Decentralized Analysis of Stability of Stochastically Varying Formations of Dynamic Agents

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Abstract

We analyze a network of dynamic agents where the topology of the network specifies the information flow between the agents. We present an analysis method for such a system for both consensus and formation stabilization problems. We consider the case of agent dynamics being a single integrator in more detail to show the general features introduced by the information flow. Then we show that the method of analysis can be extended to more general cases of complicated agent dynamics, non-ideal links for information flow, etc. We also consider the case when the topology of the network is changing over time. The focus of the paper is on obtaining conditions for the stability of the formation that can be checked in a decentralized way. Some simple examples are also presented.

1 Introduction and Motivation

Control of dynamic agents coupled to each other through an information flow network has emerged as a topic of major interest in recent years. Such a setting can be used to model many real-life situations, such as air traffic control, satellite clusters, swarms of robots, UAV formations, and potentially such applications as the Internet. Compared with more traditional applications of control theory, there are fundamentally new features introduced in this problem. The topology of the information network can have many effects. On one hand, it might introduce problems of instability if the information being fed through the network adds on constructively to the disturbance at a node; on the other, intuitively, it should serve as a means for better noise rejection for the network as a whole.

As a result of the above-mentioned properties, this problem has been garnering increasing attention. Fax and Murray [4, 5] obtained a Nyquist-like condition for stability of a formation relating the individual plant transfer function and the Laplacian of the graph generated by the topology of the information flow network. Olfati-Saber and Murray [11, 12] considered the average-consensus problem for the case of single integrators. Chaves et al. [2] considered the case of achieving a regular formation in vehicle networks for a milieu in which information is being lost stochastically. Differential geometric and algebraic conditions were presented in [14] and [13] to determine feasibility of directed formations. Jadbabaie et al. [9] considered the coordination of a group of autonomous agents when the graph topology changed over time and presented stability results for the case when the switching rule satisfies certain properties. Cremean and Murray [3] presented a general framework for stability analysis of interconnected non-linear systems using a Lyapunov function approach.

In this paper, we present a general framework for analysis of both formation stabilization and consensus problems. We also consider the case when formations might be switching according to a Markov chain. This can model disruptions in the information flow topology and can also be used to model intentional change of one formation to another. We consider general plant dynamics and come up with sufficient conditions for stability that can be checked in a decentralized way, without each node needing to know the Laplacian of the entire graph.

We also put in delays in the communication links to model more practical scenarios.

The outline of the paper is as follows. In the next section, we give the notational conventions used and formulate the problem. Then we address a few mathematical preliminaries. In subsequent sections, we first deal with the single integrator dynamics case used for the consensus problem in detail, and then consider the case of general plant dynamics. Finally we consider the effect of noise on the analysis. We end with conclusions and present some avenues for further work.

2 Notations and Conventions

We refer to the individual dynamic agents variously as plants, vehicles or nodes and the information links as channels, edges or links. In particular the node-edge terminology arises naturally from a graph in which the agents are the vertices and the links are edges. We call a link from vehicle i to vehicle j *directed* if the vehicle j can sense information about vehicle i, but not vice-versa. In the graph, we show this by a directed edge from node j to node i. Thus information flows opposite to the edge direction indicated on the graph. We assume that each vehicle can get information about its own state, even though we show no loop from each node to itself in the corresponding graph. For simplicity, we also assume that this information can be obtained without any delay. We denote the state of the ith individual agent at time step k by $x_i[k]$ in the scalar case and $X_i[k]$ in the vector case. We also denote the state of the whole network obtained by stacking the states of each individual agents by $x[k]$ or $X[k]$, in the two cases. At each node, there are two types of information available:

- absolute measurement of a node's own state $x_i[k]$, and
- relative measurements of the state of other nodes with respect to its own state, $x_j[k] - x_i[k]$.

In the control laws that we study, no node should need to access absolute values of the states of other nodes.

We say a matrix A is stable if it has no eigenvalues outside the unit circle on the complex plane, and all eigenvalues on the unit circle are simple. If all the eigenvalues lie inside the unit circle, we call it asymptotically stable. Let A be a matrix and $(\lambda_1, \lambda_2, \dots)$ be its eigenvalues arranged in decreasing order of magnitude. Also let $(\sigma_1, \sigma_2, \dots)$ be the singular values in decreasing order of magnitude. We call $|\lambda_1|$ the spectral radius of A and $|\sigma_1|$ the spectral norm of A. We denote the spectral norm of matrix A by $\sigma_{\max}(\mathrm{A})$ and the spectral radius by $\rho(\mathrm{A})$. We denote the ∞-norm of A by $\|A\|_\infty$. I represents the identity matrix and 0 the zero matrix. The dimensions are apparent from the context. We denote the set of all out-neighbors of a node i by $\mathcal{N}^i_{\text{out}}$.

Problem Formulation

We wish to solve problems which involve asking a group of vehicles to move into a formation, alter a formation, or agree about a common point such as the average of their positions, etc. We wish to do these tasks in a decentralized fashion. No vehicle should need to access the absolute values of other vehicles' states. We see immediately that there are two main issues involved:

- How to move into a specified formation. Suppose the task for the vehicles is to move into a hexagon and maintain their relative positions. This task can be done by specifying the inter-vehicle distance for all (or a subset of) vehicles. The issue here is formation stability corresponding to a particular information flow topology specified.
- How to reach consensus about a particular common point. Suppose the task is to move towards the common center of formation. The vehicles first need to agree which is the center and then they can fulfill the task.

It may be noted that purely from a stability viewpoint, the two issues can be decoupled. Suppose the task for the vehicles moving in a hexagonal formation is to switch into a different formation about the center point. This task can either be viewed as switching into a different formation with new inter-vehicle distances specified. Or it can be viewed as a two-step process. First all the vehicles reach a consensus about the center point. Then each vehicle calculates its position with respect to the center point and calculates the control input required to move to that position. This separation allows us to consider a simple dynamics (the single integrator dynamics) for consensus problems. For formation stability problems, we have to consider the actual dynamics of the plants. Of course, the disturbance rejection properties of the two approaches would be different. An important point to be noted here is that since we are considering linear systems, we need to consider only the stability of the system at the origin. That translates to all inter-vehicle distances turning out to be zero. To consider any particular formation, we can simply translate the origin into the offsets desired. As long as the offsets are bounded, our stability analysis will hold.

3 Mathematical Preliminaries

3.1 Some Results in Graph Theory

In this section, we define some basic notations and results in graph theory. A more complete treatment is available in [1]. A directed graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where an edge is an ordered pair of distinct vertices $a = (v_i, v_j)$. The vertex v_i is called the tail of a and the vertex v_j is called the head of a. If the edge is *undirected*, we mean that the edge consists of two distinct directed edges $a_1 = (v_i, v_j)$ and $a_2 = (v_j, v_i)$. The number of edges incident into a vertex v_i is its in-degree. Similarly the out-degree of v_i, denoted by O_i is the number of edges leaving the node. $O_{\max}$ represents the maximum out-degree among all nodes in a graph. We call a directed graph *balanced* if for each node, the out-degree is the same as the in-degree. We call a directed graph *symmetric* if every directed edge from node i to node j implies a directed edge from node j to node i.

The *adjacency matrix* of a graph G having n nodes, denoted by $A = A(G)$ is a square matrix of size $n \times n$, defined as follows

$$A_{ij} = \begin{cases} 1 & (v_i, v_j) \,\epsilon\, E(G) \\ 0 & \text{otherwise.} \end{cases}$$

The *degree matrix* is also a square matrix of size $n \times n$, defined as follows

$$D_{ij} = \begin{cases} O_i & i = j \\ 0 & \text{otherwise.} \end{cases}$$

We define the *Laplacian* of a graph by the following equation

$$L = D - A.$$

An important property of L is that all its row sums are zero and thus 0 is an eigenvalue corresponding to the eigenvector $e = (1, 1, ..., 1)^T$. Also we note the following property [1, 7]: if a graph has k connected components, then $\text{rank}(L) = n - k$. Moreover, for an undirected connected graph, all the eigenvalues of L are strictly positive except for exactly one eigenvalue at 0. For any graph, Gershgorin's theorem implies that all the eigenvalues of L lie within a disk of radius $O_{\max}$, centered at the point $(O_{\max}, 0)$ on the positive real axis of the complex plane.

3.2 Some Results about Kronecker Products

The Kronecker product is a binary matrix operator that maps two arbitrarily dimensioned matrices into a larger matrix with special block structure. Given the $n \times m$ matrix $A = [a_{ij}]_{n\times m}$ and the $p \times q$ matrix $B = [b_{ij}]_{p\times q}$, their Kronecker product, denoted by $A \otimes B$, is the $np \times mq$ matrix with the block structure

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1,m}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \dots & a_{n,m}B \end{bmatrix}_{np\times mq} .$$

The following properties can be proved readily from the definition [15]. In the following we assume the matrices A, B, C and D to be real and of appropriate dimensions.

- For a scalar α, $A \otimes (\alpha B) = (\alpha A) \otimes B = \alpha (A \otimes B)$.
- $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$, $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
- $A \otimes B \neq B \otimes A$ in general.
- $(A \otimes B)^T = (A^T) \otimes (B^T)$, $(A \otimes B)^{-1} = (A^{-1}) \otimes (B^{-1})$.
- $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. Thus, if λ_i be the eigenvalues of A corresponding to eigenvectors X_i, $i = 1, 2, \dots m$, and μ_j be the eigenvalues of B with eigenvectors Y_j, $j = 1, 2, \dots n$; the mn eigenvalues of $A \otimes B$ are given by $\lambda_i \mu_j$ with eigenvectors $X_i \otimes Y_j$, for various values of i and j.
- If the ∞-norm of A is given by a and that of B by b, then $\|A \otimes B\|_\infty \leq ab$.

4 Single Integrator Dynamics

In this section, we consider plants modeled by a single integrator dynamics. After studying such plants in detail, we move on to consider more complicated agent dynamics in the next section. This approach makes the issues connected with the information-flow topology clearer, without involving complications arising from the plant dynamics itself. Also single integrator systems can be used to simulate the effect of consensus problems being solved by a group of interconnected agents. We look at the system in discrete-time domain since this allows us to study the formation changes by invoking the results already existing for stabilization of systems changing according to a Markov chain. An important parameter when we discretize a continuous time system is the step size h. We will see that the stability properties are intimately connected to this parameter. This parameter can be varied within bounds imposed by physical limitations of the dynamic agents to optimize the stability properties and the performance.

4.1 Ideal Communication Links

We consider a single integrator system,

$$\dot{x}_i = -u_i.$$

In the beginning, we consider no time delays and no channel distortions except total loss of communication modeled by no link between two nodes. We consider the topology to be fixed in time. Consider the control law proposed in [12], which says that the ith agent applies the control law given by

$$u_i = \sum_{j \in \mathcal{N}_i^{\text{out}}} (x_i - x_j). \tag{1}$$

Thus the whole system can be written in the following form:

$$\begin{aligned} \dot{x} &= -u \\ u &= Lx. \end{aligned}$$

On discretizing the system with a step size h, we obtain for each node,

$$x_i[k+1] = x_i[k] - h \times u_i[k].$$

where the control law $u_i[k]$ is the standard averaging control law, given in equation 1. For the whole system, we thus obtain

$$\begin{bmatrix} x[k+1] \\ u[k] \end{bmatrix} = \begin{bmatrix} (I - h \times L) & 0 \\ L & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k-1] \end{bmatrix}.$$

Here I is an identity matrix of suitable dimension.

This is a system of the form

$$z[k+1] = \Phi z[k],$$

which is stable iff Φ is stable. The eigenvalues of Φ are the same as those of $I - h \times L$. Thus if λ_i are the eigenvalues of L, the system is asymptotically stable iff $1 - h \times \lambda_i$ lie within the unit circle for all i. The system can never be stable if L has negative eigenvalues or non-simple zero eigenvalues. A connected undirected graph, e.g, has its Laplacian as a positive semi-definite matrix, hence the eigenvalues are always real and positive, except for a single zero eigenvalue. For such a case we have the stability condition [11]

$$h < 2/\rho(L),$$

where $\rho(L)$ denotes the spectral radius of L. In general, we note the following result.

Proposition 1 *A sufficient condition for stability of the formation is* $h < \frac{1}{O_{max}}$, O_{max} *denoting the maximum outdegree of the nodes. For connected undirected graphs, a necessary condition is* $h < \frac{2}{O_{max}}$.

Proof: By Gershgorin's theorem the spectral radius of L is bounded by twice the maximum outdegree of the nodes. Thus all eigenvalues $1 - h\lambda$ would lie in the unit circle if $h < \frac{1}{O_{\max}}$. Thus a sufficient condition for stability is $h < 1/O_{\max}$. For connected undirected graphs, the maximum eigenvalue of a Laplacian matrix L is never smaller than the maximum outdegree of a node in the graph [6]. Thus a necessary condition for stability is $h < 2/O_{\max}$. ■

Note that the higher the connectivity of the graph, the smaller the time step should be. This is not an intuitive result. Also we note that for balanced and undirected graphs, the column sum of L is zero. Thus the column sum of $I - hL$ is 1, and hence the average of $x[k]$ is invariant. Thus the system would converge to the average value of $x[0]$ if the system is asymptotically stable and the average-consensus problem is solved (This result was also proved in [11]).

For a directed graph, we note that the Laplacian is no longer necessarily symmetric. However Gershgorin's theorem still bounds the eigenvalues to lie within a disk of radius $O_{\max}$, centered on the point $(O_{\max}, 0)$ on the positive real axis. Thus the sufficient condition given above still holds. However, the nodes would agree to the average only if the average is invariant. For this, we require the graph to be balanced. Also note that the requirement of eigenvalues on the periphery of the unit circle should be simple requires the graph to be strongly connected.

For consensus problems from now on, we consider only undirected graphs or balanced directed graphs. Formation stability, of course, does not require the column sum of state transition matrix to be unity.

4.2 Formation Switching

Let us now consider the case of formation switching between two topologies as a Markov chain, with the Markov state known. The case for more Markov states is similar. One situation that can be modeled using Markov chains is if the formation change happens after every vehicle has been informed of the impending change and there is a probability associated with every vehicle getting the message in a single time step. Another case is if a formation is flying in an area and on meeting an obstruction it changes into another formation. The obstructions might be modeled as arising and going away with a certain probability per time step. In the scenario of control across communication channels we can deal with neighbors being assumed lost due to communication problems by this approach. The simplest model for a wireless channel is the Gilbert channel model [16], which consists of two states. The 'Bad' state is the one in which the transmitter is in deep fade and hence cannot send out data. For undirected links, this would also mean that the vehicle cannot receive data. The 'Good' state means the vehicle can transmit and receive data correctly. The transition probabilities of the two states are usually experimentally measured for various environments. In such a scenario, we can model the network as a collection of nodes which transition independently between these states.

Again assume that the communication links are ideal and two vehicles are either in perfect communication state or the link between them is totally broken. In such situations, the assumption that the Markov state is known becomes justified as the control law itself implicitly assumes the knowledge of existing neighbors. In the case of a jump linear Markov system, we refer to stability in the sense of stability of the covariance of the system. If we denote the Markov state at time step k as r_k and define the conditional covariance as

$$P_j[k] = E[z_k z_k^T \, | r_k = j],$$

then the covariance of the system is defined by

$$P[k] = \sum P_j[k] \times \text{Prob}(r_k = j),$$

where the summation is taken over all the Markov states and the expectation over all the uncertainity present in the system. If the Markov state is known, the stability of the system depends on the stability of the matrix $(Q^T \otimes I)\text{diag}(A_i)$ [10], where

$$A_i = (I - h_i L_i) \otimes (I - h_i L_i).$$

Here i denotes the Markov state while Q is the transition probability matrix of the Markov chain.

For simplicity, let us consider a Markov chain with only two states. A similar analysis can be carried out for a larger number of Markov states. We also denote

$$\text{diag}(A_i) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}.$$

The matrix to be looked at for stability is

$$\mathrm{M} = \begin{bmatrix} q_{11}U_1 & q_{21}U_2 \\ q_{12}U_1 & q_{22}U_2 \end{bmatrix},$$

where the matrices U_1 and U_2 are given by

$$\begin{aligned} U_1 &= (I - h_1 L_1) \otimes (I - h_1 L_1) \\ U_2 &= (I - h_2 L_2) \otimes (I - h_2 L_2). \end{aligned}$$

We have the following result.

Proposition 2 *In the case of undirected graphs or balanced directed graphs, if the origin is stable for both the Markov states individually, it is stable for the Markov chain as well, provided $h < 1/O_{max}$.*

Proof: We note that the matrix M can be written as the product of the two matrices

$$\begin{aligned} M_1 &= \begin{bmatrix} q_{11}I & q_{21}I \\ q_{12}I & q_{22}I \end{bmatrix} \\ M_2 &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}. \end{aligned}$$

I is identity matrix of suitable dimensions and U_i have been defined above. We consider the 1-norm of the two matrices, which is sub-multiplicative. For M_1, it is always 1. For M_2, we have that if $h < 1/O_{\max}$, all the terms of the matrix $I - hL$ are positive. Also the column sum of matrix $I - hL$ is 1 for undirected graphs or balanced directed graphs. Thus the column sum of matrix $(I - hL) \otimes (I - hL)$ is also 1. So we find that the absolute column sum of M_2 is also 1. Hence, the 1-norm of the original matrix is bounded by 1. But the spectral radius of a matrix is bounded by its 1-norm [8]. Thus all the eigenvalues of the original matrix lie within the unit circle and the Markov chain is stable. ∎

It is however apparent that even if one of the Markov states is unstable, the Markov chain might still be stable. As a trivial example consider a 2-state system with 3 nodes. Let the Laplacian matrices corresponding to the two states be

$$
\begin{aligned}
L_1 &= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\
L_2 &= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Let the parameter h be the same for both Markov states and be equal to 0.3. We can see that the second Markov state does not correspond to a stable system. However for any transition probability matrix allowing for system to go into the Markov state 1 with any non-zero probability, we obtain a stable Markov chain. This can also be confirmed using the eigenvalues of the stability matrix of the Markov system. The eigenvalues of that matrix also yield an indication about how fast do the system node values converge to the average.

If each node can independently go into one among a number of Markov states, the size of the matrix to be checked for stability grows exponentially in the number of nodes. However for sparsely connected graphs, the coefficient multiplying the exponential term is reasonably small and the computations can be managed.

4.3 Transformation of One Formation into Another

Let us now consider the previous result for a chain $P_1 \to P_2 \to P_3$ of formations. Such a situation might arise if the vehicles want to change from one formation to another formation, but this maneuver involves going through a potentially unstable formation. We want to characterize all the maneuvers that yield a stable chain. For simplicity, we are considering only one intermediate formation. The analysis can however be easily generalized. We consider a transition probability matrix of the form

$$
Q = \begin{bmatrix} (1-p_1) & p_1 & 0 \\ 0 & (1-p_2) & p_2 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Now the matrix $(Q^T \otimes I)\mathrm{diag}A_i$ would be

$$
\begin{bmatrix} q_{11}U_1 & q_{21}U_2 & q_{31}U_3 \\ q_{12}U_1 & q_{22}U_2 & q_{32}U_3 \\ q_{13}U_1 & q_{23}U_2 & q_{33}U_3 \end{bmatrix},
$$

where

$$
U_i = (I - h_i L_i) \otimes (I - h_i L_i).
$$

Thus we have to look at the eigenvalues of the matrix

$$
\begin{bmatrix} (1-p_1)U_1 & 0 & 0 \\ p_1 U_1 & (1-p_2)U_2 & 0 \\ 0 & p_2 U_2 & U_3 \end{bmatrix}.
$$

This is a lower triangular block form. The eigenvalues would be those of matrices $(1-p_1)(I - h_1 L_1) \otimes (I - h_1 L_1)$, $(1-p_2)(I - h_2 L_2) \otimes (I - h_2 L_2)$ and $(I - h_3 L_3) \otimes (I - h_3 L_3)$.

Remarks:

1. If the entire path is through stable formations, the Markov chain is stable.
2. If an intermediate formation is unstable, the transition probability matrix begins to play an important role in determining whether the formation would be stable or not. In particular a high value of p_2 would mean lower eigenvalues of the system. This corresponds to higher probability of moving away from the unstable system, which is an intuitive result. In fact we can get a bound on p_2 to ensure that the matrix $(1-p_2)(I - h_2 L_2) \otimes (I - h_2 L_2)$ is always stable.

Proposition 3 *A sufficient condition for the matrix $(1-p)(I - hL) \otimes (I - hL)$ to be stable is that*

$$
p > \frac{4O_{max}h(O_{max}h - 1)}{(1 - 2O_{max}h)^2}.
$$

Proof: As noted above, the eigenvalues of the Laplacian matrix L lie inside a disk centered on the point $(O_{\max}, 0)$ and a radius of $O_{\max}$. Thus the eigenvalues of the matrix $I - hL$ would lie inside a desk centered at (1-$hO_{\max}$, 0) and of radius $hO_{\max}$. The eigenvalue with the maximum possible magnitude can hence either be 1 or $1 - 2O_{\max}h$. So we obtain that for the matrix $(1-p)(I-hL)\otimes(I-hL)$, the eigenvalue with the maximum possible magnitude can be either $(1-p)$ or $(1-p)(1-2O_{\max}h)^2$. If it is the former, the matrix is always stable since $0 < p < 1$. If it is the latter, we obtain as the condition

$$(1-p)(1-2O_{\max}h)^2 < 1,$$

which implies

$$p > \frac{4O_{\max}h(O_{\max}h-1)}{(1-2O_{\max}h)^2}.$$

Note that if $O_{\max}h < 1$, this condition is always satisfied. This corresponds to the case where the maximum possible magnitude of an eigenvalue is 1. ■

Remark: Note that in the above discussion, we have treated p as constant, since we were only interested in evaluating the effect of the graph topology on stability. Actually p might be considered to be an implicit function of h. If we assume that the discrete-time Markov chain has arisen from the sampling of a continuous-time Markov process, then p increases with h. We can thus get bounds on the rates of underlying Markov process.

Recalling that a sufficient condition for stability of a formation is that $h < 1/O_{\max}$, $O_{\max}$ being the maximum out-degree of the nodes, we obtain a sufficient condition for stability of the formation as stated in the following proposition.

Proposition 4 *Consider all the formations in the chain. If $O_{max,i}$ denotes the maximum out-degree of the nodes in the ith formation, and O_{max} denotes the maximum value among all the $O_{max,i}$'s, then $h < 1/O_{max}$ is a sufficient condition for stability of the formation change. An even looser condition for stability is that the maximum out-degree of the nodes should not increase, but the graph should remain connected and the condition $h < 1/O_{max,i}$ should be satisfied for the initial and last formations.*

4.4 Same Delay in All Communication Links

A more practical case is when the communication links are not ideal. We start with the simplest case, where all the links in the network introduce the same delay.

Constant time delay of one time step: Consider first the case when the time delay is exactly one time step. We generalize the treatment in the next section.

We obtain for each node,

$$x_i[k+1] = x_i[k] + h \times u_i[k].$$

where the control law $u_i[k]$ is the standard averaging control law. Since the relative measurements available at the controller are delayed measurements, the averaging is done for delayed measurements.

$$u_i[k] = \sum_{j \,\in\, \mathcal{N}_i^{\mathrm{out}}} (x_j[k-1] - x_i[k-1]).$$

For the whole system, we thus obtain

$$\begin{bmatrix} x[k+1] \\ x[k] \\ u[k] \end{bmatrix} = \begin{bmatrix} I & -h\times L & 0 \\ I & 0 & 0 \\ 0 & -L & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ x[k-1] \\ u[k-1] \end{bmatrix}.$$

Thus the matrix which should be stable in this case is not $(I - hL)$, rather it is

$$\begin{bmatrix} I & -h\times L \\ I & 0 \end{bmatrix}.$$

One thing we note immediately is that now the stability requirements are more stringent. Particularly the requirement that $h < 2/\rho(L)$ for stability no longer holds. We note the following:

Proposition 5 *A necessary and sufficient condition for stability of the matrix*

$$\Phi = \begin{bmatrix} I & -h\times L \\ I & 0 \end{bmatrix}$$

is

$$h < 1/\rho(L).$$

Proof: Let μ be an eigenvalue of L and λ be an eigenvalue of the matrix Φ. Also let X be an eigenvector of the matrix $\lambda I - hL$ corresponding to the eigenvalue $\lambda - h\mu$. Now we claim that the vector

$\begin{bmatrix} \lambda X \\ X \end{bmatrix}$ is an eigenvector of Φ with eigenvalue λ. For this to be true, we must have

$$\begin{bmatrix} I & -h \times L \\ I & 0 \end{bmatrix} \begin{bmatrix} \lambda X \\ X \end{bmatrix} = \lambda \begin{bmatrix} \lambda X \\ X \end{bmatrix}.$$

Thus, we need to satisfy the equation

$$\lambda X - hLX = \lambda^2 X.$$

Since X is an eigenvector of $\lambda I - hL$ with eigenvalue $\lambda - h\mu$, this implies that our claim is true and further, λ satisfies the equation

$$\lambda - h\mu = \lambda^2.$$

Thus we obtain λ to be equal to $\frac{1 \pm \sqrt{1-4h\mu}}{2}$. From this we can easily derive that a necessary and sufficient condition for stability is that $h < 1/|\mu|$. Since μ denotes any eigenvalue of L, this implies the condition to be $h < 1/\rho(L)$. ■

This result is intuitively satisfying in light of the fact that a delay of up to $\frac{\pi}{2\rho(L)}$ is allowed in the continuous time case [12].

Formation switching with constant time delay equal to one time step: The results for Markov case can be obtained by a similar treatment as in the case without delays. The matrix $(I - h \times L)$ is replaced by the matrix

$$\begin{bmatrix} I & -h \times L \\ I & 0 \end{bmatrix}.$$

Other details remain similar. We obviously get more stringent requirements for stability even in the case of a single Markov state.

Constant time delay not exactly one time step: Let us now generalize to the case where the links have the same delay, but not exactly one time step. This situation can arise if communication delays are a fraction of the time step h. Then the nodes can take action as soon as information arrives to them from the other nodes. For simplicity assume the delay to be less than one time step. The treatment for the non-Markov case remains similar even if the delay is more than one time-step, as long as we can bound it and it is constant. However, the Markov case might have problems of re-ordering of packets if the delay is larger than one time step.

Let h be the duration of the time step and d be the delay. Thus in the case considered in the above subsection, $d = h$. As before, we obtain for each node,

$$x_i[k+1] = x_i[k] + (h-d) \times u_i[k] + d \times u_i[k-1],$$

where the control law $u_i[k]$ is the standard averaging control law, given by

$$u_i[k] = \sum_{j \,\in\, \mathcal{N}_i^{\text{out}}} \left(x_j[k] - x_i[k]\right).$$

For the whole system, we thus obtain

$$\begin{bmatrix} x[k+1] \\ x[k] \end{bmatrix} = K \begin{bmatrix} x[k] \\ x[k-1] \end{bmatrix},$$

where the matrix K, which we need to check for stability is

$$K = \begin{bmatrix} I - (h-d) \times L & -d \times L \\ I & 0 \end{bmatrix}.$$

The whole utility of this approach becomes clear when we see that once we obtain this matrix, the Markov extension is trivial.

Generalization to cases where delay is bounded by n time steps is obtained by considering the time evolution of the system

$$\begin{bmatrix} x[k+1] \\ x[k] \\ \vdots \\ x[k+1-n] \\ u[k] \end{bmatrix}.$$

4.5 Different Delays in Communication Links

Consider now the case when the delays in each link are different. We again consider the case when the delays are given by $d_1, d_2, ..., d_n$; each bounded by one time step. The case where the bound is larger than a single time step is an obvious extension. We define a generalized Laplacian matrix L_{gen}, in the following way. Let d_{ij} be the delay from node j to node i. Then,

$$[L_{gen}]_{ij} = \begin{cases} 0 & j \,\notin\, \mathcal{N}_i^{\text{out}} \\ -d_{ij} & j \,\in\, \mathcal{N}_i^{\text{out}} \\ \sum_{k \,\in\, \mathcal{N}_i^{\text{out}}} d_{ik} & i = j. \end{cases}$$

Obviously the standard Laplacian matrix L corresponds to the case when each d_{ij} equals 1. We apply the following control law at each node,

$$u_i(t) = \sum_{j \,\in\, \mathcal{N}_i^{\rm out}} \left(x_j(t-d_{ij}) - x_i(t-d_{ij})\right).$$

For each node i, we thus obtain

$$\begin{aligned} x_i[k+1] = x_i[k] &- \sum \left((h-d_{ij})(x_i[k]-x_j[k])\right) \\ &- \sum \left(d_{ij}(x_i[k-1]-x_j[k-1])\right) \\ = x_i[k] &- h\sum \left(x_i[k]-x_j[k]\right) \\ &+ \sum \left(d_{ij}(x_i[k]-x_j[k])\right) \\ &- \sum \left(d_{ij}(x_i[k-1]-x_j[k-1])\right). \end{aligned}$$

The summations are over all the out-neighbors j. Thus we readily obtain the following equation for the time evolution of the system.

$$\begin{bmatrix} x[k+1] \\ x[k] \end{bmatrix} = K \begin{bmatrix} x[k] \\ x[k-1] \end{bmatrix},$$

where the matrix K to be checked for stability is

$$K = \begin{bmatrix} I - hL + L_{gen} & L_{gen} \\ I & 0 \end{bmatrix}.$$

4.6 Comments

We have presented a unified framework for the stability analysis and consensus problem analysis for the information flow of an arbitrary network with possible delays. This approach has the advantage of being generic and readily extendable to the Markov switching case. We can also discuss the case for the communication link being a general FIR filter using this approach.

5 General Plant Dynamics

Although the single integrator case discussed in the above section solves the consensus problem, for other practical applications like formation stabilization, we are interested in how the above approach generalizes to the case of general plant dynamics. In this section, we consider this case. We will illustrate the results be considering examples of a network of four plants, each having a double integrator dynamics and zero-order hold. Thus, denoting the time step by h, the dynamics of the ith plant are given by

$$X_i[k+1] = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} X_i[k] + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} U_i[k].$$

We consider three interconnections for which the graph Laplacians are given by

$$\begin{aligned} L_1 &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \\ L_2 &= \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \\ L_3 &= \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}. \end{aligned}$$

Unless otherwise mentioned, we will take the numerical value of h as 0.23 for use in examples. In addition, we compare the results with the results obtained in the previous section for the case of single integrator dynamics.

5.1 Ideal Communication Links

Consider a general lth order plant described by the following dynamics

$$\dot{X}_i = AX_i + BU_i.$$

$X_i \in \mathbf{R}^l$ is the state. As before, to begin with, we consider ideal communication links whenever they exist and the topology to be fixed in time. Consider the control law which depends both on the absolute measurement of vehicle's own state and relative measurements of other vehicles' states, in general. Let the ith agent apply

$$U_i = F_i^1 X_i + F_i^2 \left(\sum_{j \,\in\, \mathcal{N}_i^{\rm out}} (X_i - X_j)\right).$$

For simplicity, we consider both F_i^1 and F_i^2 to be independent of the subscript i, and denote them by F_1 and F_2, respectively. This means that all vehicles have identical control laws. We can easily generalize to the case where this is not true. On discretizing the system with a step size h, we obtain for each node,

$$X_i[k+1] = \Phi X_i[k] + \Gamma U_i[k],$$

where the control law $U_i[k]$ is given by

$$U_i[k] = F_1 X_i[k] + F_2 \left(\sum_{j \,\in\, \mathcal{N}_i^{\rm out}} \left(X_i[k] - X_j[k]\right)\right).$$

For the whole system, we thus obtain

$$X[k+1] = I\otimes(\Phi+\Gamma F_1)X[k] + (I\otimes\Gamma F_2)(L\otimes I)X[k].$$

Here I is identity matrix of suitable dimension while L is the graph Laplacian. By using the standard rule of Kronecker product algebra

$$(a \otimes b)(c \otimes d) = (ac \otimes bd),$$

we obtain the system equation as

$$X[k+1] = [I \otimes (\Phi + \Gamma F_1) + L \otimes \Gamma F_2]\, X[k].$$

The system is stable if and only if $[I \otimes (\Phi + \Gamma F_1) + L \otimes \Gamma F_2]$ is stable. We know the eigenvalues of both $I \otimes (\Phi + \Gamma F_1)$ and $L \otimes \Gamma F_2$. However none of these matrices is necessarily Hermitian. So we cannot, in general, say anything about the eigenvalues of their sum. However, we note the following result.

Proposition 6 *Let λ_i denote the eigenvalues of the matrix L. Then the eigenvalues of the matrix $(I \otimes A + L \otimes B)$ are the same as the eigenvalues of the smaller matrices $(A + \lambda_i B)$.*

Proof: Let X be an eigenvector of L, corresponding to eigenvalue λ. Also let Y be an eigenvector of the matrix $(A + \lambda B)$, with the eigenvalue μ. Now consider the vector $(X \otimes Y)$. We have

$$\begin{aligned}(I \otimes A + L \otimes B)(X \otimes Y) =& (I \otimes A)(X \otimes Y) \\ &+ (L \otimes B)(X \otimes Y) \\ =& (X \otimes AY) + LX \otimes BY \\ =& X \otimes AY + \lambda X \otimes BY \\ =& X \otimes AY + X \otimes \lambda BY \\ =& X \otimes ((A + \lambda B)Y) \\ =& X \otimes (\mu Y) \\ =& \mu(X \otimes Y).\end{aligned}$$

Thus $(X \otimes Y)$ is an eigenvector of the matrix with the eigenvalue μ. ■

In our case, this result means that we should look at the eigenvalues of the matrix $\Phi + \Gamma F_1 + \lambda \Gamma F_2$, where λ runs through all the eigenvalues of the graph Laplacian L. We know already that any Laplacian matrix L has at least one eigenvalue being equal to zero. Thus an immediate result is the following:

- A necessary condition for the system to be stable is that the individual plant be stabilized while using only its own absolute measurements, ie, the matrix $\Phi + \Gamma F_1$ be stable.
- Since most systems we are interested in are unstable on their own (Φ is unstable), this means it is impossible to stabilize such systems using only the relative measurements.
- Although the above condition can be turned into a sufficient condition as well if we put F_2 as a zero vector, we are not interested in this case since it corresponds to a formation of vehicles each worrying about its own position without any consideration for other vehilces. This would intuitively have poorer disturbance rejection properties.
- In case all we are worried about is *relative* stability, we can allow simple eigenvalues on the unit circle for the matrix $\Phi + \Gamma F_1$.

We can consider the single integrator system discussed in detail in the last section using this approach. In that case Φ was unity, F_1 was zero while ΓF_2 reduced to scalar $-h$. Thus we were able to calculate the eigenvalues explicitly. Moreover note that the necessary condition stated above holds automatically in that case. For the double-integrator case mentioned earlier, if we check the stability using control laws

$$\begin{aligned} F_1 &= \begin{bmatrix} -1 & -1 \end{bmatrix} \\ F_2 &= \begin{bmatrix} -1 & -1 \end{bmatrix}, \end{aligned}$$

we find that all the three formations are stable. On the other hand, as soon as we use $F_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}$, corresponding to the plant being unstable with its own absolute measurements, we obtain multiple eigenvalues on the unit circle and the formations go unstable. Figure 1 shows the plot for positions of the four vehicles corresponding to L_2 with control law matrix $F_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}$ while Figure 2 shows that the formation indeed goes unstable if $F_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}$ is used.

5.2 Formation Switching

We proceed as in the single-integrator case. Following the analysis, we obtain that the condition for the 2-state Markov chain to be stable is that the following matrix be stable.

$$\begin{bmatrix} q_{11} H_1 \otimes H_1 & q_{21} H_2 \otimes H_2 \\ q_{12} H_1 \otimes H_1 & q_{22} H_2 \otimes H_2 \end{bmatrix},$$

where

$$\begin{aligned} H_1 &= [I \otimes (\Phi + \Gamma F_1) + L_1 \otimes \Gamma F_2] \\ H_2 &= [I \otimes (\Phi + \Gamma F_1) + L_2 \otimes \Gamma F_2]. \end{aligned}$$

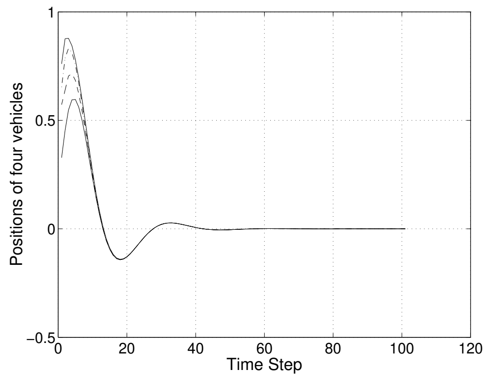

Figure 1: Formation stabilization problem for four vehicles corresponding to topology L_2, when each vehicle is stabilized using its own absolute measurements.

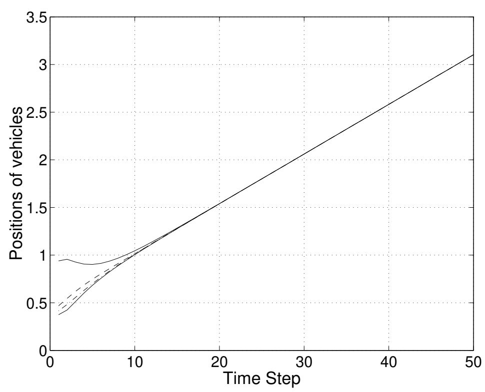

Figure 2: Formation stabilization problem for four vehicles corresponding to topology L_2, when individual vehicle is not stabilized using its own absolute measurements. We see that stability of each node using its own absolute measurements is a necessary condition for the formation stability.

Here the subscripts on the Laplacian L define the two Markov states. For the system matrices, the notation is similar to the above subsection. It is tempting to conjecture that the above matrices might have the same eigenvalues as smaller dimensional matrices where expressions of the form $[I \otimes (\Phi + \Gamma F_1) + L_1 \otimes \Gamma F_2]$ are replaced by matrices of the form $[(\Phi + \Gamma F_1) + \lambda F_2]$ with λ being an eigenvalue of L_1. However this is a false conjecture, as can be seen from the following very simple counterexample of a simple integrator system with h being equal to 1. Let the transition matrix Q be given by

$$\begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}.$$

We consider a two state Markov chain with the graph Laplacian matrices given as follows,

$$\begin{aligned} L_1 &= \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \\ L_2 &= \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}. \end{aligned}$$

It can be easily verified that 3 is an eigenvalue of L_1 and 2 is an eigenvalue of L_2. But contrary to the conjecture, the matrix

$$\begin{bmatrix} q_{11}(1-3)^2 & q_{21}(1-2)^2 \\ q_{12}(1-3)^2 & q_{22}(1-2)^2 \end{bmatrix}$$

shares no eigenvalues with the matrix

$$\begin{bmatrix} q_{11}A_1 & q_{21}A_2 \\ q_{12}A_1 & q_{22}A_2 \end{bmatrix},$$

where

$$A_1 = (I - hL_1) \otimes (I - hL_1)$$

and

$$A_2 = (I - hL_2) \otimes (I - hL_2).$$

Hence this is a counter-example to the proposed result. However, we still obtain a result for stability for the case when the formation is switching according to a Markov chain. Conclusions similar to the previous section can be drawn.

Continuing with our example of double integrator dynamics, we can see that the transition probability matrix begins to play an important role. Suppose the Markov chain consists of two states. The first state is represented by the Laplacian matrix L_1 and the second by L_2. The first state applies the control law matrix $F_1 = \begin{bmatrix} -1 & -1 \end{bmatrix}$ while the second state applies $F_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}$. The matrix F_2 for both formations is given by $F_2 = \begin{bmatrix} -1 & -1 \end{bmatrix}$. Thus we know that the first state is stable and the second is unstable. Then the stability of the Markov chain depends on the transition probability matrix Q. As particular examples, if we consider

$$Q = \begin{bmatrix} 0.7 & 0.3 \\ 0.9 & 0.1 \end{bmatrix},$$

the largest eigenvalue of the stability matrix turns out to have magnitude 0.8512; while with

$$Q = \begin{bmatrix} 0.1 & 0.9 \\ 0.05 & 0.95 \end{bmatrix},$$

it turns out to be 1.018. Thus merely by changing Q we can make the chain unstable. As dicussed before, changing Q physically corresponds to changing the time which the formation spends, on an average, in the unstable state.

5.3 Transformation of One Formation into Another

Again our analysis follows the same path as in the previous section. We consider only one intermediate formation and consider a transition probability matrix of the form

$$Q = \begin{bmatrix} (1-p_1) & p_1 & 0 \\ 0 & (1-p_2) & p_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix to be checked for stability would again be a lower block triangular matrix. Thus we would need to check the eigenvalues of the matrices $(1-p_1)H_1 \otimes H_1$, $(1-p_2)H_2 \otimes H_2$ and $H_3 \otimes H_3$, where

$$H_i = [I \otimes (\Phi + \Gamma F_1) + L_i \otimes \Gamma F_2].$$

We again note that a sufficient condition for stability is that all the intermediate states be individually stable, but that this condition is not necessary. As an example of formation change consider again the formation of 4 vehicles, each with double integrator dynamics. We consider two chains $L_1 \to L_2 \to L_3$ and $L_2 \to L_1 \to L_3$. The transition probability matrix Q in both cases is given by

$$Q = \begin{bmatrix} 1-p_1 & p_1 & 0 \\ 0 & 1-p_2 & p_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We consider p_1 as 0.33 and p_2 as 0.35. Furthermore, let the control law matrix F_1 for L_1 and L_2 be given by $F_1 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$ while for L_3, let it be given by $F_1 = \begin{bmatrix} -1 & -1 \end{bmatrix}$. F_2 is given by $\begin{bmatrix} -1 & -1 \end{bmatrix}$ for all the three formations. Thus L_1 and L_2 are both unstable, while L_3 is stable. However, chain $L_1 \to L_2 \to L_3$ turns out to be stable while the chain $L_2 \to L_1 \to L_3$ does not. Thus the order of formations encountered in the chain plays an important role.

We now wish to obtain a characterization for all the intermediate states to be stable in terms of some physical parameter like the connectivity. We note the following result.

Proposition 7 *A sufficient condition for the matrix $(I \otimes (\Phi + \Gamma F_1) + L \otimes \Gamma F_2) \otimes (I \otimes (\Phi + \Gamma F_1) + L \otimes \Gamma F_2)$ to be stable is*

$$\sigma_{max}(\Phi + \Gamma F_1) + \rho(L)\sigma_{max}(\Gamma F_2) < 1.$$

Here $\rho(A)$ is the spectral radius of A and $\sigma_{max}(A)$ is its spectral norm.

Proof: Consider the matrix H defined by $[I \otimes (\Phi + \Gamma F_1) + L \otimes \Gamma F_2]$. We know its eigenvalues are the same as the matrix $\Phi + \Gamma F_1 + \lambda \Gamma F_2$ where λ is an eigenvalue of the matrix L. Now, by definition, the spectral norm satisfies the property $\sigma_{\max}(A+B) \leq \sigma_{\max}(A) + \sigma_{\max}(B)$. Thus the norm of $\Phi + \Gamma F_1 + \lambda \Gamma F_2$ is less than the sum of norms of the matrices $\Phi + \Gamma F_1$ and $\lambda \Gamma F_2$. Since λ is a scalar, the norm of $\lambda \Gamma F_2$ is given by $|\lambda|$ times the norm of matrix ΓF_2. Thus we obtain that if $\sigma_{\max}(\Phi + \Gamma F_1) + |\lambda_{\max}(L)|\eta(\Gamma F_2) < 1$ with $\lambda_{\max}(L)$ denoting the eigenvalue of L with the highest absolute value, the norm of the matrix $\Phi + \Gamma F_1 + \lambda \Gamma F_2$ is also less than 1. But, note that spectral radius of a matrix is always less than the spectral norm. Thus this fact implies that this matrix is also stable. This, in turn implies that the matrix $[I \otimes (\Phi + \Gamma F_1) + L \otimes \Gamma F_2]$ is stable. Finally we note that the eigenvalues of a matrix $A \otimes B$ are simply the products of the various eigenvalues of A and B. Thus the matrix $[I \otimes (\Phi + \Gamma F_1) + L_1 \otimes \Gamma F_2] \otimes [I \otimes (\Phi + \Gamma F_1) + L_1 \otimes \Gamma F_2]$ would be stable as well. Also note that $|\lambda_{\max}(L)|$ is $\rho(L)$ by definition. ■

Using this theorem and the fact that $\rho(L)$ for a Laplacian matrix is always bounded by $2O_{\max}$, $O_{\max}$ denoting the maximum outdegree among the nodes, we get the following sufficient condition for stability of a formation chain.

Proposition 8 *Consider all the formations in the chain. If $O_{max,i}$ denotes the maximum out-degree of the nodes in the i-th formation, and O_{max} denotes the maximum value among all the $O_{max,i}$'s, then $\sigma_{max}(\Phi + \Gamma F_1) + 2 \times O_{max} \times \sigma_{max}(\Gamma F_2) < 1$ is a sufficient condition for stability of the formation change. An even looser condition for stability is that the maximum out-degree of the nodes should not increase, but the graph should remain connected and the condition $\sigma_{max}(\Phi + \Gamma F_1) + 2 \times O_{max,i} \times \sigma_{max}(\Gamma F_2) < 1$ should be satisfied for the initial and last formations.*

Remarks:

- Note that there is no particular reason to use the 2-norm of the matrices. The properties of the norm that were used follow mostly from the definition. The only extra fact used was that the spectral radius is less than the norm, which holds for other norms as well. So, the norm indicated can be any other norm, e.g., the 1-norm or ∞-norm.

- Also note that we have given only a sufficient condition, so it can be expected to be more strict than necessary. As an example, let us apply this condition to the single integrator plant. There, Φ is unity, Γ is $-h$, F_1 is 0 and F_2 is also unity. We immediately see that the conditions imposed by the theorem are not satisfied and we cannot say anything about the stability of the formation. However if we also introduce F_1 equal to the maximum eigenvalue of the Laplacian, we can easily ensure that the conditions of the theorem are satisfied and we obtain a provably stablizing control law. Thus, the theorem is not too restrictive. In some cases, like the double integrator dynamics, the theorem merely provides an indication of the general trend. Figure 3 shows the case of a formation corresponding to L_3 in which the vehicles apply $F_1 = \begin{bmatrix} \frac{-1}{h^2}, & \frac{1}{h^2} - \frac{2}{h} \end{bmatrix}$ and $F_2 = \begin{bmatrix} -1 & -1 \end{bmatrix}$. The quantity $\sigma_{\max}(\Phi + \Gamma F_1) + \rho(L)\sigma_{\max}(\Gamma F_2)$ never goes below unity even though the spectral radius of the matrix $(I \otimes (\Phi + \Gamma F_1) + L \otimes \Gamma F_2)$ does. However as a general trend the two quantities stay close to each other. For other dynamics, it is, in general, possible to satisfy the sufficiency condition given above and thus the test is even more useful.

5.4 Same Delay in All Communication Links

As discussed above, the extension to the case of non-ideal communication links can easily be done. We again start with the case where all the links in the network introduce the same delay.

Time delay constant with time and equal to one time step: Consider first the case when the time delay is exactly one time step. We obtain for each node,

$$X_i[k+1] = \Phi X_i[k] + \Gamma U_i[k],$$

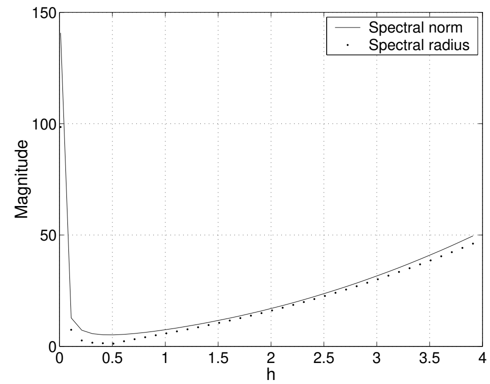

Figure 3: The spectral norm and the spectral radius for the four-vehicle formation corresponding to L_3 and the control law matrices $F_1 = \begin{bmatrix} \frac{-1}{h^2} & \frac{1}{h^2} - \frac{2}{h} \end{bmatrix}$, $F_2 = \begin{bmatrix} -1 & -1 \end{bmatrix}$. We see that the spectral norm is a good indicator of the general trend in spectral radius.

where the control law $U_i[k]$ is given by

$$U_i[k] = F_1 X_i[k] + F_2 \left(\sum_{j \in \mathcal{N}_i^{\text{out}}} \left(X_i[k-1] - X_j[k-1] \right) \right).$$

Note that since the relative measurements available at the controller are delayed, the averaging is done for delayed measurements. For the whole system, we thus obtain

$$\begin{bmatrix} X[k+1] \\ X[k] \end{bmatrix} = K \begin{bmatrix} X[k] \\ X[k-1] \end{bmatrix},$$

where the matrix K which should be checked for stablity is

$$\begin{bmatrix} I \otimes (\Phi + \Gamma F_1) & (L \otimes \Gamma F_2) \\ I & 0 \end{bmatrix}.$$

We have the following result.

Proposition 9 *The matrix*

$$\begin{bmatrix} I \otimes A & L \otimes B \\ I & 0 \end{bmatrix}$$

has the same eigenvalues as those of the smaller dimensional matrices

$$\begin{bmatrix} A & \mu B \\ I & 0 \end{bmatrix},$$

where μ is an eigenvalue of L.

Proof: Let V be an eigenvector of the matrix L corresponding to the eigenvalue μ. Consider the matrix

$$\begin{bmatrix} A & \mu B \\ I & 0 \end{bmatrix}.$$

It is obvious that the eigenvector corresponding to the eigenvalue λ of this matrix would be of the form

$$\begin{bmatrix} \lambda U \\ U \end{bmatrix},$$

where U satisfies the following equation

$$\lambda AU + \mu BU = \lambda^2 U.$$

Now we claim that

$$\begin{bmatrix} \lambda(V \otimes U) \\ (V \otimes U) \end{bmatrix}$$

is an eigenvector of the matrix

$$\begin{bmatrix} I \otimes A & L \otimes B \\ I & 0 \end{bmatrix}$$

with eigenvalue λ. For this to be true, the equation that has to hold is

$$(I \otimes A)(\lambda V \otimes U) + (L \otimes B)(V \otimes U) = \lambda^2(V \otimes U).$$

Using the rules of Kronecker product algebra and the fact that V is an eigenvector of L with eigenvalue μ, we immediately see that this equation reduces to

$$\lambda V \otimes AU + \mu V \otimes BU = \lambda^2(V \otimes U).$$

The left hand side can be rewritten as $V \otimes (\lambda AU + \mu BU)$ which is equal to $V \otimes \lambda^2 U$ or to $\lambda^2 V \otimes U$. Thus the required equation holds and our claim is true. ∎

In our case, this theorem means that we need to look only at the stability of the matrix

$$\begin{bmatrix} \Phi + \Gamma F_1 & \lambda \Gamma F_2 \\ I & 0 \end{bmatrix},$$

where λ is an eigenvalue of the Laplacian matrix L. Let us apply this critierion to our example of double integrator dynamics with the three formations L_1, L_2 and L_3 which were stable using the control law matrices F_1 and F_2 each equal to $\begin{bmatrix} -1 & -1 \end{bmatrix}$ when no delay was present. We find that only L_1 remains stable in the presence of a time delay equal to one time step. This is in accordance with the general trend that more highly connected a formation is, the harder it is to stabilize it.

Formation switching with time delay of one time step: This case can also be easily derived along the lines of the previous section. Instead of going through the formal details, let us work through an example. We again consider 4 vehicles each modeled by the double integrator dynamics. Suppose the vehicles switch between two formations L_1 and L_2. The switch is modeled as a Markov chain process with the transition probability matrix

$$\begin{bmatrix} q_1 & (1-q_1) \\ (1-q_2) & q_2 \end{bmatrix}.$$

The time step h is taken to be 0.23, and a time delay equal to one time step assumed in each communication link. This can also model a situation where the time delay is bounded by one time step but the arriving information is delayed so we always get a constant time delay. The control law matrices are taken to be $F_1 = F_2 = \begin{bmatrix} -1 & -1 \end{bmatrix}$. The matrix to be checked for stability is thus a 32×32 matrix, and the instability region is shaded in the following figure on the q_1 - q_2 plane.

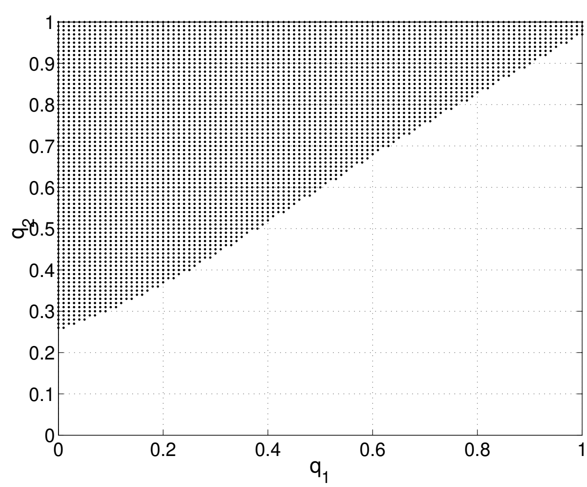

Figure 4: Formation switching with delay. Instability region is shown as shaded. $(1-q_1)$ is the probability of formation switching from L_1 to L_2, while $(1-q_2)$ the probability for switching from L_2 to L_1.

Constant time delay not exactly one time step: Let us now generalize to the case where the links have the same delay, but not exactly one time step. As in the above section, for simplicity, assume the delay to be less than one time step. Although the technique is similar to what we have already discussed for the single integrator case, we present the results so as to introduce some terminology which will be used in the next subsection.

Let h be the duration of the time step and d be the delay. For each node, we obtain

$$\begin{aligned} X_i[k+1] =& X_i[k] + \Gamma F_1 X_i[k] \\ &+\Gamma_1 F_2 \left[\sum_{j\ \in\ \mathcal{N}_i^{\text{out}}} (X_i[k-1] - X_j[k-1]) \right] \\ &+ \Gamma_2 F_2 \left[\sum_{j\ \in\ \mathcal{N}_i^{\text{out}}} (X_i[k] - X_j[k]) \right]. \end{aligned}$$

Assuming that we are using zero-order hold, the matrices Γ_1 and Γ_2 are given by

$$\begin{aligned} \Gamma_1 &= \int_{h-d}^{h} e^{As} ds B \\ \Gamma_2 &= \int_{0}^{h-d} e^{As} ds B. \end{aligned}$$

For the whole system, we thus obtain

$$\left[\begin{array}{c} X[k+1] \\ X[k] \end{array} \right] = K \left[\begin{array}{c} X[k] \\ X[k-1] \end{array} \right],$$

where the matrix K that needs to be checked for stability is equal to

$$\left[\begin{array}{ll} I \otimes (\Phi + \Gamma F_1) + L \otimes (\Gamma_2 F_2) & L \otimes (\Gamma_1 F_2) \\ I & 0 \end{array} \right].$$

Extensions to the Markov case and to the case when delay is bounded by n time steps are easily done. Let us use this result to study the effect of increasing the time delay in the communication links corresponding to the formation L_3. We know that the formation is stable using control laws $F_1 = F_2 = \left[\begin{array}{cc} -1 & -1 \end{array} \right]$ when no delay is present, and unstable when delay equal to one time step is present. The plot for the spectral radius of the stability matrix as the delay d is increased is shown in Figure 5. The x-axis shows the value of the delay while the yaxis shows the spectral radius.

5.5 Different Delays in Communication Links

The case of different delays is slightly trickier than the single integrator case. We need a generalization not of the Laplacian matrix L but rather of the Kronecker products $L \otimes (\Gamma_1 F_2)$ and $L \otimes (\Gamma_2 F_2)$. In the single integrator case, we could get away with simply generalizing L since $\Gamma_1 F_2$ and $\Gamma_2 F_2$ were scalar terms. However this is not the case here.

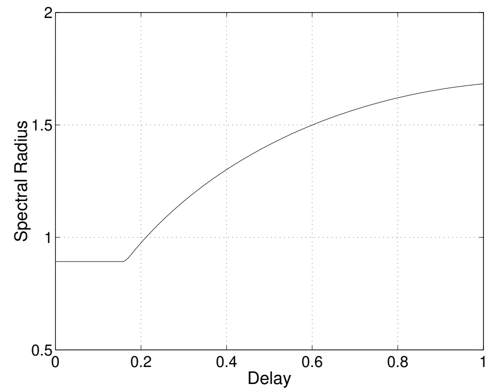

Figure 5: Spectral radius of the stability matrix for a formation of four vehicles corresponding to the topology L_3 as a function of the delay present in communication links. We see that as we increase the delay in the links, the formation tends to become unstable.

We first define Γ_1 and Γ_2 for each link between node i and node j by applying the definition and using the actual value of delay for the link in the definition. Thus we get the matrices Γ_1^{ij} and Γ_2^{ij} for each link from the node j to node i. We then calculate $\Gamma_1^{ij} F_1$ and $\Gamma_2^{ij} F_2$ for each link and finally define matrices L_{gen}^1 and L_{gen}^2 as follows. Create the adjacency matrix. Now for L_{gen}^1, replace each nonzero (i,j)th element by the matrix $-\Gamma_1^{ij} F_2$. Also replace the zero element on the diagonal by a matrix which is the sum of matrices $\Gamma_1^{ij} F_2$ over all j. The remaining zero elements would need to be replaced by zero matrices of the appropriate dimensions. This would yield the L_{gen}^1 matrix. L_{gen}^2 can be obtained by a similar procedure using $-\Gamma_2^{ij} F_1$. Thus we would obtain the following equation for the time evolution of the system.

$$\left[\begin{array}{c} X[k+1] \\ X[k] \end{array} \right] = K \left[\begin{array}{c} X[k] \\ X[k-1] \end{array} \right],$$

where the matrix K is equal to

$$\left[\begin{array}{ll} I \otimes (\Phi + \Gamma F_1) + L_{gen}^2 & L_{gen}^1 \\ I & 0 \end{array} \right].$$

Thus, stability analysis can easily be done.

5.6 Different Plant Models

In a practical scenario, all the individual vehicles will not be identical. There will be physical differences among them. An important extension of the theory we are developing would be to address these concerns.

5.6.1 Same state-space representation for every vehicle: Let us assume that each vehicle has the same state-space representation and that the differences lie in the system transition matrix Φ_i. The subscript i refers to the particular vehicle. We assume the same control laws as before, so we obtain for each node

$$X_i[k+1] = \Phi_i X_i[k] + \Gamma U_i[k].$$

For the whole system we thus obtain

$$X[k+1] = (\Phi + I \otimes \Gamma F_1 + L \otimes \Gamma F_2) X[k],$$

where Φ is a block diagonal matrix with the matrices Φ_i on the diagonal. We can immediately obtain a sufficient condition for stability.

Proposition 10 *A sufficient condition for the matrix $(\Phi + I \otimes \Gamma F_1 + L \otimes \Gamma F_2)$ to be stable is that $\|\Phi_i + \Gamma F_1\|_\infty + 2 \times O_{max} \times \|\Gamma F_2\|_\infty < 1$ for all vehicles Φ_i.*

Proof: We note that the ∞-norm of the matrix $(\Phi + I \otimes \Gamma F_1 + L \otimes \Gamma F_2)$ can be bounded as follows,

$$\begin{aligned}\|\Phi + I \otimes \Gamma F_1 + L \otimes \Gamma F_2\|_\infty \le & \|\Phi + I \otimes \Gamma F_1\|_\infty \\ & + \|L \otimes \Gamma F_2\|_\infty .\end{aligned}$$

Now, we note that $\|\Phi + I \otimes \Gamma F_1\|_\infty$ is bounded by $\|\Phi_i + \Gamma F_1\|_\infty$ for some vehicle Φ_i. Also $\|L\|_\infty$ is equal to $2O_{\max}$, $O_{\max}$ being the maximum outdegree among the nodes. Thus $\|L \otimes \Gamma F_2\|_\infty$ is always bounded by $2O_{\max}\|\Gamma F_2\|_\infty$. Thus we obtain

$$\begin{aligned}\|\Phi + I \otimes \Gamma F_1 + L \otimes \Gamma F_2\|_\infty \le & \|\Phi_i + \Gamma F_1\|_\infty \\ & + 2O_{\max}\|\Gamma F_2\|_\infty .\end{aligned}$$

Thus, if all the vehicles satisfy the condition that $\|\Phi_i + \Gamma F_1\|_\infty + 2 \times O_{\max} \times \|\Gamma F_2\|_\infty < 1$, we would obtain that the norm of the matrix $(\Phi + I \otimes \Gamma F_1 + L \otimes \Gamma F_2)$ is less than one and hence the matrix is stable. ∎

5.6.2 Different State-Space Representations for Different Vehicles: Suppose the various vehicles do not have the same state-space representation. In order to use concepts like relative measurement, the outputs of the system have to be the same. Thus the situation can be described by the following equations for the ith vehicle.

$$\begin{aligned}X_i[k+1] =& \Phi_i X_i[k] + \Gamma_i U_i[k] \\ Y_i[k] =& C_i X_i[k] \\ U_i[k] =& F_1 Y_i[k] + F_2 \left[\sum_{j \,\in\, \mathcal{N}_i^{\text{out}}} (Y_i[k] - Y_j[k]) \right].\end{aligned}$$

We express the control law in terms of state vector $X_i[k]$ to obtain

$$\begin{aligned}X_i[k+1] =& \Phi_i X_i[k] + \Gamma_i F_1 C_i X_i[k] \\ & + \Gamma_i F_2 \left[\sum_{j \,\in\, \mathcal{N}_i^{\text{out}}} (C_i X_i[k] - C_j X_j[k]) \right].\end{aligned}$$

We define the matrix $\Phi + \Gamma F_1 C$ to be a block diagonal matrix with matrices $\Phi_i + \Gamma_i F_1 C_i$ on the diagonal, as in the above subsubsection. Also we define the matrix L_{gen} by the method described in subsection 5.5 using the matrices $\Gamma_i F_2 C_j$ in place of $\Gamma_{ij} F_2$. Thus we would obtain for the whole system, the following equation.

$$X[k+1] = (\Phi + \Gamma F_1 C + L_{gen}) X[k].$$

This represents, in some sense, the most general case.

Remarks:

- This is the first time we have introduced in our formulation the notion of output as being different from the state vector. It is fairly obvious that if all the plants are the same but we do not have access to the full state, we can use outputs and state-observers. The method would still work given the usual constraints of observability.
- A slightly different notion is that of same plants but where we want only a subset of the state components to achieve a formation. This can arise, e.g., if the state includes linear motion as well as the angular motion, but for the formation purposes we are concerned with only the linear motion components. Then we can set C_i to be a zero matrix with unity on the diagonal for the components we are interested in. We can follow the above procedure again. Of course, in this case, the matrices $\Phi + \Gamma F_1 C$ and L_{gen} would be expressible in terms of Kronecker products as usual. In the specification of the formation vector, we can choose the values of the non-wanted components of the state arbitrarily.

6 Susceptibility to Noise

In practice, noise would be present in the state information passed to each node. This can be due to the fact that position estimates are obtained using a technology like GPS. Thus there is always a random noise component present. Or the noise may be due to the additive noise in the communication channel. We wish to find out whether our algorithms would hold in that case. We find immediately that we have been always able to reduce the whole system into a linear system of the form $X[k] = \Phi X[k-1]$, where $X[k]$ is obtained by stacking all the states of every node at time step k or additionally also at time steps $(k-1)$ etc. If the noise is additive, the only change in the above equation would be that $X[k-1]$ would be replaced by a corrupted estimate $X[k-1]+\Delta$, where Δ denotes the additive noise component. Thus we would still obtain the system equation as $X[k] = \Phi X[k-1] + \Phi\Delta$. Obviously as long as the noise is bounded, the system still remains stable, if it was stable for the noiseless case.

7 Conclusions and Future Work

In this paper we looked at the formation stabilization problem and the consensus problem for a group of inter-connected agents. We presented an analysis method for such a system and used it to look at several cases of different plant dynamics and channel characteristics. We came up with sufficient conditions for stability which could be checked locally at each node with access only to local information. We also looked at the problem of changing a formation into another and came up with conditions under which it can be done. We showed that the information topology plays an important role. In particular the number of agents supplying information to any node is an important parameter.

This work can be extended in several ways. We can look at the case of channel being modeled as a potentially time-varying filter and obtain its effect on stability. Another important direction is to start looking at the performance analysis question and synthesize some sort of optimal control law. It is not yet obvious what a suitable cost function should be. Also questions like how to put constraints on topology, whether the control law can be synthesized separately from the topology and how to deal with bandwidth constraints of practical channels are issues to be considered.

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