Decompositions for Control Systems on Manifolds with an Affine Connection*

Andrew D. Lewis† Richard M. Murray‡

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Abstract

In this letter we present a decomposition for control systems whose drift vector field is the geodesic spray associated with an affine connection. With the geometric insight gained with this decomposition, we are able to easily prove some special results for this class of control systems. Examples illustrate the theory.

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AMS Subject Classifications. 53B05, 53B20, 53C22, 70E15, 70H33, 70Q05, 93B03, 93B05, 93B29

1. Introduction

In a recent paper, Lewis and Murray [1996] introduce the notion of "configuration controllability" for a class of mechanical control systems and present some results for this notion of controllability. In the statement of their results, the so-called *symmetric product* provides a valuable shorthand. For Levi-Civita connections, the symmetric product was originally seen in the work of Crouch [1981] on gradient control systems. The geometric meaning of the symmetric product on manifolds with a *general* affine connection is given by Lewis [1996b]. In this letter we combine the results of Lewis and Murray [1996] with the interpretation of the symmetric product by Lewis [1996b] to obtain a decomposition of control systems whose drift vector field is the geodesic spray associated with a general affine connection. Examples of such control systems include Lagrangian mechanical systems whose Lagrangian is the kinetic energy with respect to a Riemannian metric. However, the use of general affine connections is more than simply an easily performed abstraction. Synge [1928] gives a formulation of the constrained equations of motion for a mechanical system in terms of an affine connection which is in general not a Levi-Civita connection. Other authors have done similar work along these lines since the work of Synge. In the control context, we mention a recent paper of Bloch and Crouch [1995]. This puts the associated constrained control systems in the framework of the present paper and so provides impetus to study control systems associated with general affine connections. The decompositions

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†Research Fellow, Control and Dynamical Systems, Mail Code 104–44 California Institute of Technology, Pasadena, CA 91125, U.S.A.
E-mail: andrew@indra.caltech.edu, URL: http://avalon.caltech.edu/~andrew/
Research in part supported by NSF Grant CMS–9502224
‡Assistant Professor, Control and Dynamical Systems, Mail Code 104–44 California Institute of Technology, Pasadena, CA 91125, U.S.A.
E-mail: murray@indra.caltech.edu, URL: http://avalon.caltech.edu/~murray/

we obtain are entirely analogous to the decompositions one obtains for nonlinear control systems which reduce to the Kalman controllability decomposition for linear systems.

In Section 2 we present a review of the necessary concepts from affine connections and also review the results of Lewis [1996b]. In Section 3 we present the control systems we will consider. These systems have a phase space which is the tangent bundle of a manifold. It is common to be interested in only the control of the points on the manifold and not be concerned with their velocities. Apropos to this, we review the controllability definitions of Lewis and Murray [1996]. With the results on the symmetric product presented in Section 2, we are able to concisely restate the main controllability results of Lewis and Murray [1996]. When the system is not controllable (by our definition), we are able to succinctly describe the reachable sets in terms of geometry of the affine connection. In Section 4 we use our main results of Section 3 to arrive at some natural corollaries. Thus the contribution of the paper is threefold. First, we make clear the geometry of the results of Lewis and Murray [1996]; second, we describe a natural decomposition of the systems we are considering in cases when they are not controllable; third, we present some new results which are easily derived from the general theory. Examples which illustrate our results are given in Section 5.

2. Affine Connections

We refer the reader to Kobayashi and Nomizu [1963] for a discussion of affine connections. Let M be an n-dimensional manifold. We denote by $C^\infty(M)$ the set of C^∞ functions on M and by $\mathscr{T}(M)$ the set of C^∞ vector fields on M. An *affine connection* on M assigns to every pair of vector fields X and Y on M a vector field $\nabla_X Y$ with the assignment satisfying

AC1. the map $(X,Y) \mapsto \nabla_X Y$ is $\mathbb{R}$-bilinear,

AC2. $\nabla_{fX} Y = f\nabla_X Y$ for $f \in C^\infty(M)$ and $X, Y \in \mathscr{T}(M)$, and

AC3. $\nabla_X(fY) = f\nabla_X Y + (\mathscr{L}_X f)Y$ for $f \in C^\infty(M)$ and $X, Y \in \mathscr{T}(M)$.

If we define $\nabla_X f = \mathscr{L}_X f$ for $f \in C^\infty(M)$ and $X \in \mathscr{T}(M)$, then we may extend ∇_X to a derivation of the tensor algebra of M in a unique manner. The vector field $\nabla_X Y$ is called the *covariant derivative* of Y along X. If $c\colon [a,b] \to M$ is a piecewise smooth curve, then Y is *parallel* along c if $\nabla_{\dot c(t)} Y(c(t)) = 0$ for each $t \in [a,b]$. If c is a curve such that its tangent vector field is parallel along the curve itself (i.e., $\nabla_{\dot c(t)}\dot c(t) = 0$) then c is a *geodesic* of ∇. The *torsion* of ∇ is the $(1,2)$ tensor field T defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for $X, Y \in \mathscr{T}(M)$. If $T = 0$ then we say ∇ is *torsion-free*.

Let g be a Riemannian metric on M. There exists a unique affine connection ∇ on M with the properties

LC1. ∇ is torsion-free, and

LC2. $\nabla_X g = 0$ for $X \in \mathscr{T}(M)$.

This affine connection is called the *Levi-Civita* connection associated with g. The geodesics of the Levi-Civita connection have the property that they locally determine the paths of shortest length between two points.

2.1 Remark: The link between geodesics of a Levi-Civita connection and Lagrangian mechanics is as follows. Let g be a Riemannian metric on M and define a Lagrangian function on TM by $L(v) = \frac{1}{2}g(v, v)$ (i.e., the kinetic energy with respect to the Riemannian metric). Then it is easy to show that a curve $c\colon [a, b] \to M$ is a solution of Lagrange's equations with the Lagrangian L if and only if c is a geodesic for the Levi-Civita connection associated with g. □

In a set of coordinates $(x^1, \ldots, x^n)$ for M, the *Christoffel symbols*, Γ^k_{ij}, for a general affine connection ∇ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial k}.$$

One may verify that an affine connection is torsion-free if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$, $i, j, k = 1, \ldots, n$. The equations for a geodesic then have the form

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \qquad i = 1, \ldots, n.$$

These second-order equations are equivalent to the system of first-order equations on TM given by

$$\begin{aligned} \dot{x}^i &= v^i, \qquad i = 1, \ldots, n \\ \dot{v}^j &= -\Gamma^j_{kl} v^k v^l, \qquad j = 1, \ldots, n \end{aligned}$$

which defines a vector field on TM called the *geodesic spray* of ∇ which we denote by Z_g.

Now we review the concepts of Lewis [1996b]. Let D be a distribution on a manifold M with an affine connection ∇. Classically, a submanifold N of M is said to be *totally geodesic* if every geodesic whose initial velocity is tangent to N remains on N. We wish to generalise and weaken this notion. We say a distribution D is *geodesically invariant* if for every geodesic $c\colon [a, b] \to M$ of ∇, $\dot{c}(a) \in D_{c(a)}$ implies that $\dot{c}(t) \in D_{c(t)}$ for each $t \in [a, b]$. We say D is *totally geodesic* if it is integrable and geodesically invariant.

We make a few remarks which will be useful in our discussion of control theory in Section 3.

2.2 Remarks:

1. There are distributions which are geodesically invariant but which are not totally geodesic.
2. The involutive closure (i.e., the closure under Lie bracket) of a geodesically invariant distribution D yields the smallest totally geodesic distribution containing D.
3. If D is totally geodesic then each of its maximal integral manifolds inherits a natural affine connection by restriction of the affine connection on M. □

The *symmetric product* between $X, Y \in \mathscr{T}(M)$ is the vector field

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

We shall denote by $\mathscr{D}$ the set of sections of a distribution D. The main result of Lewis [1996b] is the following theorem.

2.3 Theorem: *A distribution D is geodesically invariant if and only if $\langle X : Y \rangle \in \mathscr{D}$ for every $X, Y \in \mathscr{D}$.*

3. Control Systems on Manifolds with an Affine Connection

In this section we introduce the class of control systems we will study, we present controllability definitions, and we present decompositions for these systems. Our motivation for studying systems whose drift vector field is the geodesic spray of a general affine connection comes in part from Remark 2.1. We also remind the reader of the work of Synge [1928] mentioned in the introduction which enables one to cast mechanical systems with constraints in the general formulation we consider.

Let ∇ be an affine connection on a manifold M and let $Y_1, \ldots, Y_m$ be linearly independent vector fields on M. We shall consider control systems of the form

$$\nabla_{\dot{c}(t)}\dot{c}(t) = u^a(t) Y_a(c(t)). \tag{3.1}$$

Here a is summed over $1, \ldots, m$. This equation is readily seen to be equivalent to the first order system

$$\dot{v}(t) = Z_g(v(t)) + u^a(t) Y_a^{lift}(v(t))$$

for $v \in TM$ where Z_g is the geodesic spray associated with ∇. Here Y_a^{lift} is the *vertical lift* of Y_a which is the vector field on TM defined by

$$Y_a^{lift}(v_x) = \left.\frac{d}{dt}\right|_{t=0} (v_x + tY_a(x)).$$

In this way we obtain a first-order nonlinear control system on TM whose drift vector field is the geodesic spray Z_g.

We consider controls from the sets

$$\mathscr{U}_T = \{u \colon [0,T] \to \mathbb{R}^m \mid \ u \text{ is piecewise constant}\}.$$

A *solution* of (3.1) is a pair (c, u) where $c \colon [0,T] \to M$ is a curve on M and $u \in \mathscr{U}_T$. We define the reachable sets as follows. Let $x_0 \in M$ and let U be a neighborhood of x_0 in M. Define

$$\begin{aligned}\mathcal{R}_M^U(x_0, T) = \{x \in M \mid \ & \text{there exists a solution } (c,u) \text{ of (3.1)} \\ & \text{such that } \dot{c}(0) = 0_{x_0},\ c(t) \in U \text{ for } t \in [0,T], \text{ and } \dot{c}(T) \in T_xM\}\end{aligned}$$

and denote

$$\mathcal{R}_M^U(x_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}_M^U(x_0, t).$$

Here $0_x \in T_xM$ denotes the zero vector in the tangent space. The essential feature of these definitions is that we restrict ourselves to studying the points in M (not TM) which can be reached from a state whose initial velocity is zero. The velocity of the final state is unspecified.

We may now state our versions of controllability.

3.1 Definition: We shall say that (3.1) is *locally configuration accessible* at $x_0 \in M$ if there exists $T > 0$ such that $\mathcal{R}_M^U(x_0, \leq t)$ contains a non-empty open subset of M for all neighborhoods U of x_0 and all $0 < t \leq T$. If this holds for any $x_0 \in M$ then the system is called *locally configuration accessible*. □

It is possible to make stronger controllability definitions for these systems and we refer the reader to Lewis and Murray [1996] for some definitions and results.

To state the controllability results, we need some notation. Let $\mathscr{V}$ be a family of vector fields on M. The smallest subset of $\mathscr{T}(M)$ which contains $\mathscr{V}$ and which is closed under Lie bracket we denote by $\overline{\mathrm{Lie}}(\mathscr{V})$. This is known as the *involutive closure* of $\mathscr{V}$. The smallest subset of $\mathscr{T}(M)$ containing $\mathscr{V}$ and which is closed under symmetric product we denote by $\overline{\mathrm{Sym}}(\mathscr{V})$ and call the *symmetric closure* of $\mathscr{V}$. Although the work in Lewis and Murray [1996] is presented in the context of Levi-Civita connections, the results are equally valid for general affine connections. The verification of this is simply a check of the relevant Lie bracket formulas for the control system on TM. These formulas are

$$
\begin{aligned}
[Z_g, Y_a^{lift}](0_x) &= -Y_a(q) \\
[Y_b^{lift}, [Z_g, Y_a^{lift}]] &= \langle Y_a : Y_b \rangle^{lift} \\
[[Z_g, Y_a^{lift}], [Z_g, Y_b^{lift}]](0_x) &= [Y_a^{lift}, Y_b^{lift}](q)
\end{aligned}
$$

and are easily proved with coordinate computations.

Now we are ready to state one of the main results of Lewis and Murray [1996].

3.2 Theorem: *Let $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$. We suppose that $\overline{\mathrm{Sym}}(\mathscr{Y})$ and $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathscr{Y}))$ define distributions on M of constant rank which we denote by C_{ver} and C_{hor}, respectively. Then $\mathcal{R}_M^U(x, \leq T)$ is an open subset of the maximal integral manifold of C_{hor} through x.*

It may also be shown that the set of velocities accessible from 0_x is an open subset of $(C_{\mathrm{ver}})_x$.

Now we recall our discussion regarding geodesically invariant and totally geodesic distributions in Section 2. From Theorem 2.3 it is clear that C_{ver} is the smallest geodesically invariant distribution containing the input vector fields $\mathscr{Y}$. By Remark 2.2.2, C_{hor} is the smallest totally geodesic distribution containing the inputs $\mathscr{Y}$. Theorem 3.2 may also be stated in the following form which constitutes one of the main contributions of the present paper.

3.3 Theorem: *$\mathcal{R}_M^U(x, \leq T)$ is an open subset of the maximal integral manifold through x of the smallest totally geodesic distribution containing the inputs $\mathscr{Y}$.*

All the hard work in the proof of this theorem is contained in Theorems 2.3 and 3.2. Nevertheless, Theorem 3.3 is itself important since it identifies how the geometry of the problem enters into the description of the reachable sets.

If we choose a set of coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^{n-k})$ such that the maximal integral manifolds of C_{hor} are given by $y^1 = \cdots = y^{n-k} = \mathit{constant}$, then the system (3.1) has a coordinate representation of the form

$$
\begin{aligned}
\ddot{x}^\sigma + \Gamma^\sigma_{\rho\tau}\dot{x}^\rho\dot{x}^\tau + \Gamma^\sigma_{\rho\beta}\dot{x}^\rho\dot{y}^\beta + \Gamma^\sigma_{\beta\rho}\dot{y}^\beta\dot{x}^\rho + \Gamma^\sigma_{\beta\gamma}\dot{y}^\beta\dot{y}^\gamma &= u^a Y_a^\sigma, \qquad \sigma = 1, \ldots, k \\
\ddot{y}^\alpha + \Gamma^\alpha_{\rho\beta}\dot{x}^\rho\dot{y}^\beta + \Gamma^\alpha_{\beta\rho}\dot{y}^\beta\dot{x}^\rho + \Gamma^\alpha_{\beta\gamma}\dot{y}^\beta\dot{y}^\gamma &= 0, \qquad \alpha = 1, \ldots, n-k.
\end{aligned}
\tag{3.2}
$$

Here the Christoffel symbols of the affine connection are defined by

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^\rho}} \frac{\partial}{\partial x^\tau} &= \Gamma^\sigma_{\rho\tau} \frac{\partial}{\partial x^\sigma}, & \nabla_{\frac{\partial}{\partial x^\rho}} \frac{\partial}{\partial y^\beta} &= \Gamma^\sigma_{\rho\beta} \frac{\partial}{\partial x^\sigma}, \\
\nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial x^\rho} &= \Gamma^\sigma_{\beta\rho} \frac{\partial}{\partial x^\sigma}, & \nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\gamma} &= \Gamma^\sigma_{\beta\gamma} \frac{\partial}{\partial x^\sigma}, \\
\nabla_{\frac{\partial}{\partial x^\rho}} \frac{\partial}{\partial x^\tau} &= \Gamma^\alpha_{\rho\tau} \frac{\partial}{\partial y^\alpha}, & \nabla_{\frac{\partial}{\partial x^\rho}} \frac{\partial}{\partial y^\beta} &= \Gamma^\alpha_{\rho\beta} \frac{\partial}{\partial y^\alpha}, \\
\nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial x^\rho} &= \Gamma^\alpha_{\beta\rho} \frac{\partial}{\partial y^\alpha}, & \nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\gamma} &= \Gamma^\alpha_{\beta\gamma} \frac{\partial}{\partial y^\alpha}.
\end{aligned}$$

Observe that if $\dot{y}^\alpha(0) = 0$ then $\dot{y}^\alpha(t) = 0$ for all $\alpha = 1, \ldots, n-k$ and $t > 0$. Therefore, if we start with zero initial velocities as our controllability definitions suggest, in local coordinates our system is simply governed by the equations

$$\ddot{x}^\sigma + \Gamma^\sigma_{\rho\tau} \dot{x}^\rho \dot{x}^\tau = u^a Y^\sigma_a, \qquad \sigma = 1, \ldots, k$$

which comprise a locally configuration accessible control system of the form (3.1) on the maximal integral manifolds of C_{hor}. This is consistent with our Remark 2.2.3.

It is worth pointing out that the decomposition (3.2) provides for us the same information as the Kalman controllability decomposition provides for linear systems. In the linear case, one may simplify the system to the form

$$\begin{aligned}
\dot{x} &= A_{11} x + A_{12} y + B_1 u \\
\dot{y} &= A_{22} y.
\end{aligned} \tag{3.3}$$

The reason that there are no terms of the form $\Gamma^\alpha_{\rho\tau} \dot{x}^\rho \dot{x}^\tau$ in the y-part of equation (3.2) is analogous to A_{21} being zero in (3.3).

4. Applications of the Main Results

Now we look at a few applications of Theorem 3.3.

The first situation we look at is merely academic since the author knows of no interesting examples of this type occurring in practice. Nevertheless, it is an interesting consequence of our general results. Recall that a Riemannian manifold is said to be *irreducible* if the representation of the holonomy group in $T_x M$ is irreducible (see Kobayashi and Nomizu [1963]). One may easily see that a Riemannian manifold is irreducible if and only if the only totally geodesic submanifold of M is M itself. Thus we have the following result.

4.1 PROPOSITION: *Let (M, g) be an irreducible Riemannian manifold with ∇ the corresponding Levi-Civita connection. Let $x \in M$ and suppose that $Y \in \mathscr{T}(M)$ is such that $Y(x) \neq 0$. Then the control system*

$$\nabla_{\dot{c}(t)} \dot{c}(t) = u(t) Y(c(t))$$

is locally configuration accessible at x.

Proof: It suffices to restrict consideration to a neighborhood of x where Y is non-zero. By Theorem 3.3, $\mathcal{R}_M(x, \leq T)$ is an open subset of the maximal integral manifold of C_{hor} through x. Since M is irreducible, $C_{\text{hor}} = TM$ and so the only maximal integral manifold of C_{hor} is M. ■

Next we consider a situation where the system has a natural geodesically invariant distribution. Again we consider the case when M comes equipped with a Riemannian metric g. Now we suppose that H is a Lie group which acts on M from the left by isometries. For $h \in H$ we denote by Φ_h the associated diffeomorphism of M induced by the group action. For $\xi \in \mathfrak{h}$, the Lie algebra of H, we define the *infinitesimal generator* associated with ξ to be the vector field on M given by

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(x).$$

Now define the *momentum map* corresponding to this group action as the map $\boldsymbol{J}\colon TM \to \mathfrak{h}^*$ determined by

$$\langle \boldsymbol{J}(v_x); \xi \rangle = g(\xi_M(x), v_x).$$

We refer the reader to Marsden and Ratiu [1994] for a discussion of the above concepts. There the reader will also find a proof of the fact that for $\mu \in \mathfrak{h}^*$, $\boldsymbol{J}^{-1}(\mu)$ is invariant under the geodesic flow on TM. This is in particular true for $\mu = 0$. Furthermore, we note that since $\boldsymbol{J}$ is linear in the fibre variables on TM, $\boldsymbol{J}^{-1}(0)$ is a distribution on M (assuming that 0 is a regular value of $\boldsymbol{J}$). Therefore $D \triangleq \boldsymbol{J}^{-1}(0)$ is a geodesically invariant distribution. There are many examples where D is not integrable and hence not totally geodesic.

One may readily verify that $X \in \mathscr{T}(M)$ is such that X^{lift} is tangent to $D \subset TM$ if and only if $X \in \mathscr{D}$, the set of sections of D. Therefore, any actuation of the system by control vector fields in D will maintain the conservation law $\boldsymbol{J} = 0$. Note that this implies that, with this type of actuation, the system cannot be locally accessible in the phase space TM from points on $\boldsymbol{J}^{-1}(0)$. Nevertheless, it is possible that the system be locally *configuration* accessible. This is illustrated by the robotic leg example in Section 5.

Let us expand on this a bit further. If we use inputs which span D, or equivalently, use a maximal set of inputs which maintain the conservation law, then $\mathcal{R}^U_M(x, \leq T)$ will be an open subset of a maximal integral manifold of the involutive closure of D. In particular, if D is maximally involutive (i.e., $\overline{\text{Lie}}(\mathscr{D}) = \mathscr{T}(M)$), then the control system will be locally configuration accessible.

We summarise the above discussion with the following result.

4.2 PROPOSITION: *Let $\boldsymbol{J}\colon TM \to \mathfrak{h}^*$ be the momentum map associated with a left action of a Lie group H on M. Then:*

(i) $D \triangleq \boldsymbol{J}^{-1}(0)$ is a geodesically invariant distribution for the Levi-Civita connection;

(ii) if the control vector fields take their values in D, then the trajectories of the control system whose initial velocities are zero are tangent to $D \subset TM$;

(iii) if $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$ generates $\mathscr{D}$, then the control system is locally configuration accessible if D is maximally involutive.

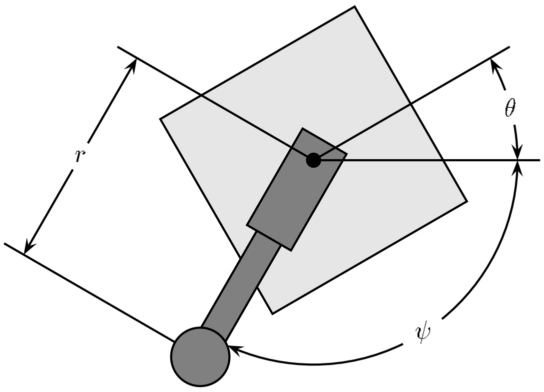

FIGURE 1. The robotic leg

5. Examples

In this section we present decompositions for a few examples. Some of these examples are presented by Lewis and Murray [1996]. However, in that work the emphasis was on deciding whether the systems were locally configuration accessible. Here we are interested in the case when the systems are *not* locally configuration accessible and in studying the associated decompositions. In the robotic leg example, we also make connections with Proposition 4.2.

5.1. The Robotic Leg. We consider a system which consists of a rigid body pinned at its centre of mass and attached is an extendible massless leg with a point mass on its tip. The configuration space is $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^+$. We use the coordinates (θ, ψ, r) as illustrated in Figure 1. We take ∇ to be the Levi-Civita connection associated with the Riemannian metric

$$g = Jd\theta \otimes d\theta + mr^2 d\psi \otimes d\psi + mdr \otimes dr$$

on M. Here J is the moment of inertia of the body and m is the value of the point mass on the end of the leg. The non-zero Christoffel symbols are

$$\Gamma^{\psi}_{\psi r} = \Gamma^{\psi}_{r\psi} = \frac{1}{r}, \qquad \Gamma^{r}_{\psi\psi} = -r.$$

As inputs we choose a force which extends the mass on the end of the leg and a torque which actuates the relative angle between the leg and the body. The corresponding input vector fields are

$$Y_1 = \frac{1}{J}\frac{\partial}{\partial\theta} - \frac{1}{mr^2}\frac{\partial}{\partial\psi}, \qquad Y_2 = \frac{1}{m}\frac{\partial}{\partial r}.$$

This system admits an action of $H = \mathbb{S}^1$ which preserves the metric. The action is given by

$$(\alpha, (\theta, \psi, r)) \mapsto (\theta + \alpha, \psi + \alpha, r).$$

The corresponding momentum map is

$$\boldsymbol{J}(\theta, \psi, r, v_\theta, v_\psi, v_r) = J v_\theta + m r^2 v_\psi.$$

One may readily verify that Y_1 and Y_2 are sections of $D \triangleq \boldsymbol{J}^{-1}(0)$. Thus, by Proposition 4.2(ii), trajectories of the control system whose initial conditions lie on D will remain on D. In particular, those trajectories of the control system whose initial velocities are zero will evolve on D. Therefore, the system cannot be locally accessible (in TM). Nevertheless, in Lewis and Murray [1996] it is shown that the system is locally *configuration* accessible with both inputs. (In fact, the system is locally configuration *controllable* with both inputs.) This turns out to be a situation where Proposition 4.2(iii) applies.

Now we consider the case where we just use the input Y_2. In this case the system is not locally configuration accessible. One may check that C_{hor} is generated by the vector field Y_2 itself. The decomposition given by (3.2) takes the form

$$\begin{aligned}
\ddot{r} - r\dot{\psi}^2 &= \frac{1}{m} u \\
\ddot{\theta} &= 0 \\
\ddot{\psi} + \frac{2}{r}\dot{r}\dot{\psi} &= 0.
\end{aligned}$$

Observe that if we have zero velocity initial conditions, the values of θ and ψ are constant so the dynamics are effectively governed by the equation

$$\ddot{r} = \frac{1}{m} u.$$

This is clearly a locally configuration accessible system on each maximal integral manifold of C_{hor}. Physically, this motion is simply one of extending and retracting the mass on the end of the leg while the body itself remains stationary.

5.2. The Planar Rigid Body. In this section we study the planar rigid body. The configuration space for the system is the Lie group $SE(2)$. To establish the correspondence between the configuration of the body and $SE(2)$, fix a point $O \in \mathbb{R}^2$ and let $\{\boldsymbol{e}_1 = \frac{\partial}{\partial x}, \boldsymbol{e}_2 = \frac{\partial}{\partial y}\}$ be the standard orthonormal frame at that point. Let $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ be an orthonormal frame attached to the body at its centre of mass. The configuration of the body is determined by the element $g \in SE(2)$ which maps the point O with its frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ to the position, P, of the centre of mass of the body with its frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. See Figure 2. The inputs for this problem consist of a force applied at an arbitrary point and pointing to the centre of mass, and a torque about the centre of mass. Without loss of generality (by redefining our body reference frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$) we may suppose that the point of application of the force is a distance h along the $\boldsymbol{f}_1$ body-axis from the centre of mass. The situation is illustrated in Figure 3.

With this convention fixed, we shall use coordinates (x, y, θ) for the planar rigid body where (x, y) describe the position of the center of mass and θ describes the orientation of the

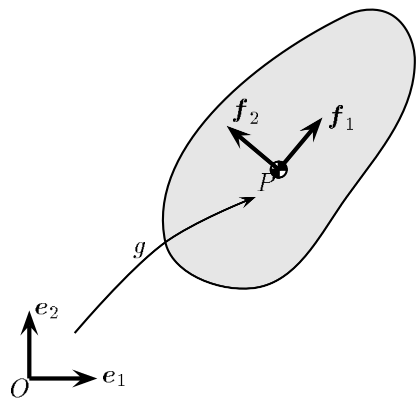

FIGURE 2. The configuration of a planar body as an element of $SE(2)$

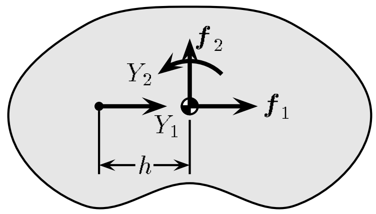

FIGURE 3. Positions for application of forces on a planar rigid body after simplifying assumptions

frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ with respect to the frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. In these coordinates, the Riemannian metric for the system is

$$g = m dx \otimes dx + m dy \otimes dy + J d\theta \otimes d\theta.$$

Here m is the mass of the body and J is its moment of inertia about the centre of mass. The inputs we consider are described by the vector fields

$$Y_1 = \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y}, \qquad Y_2 = \frac{1}{J}\frac{\partial}{\partial\theta}.$$

When both of these inputs are applied, the system may be shown to be locally configuration accessible (in fact, controllable). With each of these inputs alone, the system is not locally configuration accessible.

To describe the decomposition (3.2) when we apply the input Y_1, we need to make a change of coordinates. To this end we use coordinates

$$(\xi, \eta, \psi) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, \theta).$$

In these coordinates the non-zero Christoffel symbols are

$$\Gamma^{\xi}_{\eta\psi} = \Gamma^{\xi}_{\psi\eta} = -1, \qquad \Gamma^{\xi}_{\psi\psi} = -\xi, \qquad \Gamma^{\eta}_{\xi\psi} = \Gamma^{\eta}_{\psi\xi} = 1, \qquad \Gamma^{\eta}_{\psi\psi} = -\eta.$$

The distribution C_{hor} is generated by $\frac{\partial}{\partial\xi}$ in these coordinates. We may now write the system equations as

$$\begin{aligned}
&\ddot{\xi} - 2\dot{\eta}\dot{\psi} - \xi\dot{\psi}^2 = \frac{1}{m}u \\
&\ddot{\eta} + 2\dot{\xi}\dot{\psi} - \eta\dot{\psi}^2 = 0 \\
&\ddot{\psi} = 0.
\end{aligned}$$

Observe that when the initial velocities are zero, the values of η and ψ do not change. Thus we essentially have the motion described by the system

$$\ddot{\xi} = \frac{1}{m} u$$

which is a locally configuration accessible control system on each maximal integral manifold of C_{hor}. Physically, the motions of this control system are motions of the rigid body along the direction which corresponds to the fixed value of θ. Since the input Y_1 acts through the centre of mass, the rotational component of the motion is unaffected.

When the system has the input Y_2 then we may use the coordinates (x, y, θ) to describe the decomposition (3.2). All Christoffel symbols are zero so the equations for the system are

$$\begin{aligned} \ddot{\theta} &= \frac{1}{J} u \\ \ddot{x} &= 0 \\ \ddot{y} &= 0. \end{aligned}$$

Again we see that the top equation decouples from the last two when the initial velocities are zero. And, as the theory predicts, the top system is locally configuration accessible. The corresponding motion of the system is a rotation about the centre of mass of the body.

5.3. A Constrained Particle in $\mathbb{R}^3$. The example we consider in this section is a mechanical system with constraints. The system has a configuration space of $M = \mathbb{R}^3$ and we consider the Riemannian metric

$$g = m\left(dx \otimes dx + dy \otimes dy + dz \otimes dz\right).$$

The system is subject to the velocity constraint

$$\dot{z} = y\dot{x}. \tag{5.1}$$

In Lewis [1996a] it is shown that the equations for the constrained motion of this system are equivalent to those geodesics of a certain (non-Levi-Civita) affine connection whose initial velocities satisfy the constraint (5.1). The non-zero Christoffel symbols of this affine connection in the coordinates (x, y, z) are

$$\Gamma^x_{xy} = \frac{2y}{1+y^2}, \qquad \Gamma^x_{zy} = -\frac{1}{1+y^2}, \qquad \Gamma^z_{xy} = -\frac{1}{1+y^2}.$$

As inputs for this system we consider the single vector field

$$Y = \frac{1}{m}\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right).$$

Note that $Y \in \mathscr{D}$ and so this input will not cause the system to violate the constraints. One may verify that the corresponding control system of the form (3.1) is not locally configuration accessible. In fact, the distribution C_{hor} is generated by the vector field Y.

This is readily verified by computing $\langle Y : Y\rangle = 0$. To render the system in the form (3.2) we must choose coordinates adapted to the foliation associated with C_{hor}. If we choose

$$(\xi, \eta, \zeta) = (x, y, z - xy)$$

then one may verify that C_{hor} is generated by $\frac{\partial}{\partial \xi}$. The non-zero Christoffel symbols in these coordinates are

$$\Gamma^{\xi}_{\eta\xi} = -\frac{\eta}{1+\eta^2}, \qquad \Gamma^{\xi}_{\eta\eta} = -\frac{\xi}{1+\eta^2}, \qquad \Gamma^{\xi}_{\zeta\eta} = -\frac{1}{1+\eta^2},$$
$$\Gamma^{\zeta}_{\xi\eta} = -\xi, \qquad \Gamma^{\zeta}_{\eta\xi} = 1-\xi, \qquad \Gamma^{\zeta}_{\eta\eta} = \frac{\xi\eta}{1+\eta^2}, \qquad \Gamma^{\zeta}_{\zeta\eta} = \frac{\eta}{1+\eta^2}$$

from which we may derive the governing equations in these coordinates as

$$\begin{aligned}
&\ddot{\xi} - \frac{\eta}{1+\eta^2}\dot{\xi}\dot{\eta} - \frac{\xi}{1+\eta^2}\dot{\eta}^2 - \frac{1}{1+\eta^2}\dot{\eta}\dot{\zeta} = \frac{1}{m}u \\
&\ddot{\eta} = 0 \\
&\ddot{\zeta} + (1-2\xi)\dot{\xi}\dot{\eta} + \frac{\xi\eta}{1+\eta^2}\dot{\eta}^2 + \frac{\eta}{1+\eta^2}\dot{\eta}\dot{\zeta} = 0.
\end{aligned}$$

As the general theory predicts, when we have zero initial velocity, the values of η and ζ are not affected by the controls. Thus the system effectively reduces to

$$\ddot{\xi} = \frac{1}{m}u.$$

6. Discussion

In this letter we have presented a decomposition of control systems whose drift vector field is the geodesic spray of a given affine connection. Our results illuminate how the geometry of the affine connection interacts with the inputs to specify the reachable sets (Theorem 3.3). It is our belief that this enhanced geometric insight will lead to a better understanding of how one might perform control design for these systems. Furthermore, the possibility of writing the equations of motion for constrained systems in terms of affine connections opens up new possibilities for applying the theory of this paper, and indeed motivates our presentation in terms of general affine connections rather than just Levi-Civita connections. We have presented a simple example with constraints to verify that their presence does not affect our conclusions.

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