \rho(x_0, S_\tau u, t_1 + \tau, t_0 + \tau)$.

In particular, a system of the form

$$\dot{x} = Ax + Bu \tag{8}$$

$$y = Cx + Du \tag{9}$$

is linear and time invariant. Here (A, B, C, D) are matrices of appropriate dimensions. If the system is controllable, we can put it in Brunovsky normal form by a linear coordinate transformation:

$$y_i^{(l_i)} = u_i. \tag{10}$$

Definition 13 (Feedback linearizability)**.** The time invariant nonlinear system

$$\dot{x} = f(x, u) \tag{11}$$

is *feedback linearizable* if there is a dynamic feedback

$$\dot{z} = \alpha(x, z, v) \tag{12}$$

$$u = \beta(x, z, v) \tag{13}$$

and new coordinates $\xi = \phi(x, z)$ and $\eta = \psi(x, z, v)$ such that in the new coordinates the system has the form:

$$\dot{\xi} = A\xi + B\eta \tag{14}$$

and the mapping ϕ maps onto a neighborhood of the origin. If $\dim z = 0$ then we say the system is *static* feedback linearizable.

The form in equation (13) is the standard form in linear systems theory. It is useful if one wants to design controllers for nonlinear systems around equilibrium points.

It might be that the system can be put in the form (13) but that the coordinate transformation is not valid in a neighborhood of the origin of the target system. In that case we can shift the origin of the linear system to put it in the form

$$\dot{\xi} = A\xi + B\eta + E \tag{15}$$

We will call this a *state space affine* form. This form is called linear in [18], but most results in linear systems theory cannot be applied since the origin is not an equilibrium point. However, it is still useful in the context of trajectory generation. For example, a nonholonomic system in chained form ([19]) can be transformed to this state space affine form.

It is clear that all feedback linearizable (by static or dynamic feedback) systems are flat, since we can put them into Brunovsky normal form. The converse only holds in an open and dense set, as is shown by the following theorem. An analogous result was proven by Martin in a differentially algebraic setting [16, 17].

Theorem 5. *Every differentially flat system can be put in Brunovsky normal form in an open and dense set through regular endogenous feedback.*

Proof. Let J, J_t be the Cartan prolongations of I, I_t respectively. Then by Theorem 3, on an open and dense set, there is a prolongation by differentiation of J_t that is also a prolongation by differentiation of I_t, say J_{t1}. Let J_1 be the corresponding Cartan prolongation of J. Then J_1 is equivalent to J_{t1}, which is in Brunovsky normal form. In particular, since J_1 is a Cartan prolongation, it can be realized by regular endogenous feedback. $\square$

This proof relies on Theorem 3 which restricts its validity to an open and dense set. We conjecture that the result holds everywhere, but the above proof technique does not allow us to conclude that. The obstruction lies in certain prolongations that we cannot prove to be regular.

We emphasize here that even though flatness implies that we can find coordinates that put the system into the linear form (13) we do not require the underlying manifolds to be linear spaces. In this sense, flatness is an intrinsic property of a control system defined on a smooth manifold.

5. Flatness for Single Input Systems

For single input control systems, the corresponding differential system has codimension 2. There are a number of results available in codimension 2 which allow us to give a complete characterization of differentially flat single input control systems. In codimension 2 every Cartan prolongation is a total prolongation around every point of the fibered manifold ([22]), given our regularity assumptions 1, 2. This allows us to prove the following

Theorem 6. *Let I be a time invariant control system:*

$$I = \{dx_1 - f_1(x,u)dt, \dots, dx_n - f_n(x,u)dt\},$$

where u is a scalar control, i.e., the system has codimension 2. If I is time-independent differentially flat around an equilibrium point, then I is feedback linearizable by static time invariant feedback at that equilibrium point.

Proof. Let I be defined on M with coordinates (x,u,t), let the trivial system I_t be defined on B_t with coordinates (y_0, t), let the prolongation of I_t be J_t, and let J_t be defined on M_t. This is illustrated below :

$$\begin{tikzcd} J \arrow[d, "\pi"'] \arrow[dr, "\phi_t"] & J_t \arrow[d, "\pi_t"] \arrow[dl, "\phi"'] \\ I & I_t = \{\} \end{tikzcd}$$

First we show that J_t can be taken as a Goursat normal form around the equilibrium point. In codimension 2, every Cartan prolongation is a repeated total prolongation in a neighborhood of every point of the fibered manifold ([22], Theorem 5). Let $I_{t0} = I_t, I_{t1}, I_{t2}, \dots$ denote the total prolongations starting at I_t, defined on fibered manifolds $B_{t0} = B_t, B_{t1}, \dots$. If y_1 denotes the fiber coordinate of B_{t1} over B_{t0}, then I_{t1} has the form $\lambda dt + \mu dy_0$, where either λ or μ depends non trivially on y_1. Since the last derived system of I does not drop rank at the equilibrium, neither does I_{t1} and we have that not both λ and μ vanish at the equilibrium. Now, $\mu \neq 0$ at the equilibrium point, since $y_0 \equiv c$ is a solution curve to I_t, which would not have a lift to I_{t1} if $\mu = 0$, since dt is required to remain the independence condition of all Cartan prolongations. From continuity $\mu \neq 0$ around the equilibrium point. So we can define $y_1 := -\lambda/\mu$, and I_{t1} can be written as $dy_0 - y_1 dt$. We can continue this process for every Cartan prolongation, both of I_t and of I. This brings J_t in Goursat normal form in a neighborhood of the equilibrium point.

Now we will argue that we don't need to prolong I to establish equivalence. Since J is a Cartan prolongation, and therefore a total prolongation, its first derived system will be equivalent to the first derived system of J_t. Continuing this we establish equivalence between I and I_{tn}, where $I_{tn} = \{dy_0 - y_1 dt, \dots, dy_{n-1} - y_n dt\}$. So we have $y = (y_0, \dots, y_n) = y(x, u, t)$.

Next we will show that $y_0, \dots, y_n$ are independent of time, and that $y_0, \dots, y_{n-1}$ are independent of u. By assumption y_0 is independent of time. Since the corresponding derived systems on each side are equivalent, $dy_0 - y_1 dt$ is equivalent to the last one-form in the derived flag of I. Since the differential du does not appear in this one-form, y_0 is independent of u. Analogously, $y_i, i = 1, \dots, n-1$ are all independent of u. Since the $y_i, i = 1, \dots, n$ are repeated derivatives of y_0, and since I is time invariant, these coordinates are also independent of time.

We still have to show that the mapping $x \mapsto y$ is a valid coordinate transformation. Suppose $dy_0, \dots, dy_{n-1}$ are linearly dependent at the equilibrium. Then, J_t drops rank at the equilibrium, and since we have equivalence, so would I. But form the form of I we can see this is not the case.

Therefore $y_i = y_i(x), i = 0, \dots, n-1, y_n = y_n(x, u)$ and the system J_t is just a chain of integrators with input y_n. The original system I is equivalent to this linear system by a coordinate transformation on the states and a state dependent and time invariant feedback. This coordinate transformation is well defined around the equilibrium point. It is therefore feedback linearizable by a static feedback that is time invariant. Note that $\partial y_n / \partial u \neq 0$ because y_n is the only of the y variables that depends on u. □

Example 4. Notice that in our definition the system

$$\begin{aligned} \dot{x}_2 &= u \\ \dot{x}_1 &= x_2^3 \\ y &= x_1 \end{aligned} \tag{16}$$

is not flat around the origin, because we get $u = \frac{\ddot{y}}{3\,\dot{y}^{2/3}}$ so that curves with $\dot{y} = 0$ and $\ddot{y} \neq 0$ have no lift. It is also not feedback linearizable at the origin.

We will now show that in the case of a time invariant system, we don't need the assumption of time invariant flatness to conclude static feedback linearizability. We will require the following preliminary result, which appeared in a proof in [23].

Lemma 2. *Given a one-form $\alpha = A_i(x, u)dx_i - A_0(x, u)dt$ (using implicit summation) on a manifold M with coordinates (x, u, t), and suppose we can write $\alpha = dX(x, u, t) - U(x, u, t)dt$. Then we can also write α as $\alpha = dY(x) - V(x, u)dt$, i.e., we can take the function X independent of time and the input, and we can take U independent of time. If we know in addition that $\alpha = A_i(x)dx_i - A_0(x)dt$,* *then we can scale α as $\alpha = dY(x) - V(x)dt$, i.e., we can take V independent of u as well.*

Proof. See [23]. □

The following theorem seems to implied in [23], but the proof there refers to a general discussion of Cartan's method of equivalence as applied to control systems in [12]. We work out the proof for this special case.

Theorem 7. *A single input time invariant control system is differentially flat if and only if it is feedback linearizable by static, time invariant feedback.*

Proof. Sufficiency is trivial, so we shall only prove necessity. Let the control system be $I = \{dx_1 - f_1(x,u)dt, \dots, dx_n - f_n(x,u)dt\}$, where u is a scalar control, i.e. the system has codimension 2. Let $\{\alpha^i, i = 1, \dots, n\}$ and $\{\alpha_t^i, i = 1, \dots, n\}$ be one-forms adapted to the derived flag of I, I_t respectively. Thus, $I^{(i)} = \{\alpha^1, \dots, \alpha^{n-i}\}$ and $I_t^{(i)} = \{\alpha_t^1, \dots, \alpha_t^{n-i}\}$. Since I does not contain the differential du, the forms $\alpha^1, \dots, \alpha^{n-1}$ can be taken independent of u. Since I is time invariant, the forms $\alpha_1, \dots, \alpha_n$ can be chosen independent of time. We can thus invoke the second part of Lemma 2 for the forms $\alpha^1, \dots, \alpha^{n-1}$.

Assume $n \geq 2$. As in Theorem 6 we have equivalence between α^1 and $\alpha_t^1 = dy_0(x,t) - y_1(x,t)dt$ (if $n = 1$ we have $y_n = y_n(x,u,t)$, which we will reach eventually). Since I is time invariant we can choose α^1 time independent: $\alpha^1 = A_i(x)dx_i - A_0(x)dt$. From Lemma 2 we know that we can write α^1 as $dY_0 - Y_1 dt$ where Y_0, Y_1 are functions of x only.

Again according to Lemma 2, we can write $\alpha^2 = dV(x) - W(x)dt$. Now from,

$$\begin{aligned} 0 &= d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 \\ &= -dY_1 \wedge dt \wedge dY_0 \wedge dV \end{aligned}$$

we know $V = V(Y_1, Y_0)$. And from

$$\begin{aligned} 0 &\neq d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \\ &= -dW \wedge dt \wedge dY_0 \wedge dV \end{aligned}$$

we know that $\gamma_1 := \partial V/\partial Y_1 \neq 0$. Then, writing $\gamma_0 := \partial V/\partial Y_0$, (and $\simeq$ denotes equivalence in the sense that both systems generate the same ideal),

$$\begin{aligned} \{\alpha^1, \alpha^2\} &\simeq \{dY_0 - Y_1 dt, \gamma_1 dY_1 + \gamma_0 dY_0 - W dt\} \\ &\simeq \{dY_0 - Y_1 dt, \gamma_1 dY_1 + \gamma_0 Y_1 dt - W dt\} \\ &\simeq \{dY_0 - Y_1 dt, dY_1 - (-\gamma_0 Y_1 + W)/\gamma_1 dt\} \\ &:= \{dY_0 - Y_1 dt, dY_1 - Y_2 dt\}. \end{aligned} \tag{17}$$

Where Y_2, defined to be $Y_2 = (-\gamma_0 Y_1 + W)/\gamma_1$, is independent of (t, u) since $(\gamma_1, \gamma_0, Y_1, W)$ are. One can continue this procedure, at each step defining a new coordinate Y_i. In the last step the variable $W = W(x,u)$ (this will also be the first step if $n = 1$), and therefore Y_n depends on u nontrivially. Hence we obtain equivalence between I and $\{dY_0 - Y_1 dt, \dots, dY_{n-1} - Y_n dt\}$ with $Y_i = Y_i(x), i = 0, \dots, n-1$, and $Y_n = Y_n(x,u)$, i.e., feedback linearizability by static time invariant feedback. □

Corollary 2. *If a time invariant single input system is differentially flat we can always take the flat output as a function of the states only: $y = y(x)$.*

None of these results extend easily to higher codimensions. The reason for this is that only in codimension two we can find regularity assumptions on the original system such that every Cartan prolongation is a total prolongation. This is related to the well known fact that for SISO systems static linearizability is equivalent to dynamic linearizability. For MIMO systems we cannot express these regularity conditions on the original system: we have to check regularity on the prolonged systems.

6. Concluding remarks

We have presented a definition of flatness in terms of the language of exterior differential systems and prolongations. Our definition remains close to the original definition due to Fliess [8, 9], but it involves the notion of a preferred coordinate corresponding to the independent variable (usually time).

Using this framework we were able to recover all results in the differential algebra formulation. In particular we showed that flat systems can be put in linear form in an open and dense set. This set need not contain an equilibrium point, and this linearizability therefore does not allow one to use most methods from linear systems theory. In other words, although flatness implies a linear *form*, it does not necessarily imply a linear *structure*. For a SISO flat system we resolved the regularity issue, and established feedback linearizability around an equilibrium point. We also resolved the time dependence of flat outputs in the SISO case.

The most important open question is a characterization of flatness in codimension higher than two.

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